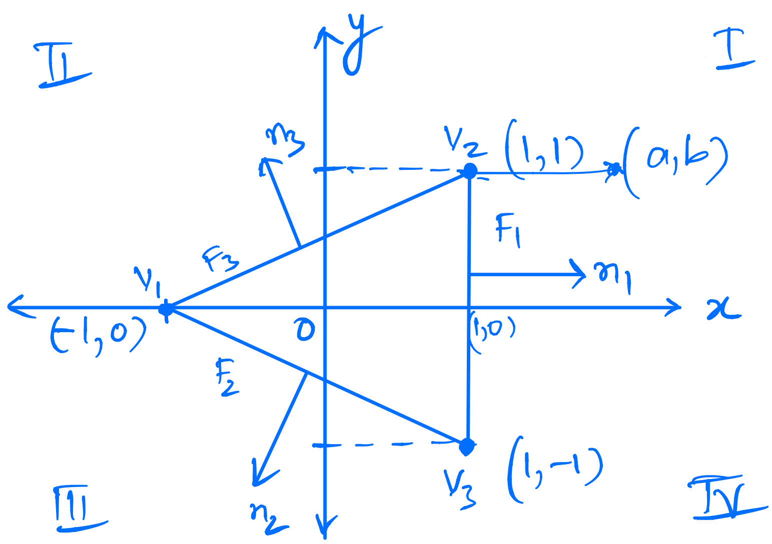
HW3: Control to facet controller

system $\dot{x} = \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} x + \begin{bmatrix} -2 \\ -2 \end{bmatrix} u + \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Given polytope with vertices:

$V_1 = (-1, 0)^T$, $V_2 = (1, 1)^T$, $V_3 = (1, -1)^T$



polytope for given vertices

1. Let n_1, n_2, n_3 be the normal vectors to each of the facets F_1, F_2, F_3.

Let $\bar{f_1}, \bar{f_2}, \bar{f_3}$ be the vectors corresponding to each of the facets F_1, F_2, F_3, respectively.

We know that slope $(\bar{f_1})$. slope $(n_1) = -1$

or $m_{F_1} \cdot m_{n_1} = -1$

Now, $m_{F_1} = \dfrac{0-1}{1-1} \ldots \left(\dfrac{y}{x}\right)$

$\Rightarrow \quad m_{n_1} = -1\left(\dfrac{1-1}{0-1}\right) = \dfrac{0}{1} \ldots \left(\dfrac{y}{x}\right)$

$\Rightarrow \quad n_1 = (1, 0)$

$\Rightarrow \quad \boxed{n_1 = \dfrac{1}{1}, \dfrac{0}{1} = (1, 0)^T}$ (unit normal $\bar{v}$ to F_1)

Similarly, $m_{F_2} \cdot m_{n_2} = -1$

$m_{F_2} = \dfrac{0-(-1)}{-1-1} = \dfrac{1}{-2} \ldots \left(\dfrac{y}{x}\right)$

$\Rightarrow \quad m_{n_2} = \dfrac{-1}{\left(-1/2\right)} = \dfrac{2}{1} \ldots \left(\dfrac{y}{x}\right)$

$\Rightarrow \quad n_2 = (1, 2)^T$

$\Rightarrow \quad n_2 = \left(\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)^T$

n_2 is directed towards I quadrant but we need normal $\bar{v}$ to III quadrant for

n_2 to be going outside facet F_2.

Thus, $\boxed{n_2 = \left(\frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)}$ (unit normal $\bar{v}$ to F_2)

Similarly, $m_{F_3} \cdot m_{n_3} = -1$

$$m_{F_3} = \frac{0-1}{-1-1} = \frac{-1}{-2} = \frac{1}{2} \cdots \frac{y}{x}$$

$$\therefore \quad m_{n_3} = \frac{-2}{1} \cdots \frac{y}{x}$$

$$\Rightarrow \quad n_3 = (1, -2)^T$$

$$\Rightarrow \quad n_3 = \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)^T$$

n_3 is directed towards quadrant IV but for n_3 to go outwards of facet F_3, it must be directed towards quadrant II.

$\therefore$ We choose $\boxed{n_3 = \left(\frac{-1}{\sqrt{5}}, +\frac{2}{\sqrt{5}}\right)^T}$

2. Vertices – $v_1 \; v_2 \; v_3$; $M=3$

Facets – $F_1 \; F_2 \; F_3$

V_i = set of vertices of facet F_i

$V_1 = \{v_2, v_3\}$

$V_2 = \{v_1, v_3\}$

$V_3 = \{v_1, v_2\}$

W_j = set of facets for which v_j is vertex

$W_1 = \{F_2, F_3\}$

$W_2 = \{F_1, F_3\}$

$W_3 = \{F_1, F_2\}$

Facet F_1 is solvable $\Rightarrow \exists \; u_1, u_2, u_3 \in U$ s.t. below necessary conditions are valid.

<u>Necessary Conditions</u> :

① $\forall \; v_j \in V_1$: $V_1 = \{v_2, v_3\}$

1. $n_1^T(Av_j + Bu_j + a) > 0$;

$$\Rightarrow \boxed{\begin{aligned} n_1^T(Av_2 + Bu_2 + a) &> 0 \\ n_1^T(Av_3 + Bu_3 + a) &> 0 \end{aligned}}$$

2. $\forall\, \underbrace{F_i \in W_j \setminus \{F_1\}}$, $n_i^T(Av_j + Bu_j + a) \le 0$

$j=2$: $\{F_1, F_3\} \setminus \{F_1\} = \{F_3\}$

$$\Rightarrow \boxed{n_3^T(Av_2 + Bu_2 + a) \le 0}$$

$j=3$: $\{F_1, F_2\} \setminus \{F_1\} = \{F_2\}$

$$\Rightarrow \boxed{n_2^T(Av_3 + Bu_3 + a) \le 0}$$

2 $\forall j \in \{1,2,3\} \setminus v_1$: $\{v_1, v_2, v_3\} \setminus \{v_2, v_3\} = \{v_1\}$

1. $\forall i \in W_j$: $n_i^T(Av_j + Bu_j + a) \le 0$

$W_1 = \{F_2, F_3\}$, $\forall F_i \in W_1$,

$$\boxed{\begin{aligned} n_2^T(Av_1 + Bu_1 + a) &\le 0 \\ n_3^T(Av_1 + Bu_1 + a) &\le 0 \end{aligned}}$$

and

2. $\sum_{i \in W_j} n_i^T(Av_j + Bu_j + a) < 0$

$$\boxed{n_2^T(Av_1 + Bu_1 + a) + n_3^T(Av_1 + Bu_1 + a) < 0}$$

The above boxed equations are the set of inequalities to be satisfied for u_1, u_2, u_3 such that the controller steers any point on the polytope to go out through facet F_1.

Rearranging the inequalities, we have:

u_1:
$$n_2^T B u_1 \leq -n_2^T (Av_1 + a) \Rightarrow u_1 \leq 5/3$$
$$n_3^T B u_1 \leq -n_3^T (Av_1 + a) \Rightarrow u_1 \geq 1$$

$$n_2^T B u_1 + n_3^T B u_1 < -n_2^T (Av_1 + a) - n_3^T (Av_1 + a)$$
$$\Rightarrow u_1 < 2$$

u_2:
$$n_1^T B u_2 > -n_1^T (Av_2 + a) \Rightarrow u_2 < 1/2$$
$$n_3^T B u_2 \leq -n_3^T (Av_2 + a) \Rightarrow u_2 \geq -\frac{1}{2}$$

u_3:
$$n_1^T B u_3 > -n_1^T (Av_3 + a) \Rightarrow u_3 < 3/2$$
$$n_2^T B u_3 \leq -n_2^T (Av_3 + a) \Rightarrow u_3 \leq -1/6$$

The inequalities are solved in the attached MATLAB script.

Using these limits, we can find u_1, u_2, u_3 within the below specified regions such that the earlier mentioned necessary conditions are satisfied. Then an affine feedback control law can be computed such that sufficient conditions are also satisfied.

$u_1 \in [1, 5/3]$;

$u_2 \in [-1/2, 1/2)$;

$u_3 \leq -1/6$;

As, u_1 u_2 u_3 exist in the above mentioned regions. In our case, a controller exists such that the necessary conditions and sufficient conditions are satisfied.

3. select u_1, u_2, u_3 such that

$$u_1 \in [1, 5/3] \;;$$
$$u_2 \in [-1/2, 1/2) \;;$$
$$u_3 \leq -1/6 \;;$$

Let $u_1 = 1$, $u_2 = 0$, $u_3 = -1/6$ be the input values.

We have $u = f(x) = Fx + g$

Solve the above using eq(8) from the paper.

$$\begin{pmatrix} v_1^T & 1 \\ \vdots & \vdots \\ v_{N+1}^T & 1 \end{pmatrix} \begin{pmatrix} F^T \\ \hline g^T \end{pmatrix} = \begin{pmatrix} u_1^T \\ \vdots \\ u_{N+1}^T \end{pmatrix} \quad \ldots \; (8)$$

$$\Rightarrow \begin{pmatrix} v_1^T & 1 \\ v_2^T & 1 \\ v_3^T & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ g \end{pmatrix} = \begin{pmatrix} u_1^T \\ u_2^T \\ u_3^T \end{pmatrix}$$

$$\text{or} \quad \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ g \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{6} \end{pmatrix}$$

Solving for F and g from MATLAB, we have

$$\boxed{\begin{aligned} f_1 &= -0.5417 \\ f_2 &= 0.0833 \\ g &= 0.4583 \end{aligned}}$$

Therefore, the affine feedback controller is given by

$$\boxed{u = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + g}$$; where f_1, f_2, g are taken from above result

Now substituting for u in system equation –

$$\dot{x} = Ax + Bu + a$$

$$\dot{x} = Ax + B(Fx + g)\ a$$

$$\boxed{\dot{x} = (A + BF)x + (Bg + a)}$$

The above equation could be forward integrated using ODE 45 to get the state trajectory of the system and this trajectory must exit the polytope from facet F_1.

This equation has been implemented in MATLAB to show that the affine control law steers the system to exit through facet F_1.