CHAPTER 8 NUMERICAL INTEGRATION

SOLVED EXAMPLE PROBLEMS

for

NUMERICAL METHODS FOR SCIENTISTS AND ENGINEERS With Pseudocodes

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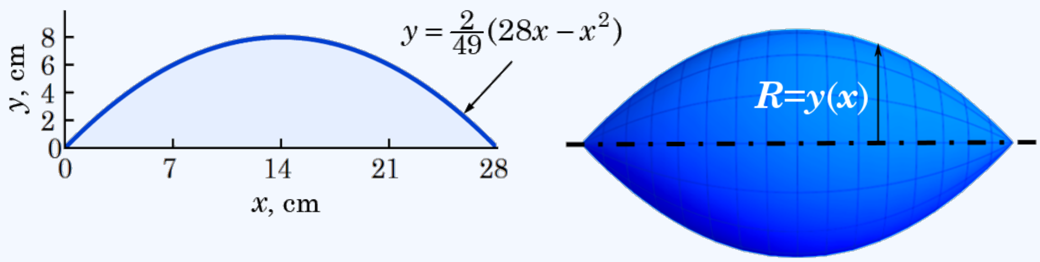
EXAMPLE 8.1: Numerical integration

A professional American football ball can be generated by rotating the parabola $y = 2(28x - x^2)/49$ around the horizontal axis.



Noting that the cross-section area of the ball is $A(x) = \pi R^2(x)$ where the radius is $R(x) = y(x)$. Then the volume of the ball can be calculated by the following integral:

$$V = \int_0^{28} A(x)dx = \pi \int_0^{28} y^2 dx$$

Divide the integration range into 10 equally spaced panels and use trapezoidal and Simpson's 1/3 rules to estimate the volume of the ball. Discuss the accuracy of the numerical estimates with both methods.

SOLUTION:

The ball profile is given as a continuous function; thus, to estimate the integral with the N-panel trapezoidal and Simpson's rules, a set of 10-panel discrete data ($\Delta x = 28/10 = 2.8$) needs to be generated from y^2.

The weights are set as $w_0 = w_{10} = 1$ and $w_i = 2$ for $i = 1$ through 9 for the trapezoidal rule, and $w_i = 4$ for $i = 1, 3, 5, 7, 9$, $w_i = 2$ for $i = 2, 4, 6, 8$, and $w_0 = w_{10} = 1$ for Simpson's 1/3 rule. The weights as well as the discrete function and $w_i f_i$ products for trapezoidal and Simpson's rules were calculated and tabulated in **Table 8.1**.

Table 8.1

			Trapezoidal Rule		Simpson's 1/3 Rule	
i	x_i	f_i	w_i	$w_i f_i$	w_i	$w_i f_i$
0	0	0	1	0	1	0
1	2.8	8.2944	2	16.5888	4	33.1776
2	5.6	26.2144	2	52.4288	2	52.4288
3	8.4	45.1584	2	90.3168	4	180.6336
4	11.2	58.9824	2	117.9648	2	117.9648
5	14.0	64	2	128	4	256
6	16.8	58.9824	2	117.9648	2	117.9648
7	19.6	45.1584	2	90.3168	4	180.6336
8	22.4	26.2144	2	52.4288	2	52.4288
9	25.2	8.2944	2	16.5888	4	33.1776
10	28	0	1	0	1	0
			$\Sigma w_i f_i =$	**682.5984**	$\Sigma w_i f_i =$	**1024.4096**

The resulting integral involves a simple polynomial integration, which results in 14336/15, and the volume becomes $V = 14336\pi/15$ (or 3002.52482). Also note that the weighted sums are also computed and tabulated in the fifth and seventh columns for trapezoidal and Simpson's rules, respectively.

This leads to

$$\int_0^{28} y^2 dx \approx T_{10} = \frac{\Delta x}{2}\sum_{i=0}^{10} w_i f_i = \frac{2.8}{2}(682.5984) = 955.63776, \quad E_{abs} = 0.0955733$$

$$\int_0^{28} y^2 dx \approx S_{10} = \frac{\Delta x}{3}\sum_{i=0}^{10} w_i f_i = \frac{2.8}{3}(1024.4096) = 956.11563, \quad E_{abs} = 0.382293$$

where E_{abs} is the absolute error.

Note that, contrary to our expectations, for $N = 10$ panels, the estimate found by the trapezoidal rule is better than that found by the Simpson's rule. If we repeat the approximations for $N = 20$, we obtain

$$T_{20} = 955.72736, \qquad E_{abs} = 0.0059733$$
$$S_{20} = 955.75723, \qquad E_{abs} = 0.0238967$$

It is observed that increasing the number of panels reduces the absolute errors; however, the estimate with Simpson rule does not show any improvement compared to the trapezoidal rule. The estimated volume for 10 panels is $V = \pi T_{10} = 3002.2246$ with an absolute error of 0.30.

Discussion: The trapezoidal rule can sometimes give better approximations than Simpson's rule, even though Simpson's rule, a fourth-order method, is generally more accurate for smooth functions. This is because Simpson's rule assumes a quadratic fit, which can introduce unnecessary curvature on intervals where the function is close to a straight line.

In **Figure 8.1**, the distribution of the integrand $f(x) = y^2$ is depicted. Note that the integrand is almost linear in the intervals highlighted by the marked bands (*see Fig. 8.1*). That is why, as the number of panels is increased, we do not obtain greater accuracy in the estimated integral with Simpson's rule compared to the trapezoidal rule.

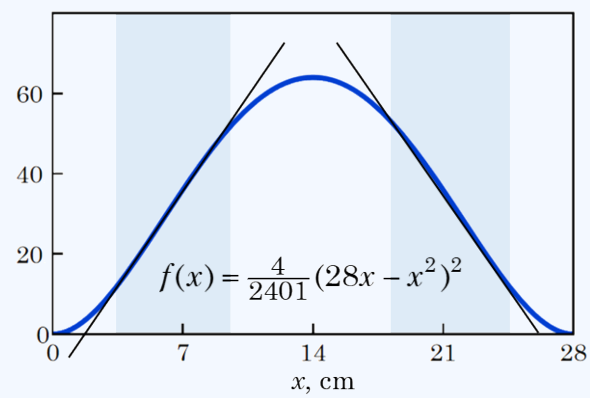


Figure 8.1

The trapezoidal rule, as in this example, can sometimes give better estimates than Simpson's rule, but this is rare and depends on the nature of the integrand. For instance, the trapezoidal rule can also capture the behavior of oscillatory functions better, especially if the interval is sufficiently small and the function behaves nearly linearly within each panel. However, in most cases, Simpson's rule provides better estimates because it uses quadratic interpolation, leading to an error proportional to $\mathcal{O}(h^4)$, whereas the trapezoidal rule has an error of $\mathcal{O}(h^2)$. So for smooth, non-linear functions, Simpson's rule is usually the better choice. It is therefore important to analyze the integrand in terms of smoothness, discontinuities, oscillations, etc., before applying any numerical method.

EXAMPLE 8.2: Romberg integration

Consider **Example 8.1**. Evaluate the integral to at least five decimal places ($\varepsilon = 0.5 \times 10^{-5}$).

SOLUTION:

We will set $f(x) = y^2 = 4\left(28x - x^2\right)^2/2401$ and integrate it from $a = 0$ to $b = 28$. The volume is then found by multiplying the integral by π. One-panel trapezoidal rule gives

$$R_{1,1} = \frac{28-0}{2}\big(f(0)+f(28)\big) = 14(0+0) = 0$$

Then, using Eq. (8.50), we find the second entry of the first column of the Romberg table as follows:

$$R_{2,1} = \frac{R_{1,1}}{2} + \frac{b-a}{2} f\left(a + \frac{b-a}{2}\right) = \frac{0}{2} + \frac{28-0}{2} f\,(14) = 0 + 14(64) = 896$$

The second column ($n = 2$) element corresponding to Simpson's rule can be readily calculated using Eq. (8.51) as

$$R_{2,2} = \frac{4R_{2,1} - R_{1,1}}{3} = \frac{4(896) - 0}{3} = 1194.666667$$

Since $|R_{2,2} - R_{1,1}| > \varepsilon$, we will continue to find the next estimate with the trapezoidal rule.

$$\begin{aligned} R_{3,1} &= \frac{R_{2,1}}{2} + \frac{b-a}{4}\left(f\left(a + \frac{b-a}{4}\right) + f\left(a + \frac{3(b-a)}{4}\right)\right) \\ &= \frac{896}{2} + \frac{28-0}{4}\Big(f\,(7) + f\,(21)\Big) = 448 + 7(36+36) = 952 \end{aligned}$$

The second element of the second column is calculated using Eq. (8.51) as follows:

$$R_{3,2} = \frac{4R_{3,1} - R_{2,1}}{3} = \frac{4(952) - 896}{3} = 970.666667$$

Now that we have $R_{2,2}$ and $R_{3,2}$, we may proceed to calculate $R_{3,3}$, which should be a sixth-order accurate estimate.

$$R_{3,3} = \frac{4^2 R_{3,2} - R_{2,2}}{4^2 - 1} = \frac{16(970.666667) - 1194.666667}{15} = 955.333333$$

Noting that $|R_{3,3} - R_{2,2}| > \varepsilon$, we repeat the integration and Richardson extrapolations in the same manner for the next row. The results are summarized in the table below. We observe that $|R_{4,4} - R_{3,3}| = 0$; in other words, $R_{3,3}$ is the true value of the integral.

h	$\mathcal{O}(h^2)$	$\mathcal{O}(h^4)$	$\mathcal{O}(h^6)$	$\mathcal{O}(h^8)$
28	0			
14	896	1194.66667		
7	952	970.666667	955.3333333	
3.5	955.5	956.666667	955.3333333	955.3333333

Discussion: It should be pointed out that the integrand is a fourth-order polynomial. The third column entries, $R_{k,3}$ (k = 3, 4, 5, etc.), correspond to so-called *Boole's rule*, which has the order of error of $\mathcal{O}(h^6)$. This means that the error term is proportional to the sixth derivative of $f(x)$. Since $f^{(6)}(x) = 0$, $R_{3,3}$ should give the true value (=14336/15) if no truncation error is introduced, which is verified above. The volume can then be determined as $V = \pi R_{3,3} = 14336\pi/15$.

EXAMPLE 8.3: Integration of Discrete Functions

Data obtained from the motion of a self-balancing two-wheeled robot with a combination of constant acceleration and 'periodic turbulence' for the first 11 seconds are presented below:

i	0	1	2	3	4	5	6	7	8	9	10	11
t_i (s)	0	1	2	3	4	5	6	7	8	9	10	11
v_i (m/s)	1.0	0.6	1.24	1.08	1.96	2.04	3.16	3.48	4.84	5.4	7.0	7.8

Use a suitable method to estimate the accumulated distance the robot traveled in the first 11 seconds.

SOLUTION:

To find the accumulated distance (s), we need to integrate the velocity over time: $s = \int_0^{11} v(t)dt$. Faced with this problem, our first instinctive reaction may be to apply Simpson's rule. We know that Simpson's 1/3 rule requires an even number of panels. However, the number of panels in the dataset is odd and is not a multiple of 3 either. In other words, Simpson's 3/8 rule can be applied to the entire data set if the number of panels is odd and a multiple of 3. An alternative is to apply Simpson's 1/3 rule to the first 8 panels and Eq. (8.37) to the last three panels or apply Simpson's 1/3 rule to the first 10 panels and Eq. (8.64) to the last panel.

The preceding arguments are valid provided that the distribution is smooth and continuous. For a function defined on a set of discrete points, determining smoothness and continuity is different from the continuous case. With the aid of a discrete data plot, we can visually and numerically check continuity and smoothness. Non-smooth functions depict sharp corners, zigzags, or abrupt changes in direction.

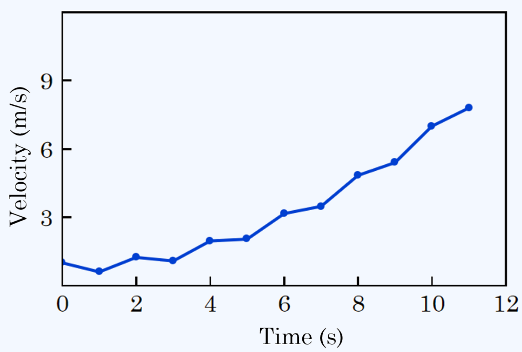


Figure 8.2

When the data set is plotted as shown in **Figure 8.2**, we observe that the data set is not smooth; that is, Simpson's rule clearly is not suitable. Therefore, we will apply the trapezoidal rule, which can easily handle the data with zigzags. Noting that $\Delta t = 1$ s, we implement the trapezoidal rule as follows:

$$s = \frac{\Delta t}{2}\left(v_0 + v_{11} + 2\sum_{i=1}^{10} v_i\right) = \frac{1}{2}\big(1 + 7.8 + 2(30.8)\big) = 35.2\,\mathrm{m}$$

Discussion: By applying the trapezoidal rule to the given problem, the estimated travel distance is 35.2 m, the same as the exact distance traveled. When we apply Simpson's 1/3 rule to the first 10 panels and Eq. (8.64) to the last panel, the estimated travel distance is found to be 34.4 m with an error of 0.8 m. Since Simpson's rule unnecessarily corrected the zigzag lines of any two adjacent intervals with a quadratic curve, it led to the error in the estimated distance. On the other hand, the trapezoidal rule is a perfect fit for this case of 11 different-sized trapezoids. As a final note for those working with discrete data sets: do not blindly select and apply a method without visually inspecting the distribution.

EXAMPLE 8.4: Integration with Gauss-Legendre quadrature

A light spring of length L is rigidly clamped to its lower end B and is initially vertical. A downward force P is applied at the free end (*see* **Figure 8.3**), causing the spring to bend. If θ denotes the angle of the slope at any point and s is the distance along the spring measured from A, the integration of the exact governing equation leads to the following integral:

$$L=\sqrt{\frac{EI}{P}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-\sin^2(\alpha/2)\sin^2 x}}$$

where E is the Young's modulus, I is the moment of inertia, and α is the value of θ at A. For $\alpha=\pi/4$, calculate the definite integral using $N=2$, 3, 4, and 5-point Gauss-Legendre quadrature.

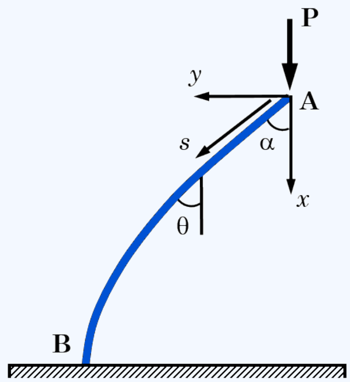


Figure 8.3

SOLUTION:

Noting that $\sin^2(\pi/8)=(2-\sqrt{2})/4$, the definite integral can be expressed as follows:

$$I=\int_0^{\pi/2}\frac{2dx}{\sqrt{4-(2-\sqrt{2})\sin^2 x}}$$

The method, also called *Gauss Rule*, is designed for integrals over $[-1,1]$. For integrals on other intervals, $[a,b]$, a transformation is needed. We start by setting $f(x)=2/\sqrt{4-(2-\sqrt{2})\sin^2 x}$, and the numerical estimates are found through the Gauss-Legendre sums:

$$I=\int_{x=0}^{\pi/2} f(x)dx=\int_{u=-1}^{+1} f\Big(\frac{\pi}{4}(u+1)\Big)\frac{\pi}{4}du\approx G_N=\sum_{i=1}^{N} w_i f(x_i)+R_N$$

where

$$x_i=\frac{\pi}{4}\left(u_i+1\right),\qquad w_i=\frac{\pi}{2}w_{ui}$$

and (u_i,w_{ui}) is the quadrature set on $[-1,1]$ (*see* Appendix B), and R_N is the global error given by

$$R_N=\frac{\pi^{2N+1}(N!)^4}{(2N+1)[(2N)!]^3}f^{(2N)}(\xi),\qquad 0<\xi<\pi$$

The upper and lower bounds of the global error ($R_{N,max}$ or $R_{N,min}$) are estimated using the absolute maximum and minimums of $f^{(2N)}(\xi)$.

Table 8.2

i	u_i	w_{ui}	x_i	$f(x_i)$	w_i	$w_i f(x_i)$
1	−0.861136	0.347855	0.109063	1.000869	0.273204	0.273442
2	−0.339981	0.652145	0.518377	1.018475	0.512193	0.521656
3	0.339981	0.652145	1.052418	1.060293	0.512193	0.543075
4	0.861136	0.347855	1.461732	1.081294	0.273204	0.295414
					$\sum w_i f_i$ =	**1.633588**

For $N = 4$, the Gauss-Legendre quadrature (u_i, w_{ui}), modified quadratures (x_i, w_i), integrands at x_i's, $f(x_i)$, and $w_i f(x_i)$ products are calculated and presented in **Table 8.2**. The numerical values were obtained using high-precision arithmetic operations but are rounded to six decimal places to fit into the table. The sum of the last column results in the following ten-decimal-place accurate weighted sum:

$$G_4 = \sum_{i=1}^{4} w_i f(x_i) = 1.6335880828$$

Discussion: The integral in question has a true solution: $K\left((2-\sqrt{2})/4\right) = 1.6335863075$, where $K(k)$ is the complete elliptic integral of the first kind. For $N = 2$, 3, 4, and 5, the estimated results using the Gauss-Legendre quadrature and the true absolute errors, $|I - G_N|$, are summarized in **Table 8.3**.

Table 8.3

N	2	3	4	5
G_N	1.6340460	1.6335444	1.6335881	1.6335863
$\|I - G_N\|$	4.597×10^{-4}	4.192×10^{-5}	1.775×10^{-6}	2.345×10^{-8}

We note that we obtain very good results even for a small number of N, that is, a few integration points. This example illustrates how quickly the Gauss-Legendre method converges towards the true value. This is because the method achieves *exponential convergence* for smooth functions.

Compared to the trapezoidal or Simpson's rule, it requires fewer function evaluations for a given level of accuracy. In this regard, the G_5 estimate resulted in a seven-decimal-place accurate estimate with only five function evaluations.

For polynomials of degree $2N - 1$, the exact integral is calculated exactly using only N points. However, if the integrand has discontinuities, sharp peaks, or singularities, it may not converge well. On the other hand, unlike the methods that use uniform spacing, Gauss-Legendre requires precomputed roots of Legendre polynomials, making implementation a bit more complex. However, a quadrature generator, Pseudocode 8.6, can be used to generate Gauss-Legendre quadratures.

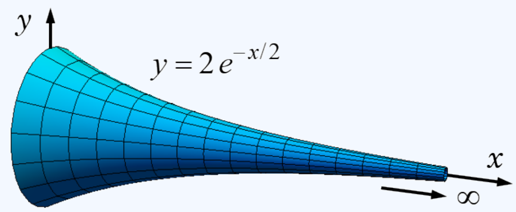
Here, we presented an integration example, whose true solution is known, to highlight the importance and effectiveness of the method. However, in cases where the exact solution cannot be found, we almost always resort to numerical integration. In such cases, it is important to know the level of accuracy of the resulting numerical answer. Preestimating the global error, R_N, can be practical only if the $2N$'th derivative of the integrand can be easily found. Otherwise, finding $f^{(2N)}(x)$ and its absolute maximum can become very complicated and impractical. Therefore, we generally rely on practical error estimation techniques. A common approach is to compute the integral using two different values of N (e.g. N_1 and N_2) and use the difference as an error estimate: $E = |G_{N_1} - G_{N_2}|$. If the difference is small, the estimate is considered accurate.

EXAMPLE 8.5: Improper Integrals

Consider $y = 2e^{-x/2}$ over the interval [0,∞]. The surface area of the surface of revolution formed by revolving the graph of y around the $x-$axis is given by

$$S = 2\pi \int_0^\infty y\sqrt{1+(y')^2}dx$$

Calculate the surface area using Gauss-Laguerre quadrature with N = 2, 3, 4, and 5.



SOLUTION:

Noting that $y' = -e^{-x/2}$, the definite integral becomes

$$S = 2\pi \int_0^\infty 2e^{-x/2}\sqrt{1+e^{-x}}dx$$

The integral can be rearranged as follows:

$$S = 8\pi \int_0^\infty e^{-x}\sqrt{1+e^{-2x}}dx$$

The integral term is a Gauss-Laguerre type with $f(x) = \sqrt{1+e^{-2x}}$. We can then approximate the integral term as follows:

$$I = \int_0^\infty e^{-x}\sqrt{1+e^{-2x}}dx \approx GL_N = \sum_{i=1}^{N} w_i f(x_i)$$

where I has a true solution of $(\sqrt{2}+\sinh^{-1}1)/2$.

Since the integrand is cast in $e^{-x}f(x)$, the Gauss-Laguerre method is a numerical method that can inherently handle the improper integral on the interval [0,∞). For N = 2, 3, 4, and 5, the estimated results using the Gauss-Laguerre quadrature and the true absolute errors, $|I - GL_N|$, are summarized in **Table 8.4**. We note that the method converges to 3 to 4 decimal place accurate estimates with a few quadrature points.

Table 8.4

N	2	3	4	5
GL_N	1.12341735	1.14225950	1.1469957	1.14787623
$\lvert I - GL_N\rvert$	2.438×10^{-2}	5.534×10^{-3}	7.979×10^{-4}	8.265×10^{-5}

Discussion: The Gauss-Laguerre quadrature method is optimal for functions that involve an exponential decay term (e.g., e^{-x} or similar terms). Many integrals in science and engineering have this form, making Gauss-Laguerre an efficient choice. Approximations with Gauss-Laguerre quadrature converge rapidly for integrals over semi-infinite domains. The method also handles integrals with singularities occurring near the origin. For smooth integrands (without sharp peaks or other irregular behavior), the method provides very accurate results with relatively few evaluation points, which makes it computationally efficient when high accuracy is required.

EXAMPLE 8.6: Integrals with Singularity

Consider the following integral with singularity at $x = 0$:

$$I = \int_0^{\pi/2} \frac{\cos x}{\sqrt{x}} dx$$

Find numerical approximations for the integral using $N = 2$, 3, 4, 5, 8, and 10 point the Gauss-Chebyshev quadrature.

SOLUTION:

We note that the following integral has a singularity at $x = 0$. Traditional methods (trapezoidal, Simpson, Romberg's rules) with uniform panel size cannot be employed. On the other hand, the integral is not *exactly* of the Gauss-Chebyshev type integral. Multiplying the numerator and the denominator by $\sqrt{\pi/2 - x}$, we put the integral into the desired form as follows:

$$I = \int_0^{\pi/2} \frac{\cos x\sqrt{\pi/2 - x}}{\sqrt{x(\pi/2 - x)}} dx$$

Then,

$$\int_0^{\pi/2} \frac{f(x)}{\sqrt{x(\pi/2 - x)}} dx \approx C_N = \sum_{i=1}^{N} w_i f(x_i)$$

where $f(x) = \cos x\sqrt{\pi/2 - x}$, and $x_i = \cos((2i - 1)\pi/2N)$ and $w_i = \pi/N$ for $i = 1, 2, ..., N$ are the Gauss-Chebyshev quadrature.

The given integral has a true solution: $I = \sqrt{2\pi}\, C(1)$, where $C(x) = \int_0^x \cos t^2 dt$ is so-called the Fresnel integral. For $N = 2$, 3, 4, 5, 8, and 10, the approximate values using the Gauss-Chebyshev quadrature and the true absolute errors, $|I - C_N|$, are summarized in **Table 8.5**. We note that the method converges to five decimal place accurate estimates with 8 and 10 quadrature points.

Table 8.5

N	2	3	4	5	8	10
C_N	1.9427140	1.9526458	1.9541965	1.9546165	1.9548598	1.9548853
$\lvert I - C_N \rvert$	0.012189	0.002257	0.000706	0.000286	4.31×10^{-5}	1.76×10^{-5}

Discussion: Gauss-Chebyshev quadrature is a numerical integration method specifically designed to integrate functions containing $f(x)/\sqrt{1 - x^2}$ on the interval $[-1,1]$. However, the method can be easily extended to integration of functions $f(x)/\sqrt{(x - a)(b - x)}$ over the interval $[a, b]$.

Gauss-Chebyshev quadrature is highly efficient for integrals involving Gauss-Chebyshev types, offering high accuracy with fewer sample points compared to standard numerical methods such as the trapezoidal, Simpson, and Romberg's rules. Gauss-Chebyshev quadrature is particularly useful for integrals involving functions with singularities at the endpoints of the integration interval. The method's ability to handle such singularities makes it superior to many other quadrature methods. The Gauss-Chebyshev quadrature significantly reduces the computational complexity. This is because it integrates efficiently by mapping the integrand to the Chebyshev nodes, which are distributed in a way that minimizes numerical error.

EXAMPLE 8.7: Double Integration

Consider the following double integral:

$$I = \int_{x=0}^{1} \int_{y=x}^{2x} y \cos \sqrt{x^2 + y^2} dydx$$

Find numerical approximations of I using $N = 2$, 3, and 4 point the Gauss-Legendre quadrature for both x- and y-variables.

SOLUTION:

First, we convert the integral over the y-variable to an integral with fixed lower and upper bounds. One of the easiest ways to achieve this is to apply the $y = xt$ substitution ($dy = xdt$), which leads to

$$\int_{t=1}^{2} x^2 t \cos(x\sqrt{1+t^2} dt$$

Substituting this into the double integral results in

$$I = \int_{x=0}^{1} \int_{t=1}^{2} x^2 t \cos(x\sqrt{1+t^2}) dtdx$$

which can be evaluated analytically, leading to the following true solution:

$$I = \frac{2}{\sqrt{5}} \sin\sqrt{5} - \cos\sqrt{5} + \cos\sqrt{2} - \sqrt{2}\sin\sqrt{2} = 0.07999438970072$$

The Gauss-Legendre quadrature for the integrations over x- and t-variables are computed as follows:

$$x_i = \frac{1}{2}(u_i + 1), \quad w_i = \frac{1}{2} w_{ui} \quad \text{and} \quad t_i = \frac{1}{2}(u_i + 3), \quad w_{ti} = \frac{1}{2} w_{ui}$$

where (u_i, w_{ui}) is the Gauss-Legendre quadrature set on $[-1,1]$ (*see* Appendix B).

The double integral can then be evaluated as

$$I_{N,N} = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_{tj} f(x_i, t_j), \qquad f(x,t) = x^2 t \cos(x\sqrt{1+t^2})$$

where N quadrature points have been used for both variables; however, the number of points does not need to be the same. The number of integration points can be chosen to be larger in one variable depending on the length of the interval size or non-smoothness of the integrand with the considered variable.

For $N = 2$, 3, and 4, the numerical approximations as well as the true absolute errors, $|I - I_{N,N}|$, are obtained and tabulated in **Table 8.6**. We note that the method converges to five-decimal-place accurate estimates with 8 and 10 quadrature points.

Table 8.6

N	2	3	4
$I_{N,N}$	0.07999439	0.07993918	0.07999504
$\lvert I - I_{N,N}\rvert$	3.12×10^{-4}	5.52×10^{-5}	6.48×10^{-7}

Discussion: Note that $I_{4,4}$ is a five-decimal-place accurate estimate, which required 16 function, $f(x,t)$, evaluations.

EXAMPLE 8.8: Solving Integral Equations

Consider the following integral equation:

$$y(x) = 2x + \int\limits_{t=0}^{1} (6x^5 - 7xt)y(t)dt$$

where the unknown function $y(x)$ is sought. Equations of this form are called *Fredholm integral equation of the second kind*. Apply 10- and 20-panel trapezoidal and Simpson's rules to obtain the numerical approximations to $y(x)$. The true solution is given as follows: $y(x) = 2x^5$.

SOLUTION:

The general form of a Fredholm integral equation of the second kind is given by

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt$$

where $y(x)$ is the unknown function, $f(x)$ is the given function, $K(x,t)$ is the kernel, and λ is a parameter.

Here we will describe a method (referred to as *Nyström's Method*) that approximates the integral term using numerical quadrature, leading to a system of linear equations. In this method, we first choose an integration method (Newton-Cotes or Gauss quadrature rules, etc.) and determine quadrature points t_j and weights w_j. Then, we approximate the integral as

$$\int_a^b K(x,t)y(t)dt \approx \sum_{j=0}^{N} w_j K(x,t_j)y_j$$

Then we evaluate the above expression at discrete points x_j's, which results in a linear system.

$$y_i = f_i + \lambda \sum_{j=0}^{N} w_j K(x_i,t_j)y_j \quad \text{for } i = 0, 1, ..., N$$

The system can then be solved for y_j's using matrix inversion, Gauss elimination, or iterative methods.

Now let's tackle the given problem by replacing the integral over the variable t with the N-panel approximation expressed as a weighted sum:

$$y(x) = 2x + \sum_{j=0}^{N} w_j K(x,t_j)y_j$$

where $K(x,t) = 6x^5 - 7xt$, $t_j = jh$ for $j = 0$, 1,..., N, and $y_j = y(t_j)$, and the weights for the trapezoidal rule are calculated as $w_0 = w_N = h/2$ and $w_j = h$ for $j \neq 0$ and N. For Simpson's rule, the weights are set to $w_0 = w_N = h/3$, $w_j = 4h/3$ for odd and $w_j = 2h/3$ for even j's.

The above equation contains $N+1$ unknowns (y_0, y_1, ..., y_N). We generate $N+1$ equations by replacing x with the integration points ($x_i = ih$ for $i = 0, 1, ..., N$) which leads to

$$\begin{aligned} y_0 &= \quad 2x_0 + \textstyle\sum_{j=0}^{N} w_j K(x_0,t_j)y_j \\ y_1 &= \quad 2x_1 + \textstyle\sum_{j=0}^{N} w_j K(x_1,t_j)y_j \\ &\vdots \qquad\qquad \vdots \\ y_N &= \quad 2x_N + \textstyle\sum_{j=0}^{N} w_j K(x_N,t_j)y_j \end{aligned}$$

This is a linear system of equations, which, by collecting the unknowns on the left-hand side, can be expressed as follows:

$$\begin{bmatrix} 1-w_0K(x_0,t_0) & -w_1K(x_0,t_1) & \cdots & -w_NK(x_0,t_N) \\ -w_0K(x_1,t_0) & 1-w_1K(x_1,t_1) & \cdots & -w_NK(x_1,t_N) \\ \vdots & \vdots & \ddots & \vdots \\ -w_0K(x_N,t_0) & -w_1K(x_N,t_1) & \cdots & 1-w_NK(x_N,t_N) \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix} = 2 \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}$$

Then this system was solved for $N=10$ and 20 using the Gauss elimination algorithm.

Discussion: The accuracy of the numerical method described here depends heavily on the numerical integration method used. For smooth kernels, a high-order closed Newton-Cotes rule (Simpson's or Boole's rule) is often a good choice because it provides high accuracy with relatively few points. Another consideration is the convergence of the method. As the number of panels or quadrature points (N) increases, the numerical estimates should converge to the true solution, provided the weighted sum converges for the integral in the equation. The convergence rate would depend on both the numerical method used and the smoothness of $y(x)$ and $K(x,t)$.

Table 8.7

		Trapezoidal Rule		Simpson's Rule	
x	$y_{\text{true}}(x)$	$N=10$	$N=20$	$N=10$	$N=20$
0	0	0	0	0	0
0.1	0.00002	1.65×10^{-3}	4.16×10^{-4}	1.33×10^{-5}	8.34×10^{-7}
0.2	0.00064	3.31×10^{-3}	8.33×10^{-4}	2.68×10^{-5}	1.68×10^{-6}
0.3	0.00486	4.99×10^{-3}	1.26×10^{-3}	4.12×10^{-5}	2.57×10^{-6}
0.4	0.02048	6.75×10^{-3}	1.70×10^{-3}	5.83×10^{-5}	3.65×10^{-6}
0.5	0.06250	8.70×10^{-3}	2.19×10^{-3}	8.17×10^{-5}	5.12×10^{-6}
0.6	0.15552	1.10×10^{-2}	2.77×10^{-3}	1.18×10^{-4}	7.37×10^{-6}
0.7	0.33614	1.39×10^{-2}	3.51×10^{-3}	1.74×10^{-4}	1.10×10^{-5}
0.8	0.65536	1.78×10^{-2}	4.50×10^{-3}	2.65×10^{-4}	1.66×10^{-5}
0.9	1.18098	2.32×10^{-2}	5.86×10^{-3}	4.05×10^{-4}	2.55×10^{-5}
1	2	3.06×10^{-2}	7.74×10^{-3}	6.15×10^{-4}	3.88×10^{-5}
	Mean error	1.22×10^{-2}	3.08×10^{-3}	1.80×10^{-4}	1.13×10^{-5}

The true solution as well as the true absolute errors obtained using the trapezoidal and Simpson's rules with $N=10$ and 20 uniform panels are tabulated in **Table 8.7**. Recall that the trapezoidal and Simpson's rules are 2'nd and 4'th order accurate. It follows that Simpson's 1/3 rule with an even number of panels (provided that the integrand is smooth) should yield much better numerical estimates than the trapezoidal rule.

The mean true absolute errors are also presented in the last row of **Table 8.7**. Using the trapezoidal rule, we see that the true absolute error in the numerical solution obtained for $N=10$ is about four times larger than the numerical solution with N=20. This is not surprising since the panel width is reduced by half, the truncation error becomes $\mathcal{O}\big((h/2)^2\big)\equiv\mathcal{O}(h^2)/4$. In the case of Simpson's rule, the absolute error in the numerical solution obtained for N=10 is about 16 times larger than that of the numerical solution with N=20 due to being a fourth-order method; i.e., $\mathcal{O}\big((h/2)^4\big)\equiv\mathcal{O}(h^4)/16$.