Lecture notes: Abstraction of continuous systems

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Outline

This lecture note is mostly based on

- Kloetzer, Marius, and Calin Belta. ”A fully automated framework for control of linear systems from LTL specifications.” HSCC. Vol. 3927. 2006. APA.
- Habets, L. C. G. J. M., and Jan H. Van Schuppen. ”A control problem for affine dynamical systems on a full-dimensional polytope.” Automatica 40.1 (2004): 21-35.

Problem statement

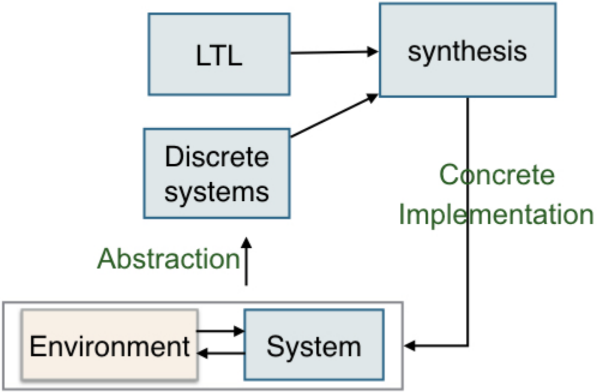
- **Given** a linear system

$$\dot{x} = Ax + a + Bu$$

  where b is a constant vector, with polyhedral control constraint U.
- a specification LTL φ over a set of **linear predicates** in X
- **Compute**: All initial states X_{INIT} and a feedback controller so that the corresponding trajectory, for any initial state in X_{INIT} of the closed-loop system satisfy φ, while staying inside a given full-dimensional polytope P.

Challenge: Infinite-state space transition systems. Require closed-loop satisfaction. Cannot use any existing planning algorithm for discrete systems.

Planning and control framework for continuous systems

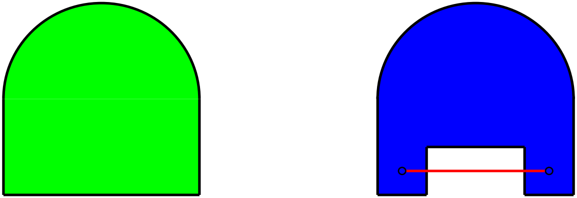


Polytopes

A full-dimensional polytope P is defined as the **convex hull** of at least $N+1$ affinely independent points in $\mathbf{R}^N$.

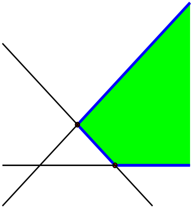
def 1: a **convex hull** of a set of points is the smallest **convex set** that containing this set of points.

def 2: a **convex set** is a subset of an affine space that is closed under convex combinations.

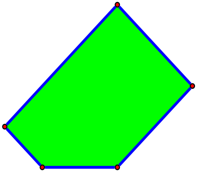


(a) (b)

A full-dimensional polytope P is defined as the **convex hull** of at least $N+1$ affinely independent points in $\mathbf{R}^N$.



(a)



(b)

Two ways to describe a polytope:

- $\mathcal{H}$ -polyhedron: The intersection of a finite number of closed half-spaces;

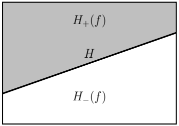
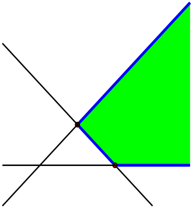


Figure 2.2: The two half-spaces determined by a hyperplane, H

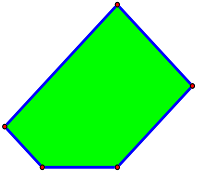
$$H_+ = \{x \mid a^T x - b \geq 0\}, \quad H_- = \{x \mid a^T x - b \leq 0\}$$

- $\mathcal{H}$- polytope: A closed $\mathcal{H}$ -polyhedron.

- $\mathcal{V}$-polytope: the convex hull of a set of points.



(a)

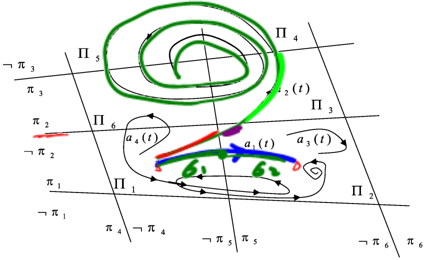


(b)

A full dimensional polytope with $N+1$ vertices and thus $N+1$ facets is called a full-dimensional **simplex.**
So... how these are related to control?

- Atomic propositions: $x \models p_i$ if $c_i^T x < d_i$, where $c_i \in \mathbf{R}^N$ and $d_i \in \mathbf{R}$. i.e., the truth/false evalution of an atomic proposition — define a hyperplane.
- A subset of AP: $x \models \{p_1, p_2\}$ defines a polyhedron.

Labeling function and behavior



$\neg\pi_4 \wedge \pi_1 \wedge \neg\pi_5 \wedge \neg\pi_6 \wedge \neg\pi_2 \wedge \pi_3$

$\{\pi_1, \pi_3\}$

$\pi_1 \wedge \neg\pi_2 \wedge \pi_3 \wedge \neg\pi_4 \wedge \pi_5 \wedge \neg\pi_6$

$\{\pi_1, \pi_3, \pi_5\}$

$\Box\pi_1 \wedge \Diamond\pi_5$

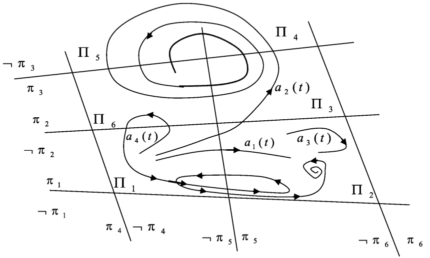
where π_i is atomic proposition.

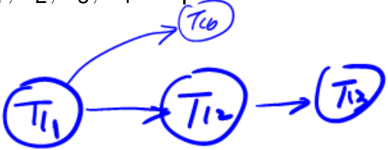
$\phi(x_0, [0, T], u)$ refers to the trajectory of the continuous system with initial condition x_0 under the control input $u(t)$ over the time interval $[0, T]$. Let $t_0, t_1, \ldots, t_N$ be times, such that

- $0 = t_0 < t_1 < \cdots < t_N = T$,
- $L(x(t)) = L(x(t_k))$, $t_k \le t < t_{k+1}$, $k = 0, \ldots, N$,
- $L(x(t_k^-)) \neq L(x(t_k^+))$, $k = 0, \ldots, N$.

The **discrete behavior**, denoted $Beh(\phi(x_0, [0, T], u))$, is the discrete word $\sigma_0\sigma_1 \ldots \sigma_{N-1} \in \Sigma^*$, where $\sigma_k = L(x(t_k))$.

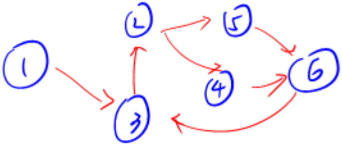
Question: What are the trajectories a_1, a_2, a_3, a_4 map to?





$\forall x \in \Pi_1.\ \exists u = f(x)$

st. $\phi(t, u, x)$ enters Π_2



Finding the transition systems

- Each polytope can be treated as a discrete state: Two polytopes share the same facet F_{12}
- A transition from polytope 1 to polytope 2 can be **enabled** by finding a controller u_{12} such that: For **any** state x_0 in 1, applying u_{12} enables the trajectory $\phi(x_0, [0, T], u_{12})$ exits through facet F_{12} in some finite time T.
- Otherwise, no transition from 1 to 2.

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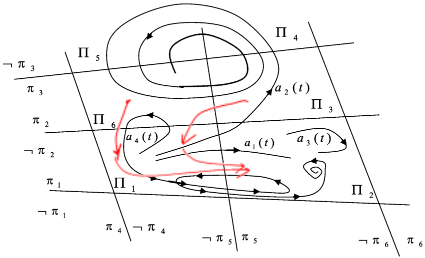
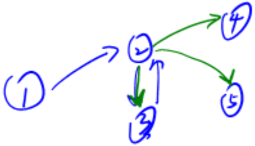
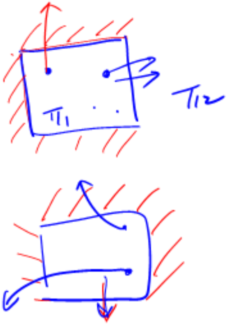
Constructing the transition system = Finding pairwise controllers for all polytopes that sharing facet —— **control to facet** problem.

Labeling function and behavior

illustrate the transition system:



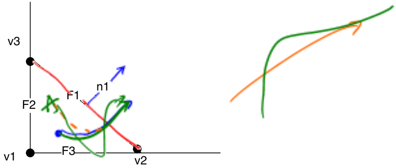




Control to facet

Problem: Consider the affine system

$$\dot{x} = Ax + a + Bu$$



on the full-dimensional polytope P ,
Let F_j be a facet of P, decide if there exists a feedback controller $u = f(x)$ such that for any $x(0) \in P$,

- $\forall t \in [0, T]$, $x(t) \in P$;
- there exists a time T, such that $x(T)$ reaches the facet F_j at a finite time T.
- and $n_j^T \dot{x}(T) > 0$, i.e., the velocity of the system at the time T has a positive component in the normal of facet n_j.

Furthermore, we aim to find a piecewise affine control

$$u = Fx + g, \quad F \in \mathbf{R}^{m\times N}, \quad , g \in \mathbf{R}^{m}.$$

Closed-loop dynamics:

Proposition:
Let P be a full-dimensional polytope with vertices $v_1, \ldots, v_M$, $M \geq N+1$, with facets $F_1, \ldots, F_K$, let V_i be the set of vertices of the facets F_i, and W_j be the set of facets for which v_j is a vertex. If the problem of exiting facet F_1 is solvable, then there exists $u_1, \ldots, u_M \in U$ such that

1. for all $v_j \in V_1$:
   1. $n_1^T(Av_j + Bu_j + a) > 0$;
   2. for all $F_i \in W_j \setminus \{F_1\}$, $n_i^T(Av_j + Bu_j + a) \leq 0$.
2. for all $v_j \in V \setminus V_1$:
   1. for all $F_i \in W_j$, $n_i^T(Av_j + Bu_j + a) \leq 0$.
   2. $\sum_{F_i \in W_j} n_i^T(Av_j + Bu_j + a) < 0$.

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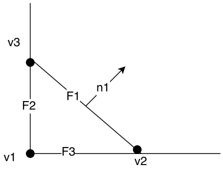


Theorem: Let P be a full-dimensional polytope with vertices $v_1, \ldots, v_M$, $M \geq N+1$, with facets $F_1, \ldots, F_K$, let V_i be the set of vertices of the facets F_i, and W_j be the set of facets for which v_j is a vertex. If there exists a Lipschitz function $f : P \to U$ such that

1. $\forall x \in P$, $n_1^T(Ax + Bf(x) + a) > 0$;
2. $\forall i = \{2, \ldots, K\}, \forall x \in F_i$: $n_i^T(Ax + Bf(x) + a) \leq 0$

proof:

Controller design

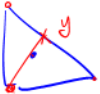
First, a property of polytope: there exists a piecewise-affine function $\xi : P \to \Lambda$ where Λ is a convex set defined by

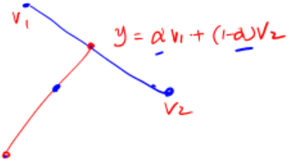
$$\Lambda = \{(\lambda_1, \ldots, \lambda_M) \in [0,1]^{N+1} \mid \sum_{j=1}^{N+1} \lambda_j = 1\}$$

For any $x \in P$,

$$x = \sum_{i=1}^{N+1} \xi(x)_j v_i$$

$$x = \sum_{i=1}^{3} \underline{\underline{\xi_i(x)}}\, v_i$$





$\underline{\underline{u_i}}$:

$$f(x) = \sum_{i=1}^{3} \xi_i(x)\, u_i$$

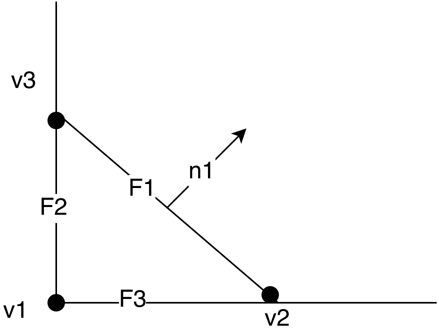
$$n_1^T (Ax + Bf(x) + a) > 0$$

$$\forall i \neq 1, \quad \forall x \in F_i, \quad n_i^T (Ax + Bf(x) + a) \leq 0$$

$$n_1^T \left(A \sum_{i=1}^{3} \xi_i(x) V_i + B \sum_{i=1}^{3} \xi_i(x) u_i + \underbrace{a}_{\sum_{i=1}^{3} \xi_i(x) a} \right)$$

$$\sum_{i=1}^{3} \xi_i(x) \left[n_1^T (A V_i + B u_i + a) \right] > 0$$

idea: A polytope can be triangulated into simplex. Every point in a simplex can be represented as linear combination of its vertices.



Controller design

Controller generated from the necessary condition: if $x = \sum_{j=1}^{N+1} \xi_j(x) v_j$, for each vertex v_j, there is a control input u_j that satisfies the constraints.

$$f(x) = \sum_{j=1}^{N+1} \xi_j(x) u_j$$

proof: For every state x, there exists $\xi_j(x)$ such that $x = \sum_{j=1}^{N+1} \xi_j(x) v_j$.

$$\begin{aligned}
& n_1^T(Ax + Bf(x) + a) \\
=& n_1^T(A \sum_{j=1}^{N+1} \xi_j(x) v_j + B \sum_{j=1}^{N+1} \xi_j(x) u_j + \sum_{j=1}^{N+1} \xi_j(x) a) \\
=&
\end{aligned}$$

From necessary to sufficient conditions.

Abstraction of transition system

1. Partitioning the state space using atomic propositions defined hyperplanes.
2. For each polytope, for each facet of the polytope, decide the existance of a control-to-facet controller.
3. A transition between Π_i and Π_j is enabled if there exists a control-to-facet controller u_{ij} for facet F_{ij}, belonging to Π_i and Π_j.
4. Construct the transition system with all **enabled** transitions.
5. If there exists a path in the TS to satisfy the specification —- there exists a piecewise continuous controller to satisfy the specification in the original continuous system.