Assignment II

MEMAD

August 24, 2025

Abstract

The goal is to revise some relevant theory and key results from vector calculus, critical values of functions and numerical optimization. Please upload your solutions in one compressed file to Classroom before September 2nd.

Problem 1

For each of the following functions, find the maximum and minimum values on the indicated intervals, by finding the points in the interval where the derivative is 0, and comparing the values at these points with the values at the endpoints.

(a) $f(x) = x^3 - x^2 - 8x + 1 \quad \text{on } [-2, 2].$

(b) $f(x) = \frac{x}{x^2-1} \quad \text{on } [0, 5].$

Problem 2

If $a_1 < a_2 < \cdots < a_n$, find the minimum value of

$$f(x) = \sum_{i=1}^{n} (x - a_i)^2.$$

Problem 3

What is the relationship between the critical points of f and those of f^2?

Problem 4

Find, among all right circular cylinders of fixed volume V, the one with smallest surface area.

Problem 5

Show that the sum of a positive number and its reciprocal is at least 2.

Problem 6

For each of the following functions, find their critical points and classify them. Then use Python to make contour plots of them to verify your results.

(a) $f(x, y) = \ln(x^2 + y^2 + 1)$.

(b) $f(x, y) = x^2 + y^2 - x - y + 1$.

(c) $f(x, y) = e^x \cos(y)$.

(d) $f(x, y) = (x^2 + y - 11)^2 + (x + y^2 - 7)^2$.

Problem 7

Show that of all rectangular parallelepipeds with a given surface, the cube has the greatest volume.

Problem 8

Find out if the following matrices are positive definite.

(a)

$$A = \begin{pmatrix} 16 & -8 & -4 \\ -8 & 29 & 12 \\ -4 & 12 & 41 \end{pmatrix}.$$

(b)

$$A = \begin{pmatrix} 9 & 0 & -8 \\ 6 & -5 & -2 \\ -9 & 3 & 3 \end{pmatrix}.$$

(c)

$$A = \begin{pmatrix} -9 & 0 & -8 \\ 6 & -5 & -2 \\ -9 & 3 & 3 \end{pmatrix}.$$

Problem 9

Consider the following functions:

$$f(\mathbf{x}) = \sum_{i=1}^{n} x_i^2. \tag{1}$$

$$f(x, y) = (x^2 + y - 11)^2 + (x + y^2 - 7)^2. \tag{2}$$

$$f(\mathbf{x}) = \sum_{i=1}^{n-1} \left[100(x_{i+1} - x_i^2)^2 + (x_i - 1)^2\right]. \tag{3}$$

(a) Make a script in Python to approximate their gradients and Hessians, evaluated in a particular point $\mathbf{x}$, utilizing finite differences.

(b) Find their minimum points by inspection.

(c) Use your script from (a) to verify numerically that the points you have found in (b) are minimum points of their corresponding functions. For functions (1) and (3), you can assume $n = 5$.

Problem 10

Consider Newton's method for function minimization given by the iterative formula:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - [\nabla^2 f(\mathbf{x}_k)]^{-1} \nabla f(\mathbf{x}_k), \quad k = 0, 1, 2, \cdots, K,$$

Code Newton's method in a script in Python. Then use it to optimize the functions from problem 9 using close enough starting points (propose them and also propose a value for K). For functions (1) and (3), you can assume $n = 5$. For each function, make a plot of how the function's value decreases as the algorithm iterates (horizontal axis corresponds to k and vertical axis to $f(\mathbf{x}_k)$). Finally, print in your script the approximation $\mathbf{x}_K$ you achieve.