

Solving the three-dimensional Black-Scholes equation using a finite different method

Xi Chen

Dissertation submitted for the MSc in Mathematical Finance

Department of Mathematics

University of York

11 September 2020

Supervisor: Doctor Christian Litterer

Contents

1	Introduction	3
2	Multi-dimensional Black-Scholes Model	3
2.1	The stochastic calculus	3
2.2	Black-Scholes Model[1]	5
3	Black-Scholes Partial Differential Equation[5]	8
3.1	One dimension	8
3.2	Three dimension	9
4	Numerical Method	11
4.1	Discretization	11
4.2	Explicit and Implicit Euler method[17, 2]	12
4.3	Operating Splitting Method	12
4.4	Model modification	13
5	Mathematical Algorithm	15
5.1	Thomas Algorithm[7]	16
5.2	The implementation in C++[3, 4]	17
6	Numerical experiment	18
6.1	Example1:European Call Option	18
6.2	Example2:Basket Option	19
7	Conclusion	20

1 Introduction

Since Black and Scholes[9], and Merton[10] firstly introduce the Black-Scholes(BS) model, which has been wide-spread used in financial engineering to price equity-linked securities (ELS). This kind of financial derivatives has become more and more popular in the financial market, and its return is linked to the performance of the underlying assets[13, 14]. But, by the derivatives becoming more and more complex, the simple Black-Scholes Model is hard to apply.

Now there are two broad categories on numerical methods for financial derivatives. The one is probabilistic methods based on Monte-Carlo simulations, which generates enough samples randomly based on a specific probability model. This feature is from the law of large number, which ensures that the estimation will more converge to the correct as of the number of samples increases[11]. However, the original Monte-Carlo method cannot achieve a high level of accuracy within a reasonable amount of time. Another disadvantage is that the method always yields an error bound [12].

Another one is the method for solving PDEs. Usually, the method is finite difference method(FDM) like alternating direction implicit(ADI) and operator splitting method (OSM), which has been extensively used in practice. Compared with Monte-Carlo method, it can more quickly converge to a more stable value, but it also needs lots of computational sources and mathematical calculations[3].

For multi-dimensional models, this dissertation is to present and implement a practical FDM in the framework of three-dimensional Black-Scholes Partial Differential Equation(PDE).

2 Multi-dimensional Black-Scholes Model

This section is to present part of necessities mathematical tools of stochastic *Itô* stochastic, and then we will introduce the multi-dimensional Black-Scholes model.

2.1 The stochastic calculus

First, in the BS model, there are two important random variables-stochastic process and Brownian Motion, which are the base of the model. The following are about their simple concepts.

Definition 2.1 (Stochastic process). On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a parametrised family of random variables

$$\{X_t\}_{t \in S}$$

take values in \mathbb{R} , where S is either $S = [0, T], T > 0$ or $S = [0, \infty)$.

Definition 2.2 (Brownian Motion). For a stochastic process $(W_t)_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if it satisfies the following properties

1. $W_0 = 0$ a.s.
2. Independent increments: the Following the random variables (r.v.)

$$W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \dots, W_{t_k} - W_{t_{k-1}}$$

are independent for all $k \in \mathbb{N}$ and all $0 = t_0 < t_1 < \dots < t_k$.

3. for all $t > s \geq 0$, $W_t - W_s \sim N(0, t - s)$
4. The trajectories are continuous a.s.

Then We will introduce the *Itô* lemma, which is the core of Black-Scholes Model, the following is the necessary tools.

Definition 2.3. For a stochastic process $M = (M_t)_{t \geq 0}$, if it satisfies

1. M is adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$
2. $\mathbb{E}(|M_t|) < \infty$ for all $t \geq 0$
3. $\mathbb{E}(M_t | \mathcal{F}_s) = M_s$ a.s. for all $0 \leq s < t$.

Definition 2.4. If an *Itô* process with characteristics a and b defined $(\xi(t))_{t \geq 0}$ is a stochastic process such that

1. $(\xi(t))_{t \geq 0}$ is adapted to $(\mathcal{F}_t)_{t \geq 0}$
2. $(\xi(t))_{t \geq 0}$ has continuous or piecewise continuous trajectories a.s.
- 3.

$$\xi(t) = \xi(0) + \int_0^t a(s) ds + \int_0^t b(s) dW(s), t \geq 0,$$

where

$$\mathbb{E} \int_0^T |a(s)| ds < \infty, \mathbb{E} \int_0^T |b(s)|^2 ds < \infty$$

for any $T > 0$

Definition 2.5. [Itô Lemma in simple form] $f(t, x)$ is in $C^{1,2}(\mathbb{R}^2)$, i.e. so that it has the derivatives $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$ which are continuous. We define a process η by

$$\eta(t) = f(t, W(t)), t \geq 0.$$

For $t \geq 0$, then $\eta(t)$ is an *Itô* process and

$$\eta(t) = f(0, 0) + \int_0^t \left[\frac{\partial f}{\partial s}(s, W(s)) + 1/2 \frac{\partial^2 f}{\partial x^2}(s, W(s)) \right] ds + \int_0^t \frac{\partial f}{\partial x}(s, W(s)) dW(s)$$

From the definitions 2.4 and 2.5, we can obtain a general form of Itô lemma.

Definition 2.6 (Itô lemma in general form). the Itô process $(\eta(t))_{t \geq 0}$ has characteristics $(a(t))_{t \geq 0}$ and $(b(t))_{t \geq 0}$, i.e.

$$\eta(t) = \eta(0) + \int_0^t a(s)ds + \int_0^t b(s)dW(s) \quad a \in M_{loc}^1(0, \infty) \quad b \in M_{loc}^2(0, \infty).$$

Same with the definition 2.5, let $f(t, x)$ is in $C^{1,2}(\mathbb{R}^2)$, i.e. Its derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ exist and are continuous. Then, define a process $\eta(t)$ by

$$\eta(t) := f(t, \xi(t)), t \geq 0.$$

For $t \geq 0$, then $\eta(t)$ is an Itô process and

$$\begin{aligned} \eta(t) = f(0, 0) &+ \int_0^t \left[\frac{\partial f}{\partial s}(s, \xi(s)) + \frac{\partial f}{\partial s}(s, \xi(s))a(s) \right. \\ &\left. + 1/2 \frac{\partial^2 f}{\partial x^2}(s, \xi(s))b^2(s) \right] ds + \int_0^t \frac{\partial f}{\partial x}(s, \xi(s))b(s)dW(s) \end{aligned}$$

We can also say $\eta(t)$ is an Itô process with characteristics

$$\begin{aligned} \tilde{a}(s) &= \frac{\partial f}{\partial t}(s, \xi(s)) + \frac{\partial f}{\partial x}(s, \xi(s))a(s) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, \xi(s))b^2(s) \\ \tilde{b}(s) &= \frac{\partial f}{\partial x}(s, \xi(s))b(s). \end{aligned}$$

2.2 Black-Scholes Model[1]

Folowing we will introduce the simple Black-Scholes Model and extend it to multi-dimensional BS Model. This part is the fundament of the Black-Scholes partial difference equation.

Assumption 2.7. We assume that there are two kinds of underlying securities in the financial market.

- The **risk-free** asset (money -market account) with a deterministic function

$$dB(t) = rB(t)dt \quad (2.1)$$

which provide the risk-free rate r .

- The **risky** assets $S_i (i = 1, 2, \dots)$, which satisfy the the Black-Scholes model

$$dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i(t)dW_i(t), \quad i = 1, 2, 3, \dots \quad (2.2)$$

where μ_i, σ_i and $W_i(t)$ are expected instantaneous rate of return, constant volatility and standard Brownian Motion on the underlying asset S_i , respectively. The initial price $S_0 > 0$.

For simplifying the model, we also make some assumption.

Assumption 2.8. *Generally, we assume we have a frictionless market, so that*

1. *Both of assets above can be traded in arbitrary amounts with no transaction costs.*
2. *We can hold a short position, which means we can invest in, or borrow from, the riskless account at risk-free rate r .*

The core idea of building a framework portfolio arbitrage model is to construct an asset portfolio so that the risk of the option can be fully hedged at maturity and then the option is price of the asset portfolio. Then we should construct a risk-free arbitrage.

Definition 2.9 (Trading strategy). (ψ, φ) is a pair adapted processes, and both have continuous (or piecewise continuous) trajectories a.s., such that for each $T > 0$

$$\int_0^T |\psi(s)| ds + \int_0^T |\varphi(s)|^2 ds < \infty \quad (2.3)$$

Then, the corresponding wealth process $W(t)$, $t \geq 0$ of this portfolio, is defined by

$$X(t) := \psi(t)B(t) + \varphi(t)S(t), \quad t \geq 0; \quad (2.4)$$

if the corresponding wealth process satisfies

$$dX_t = \psi(t)dB_t + \varphi(t)dS_t, \quad t \geq 0 \quad (2.5)$$

The value of strategy (x_i, y) is the process $V_{x_i, y} = (V_{x_i, y}(t))_{t \in [0, T]}$ defined by

$$V_{x_i, y}(t) = x_i(t)dS_i(t) + y(t)dA(t), \quad i = 1, 2, 3, \dots \quad (2.6)$$

we can simply write $V(t)$, when the mea, and then of strategy is clear.

Definition 2.10. A strategy (x_i, y) replicates the derivative security with payoff $H = V_{x_i, y}(T)$. And the market model is complete if every \mathcal{F}_T^S -measurable random variable H can be replicated.

Theorem 2.11. *If H is replicated by $(x_i(t), y(t))$ and its prices process $H(t)$ is an Itô process, the No arbitrage principle implies that for all t in $[0, T]$, $H(t) = V_{x_i, y}(t)$*

Now, Extending the model to multi-dimensional model

Definition 2.12. [Multi-dimension] We assume a vector of Wiener processes $W = (W_1, \dots, W_n)$ defined on a product space $\Omega = \Omega_1 \times \dots \times \Omega_n$.

Then in the muliti-dimensional Black-Scholes market, there are the **risk-free asset**

$$dB(t) = rB(t)dt, \quad t \geq 0$$

with $B(0) > 0$, where $r > 0$ is also the risk-free and **n risky assets**

$$dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i(t)dW_i(t), \quad i = 1, \dots, n.$$

Again, we will constuct a self-financing trading strategy

Assumption 2.13. *We assume that the adapted strategy process*

$$(\psi(t), \varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))$$

satisfies

$$V(t) = \psi(t)dB(t) + \sum_{i=1}^n \varphi_i(t)dS_i(t)$$

where $V(t)$ is the wealth process, and it also satisfies

$$dV(t) = \psi(t)dB(t) + \sum_{i=1}^n \varphi_i(t)dS_i(t).$$

Correspondingly, for building three-dimensional BSPDE, we also need to develop multi-dimensional Itô lemma.

Proposition 2.14. *[multi-dimensional Itô lemma in general form] Taking integrable processes a_i and a matrix $[b_{ij}(t)]$, $i = 1, \dots, k$, $j = 1, \dots, n$ of \mathcal{P}^2 -processes, all with continuous paths. In addition, we define a multi-dimensional Itô processes $X(t) = (X_1(t), X_2(t), \dots, X_k(t))$, which satisfies*

$$dX_i = a_i dt + \sum_{j=1}^n b_{ij} dW_j, i = 1, \dots, k$$

If $F : [0, \infty) \times \mathbb{R}^k \rightarrow \mathbb{R}$ with first variable in class C^1 and other variables in class C^2 . Then we define an Itô process $Y(t) = F(t, X(t))$ with

$$\begin{aligned} dY(t) = & F_t(t, X(t))dt + \sum_{i=1}^k F_{x_i}(t, X(t))a_i(t)dt \\ & + \sum_{i=1}^k F_{X_i}(t, X(t)) \sum_{j=1}^n b_{ij}(t)dW_j(t) \end{aligned}$$

The following theorem establish an important connection between SDE solutions and partial differential quation(in this paper, it means the second order, linear, parabolic PDEs).It will allow us to compute the expected value of a SDE solution function to solve a PDE.

Theorem 2.15 (Feynman-Kac formula). *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be given function and let X be an Itô process defined as the solution of the SDE*

$$dX(t) = a(t, X_t)dt + b(t, X_t)dW(t) \quad (2.7)$$

Let $F(t, X_t)$ be a funtion of two variables in $C^{1,2}$ that satisfies

$$\begin{aligned}\frac{\partial F}{\partial t}(t, x) &= -\frac{1}{2}b^2(t, x)\frac{\partial^2 F}{\partial x^2}(t, x) - a(t, x)\frac{\partial F}{\partial x}(t, x) \\ F(t, x) &= \phi(x)\end{aligned}$$

Moreover, we assume that

$$\frac{\partial F}{\partial x}(t, X(t))b(t) \in M_{loc}^2 \quad (2.8)$$

Then

$$F(t, X(t)) = \mathbb{E}(\phi(X(T))|\mathcal{F}_t), \quad (2.9)$$

where $(\mathcal{F}_t)_{t \geq 0}$ is any filtration such that $F(t, X(t))$ is \mathcal{F}_t -measurable for all $t > 0$ and the filtration $(\mathcal{F}_t)_{t \geq 0}$ can be enlarged to a filtration $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ s.t. W is Brownian Motion with respect to $(\tilde{\mathcal{F}}_t)_{t \geq 0}$

Remark 2.16. Feynman-Kac theorem guarantee that the price of derivatives can also be obtained by solving PDE although the stock price does not obey the brownian motion.

3 Black-Scholes Partial Differential Equation[5]

In this part, we will first discuss the one-dimensional PDE, and then extend it to three-dimensional one, which has a similar structure. In addition, we also use transfer the backwards-in-time PDE into forward-in-time PDE by using $\tau = T - t$

3.1 One dimension

Proposition 3.1 (one-dimensional Black-Scholes Partial Differential Equation). *Let $u(t, S(t))$ be a function of two variables in $C^{1,2}$, and it satisfies*

$$\begin{aligned}u_t(t, S(t)) &= -\frac{1}{2}\sigma^2 S^2 u_{zz}(t, z) - rzu_z(t, z) + ru(t, z) \quad \text{for } 0 < t < T, z \in \mathbb{R} \\ u(T, z) &= H(z)\end{aligned}$$

By using $\tau = T - t$, the forward-in-time PDE is:

$$\begin{aligned}u_\tau(\tau, S) &= \frac{1}{2}\sigma^2 S^2 u_{zz}(\tau, z) - rzu_z(\tau, z) + ru(\tau, z) \quad \text{for } 0 < \tau < T, z \in \mathbb{R} \\ u(0, z) &= H(z)\end{aligned}$$

Proof. First, the stock price satisfies the formula 4.6 and the option price $u(S_t, t)$ can be seen a function of the price of stock S_t and time t . By the Itô lemma,

$$du_{(S_t, t)} = \left(\frac{\partial u}{\partial t} + \mu S_t \frac{\partial u}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 u}{\partial S^2} \right) dt + \sigma S_t \frac{\partial u}{\partial S} dW_t \quad (3.1)$$

And according to the trading strategy, we construct a replicated strategy(x,y)

$$V(S_t, t) = xS_t + yu_{(S_t, t)} \quad (3.2)$$

where x and y are the number of stocks and the option, respectively. If there are no arbitragy opportunities existing , then it is a self-financing strategy

$$\begin{aligned} dV_{S_t, t} &= x dS_t + y du_{S_t, t} \\ &= x(\mu S_t dt + \sigma S_t dW_t) + y[(\frac{\partial u}{\partial t} + \mu S_t \frac{\partial u}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 u}{\partial S^2})dt + \sigma S_t \frac{\partial u}{\partial S} dW_t] \\ &= [x\mu S_t + y(\frac{\partial u}{\partial t} + \mu S_t \frac{\partial u}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 u}{\partial S^2})]dt + (x\sigma S_t + y\sigma S_t \frac{\partial u}{\partial S})dW_t \end{aligned}$$

Two assets with the same value at every moment in the future can hedge each other. The present value (current price) of such two assets at every moment must be equal, which means it is a risk-free arbitrage so that $\frac{dV}{V} = rdt$. Now, we need to cancel the item dW_t , and the strategy should satisfy $x\sigma S_t + y\sigma S_t \frac{\partial u}{\partial S} = 0$. Let $y = -1$, then $x = \frac{\partial u}{\partial S}$. Then the formula can be transformed into

$$\begin{aligned} dV &= [\mu S_t \frac{\partial C}{\partial S} - (\frac{\partial u}{\partial t} + \mu S_t \frac{\partial u}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 u}{\partial S^2})]dt \\ &= r(S_t \frac{\partial u}{\partial S} - u_{S_t, t})dt \end{aligned}$$

Then we arrange this equation

$$\frac{\partial u}{\partial t} = -\frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 u}{\partial S^2} r S_t \frac{\partial u}{\partial S} + r u_{S_t, t} \quad (3.3)$$

together with the condition $u(T, S) = H(S)$ and according to the Feynman-kac theorem, the proof is complete. \square

3.2 Three dimension

Proposition 3.2 (Multi-dimensional Itô lemma). *Define a multi-dimensional Itô process $X(t) = (X_1(t), \dots, X_k(t))$ and it satisfies*

$$dY(t) = a_i dt + \sum_{j=1}^n b_{ij} dW_j, \quad i = 1, \dots, k. \quad (3.4)$$

If $F : [0, \infty) \times \mathbb{R}^k \rightarrow \mathbb{R}$, of class of C^1 in the first variable and of class C^2

in the others, and if $Y(t) = F(t, X(t))$, then Y is an Itô process with

$$\begin{aligned} dY(t) = & F_t(t, X(t))dt + \sum_{i=1}^k F_{x_i}(t, X(t))a_i(t)dt \\ & + \sum_{i=1}^k F_{x_i}(t, X(t)) \sum_{j=1}^n b_{ij}(t)dW_j(t) \\ & + \frac{1}{2} \sum_{j=1}^n \sum_{l=1}^k F_{x_i, x_l}(t, X(t))b_{ij}(t)b_{lj}(t)dt \end{aligned}$$

Proposition 3.3 (Three-dimensional Black-Scholes PDE). *The multivariable Black-Scholes PDE formula has a similar structure with one-dimensional. According the Itô formula, the equation is*

$$\frac{\partial u(S, t)}{\partial t} + \sum_{i=1}^d rS_i \frac{\partial u(S, t)}{\partial S_i} + \frac{1}{2} \sum_{i=1}^3 \rho_i \sigma_i^2 S_i^2 \frac{\partial^2 u(S, t)}{\partial S_i^2} - ru(S, t) = 0 \quad (3.5)$$

$$u(S, T) = H(S) \quad (3.6)$$

where T is the expiry time and $u(S, t)$ is the value of the option on n underlying assets. In addition, σ_i is the volatility of asset i and r is the risk-free interest rate from 2.1

The proof is similar with the one-dimensional Black-Scholes PDE, so the following is the simple proof and other details is same with previous proof [?]

Proof. According to multi-dimensional Black-Scholes Model (definition 2.12), we construct a self-financing trading strategy (assumption 2.13) (x_1, x_2, x_3) . It satisfies

$$V(S_1^t, S_2^t, S_3^t, t) = \sum_{i=1}^3 x_i S_i^t + yu(S_1^t, S_2^t, S_3^t, t), \quad (3.7)$$

so V solves the Black-Scholes PDE. Then using the multi-dimensional Itô lemma (proposition 2.14)

$$\begin{aligned} & \frac{\partial V}{\partial t}(t, x_1, x_2, x_3) + r \sum_{i=1}^3 x_i \frac{\partial V}{\partial x_i}(t, x_1, x_2, x_3) \\ & + \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \rho_{ij} x_i x_j \frac{\partial^2 V}{\partial x_i \partial x_j}(t, x_1, x_2, x_3) - rV(t, x_1, x_2, x_3) \\ & = 0 \end{aligned}$$

Where we assume that $dW_i dW_j = \rho_{ij} dt$ for $i, j = 1, 2, 3$ and ρ_{ij} means the correlation between the two Brownian Motion W_i and W_j . Particularly, $\rho_{ij} = 1$ when $i = j$ and $\rho_{ij} = 0$ for $i \neq j$ if W_i and W_j are independent. \square

Remark 3.4. The multivariable Black-Scholes PDE is one special case of the general linear parabolic PDE in n dimensions ([5]), which is

$$\begin{aligned}\frac{\partial u}{\partial t} &= Lu \\ Lu &\equiv \sum_{i,j=1}^n a_{i,j}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^n b_j(x,t) \frac{\partial u}{\partial x_j} + cu\end{aligned}$$

4 Numerical Method

Duff considered the finite difference method as an numerical method[5]. Jeong, D. find an accurate and efficient numerical method for Black-Scholes method-operating splitting method(OSM) and use it to solve the two-dimensional Black-Scholes PDE [15, 16]. Jeong and Kim compared the alternating direction implicit(ADI) method and OSM in 2013[4]. Then Junseok Kim et al. also use OSM to solve three-dimensional Black-Scholes equation[3], we follow them and fix their mistakes.

4.1 Discretization

Let x, y and z be three underlying asset's prices and t be a time variable. For $(x, y, z) \in \Omega$ and $t \in T$, the option price $u(x, y, z, \tau)$ follow the Black-Scholes PDE, like the formula 3.5.

But we transfer the backward-in-time PDE into forward-in-time by using $\tau = T - t$, so that

$$u_\tau = rxu_x(x, y, z, \tau) + ryu_y(x, y, z, \tau) + rzu_z(x, y, z, \tau) \quad (4.1)$$

$$+ \frac{1}{2} \sigma_x^2 x^2 u_{xx}(x, y, z, \tau) + \frac{1}{2} \sigma_y^2 y^2 u_{yy}(x, y, z, \tau) + \frac{1}{2} \sigma_z^2 z^2 u_{zz}(x, y, z, \tau) \quad (4.2)$$

$$+ \rho_{xy} \sigma_x \sigma_y xy u_{xy}(x, y, z, \tau) + \rho_{yz} \sigma_y \sigma_z yz u_{yz}(x, y, z, \tau) + \rho_{zx} \sigma_z \sigma_x zx u_{zx}(x, y, z, \tau) \quad (4.3)$$

$$- ru(x, y, z, \tau), \quad (x, y, z, \tau) \in \omega \times (0, T] \quad (4.4)$$

and

$$u(x, y, z, 0) = \Phi(x, y, z)$$

Now, we define the domain as $\Omega = [0, L] \times [0, M] \times [0, N]$ and discretize it with positive non-uniform space step $h_{i-1}^x = x_i - x_{i-1}$, $h_{j-1}^y = y_j - y_{j-1}$ and $h_{k-1}^z = z_k - z_{k-1}$ ($x_0 = y_0 = z_0 = 0$, $x_{N_x} = L$, $y_{N_y} = M$ and $z_{N_z} = N$). Especially, we define $x_{N_x+1} = x_{N_x} + h_{N_x-1}$, $y_{N_y+1} = y_{N_y} + h_{N_y-1}$ and $z_{N_z+1} = z_{N_z} + h_{N_z-1}$.

In addition, a time step size is $\Delta\tau = T/N_\tau$. Then, the numbers of grid points in the x -, y -, z - and τ -directions are defined by N_x, N_y, N_z and N_τ , respectively[3].

4.2 Explicit and Implicit Euler method[17, 2]

In one dimensional Model, we need to discretise forumula (??) to compute $V(t_i, x_j)$. We can choose two different approximation using Taylor's expansion-Explicit and Implicit Euler method.

- The explicit method follow from Taylor's expansion of $V(t, x)$ at (t_i, x_j)

$$\begin{aligned}\frac{\partial V(t_i, x_j)}{\partial t} &\approx \frac{V_{i,j} - V_{i-1,j}}{\Delta t} \\ \frac{\partial V(t_i, x_j)}{\partial x} &\approx \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta x} \\ \frac{\partial^2 V(t_i, x_j)}{\partial x^2} &\approx \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta x^2}\end{aligned}$$

- The implicit method follow from Taylor's expansion of $V(t, x)$ at (t_{i-1}, x_j)

$$\begin{aligned}\frac{\partial V(t_{i-1}, x_j)}{\partial t} &\approx \frac{V_{i,j} - V_{i-1,j}}{\Delta t} \\ \frac{\partial V(t_{i-1}, x_j)}{\partial x} &\approx \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta x} \\ \frac{\partial^2 V(t_{i-1}, x_j)}{\partial x^2} &\approx \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta x^2}\end{aligned}$$

In this paer,we will combine the two methods and use semi-implicit scheme introudced in the following section

4.3 Operating Splitting Method

There, we can use the operator splitting method to divide each step into fractional time steps with simpler operator $\mathcal{L}_{BS}^x, \mathcal{L}_{BS}^y$ and \mathcal{L}_{BS}^z , which is

$$\frac{u_{ijk}^{n+1} - u_{ijk}^n}{\Delta \tau} = \mathcal{L}_{BS}^x u_{ijk}^{n+\frac{1}{3}} + \mathcal{L}_{BS}^y u_{ijk}^{n+\frac{2}{3}} + \mathcal{L}_{BS}^z u_{ijk}^{n+1} \quad (4.5)$$

This method has deduce the multi-dimensional equations into multi one-dimensional problems.

The scheme divide the time zone $(n, n+1]$ divide into $n, n + \frac{1}{3}$ and $n + \frac{2}{3}$ three time levelsl, and the operator can be defined by

$$\begin{aligned}\frac{u_{ijk}^{n+\frac{1}{3}} - u_{ijk}^n}{\Delta \tau} &= \mathcal{L}_{BS}^x u_{ijk}^{n+\frac{1}{3}} \\ \frac{u_{ijk}^{n+\frac{2}{3}} - u_{ijk}^{n+\frac{1}{3}}}{\Delta \tau} &= \mathcal{L}_{BS}^y u_{ijk}^{n+\frac{2}{3}} \\ \frac{u_{ijk}^{n+1} - u_{ijk}^{n+\frac{2}{3}}}{\Delta \tau} &= \mathcal{L}_{BS}^z u_{ijk}^{n+1}\end{aligned}$$

where,

$$\begin{aligned}\mathcal{L}_{BS}^x u_{ijk}^{n+\frac{1}{3}} &= \frac{(\sigma_x x_i)^2}{2} D_{xx} u_{ijk}^{n+\frac{1}{3}} + r x_i D_x u_{ijk}^{n+\frac{1}{3}} + \frac{1}{3} \sigma_x \sigma_y \rho_{xy} x_i y_j D_{xy} u_{ijk}^n \\ &\quad + \frac{1}{3} \sigma_y \sigma_z \rho_{yz} y_j z_k D_{yz} u_{ijk}^n + \frac{1}{3} \sigma_z \sigma_x \rho_{zx} z_k x_i D_{zx} u_{ijk}^n - \frac{1}{3} r u_{ijk}^{n+\frac{1}{3}}\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{BS}^y u_{ijk}^{n+\frac{2}{3}} &= \frac{(\sigma_y y_j)^2}{2} D_{yy} u_{ijk}^{n+\frac{2}{3}} + r y_j D_y u_{ijk}^{n+\frac{2}{3}} + \frac{1}{3} \sigma_x \sigma_y \rho_{xy} x_i y_j D_{xy} u_{ijk}^{n+\frac{1}{3}} \\ &\quad + \frac{1}{3} \sigma_y \sigma_z \rho_{yz} y_j z_k D_{yz} u_{ijk}^{n+\frac{1}{3}} + \frac{1}{3} \sigma_z \sigma_x \rho_{zx} z_k x_i D_{zx} u_{ijk}^{n+\frac{1}{3}} - \frac{1}{3} r u_{ijk}^{n+\frac{2}{3}}\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{BS}^z u_{ijk}^{n+1} &= \frac{(\sigma_z z_k)^2}{2} D_{zz} u_{ijk}^{n+1} + r z_k D_z u_{ijk}^{n+1} + \frac{1}{3} \sigma_x \sigma_y \rho_{xy} x_i y_j D_{xy} u_{ijk}^{n+\frac{2}{3}} \\ &\quad + \frac{1}{3} \sigma_y \sigma_z \rho_{yz} y_j z_k D_{yz} u_{ijk}^{n+\frac{2}{3}} + \frac{1}{3} \sigma_z \sigma_x \rho_{zx} z_k x_i D_{zx} u_{ijk}^{n+\frac{2}{3}} - \frac{1}{3} r u_{ijk}^{n+1}\end{aligned}$$

So that we take $\mathcal{L}_{BS}^x u_{ijk}^{n+\frac{1}{3}}$ as an example,

$$\begin{aligned}\frac{u_{ijk}^{n+\frac{1}{3}} - u_{ijk}^n}{\Delta\tau} &= \frac{(\sigma_x x_i)^2}{2} D_{xx} u_{ijk}^{n+\frac{1}{3}} + r x_i D_x u_{ijk}^{n+\frac{1}{3}} + \frac{1}{3} \sigma_x \sigma_y \rho_{xy} x_i y_j D_{xy} u_{ijk}^n \quad (4.6) \\ &\quad + \frac{1}{3} \sigma_y \sigma_z \rho_{yz} y_j z_k D_{yz} u_{ijk}^n + \frac{1}{3} \sigma_z \sigma_x \rho_{zx} z_k x_i D_{zx} u_{ijk}^n - \frac{1}{3} r u_{ijk}^{n+\frac{1}{3}} \quad (4.7)\end{aligned}$$

the others are same.

4.4 Model modification

In the reference [3], the author present a decent solution but there exists some minor problems, which will be solved in the following part. To distinguish the formula before and after correction, I will add note "(reference)" in the formula from reference.

Then, given u_{ijk}^n , we can rewritten the formula 4.6

$$\alpha_i u_{i-1,jk}^{n+\frac{1}{3}} + \beta_i u_{ijk}^{n+\frac{1}{3}} + \eta_i u_{i+1,jk}^{n+\frac{1}{3}} = f_{ijk} \quad (4.8)$$

Now, for solving yhe α_i , β_i and η_i , we use the semi-implicit method and the

discretization of the spatial variable in 4.6 will be

$$D_x u_{ijk} = -\frac{h_i^x}{h_{i-1}^x(h_{i-1}^x + h_i^x)} u_{i-1,jk} + \frac{h_i^x - h_{i-1}^x}{h_{i-1}^x h_i^x} u_{ijk} \quad (4.9)$$

$$+ \frac{h_{i-1}^x}{h_i^x(h_{i-1}^x + h_i^x)} u_{i+1,jk} \quad (4.10)$$

$$D_{xx} u_{ijk} = \frac{2}{h_{i-1}^x(h_{i-1}^x + h_i^x)} u_{i-1,jk} - \frac{2}{h_{i-1}^x h_i^x} u_{ijk} \quad (4.11)$$

$$+ \frac{2}{h_i^x(h_{i-1}^x + h_i^x)} u_{i+1,jk} \quad (4.12)$$

$$D_{xy} u_{ijk} = \frac{u_{i+1,j+1,k} - u_{i-1,j+1,k} - u_{i+1,j-1,k} + u_{i-1,j-1,k}}{h_i^x h_j^y + h_{i-1}^x h_j^y + h_i^x h_{j-1}^y + h_{i-1}^x h_{j-1}^y} \quad (4.13)$$

$$D_{yz} u_{ijk} = \dots \quad (4.14)$$

$$D_{zx} u_{ijk} = \dots \quad (4.15)$$

In the reference [3], the author give the formula are

$$\begin{aligned} \alpha_i(\text{reference}) &= -\frac{(\sigma_x x_i)^2}{h_{i-1}^x(h_{i-1}^x + h_i^x)} + r x_i \frac{h_i^x}{h_{i-1}^x(h_{i-1}^x + h_i^x)} \\ \beta_i(\text{reference}) &= \frac{(\sigma_x x_i)^2}{h_{i-1}^x h_i^x} - r x_i \frac{h_i^x - h_{i-1}^x}{h_{i-1}^x h_i^x} + \frac{1}{\Delta\tau} \\ \eta_i(\text{reference}) &= -\frac{(\sigma_x x_i)^2}{h_{i-1}^x(h_{i-1}^x + h_i^x)} - r x_i \frac{h_i^x}{h_i^x(h_{i-1}^x + h_i^x)} \\ f_{ijk}(\text{reference}) &= \frac{1}{3} \sigma_x \sigma_y \rho_{xy} x_i y_j D_{xy} u_{ijk}^n + \frac{1}{3} \sigma_y \sigma_z \rho_{yz} y_j z_k D_{yz} u_{ijk}^n \\ &\quad + \frac{1}{3} \sigma_z \sigma_x \rho_{zx} z_k x_i D_{zx} u_{ijk}^n - u_{ijk}^{n+\frac{1}{3}} \end{aligned}$$

but for the β_i and f_{ijk} , they exists mistakes

Proof. First, take the 4.9 into 4.6

$$\begin{aligned} \frac{u_{ijk}^{n+\frac{1}{3}} - u_{ijk}^n}{\Delta\tau} &= \frac{(\sigma_x x_i)^2}{2} \left(\frac{2}{h_{i-1}^x(h_{i-1}^x + h_i^x)} u_{i-1,jk} - \frac{2}{h_{i-1}^x h_i^x} u_{ijk} + \frac{2}{h_i^x(h_{i-1}^x + h_i^x)} u_{i+1,jk} \right) \\ &\quad + r x_i \left(-\frac{h_i^x}{h_{i-1}^x(h_{i-1}^x + h_i^x)} u_{i-1,jk} + \frac{h_i^x - h_{i-1}^x}{h_{i-1}^x h_i^x} u_{ijk} + \frac{h_{i-1}^x}{h_i^x(h_{i-1}^x + h_i^x)} u_{i+1,jk} \right) \\ &\quad + \frac{1}{3} \sigma_x \sigma_y \rho_{xy} x_i y_j D_{xy} u_{ijk}^n + \frac{1}{3} \sigma_y \sigma_z \rho_{yz} y_j z_k D_{yz} u_{ijk}^n + \frac{1}{3} \sigma_z \sigma_x \rho_{zx} z_k x_i D_{zx} u_{ijk}^n \\ &\quad - \frac{1}{3} r u_{ijk}^{n+\frac{1}{3}} \end{aligned}$$

Then we arrange the formula,

$$\begin{aligned}
& u_{ijk}^{n+\frac{1}{3}} + \frac{1}{3}\sigma_x\sigma_y\rho_{xy}x_iy_jD_{xy}u_{ijk}^n + \frac{1}{3}\sigma_y\sigma_z\rho_{yz}y_jz_kD_{yz}u_{ijk}^n + \frac{1}{3}\sigma_z\sigma_x\rho_{zx}z_kx_iD_{zx}u_{ijk}^n \\
& = \left(-\frac{(\sigma_x x_i)^2}{h_{i-1}^x(h_{i-1}^x + h_i^x)} + rx_i \frac{h_i^x}{h_{i-1}^x(h_{i-1}^x + h_i^x)}\right)u_{i-1,jk}^{n+\frac{1}{3}} \\
& \quad + \left(\frac{(\sigma_x x_i)^2}{h_{i-1}^x h_i^x} - rx_i \frac{h_i^x - h_{i-1}^x}{h_i^x h_{i-1}^x} + \frac{1}{\Delta\tau} + \frac{1}{3}r\right)u_{ijk}^{n+\frac{1}{3}} \\
& \quad + \left(-\frac{(\sigma_x x_i)^2}{h_i^x(h_{i-1}^x + h_i^x)} - rx_i \frac{h_{i-1}^x}{h_i^x(h_{i-1}^x + h_i^x)}\right)u_{i+1,jk}^{n+\frac{1}{3}}
\end{aligned}$$

so that, according the reference[3],

$$\alpha_i = -\frac{(\sigma_x x_i)^2}{h_{i-1}^x(h_{i-1}^x + h_i^x)} + rx_i \frac{h_i^x}{h_{i-1}^x(h_{i-1}^x + h_i^x)} \quad (4.16)$$

$$\beta_i = \frac{(\sigma_x x_i)^2}{h_{i-1}^x h_i^x} - rx_i \frac{h_i^x - h_{i-1}^x}{h_{i-1}^x h_i^x} + \frac{1}{\Delta\tau} + \frac{1}{3}r \quad (4.17)$$

$$\eta_i = -\frac{(\sigma_x x_i)^2}{h_{i-1}^x(h_{i-1}^x + h_i^x)} - rx_i \frac{h_i^x}{h_i^x(h_{i-1}^x + h_i^x)} \quad (4.18)$$

$$f_{ijk} = \frac{1}{3}\sigma_x\sigma_y\rho_{xy}x_iy_jD_{xy}u_{ijk}^n + \frac{1}{3}\sigma_y\sigma_z\rho_{yz}y_jz_kD_{yz}u_{ijk}^n \quad (4.19)$$

$$+ \frac{1}{3}\sigma_z\sigma_x\rho_{zx}z_kx_iD_{zx}u_{ijk}^n + \frac{1}{\Delta\tau}u_{ijk}^n \quad (4.20)$$

Finally, by comparison, we know the α_i lack of the item $\frac{1}{3}r$ and the sign of item $u_{ijk}^{n+\frac{1}{3}}$ for f_{ijk} should be positive. \square

5 Mathematical Algorithm

Then, we can construct a tridiagonal system and the equation 4.8 can be rewritten

$$A_x u_{1:N_x,jk}^{n+\frac{1}{3}} = f_{1:N_x,jk}, \quad (5.1)$$

where

$$A_x = \begin{bmatrix} \beta_1 & \eta_1 & 0 & \cdots & 0 & 0 \\ \alpha_2 & \beta_2 & \eta_2 & \cdots & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{N_x-1} & \eta_{N_x-1} \\ 0 & 0 & 0 & \cdots & \alpha_{N_x} - \eta_{N_x} & \beta_{N_x} + 2\eta_{N_x} \end{bmatrix} \quad (5.2)$$

with the zero Dirichlet ($u_{0jk}^{n+\frac{1}{3}} = 0$ at $x = 0$) and linear boundary ($u_{N_x+1,jk}^{n+\frac{1}{3}} = 2u_{N_x,jk}^{n+\frac{1}{3}} - u_{N_x-1,jk}^{n+\frac{1}{3}}$ at $x = L$) conditions.

If the index j and k are fixed, by solving the tridiagonal system, we can obtain the solution vector

$$u_{1:N_x,jk}^{n+\frac{1}{3}} = [u_{1jk}^{n+\frac{1}{3}} \quad u_{2jk}^{n+\frac{1}{3}} \cdots u_{N_xjk}^{n+\frac{1}{3}}]^T \quad (5.3)$$

For $u_{i,1:N_y,k}^{n+\frac{2}{3}}$ and $u_{ij,1:N_z}^{n+1}$, we can use same way.

$$\begin{aligned} u_{i,1:N_y,k}^{n+\frac{2}{3}} &= [u_{i,1,k}^{n+\frac{2}{3}} \quad u_{i,2,k}^{n+\frac{2}{3}} \cdots u_{i,N_y,k}^{n+\frac{2}{3}}]^T \\ u_{ij,1:N_z}^{n+1} &= [u_{ij,1}^{n+1} \quad u_{ij,2}^{n+1} \cdots u_{ij,N_z}^{n+1}]^T \end{aligned}$$

5.1 Thomas Algorithm[7]

This part is to introduce thomas Algotirhm which will be used to solve the flowing tridiagonal matrix. According to the equations 5.1 and 5.2, we can obtain

$$\begin{bmatrix} \beta_1 & \eta_1 & 0 & \cdots & 0 & 0 \\ \alpha_2 & \beta_2 & \eta_2 & \cdots & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{N_x-1} & \eta_{N_x-1} \\ 0 & 0 & 0 & \cdots & \alpha_{N_x} - \eta_{N_x} & \beta_{N_x} + 2\eta_{N_x} \end{bmatrix} \begin{bmatrix} u_{1jk}^{n+\frac{1}{3}} \\ u_{2jk}^{n+\frac{1}{3}} \\ u_{3jk}^{n+\frac{1}{3}} \\ \vdots \\ u_{N_xjk}^{n+\frac{1}{3}} \end{bmatrix} = \begin{bmatrix} f_{1jk} \\ f_{2jk} \\ f_{3jk} \\ \vdots \\ f_{N_xjk} \end{bmatrix}$$

Then, according to the following new coefficient

$$\begin{aligned} \eta_i^* &= \begin{cases} \frac{\eta_1}{\beta_1} & i = 1 \\ \frac{\frac{\eta_1}{\beta_1} \eta}{\beta_i - \eta_{i-1}^* \alpha_i} & i = 2, 3, \dots, N_x - 1 \\ \beta_{N_x} + 2\eta_{N_x} - \eta_{N_x-1}^* (\alpha_{N_x} - \eta_{N_x}) & i = N_x \end{cases} \\ f_{ijk}^* &= \begin{cases} \frac{f_{1jk}}{\beta_1} & i = 1 \\ \frac{\frac{f_{1jk}}{\beta_1} - f_{i-1,jk}^* \alpha_i}{\beta_i - \eta_{i-1}^* \alpha_i} & i = 2, 3, \dots, N_x - 1 \\ \frac{f_{N_xjk} - f_{N_x-1,jk}^* (\alpha_{N_x} - \eta_{N_x})}{\beta_{N_x} + 2\eta_{N_x} - \eta_{N_x-1}^* (\alpha_{N_x} - \eta_{N_x})} & i = N_x \end{cases} \end{aligned}$$

the matrix equation can be rewritten as ;

$$\begin{bmatrix} 1 & \eta_1^* & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \eta_2^* & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \eta_3^* & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \cdots & 1 & \eta_{N_x-1}^* \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{1jk}^{n+\frac{1}{3}} \\ u_{2jk}^{n+\frac{1}{3}} \\ u_{3jk}^{n+\frac{1}{3}} \\ \vdots \\ u_{N_xjk}^{n+\frac{1}{3}} \end{bmatrix} = \begin{bmatrix} f_{1jk}^* \\ f_{2jk}^* \\ f_{3jk}^* \\ \vdots \\ f_{N_xjk}^* \end{bmatrix} \quad (5.4)$$

So that we can obtain:

$$\begin{aligned} u_{N_x,jk}^{n+\frac{1}{3}} &= f_{N_x,jk}^* & i = N_x \\ u_{ijk}^{n+\frac{1}{3}} &= f_{ijk}^* - \eta_i^* u_{i+1,jk}^{n+\frac{1}{3}} & i = 1, 2, 3, \dots, N_x - 1 \end{aligned}$$

5.2 The implementation in C++[\[3, 4\]](#)

It is time to solve the Black-Scholes PDE. Simply, we want to finish a loop from time n to $n + 1$ and iterate the loop to obtain the solution $V(T, S_1, S_2, S_3)$ like the following steps

- Step 1: For fixed j and k , we can find the vector $u_{1:N_x, jk}^{n+\frac{1}{3}}$ by solving the tridiagonal system

$$A_x u_{1:N_x, jk}^{n+\frac{1}{3}} = f_{1:N_x, jk}$$

then we can implement the method in a loop over x-direction:

Algorithm 1 Pseudocode of Operating splitting method

Input: Previous data u^n

Output: Find the solution $u^{n+\frac{1}{3}}$

```

1: for  $dk = 1; k \geq N_z; k++$  do
2:   for  $dj = 1; j \geq N_y; j++$  do
3:     for  $di = 1; i \geq N_x; i++$  do
4:       Set  $\alpha_i, \beta_i, \eta_i$  and  $f_{ijk}$  by Eqs.(4.16)
5:     end for
6:     Solve  $A_x u_{0:N_x, jk}^{n+\frac{1}{3}} = f_{1:N_x, jk}$  by using Thomas algorithm
7:   end for
8: end for
```

- Step 2: Fixing i and k to find $u_{i, 1:N_y, k}^{n+\frac{2}{3}}$

$$A_y u_{i, 1:N_y, k}^{n+\frac{2}{3}} = f_{i, 1:N_y, k}$$

then we can implement the method in a loop over y-direction:

Algorithm 2 Pseudocode of Operating splitting method

Input: Previous data $u^{n+\frac{1}{3}}$

Output: Find the solution $u^{n+\frac{2}{3}}$

```

for  $di = 1; i \geq N_x; i++$  do
  for  $dk = 1; k \geq N_z; k++$  do
    for  $dj = 1; j \geq N_y; j++$  do
      Set  $\alpha_i, \beta_i, \eta_i$  and  $f_{ijk}$ 
    end for
    Solve  $A_y u_{i, 1:N_y, k}^{n+\frac{1}{3}} = f_{i, 1:N_y, k}$  by using Thomas algorithm
  end for
end for
```

- Step 3: Fixing i and j to find $u_{ij, 1:N_z}^{n+1}$.

$$A_z u_{ij, 1:N_z}^{n+1} = f_{ij, 1:N_z}$$

then we can implement the method in a loop over z-direction:

Algorithm 3 Pseudocode of Operating splitting method

Input: Previous data $u^{n+\frac{2}{3}}$

Output: Find the solution u^{n+1}

```
for do  $i = 1; i \geq N_x; i++$  do
  for do  $j = 1; j \geq N_y; j++$  do
    for do  $k = 1; k \geq N_z; k++$  do
      Set  $\alpha_i, \beta_i, \eta_i$  and  $f_{ijk}$ 
    end for
    Solve  $A_z u_{ij,0:N_z}^{n+\frac{1}{3}} = f_{ij,1:N_z}$  by using Thomas algorithm
  end for
end for
```

6 Numerical experiment

Until now, we have introduced thoroughly the method, then we will perform some numerical experiments to show the accuracy and efficiency of the proposed method. In all numerical tests of this paper, we use the same parameters $r = 0.03$, $\sigma = \sigma_x = \sigma_y = \sigma_z = 0.1$ unless otherwise stated.

6.1 Example1: European Call Option

The first example is to consider uncorrelated assets and an option whose payoff consists of three standard European Call options on each of these assets. We also compute the price of such an option and compare it with the sum of the prices of the three calls with same parameters.

Definition 6.1. A European call option is a contract which gives the holder the right to buy the underlying; each discretisation asset according to the calibrated price [18]. When the price of the asset exceeds the exercise price, the holder can choose to buy the assets to profit. Then the payoff of European Call option is

$$u(S, T) = \begin{cases} S - K & S \geq K \\ 0 & \text{otherwise} \end{cases}$$

where T is the exercise time, S the price at time T and K is the exercise price.

Then we consider a forward-in-time framework of three-dimensional Black-Scholes model (4.1), so the payoff is

$$u(S_i, 0) = \begin{cases} S_i - K_i & S_i \geq K_i \\ 0 & \text{otherwise} \end{cases}$$

In addition, we continue to consider that the assets are uncorrelated, which means $W = (W_1, W_2, W_3)$ are pairwise independent. In other words, the random variable ρ_{ij} satisfies

$$\rho_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (6.1)$$

Take it into the equation (4.1) so that we can obtain

$$\begin{aligned} u_\tau = & rxu_x(x, y, z, \tau) + ryu_y(x, y, z, \tau) + rzu_z(x, y, z, \tau) \\ & + \frac{1}{2}\sigma_x^2 x^2 u_{xx}(x, y, z, \tau) + \frac{1}{2}\sigma_y^2 y^2 u_{yy}(x, y, z, \tau) + \frac{1}{2}\sigma_z^2 u_{zz}(x, y, z, \tau) \\ & - ru(x, y, z, \tau), \quad (x, y, z, \tau) \in \omega \times (0, T] \end{aligned}$$

and

$$f_{ijk} = \frac{1}{\Delta\tau} u_{ijk}^{n+\frac{1}{3}} \quad (6.2)$$

Then we compare the result with the sum of three one-dimensional European Call Option. We assume there are three underlying assets with the same parameters, each stock has the initial value $S_0=100$, $\sigma=0.1$ and the exercise price $K=100$.

With the help of the project in the book [2], we can calculate the value of option is 1.27868, so the sum of these three options value is 3.83604.

We set the time discretization is $\Delta\tau = 1/1440$, $\Delta\tau = 1/720$, $\Delta\tau = 1/3600$, and $h = 2, h = 4, h = 8$ so that $m = 49$, $m = 24$, $m = 11$. The result shows it converge the correct value as the larger of mesh grid like the larger time discretization or the larger domain discretisation.

But it is worth to notice that it may doesnot work for some sets of data, like when the sigma is larger, it will show that the value is nan.

6.2 Example2:Basket Option

Now we develop the numerical examples of pricing basket options on correlated stocks in three-dimensional Black-Scholes model.

Definition 6.2 (Basket Option[8]). The option is related with a portfolio of assets, and there we can define a portfolio of 3 assets $S_i(t)$, $i=1,2,3$. This portfolio is determined by the weight w_i of each asset. We can define the price of the portfolio at time t

$$S(t) = \sum_{i=1}^3 w_i S_i(t) \quad (6.3)$$

and we consider the simplest option European call option, so the payoff

$$\max(S(t) - K, 0) = \max\left(\sum_{i=1}^3 w_i S_i(t) - K, 0\right) \quad (6.4)$$

Now, we take $K = 100$, and the initial value of stocks are $S_1(0) = 100$, $S_2(0) = 100$ and $S_3(0) = 100$. So we use $\Omega = [0, 200] \times [0, 200] \times [0, 200]$ as the domain again. The exercise time $T = 1/12$, and time discretization $\Delta\tau = 1/1440$. Then each direction of the domain is discretized as $[0, 100 - (m + 0.5)h, \dots, 100 - 1.5h, 100 - 0.5h, 100 + 0.5h, 100 + 1.5h, \dots, 100 + (m + 0.5)h, 200]$, where $h = 2$ and $m = \text{round}(100/h - 0.5) - 1 = 49$. Based on the previous code, we just make some minor adjustment on the option class, we can calculate the value of option is 199.549.

7 Conclusion

This paper provides efficiency and accurate method to solve the Black-Scholes PDE the FDM method-OSM transform the multi-dimensional model into the multi one-dimensional model, Cleverly solving complex problems. So it means this method is appropriate to fix complex options.

Besides, the Semi-implicit method combines both advantages of explicit and implicit ways. The Non-equal dispersion on the domain can more quickly converge to the correct value.

But it brings more difficulty on mathematical computation and the code is more comfortable to write wrong because of more random variables and more complicated formula. Correspondingly, the rate and accuracy of the level have been improved.

References

- [1] Marek Capinski, Ekkehard Kopp, The Black-Scholes Model, Mastering Mathematical Finance, Cambridge University Press 2012.
- [2] Maciej J. Capinski, Tomasz Zastawniak, Numerical Methods in Finance with C++, Mastering Mathematical Finance, Cambridge University Press 2012.
- [3] Junseok Kima, Taekkeun Kimb, Jaehyun Joo, Yongho Choia, Seunggyu Leea, Hyeongseok Hwangd, Minhyun Yood, Darae Jeong, A practical finite difference method for the three-dimensional Black-Scholes equation, European Journal of Operational Research (2015) 1-8.
- [4] Jeong, Darae, & Kim, Junseok. (2013). A comparison study of ADI and operator splitting methods on option pricing models. Journal of Computational and Applied Mathematics, 247(1), 162-171.
- [5] Duffy, D. J. (2014). Finite difference methods in financial engineering: A partial differential equation approach. Hoboken, N.J: Wiley.
- [6] Wolfram, S. (1999). The MATHEMATICA® Book Version 4 . Cambridge: Cambridge University Press .
- [7] Quantstart.com. 2020. Tridiagonal Matrix Solver Via Thomas Algorithm | Quantstart. [online] Available at: <<https://www.quantstart.com/articles/Tridiagonal-Matrix-Solver-via-Thomas-Algorithm/>> [Accessed 15 September 2020].
- [8] Duffy, Daniel J ; Kienitz, Jorg. Monte Carlo Frameworks, John Wiley & Sons, Inc, Hoboken, NJ, USA, 2012.
- [9] Black, Fischer, & Scholes, Myron. (1973). The Pricing of Options and Corporate Liabilities. The Journal of Political Economy, 81(3), 637-654.
- [10] Merton, R. C. (1973). Theory of Rational Option Pricing. The Bell Journal of Economics and Management Science, 4(1), 141-183.
- [11] Glasserman, P. (2004). Monte Carlo methods in financial engineering (Applications of mathematics ; 53). New York: Springer.
- [12] Capinski, M., Kopp, P. E., Traple, Janusz, & Kopp, Ekkehard. (2012). Stochastic calculus for finance (Mastering mathematical finance). Cambridge: Cambridge University Press.
- [13] Persson, J. , & von Persson, L. (2007). Pricing European multi-asset options using a space-time adaptive FD-method. Computing and Visualization in Science, 10 (4), 173-183 .

- [14] Rambeerich, N. , Tangman, D. Y. , Lollchund, M. R. , & Bhuruth, M. (2013). High-order computational methods for option valuation under multifactor models. *European Journal of Operational Research*, 224 (1), 219-226 .
- [15] Jeong, D. , Kim, J. , & Wee, I. S. (2009). An accurate and efficient numerical method for Black-Scholes equations. *Communications of the Korean Mathematical Society*, 24 (4), 617-628 .
- [16] Jeong, D. , Wee, I. S. , & Kim, J. (2010). An operator splitting method for pricing the ELS option. *Journal of the Korean Society for Industrial and Applied Mathematics*, 14 (3), 175-187.
- [17] Hu, Jinhao, & Gan, Siqing. (2018). High order method for Black-Scholes PDE. *Computers & Mathematics with Applications* (1987), 75(7), 2259-2270.
- [18] Applebaum, D. (2004). *Mathematics for finance, an introduction to financial engineering*, by M. Capinski, T. Zastawniak. Pp. 310. £18.95. 2003. ISBN 1 85233 330 8 (Springer-Verlag). *Mathematical Gazette*, 88(512), 389-390.