Models of Computation: Quantum Computing

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Crash Course on Physical Theories

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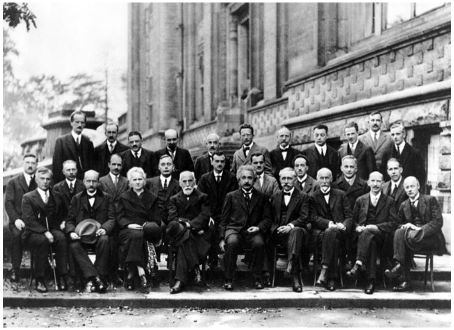

Figure: The 1927 Solvay Conference in Brussels

Computer Design

- Modern computers operate by manipulating electromagnetic processes in electronic circuits.
- However, electronic circuits become smaller and smaller and start exhibiting quantum phenomena.
- What happens when our computational hardware becomes so small that it is fully quantum?

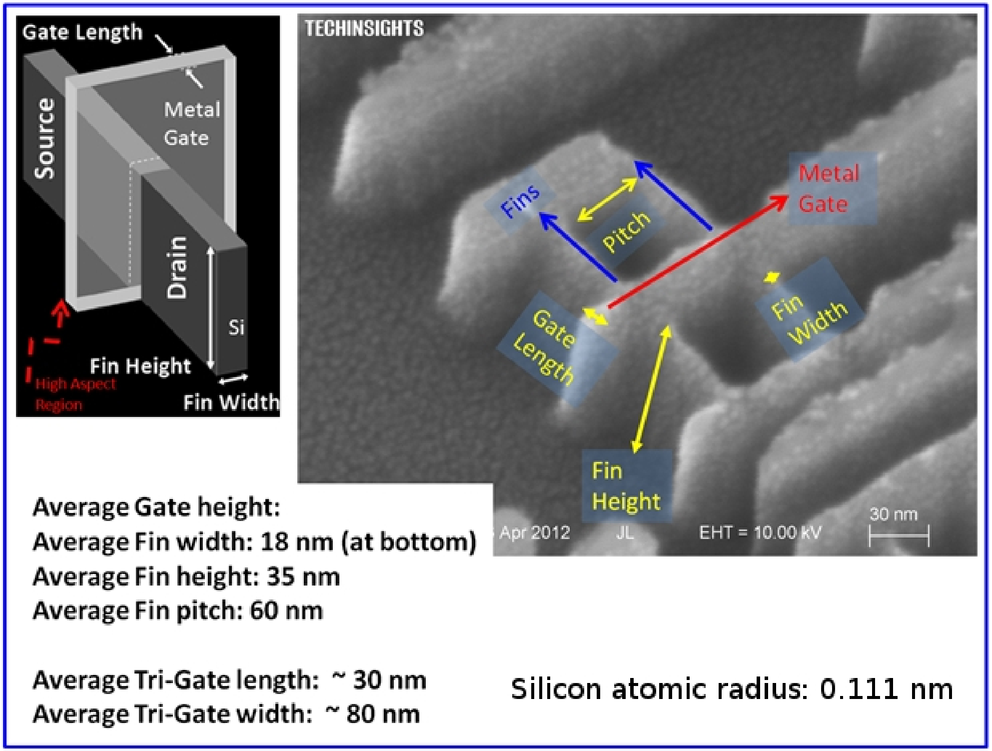

Figure: Intel 22-nm Tri-Gate device

Classical Computing

- Classical computers (laptops, phones, etc.) manipulate classical information (bits) in order to perform computation.
- Classical information is described using classical information theory which is a mathematical model that assumes the world is explained using classical physics.
- This is a perfectly reasonable assumption to make for our current hardware.

Quantum Computing

- Consider a computer so small that it can manipulate simple quantum systems called qubits (quantum bits).
- The underlying mathematical model is now different as it is based on quantum physics.
- Processing of quantum information (qubits) is as a result fundamentally different.
- The speed of certain computations is also faster in some cases.

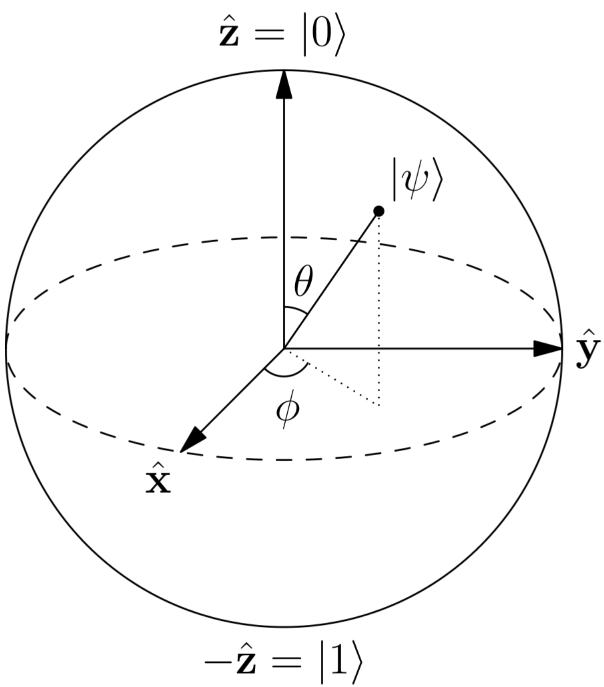

Figure: Bloch-sphere representation of a qubit state.

Quantum Entanglement – important resource

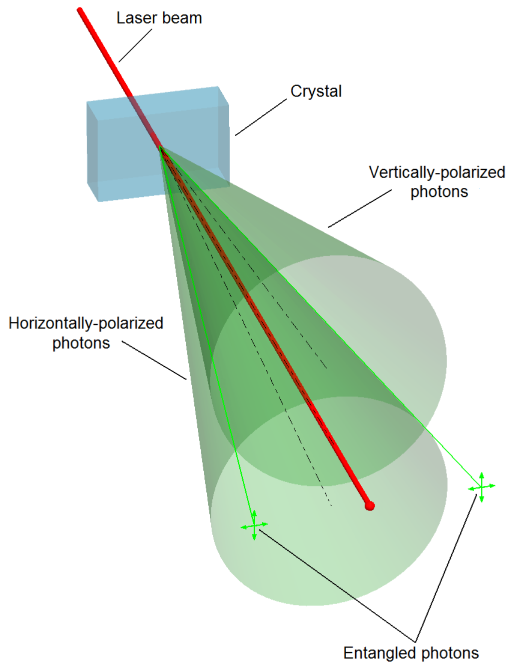

Figure: Illustration of quantum optics experiment which produces entanglement

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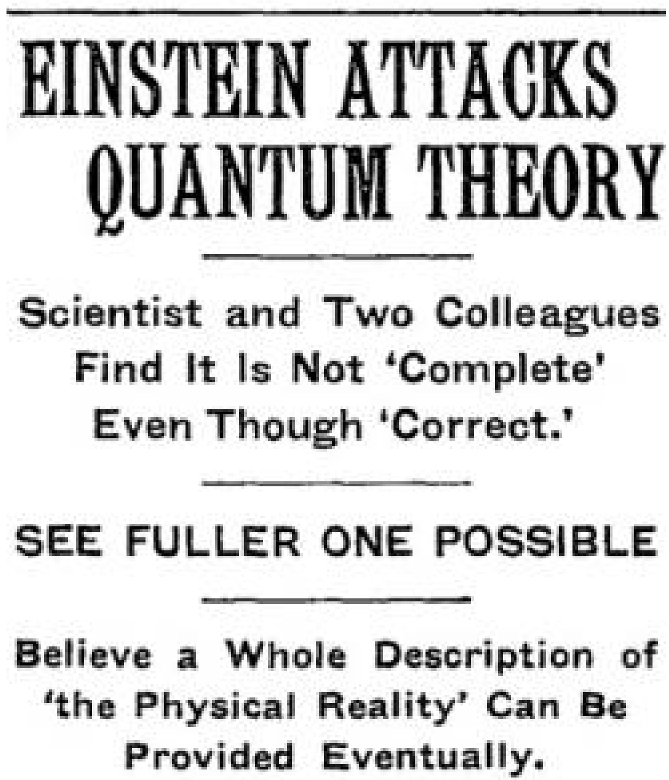

EINSTEIN ATTACKS QUANTUM THEORY

Scientist and Two Colleagues Find It Is Not 'Complete' Even Though 'Correct.'

SEE FULLER ONE POSSIBLE

Believe a Whole Description of 'the Physical Reality' Can Be Provided Eventually.

Figure: May 4, 1935 *New York Times* article headline regarding the imminent EPR paper.

Quantum Entanglement – important resource

- Quantum entanglement is a special kind of correlation between systems which allows them to exhibit similar properties, even when space-time seperated.
- Einstein famously referred to it as: "Spooky action at a distance".
- Schrödinger described it as: "I would not call entanglement one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought.".
- Quantum entanglement is a crucial resource for quantum computing and also for many quantum information security protocols.

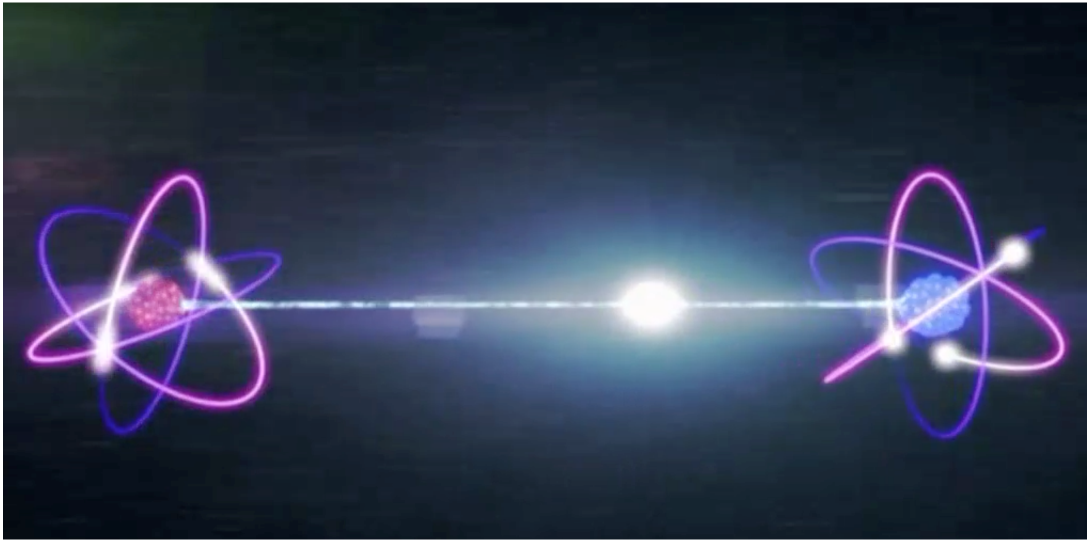

Figure: A most likely inaccurate illustration of quantum entanglement.

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- In the classical case where all actors have classical computers and use classical communication channels, we get computational security (this is the case for encryption).
- In the quantum case where all actors have quantum computers and use quantum communication channels, we get unconditional security.
- In the quantum case eavesdropping can be detected, but in the classical case it cannot.

Quantum Superposition – important resource

Very roughly speaking: a quantum system may be in many different states at the same time.

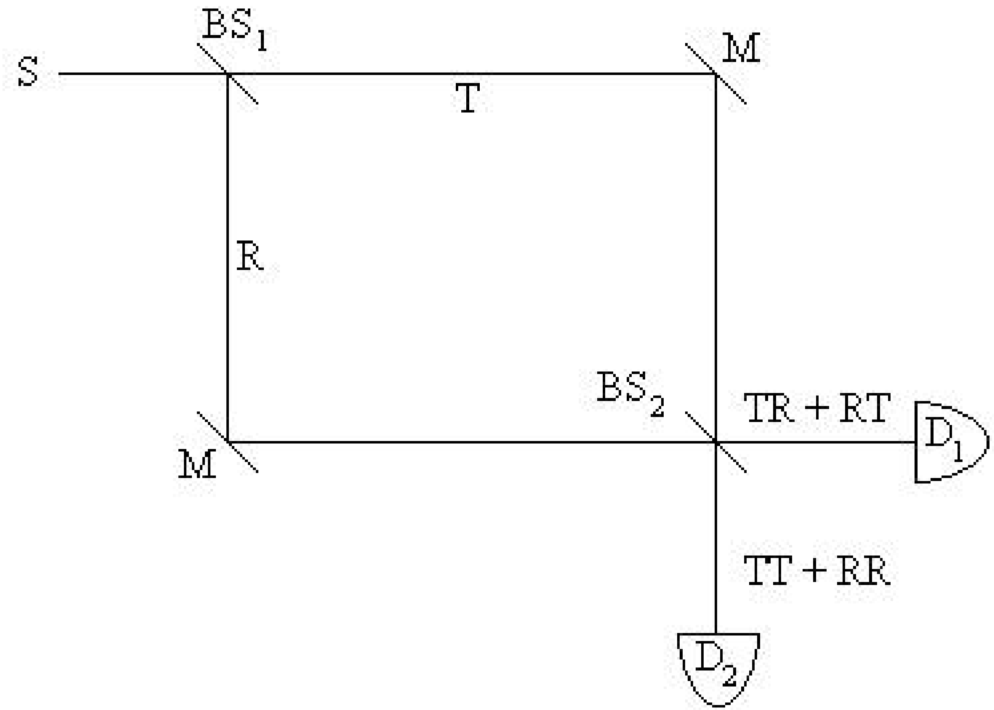

Figure: Single-photon interference performed with a Mach-Zehnder interferometer.

- Very rough analogy: allows for exponential parallelism.
- Crucial for computational speedup.

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- Improved computational complexity for many practical problems.
- Many other improved algorithms are known, but the above two are the most famous.

About the course

- Required background: some basic linear algebra.
- This course is *not* about quantum physics. We cover quantum *computation*.
  - Example: you do not have to know anything about electromagnetism to study classical computation.
- We will cover only basic concepts, but enough to get you started for more advanced study/research/work.

Some extra material

- Almost all the material you need will be on the slides.
- If you want to learn more:
  - Lecture notes from Bob Coecke: `www.cs.ox.ac.uk/people/bob.coecke/QCS.pdf`. I recommend reading the notes if there are things you do not understand from the slides/lectures.
  - Book: Bob Coecke and Aleks Kissinger: *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*. Cambridge University Press 2017.
  - Book: N.D. Mermin, *Quantum Computer Science*. Cambridge University Press 2007.
  - Book: M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press 2000.

Complex Numbers (Recap)

- Recall that a complex number is a number of the form $z = a + ib$, where $a, b \in \mathbb{R}$.
- The number a is the *real* part of z and the number b is the *imaginary* part of z.
- The *imaginary unit* is the complex number i, which satisfies $i^2 = -1$.
- Every real number a may be seen as a complex number with imaginary part 0.
- The complex numbers admit a geometric representation using cartesian coordinates in the complex plane.
- The absolute value of a complex number $z = a + ib$ is defined as $|z| \stackrel{\text{def}}{=} \sqrt{a^2 + b^2}$.
- Addition of complex numbers is given by $(a + bi) + (c + di) = (a + c) + (b + d)i$.
- Multiplication of complex numbers is given by $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.
- The conjugate complex number of $z = a + bi$ is the number $\overline{z} \stackrel{\text{def}}{=} a - bi$.
- **Euler's formula:** $e^{i\varphi} = \cos\varphi + i\sin\varphi$, for any $\varphi \in \mathbb{R}$.
- Every complex number z can also be expressed as $z = re^{i\varphi}$, where $r = |z| = \sqrt{z\overline{z}}$ and the argument φ is known as the *phase* (geometrically, it is the angle between the positive real axis and the complex number depicted on the complex plane).

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- $\overline{e^{i\varphi}} = e^{-i\varphi}$.
- $|e^{i\varphi}| = 1$.

Vector Spaces (Recap)

Definition

A *vector space* over the field of complex numbers $\mathbb{C}$ is a triple $(V, +, \cdot)$ consisting of a set V (the elements of which we refer to as *vectors*), a binary operation $+ : V \times V \to V$ called *vector addition* and a binary operation $\cdot : \mathbb{C} \times V \to V$ called *scalar multiplication* which satisfy the following axioms:

- **Commutativity.** For all vectors u and v in V, we have $u + v = v + u$.
- **Associativity.** For all vectors u, v and w in V, we have $(u + v) + w = u + (v + w)$.
- **Additive identity.** The set V contains an element, called the *zero vector* and denoted by 0, such that for any vector $v \in V$ we have $v + 0 = v$.
- **Additive inverses.** For any vector $v \in V$, there exists a vector $(-v) \in V$ which has the property that $v + (-v) = 0$.
- **Distributivity w.r.t. vector addition.** For every complex number $c \in \mathbb{C}$ and any vectors $u, v \in V$, we have $c \cdot (u + v) = (c \cdot u) + (c \cdot v)$.
- **Distributivity w.r.t. complex addition.** For every complex numbers $c, d \in \mathbb{C}$ and any vector $v \in V$, we have $(c + d) \cdot v = (c \cdot v) + (d \cdot v)$.
- **Compatability.** For all complex numbers $c, d \in \mathbb{C}$ and any vector $v \in V$, we have $c \cdot (d \cdot v) = (cd) \cdot v$.
- **Unitarity.** For any vector $v \in V$, we have $1 \cdot v = v$.

Remark

A few remarks:

- In this course we only consider finite-dimensional vector spaces over $\mathbb{C}$. From now on, this is implicitly assumed.
- The scalar multiplication $\cdot$ is usually written as juxtaposition, e.g., $3v \stackrel{\text{def}}{=} 3 \cdot v$.
- We write $u - v \stackrel{\text{def}}{=} u + (-v)$.

Vector Spaces (Recap)

Let us consider a few examples and non-examples of vector spaces.

- Any singleton set can be (uniquely) equipped with the structure of a vector space. Why?
- Can the empty set $\emptyset$ be equipped with the structure of a vector space?

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- The set of complex numbers $\mathbb{C}$ can be seen as a vector space when we define vector addition to coincide with addition of complex numbers and when we define scalar multiplication to coincide with multiplication of complex numbers.
- The set $\mathbb{C}^n$ of n-tuples of complex numbers can be equipped with the structure of a vector space when we define:

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}$$

$$c \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{pmatrix},$$

where $u, v \in \mathbb{C}^n$ and $c \in \mathbb{C}$. **This is the most important example of a vector space in this course! This structure is canonical and we will often implicitly assume it.**

Linear (in)dependence (Recap)

Definition

Given a vector space V and a finite index set I, then a set of vectors $\{v_i\}_{i \in I}$ in V is said to be *linearly dependent* if the equation

$$\sum_{i \in I} a_i v_i = 0$$

has a non-trivial solution, i.e., we can find a set of scalars $a_i \in \mathbb{C}$ which validate the above equation, such that at least one of the scalars a_i is different from 0. If a set of vectors $\{v_i\}_{i \in I}$ is not linearly dependent, then we say that this set of vectors is *linearly independent*.

Questions:

- If a set of vectors contains the zero vector, then is it linearly dependent?
- Let v_1 and v_2 be two vectors. When are these two vectors linearly dependent?
- Is a singleton set of vectors linearly independent?

Linear Spans (Recap)

- **Definition:** Given a vector space $(V, +, \cdot)$, a *linear subspace* of V is a subset $W \subseteq V$, such that $(W, +, \cdot)$ is a vector space.
- This is equivalent to requiring that $c_1 w_1 + c_2 w_2 \in W$ for every two vectors $w_1, w_2 \in W$ and $c_1, c_2 \in \mathbb{C}$.
- **Definition:** Given a vector space V, the span of a set S, denoted $\mathrm{span}(S)$, is defined to be the intersection of all subspaces of V that contain S.
- It follows that $\mathrm{span}(S)$ is a subspace of V and

$$\mathrm{span}(S) = \left\{ \sum_{i=1}^{m} c_i v_i \mid m \in \mathbb{N}, v_i \in S, c_i \in \mathbb{C} \right\}.$$

- In other words, the span of a set S is the linear subspace of V that contains all finite linear combinations of vectors in S.
- **Question:** What is the span of the set

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

when seen as a subspace of $\mathbb{C}^3$ (with its canonical vector space structure)?

Basis (Recap)

- **Definition:** A *basis* of a vector space V is a set B of linearly independent vectors whose span is V.
- **Theorem:** Every vector space has a basis. Furthermore, every two bases of the same vector space have the same cardinality.
- **Definition:** The *dimension* of a vector space V, denoted $\dim(V)$, is the cardinality of a basis of V.
- **Remark:** In this course we only consider vector space over $\mathbb{C}$ which have *finite* dimension.
- If $B = \{v_1, \ldots, v_n\}$ is a basis of V, it follows that every vector $v \in V$ can be *uniquely* expressed as a linear combination of the basis elements:

$$v = c_1 v_1 + \cdots + c_n v_n.$$

- In this situation, the ordered tuple of complex numbers c_i are called the *coordinates* of the vector v with respect to the basis B.
- When a basis is fixed, or implicitly understood, we can simply write

$$v = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

to denote this decomposition of v.

The Standard Basis of $\mathbb{C}^n$ (Recap)

- What is the dimension of the vector space $\mathbb{C}^n$?

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- What is the dimension of the vector space $\mathbb{C}^n$? $\dim(\mathbb{C}^n) = n$. Why?

The Standard Basis of $\mathbb{C}^n$ (Recap)

- What is the dimension of the vector space $\mathbb{C}^n$? $\dim(\mathbb{C}^n) = n$. Why?
- The *standard basis* of $\mathbb{C}^n$ is given by the set

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}.$$

Linear Maps (Recap)

- **Definition:** A function $f : V \to W$ between vector spaces V and W is said to be *linear* when
  - $f(v_1 + v_2) = f(v_1) + f(v_2)$; and
  - $f(a \cdot v) = a \cdot f(v)$,

  for any possible choice of a scalar $a \in \mathbb{C}$ and vectors $v, v_1, v_2 \in V$. We will also call linear functions by the names *linear maps* and *linear operators*.
- **Proposition:** Any linear map $f : V \to W$ is completely determined by its action on the basis elements. Indeed, writing v_i for the basis vectors of V, observe that:

$$f(v) = f(c_1 v_1 + \cdots + c_n v_n) = c_1 f(v_1) + \cdots + c_n f(v_n).$$

- **Proposition:** Any complex $m \times n$ matrix A determines a linear function $f_A : \mathbb{C}^n \to \mathbb{C}^m$ by setting $f_A(v) := Av$. Hint: recall how matrix multiplication works.
- **Proposition:** Conversely, every linear function $f : \mathbb{C}^n \to \mathbb{C}^m$ is completely determined by a $m \times n$ complex matrix A, such that $f_A(v) = f(v)$.
- Therefore, we may think of complex $m \times n$ matrices and linear functions $f : \mathbb{C}^n \to \mathbb{C}^m$ interchangeably and we will often do so from now on.
- **Proposition:** Any two vector spaces of the same dimension are isomorphic.

Composing Linear Maps and Matrix Multiplication (Recap)

- Let A be an $k \times m$ complex matrix and B an $m \times n$ complex matrix. Then $C = AB$, the matrix obtained via matrix multiplication, is a $k \times n$ matrix.
- Recall that the c_{ij} entry of C is given by the dot product of the i-th row of A and the j-th column of B.
- Matrix multiplication represents composition of linear functions. That is, if $f : W \to U$ is a linear map represented by the matrix A and $g : V \to W$ is a linear map represented by the matrix B, then the composition $f \circ g : V \to U$ is represented by the matrix $C = AB$.
- Composition of linear maps (and therefore multiplication of matrices) is associative, but not commutative, in general.
- **Question:** What is the matrix representation of the identity linear map on $\mathbb{C}^n$?
- **Exercise:** matrix multiplication.

Hilbert Spaces (Recap)

We want to equip vector spaces with some additional structure that will allow us to:

- measure the length of vectors.
- measure the angle between vectors.
- measure distances in the vector space.

We arrive at the concept of a Hilbert space.

Definition

A *finite-dimensional Hilbert space* is a finite-dimensional vector space $\mathcal{H}$ over the complex number field $\mathbb{C}$ which comes equipped with an *inner-product*, i.e., a map

$$\langle -, - \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C},$$

which satisfies the following properties:

- $\langle v, a_1 \cdot w_1 + a_2 \cdot w_2 \rangle = a_1 \langle v, w_1 \rangle + a_2 \langle v, w_2 \rangle$;
- $\langle v, w \rangle = \overline{\langle w, v \rangle}$;
- $\langle v, v \rangle \in \mathbb{R}$ and $\langle v, v \rangle \geq 0$;
- $\langle v, v \rangle = 0$ if and only if $v = \mathbf{0}$,

for any scalars $a_1, a_2 \in \mathbb{C}$ and any vectors $v, w, v_1, v_2, w_1, w_2 \in \mathcal{H}$.

From this definition follows:

$$\langle a_1 \cdot v_1 + a_2 \cdot v_2, w \rangle = \overline{a_1} \langle v_1, w \rangle + \overline{a_2} \langle v_2, w \rangle.$$

In other words, the inner product is linear in the second argument, but antilinear in the first.

Hilbert Spaces (Recap)

Recall that:

- The transpose of a matrix A is the matrix A^T with entries given by $a^T_{ij} \stackrel{\text{def}}{=} a_{ji}$, i.e., by swapping rows and columns.
- The conjugate of a matrix A is the matrix $\overline{A}$ with entries given by $\overline{a_{ij}} \stackrel{\text{def}}{=} \overline{a}_{ij}$, i.e., by entrywise conjugation.
- The conjugate transpose (also known as *adjoint*) of a matrix A is the matrix $A^\dagger$ given by $A^\dagger \stackrel{\text{def}}{=} \overline{A^T} = \overline{A}^T$.
- All of these definitions apply to vectors as special cases.

Proposition

The complex vector space $\mathbb{C}^n$ has the structure of a (finite-dimensional) Hilbert space when we define

$$\langle v, w \rangle := v^\dagger w.$$

Proof.

Exercise. $\square$

The Canonical Norm of a Hilbert Space

- Every Hilbert $\mathcal{H}$ space has a canonical norm $||-|| : \mathcal{H} \to \mathbb{R}_{\geq 0}$ defined by $||v|| \stackrel{\text{def}}{=} \sqrt{\langle v, v \rangle}$.
- The norm can be used to measure the length of vectors.
- This norm satisfies the usual properties of a norm, namely:
  - $||c \cdot v|| = |c|||v||$, where $c \in \mathbb{C}$ and $v \in \mathcal{H}$.
  - $||v + w|| \leq ||v|| + ||w||$, where $v, w \in \mathcal{H}$.
  - $||v|| = 0$ iff $v = 0$.

- **Exercise:** What is the norm of a vector in $\mathbb{C}^n$?

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  - $||v + w|| \leq ||v|| + ||w||$, where $v, w \in \mathcal{H}$.
  - $||v|| = 0$ iff $v = 0$.
- **Exercise:** What is the norm of a vector in $\mathbb{C}^n$? **Answer:** If $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n$ then

$$||v|| = \sqrt{\langle v, v \rangle} = \sqrt{v^\dagger v} = \sqrt{\sum_i |v_i|^2}$$

- **Definition:** A vector is said to be normalised whenever $||v|| = 1$.

Orthonormal Basis of a Hilbert Space

- An *orthonormal basis* of a Hilbert space $\mathcal{H}$ is a basis $B = \{v_1, \ldots, v_n\}$ of $\mathcal{H}$ (when seen as a vector space) such that:
$$\langle v_i, v_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$
- From now on, when we speak of a basis of a Hilbert space, we will implicitly assume the basis is orthonormal.
- **Exercise:** what is an orthonormal basis of $\mathbb{C}^n$? Are there more than one such bases? Can you think of a basis which is not orthonormal?

Adjoints

- **Theorem:** Let $f : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear map between Hilbert spaces. Then, there exists a unique linear map $f^\dagger : \mathcal{H}_2 \to \mathcal{H}_1$, such that

$$\langle v, f(w) \rangle = \langle f^\dagger(v), w \rangle$$

  for all vectors $v \in \mathcal{H}_2$ and $w \in \mathcal{H}_1$.
- The map $f^\dagger$ above is called the *adjoint* of f.
- If A is the matrix corresponding to the linear map f, then $A^\dagger$ (the conjugate transpose of A) is the matrix corresponding to the linear map $f^\dagger$.
- Note that $(f^\dagger)^\dagger = f$ and that $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$. **Exercise:** Can you prove these facts without using the matrix representation?

Unitary Maps

- **Definition:** Given a Hilbert space $\mathcal{H}$, a linear map $f : \mathcal{H} \to \mathcal{H}$ is said to be *unitary* if $f \circ f^\dagger = \mathrm{id} = f^\dagger \circ f$, where $\mathrm{id} : \mathcal{H} \to \mathcal{H}$ is the identity linear map.
- The same definition can be used to define a *unitary matrix*. That is, a complex matrix A is said to be unitary if $AA^\dagger = I = A^\dagger A$, where I is the identity matrix.
- **Theorem:** A map $f : \mathcal{H} \to \mathcal{H}$ is unitary iff f and $f^\dagger$ preserve the inner product:

$$\langle f(v), f(w) \rangle = \langle v, w \rangle \qquad \text{and} \qquad \langle f^\dagger(v), f^\dagger(w) \rangle = \langle v, w \rangle$$

- Note that this theorem implies that a unitary map preserves the norm as well.
- Unitary maps can be used to change (orthonormal) bases. That is, if $\{v_1, \ldots, v_n\}$ is an orthonormal basis of $\mathcal{H}$ and $f : \mathcal{H} \to \mathcal{H}$ is a unitary map, then $\{f(v_1), \ldots, f(v_n)\}$ is also an orthonormal basis of $\mathcal{H}$. **Exercise:** prove this.

Quantum Preliminaries

- *"Anyone who is not shocked by quantum theory has not understood it."* – Niels Bohr.
- Quantum theory was given its mathematical formalism mostly by John von Neumann in 1920s–1930s.
- This formalism is known as the "Hilbert Space Formalism" and this is what we introduce.
- We will only consider it for finite-dimensional spaces and we assume that we have full and perfect control of the underlying quantum systems. These are common assumptions in quantum computing.
- Even under those simplifying assumptions, the notion of a *quantum state* is very different from classical states. For example:
  - Quantum states can be combined via **superposition**.
  - Composite quantum systems cannot always be decomposed into simpler parts – a state of a composite system is *not* necessarily determined by the states of its components. In this situation, the state is **entangled**.
  - Quantum states cannot be copied, in general. This is known as the **no cloning theorem**.
  - Quantum states cannot be read off in the same way as classical states. One can only perform **quantum measurements** on quantum states which *change* the state that is being measured.
  - Performing the same measurement on the same state does *not* always produce the same result. The outcomes of quantum measurements are **probabilistic**.
  - Quantum systems may exhibit **non-local correlations** due to the possibility of entanglement, even when space-time separated. The resulting probability distributions *cannot* be explained via classical statistical mechanics.
  - In order to extract (classical) information from a quantum system, we have to perform quantum measurements on it, thereby changing its previous state.

Quantum bits (qubits)

The simplest (and most important) non-trivial quantum system is the *quantum bit*, often abbreviated to *qubit*.

Definition

The *state space of qubits* is given by the finite-dimensional Hilbert space $\mathbb{C}^2$. A *qubit* is described by a vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$ which is normalised in the sense that $|a|^2 + |b|^2 = 1$. Two unit (i.e. normalised) vectors $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{C}^2$ represent the same qubit iff they differ by a normalised complex multiple, i.e, if there exists $z \in \mathbb{C}$ with $|z| = 1$ such that $\mathbf{q}_1 = z \cdot \mathbf{q}_2$.

Example

The *zero qubit* is defined to be $|0\rangle \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The *one qubit* is defined to be $|1\rangle \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

These two qubits form an (orthonormal) basis of $\mathbb{C}^2$ known as the *computational basis*.

Remark

You can think of $|0\rangle$ and $|1\rangle$ as corresponding to the classical bits 0 and 1.

Exercise

How many states does a bit have? How many states can a qubit have?

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Remark

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Exercise

How many states does a bit have? How many states can a qubit have?
Answer: A bit has two possible states – 0 or 1. A qubit can be in *uncountably* many states.

Exercise: qubits

Which of the following vectors represent qubits? Which of these vectors represent the same qubit?

- $\begin{pmatrix} i \\ 0 \end{pmatrix}$
- $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$
- $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi} \\ e^{i\phi} \end{pmatrix}$, where $\phi \in [0, 2\pi)$.

Remark

Recall that $e^{i\phi} = \cos\phi + i\sin\phi$.

Superposition

Given an ONB $B = \{v_1, \ldots, v_n\}$ of a Hilbert space $\mathcal{H}$, we say that a vector of v of $\mathcal{H}$ is in *superposition* with respect to B iff the (unique) decomposition

$$v = \sum_{i=1}^{n} a_i v_i$$

has at least two non-zero coefficients a_i. Notice that the notion of superposition is relative to a basis.

Example

The *plus qubit* is defined to be $|+\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$.

The *minus qubit* is defined to be $|-\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$.

These two qubits also form an (orthonormal) basis of $\mathbb{C}^2$.
Both of these qubits are a non-trivial linear combination of $|0\rangle$ and $|1\rangle$ (and vice versa). Because of this, we say that $|+\rangle$ (and $|-\rangle$) is in superposition of $|0\rangle$ and $|1\rangle$ (and vice versa).

Exercise

- How can you express $|+\rangle$ in terms of $|0\rangle$ and $|1\rangle$?
- How can you express $|-\rangle$ in terms of $|0\rangle$ and $|1\rangle$?
- How can you express $|0\rangle$ in terms of $|+\rangle$ and $|-\rangle$?
- How can you express $|1\rangle$ in terms of $|+\rangle$ and $|-\rangle$?

Single-qubit unitary operations

- In quantum computer science, we assume that the time evolution of quantum systems are described by unitary operators and that we have full control of it.
- Therefore this evolution is *deterministic* and *reversible*.
- **Example:** Unitary operations on a single qubit are described by unitary matrices acting on $\mathbb{C}^2$.

Exercise

Consider the following matrices:

$$H \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{and} \qquad T \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}.$$

What is $H^\dagger$ and $T^\dagger$? Are these matrices unitary? Describe the action of H and T on the computational basis. Describe the action of H on the $\{|+\rangle, |-\rangle\}$ basis.

Single-qubit unitary operations

Definition

The *Hadamard gate* is a single qubit unitary operation defined by

$$H \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The T *gate* is a single qubit unitary operation defined by

$$T \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}.$$

We then have:

- $H^\dagger = H$.
- $H\left|0\right\rangle = \left|+\right\rangle$ and $H\left|1\right\rangle = \left|-\right\rangle$.
- $H\left|+\right\rangle = \left|0\right\rangle$ and $H\left|-\right\rangle = \left|1\right\rangle$.
- $T^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{pmatrix}$.
- $T\left|0\right\rangle = \left|0\right\rangle$ and $T\left|1\right\rangle = e^{i\frac{\pi}{4}}\left|1\right\rangle$.

These two unitary gates (operations) are perhaps the most important examples of single-qubit deterministic transformations. In fact, any single-qubit unitary operation may be approximated with arbitrary precision by applying a sequence of H and T gates.

Exercise: expressing other quantum operations

Exercise

Consider the following quantum operations:

$$S \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad Z \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Which of these operations are unitary? What is their action on the computational basis? What does Z do on the $\{|+\rangle, |-\rangle\}$ basis? Is it possible to express each of them as a combination of H and T? Hint: work from left to right and think in terms of basis states.

Exercise: expressing other quantum operations

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Consider the following quantum operations:

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- All of them are unitary.
- $S = TT$; $\qquad S\left|0\right\rangle = \left|0\right\rangle, S\left|1\right\rangle = i\left|1\right\rangle.$
- $Z = SS$; $\qquad Z\left|0\right\rangle = \left|0\right\rangle, Z\left|1\right\rangle = -\left|1\right\rangle;$ $\qquad Z\left|+\right\rangle = \left|-\right\rangle, Z\left|-\right\rangle = \left|+\right\rangle.$
- $X = HZH$; $\qquad X\left|0\right\rangle = \left|1\right\rangle, X\left|1\right\rangle = \left|0\right\rangle.$

Bra-ket notation

Notation

We shall often write $|\psi\rangle \in \mathbb{C}^2$ to refer to arbitrary qubits. We also write $\langle\psi| \stackrel{\text{def}}{=} |\psi\rangle^\dagger$.

Exercise

Write in matrix notation the following expressions:

- $\langle 0|$.
- $\langle 1|$.
- $\langle +|$.
- $\langle -|$.

Are the above expressions qubits in $\mathbb{C}^2$?

Exercise

Write in matrix notation the following expressions:

- $|0\rangle\langle 0|$.
- $|1\rangle\langle 1|$.
- $|0\rangle\langle 0| + |1\rangle\langle 1|$.
- $|0\rangle\langle 1| + |1\rangle\langle 0|$.

Have we seen any of them before?

Inner product of qubits

Let $|\psi\rangle$ and $|\phi\rangle$ be two vectors in a Hilbert space. Observe that their inner product is

$$\langle \psi, \phi \rangle = \langle \psi| \, |\phi\rangle$$

We will therefore often write $\langle \psi|\phi\rangle \stackrel{\text{def}}{=} \langle \psi| \, |\phi\rangle$ for the inner product as well.

Exercise

What are the following inner products? Use linearity to compute most of them.

- $\langle 0|0\rangle$.
- $\langle 0|1\rangle$.
- $\langle 1|1\rangle$.
- $\langle +|-\rangle$.
- $\langle +|+\rangle$.
- $\langle -|-\rangle$.
- $\langle 0|+\rangle$.
- $\langle 1|+\rangle$.
- $\langle 0|-\rangle$.
- $\langle 1|-\rangle$.

Quantum measurement (single-qubit system)

Definition

Let $|\psi\rangle \in \mathbb{C}^2$ be an arbitrary qubit. A *single-qubit measurement in the computational basis* on state $|\psi\rangle$ collapses the state of the system to either $|0\rangle$ or $|1\rangle$ and produces one bit of classical information to the observer performing the measurement.
The probability the state collapses to $|0\rangle$ is $\langle\psi| |0\rangle \langle 0| |\psi\rangle$ and then the observer gets bit 0 as result.
The probability the state collapses to $|1\rangle$ is $\langle\psi| |1\rangle \langle 1| |\psi\rangle$ and then the observer gets bit 1 as result.

Notice: measurements are **probabilistic** and **irreversible**.

Exercise

Assume we are given a qubit $|\psi\rangle \in \mathbb{C}^2$. An observer performs a measurement in the computational basis. Describe the probability distribution of the possible measurement outcomes when:

- $|\psi\rangle = |0\rangle$.
- $|\psi\rangle = |1\rangle$.
- $|\psi\rangle = |+\rangle$.
- $|\psi\rangle = |-\rangle$.

Exercise

Assume we are given a qubit $|\psi\rangle \in \mathbb{C}^2$. We apply a T gate to $|\psi\rangle$. Does this influence the probability of the measurement outcomes? Why? What if we instead apply an H gate?

Exercise

The probability calculation in the above definition can be equivalently expressed in a simpler way. Do you see how?

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- $|\psi\rangle = |1\rangle$.
- $|\psi\rangle = |+\rangle$.
- $|\psi\rangle = |-\rangle$.

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Assume we are given a qubit $|\psi\rangle \in \mathbb{C}^2$. We apply a T gate to $|\psi\rangle$. Does this influence the probability of the measurement outcomes? Why? What if we instead apply an H gate?

Exercise

The probability calculation in the above definition can be equivalently expressed in a simpler way. Do you see how? **Answer:** If $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, then outcome 0 occurs with probability $|\alpha|^2$ and outcome 1 occurs with probability $|\beta|^2$.

Composite quantum systems

Definition

The state space of an n-qubit system is given by $\mathbb{C}^{2^n}$. The state of an n-qubit register is a unit vector of $\mathbb{C}^{2^n}$. Two unit vectors $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{C}^{2^n}$ represent the same state iff they differ by a normalised complex multiple, i.e, if there exists $z \in \mathbb{C}$, with $|z| = 1$ such that $\mathbf{q}_1 = z \cdot \mathbf{q}_2$.

Remark

Recall that a vector $(a_1 \cdots a_n)^T \in \mathbb{C}^n$ is a unit vector when

$$\sum_i |a_i|^2 = 1.$$

Exercise

What is the state space of the smallest possible quantum register?

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Remark

Recall that a vector $(a_1 \cdots a_n)^T \in \mathbb{C}^n$ is a unit vector when

$$\sum_i |a_i|^2 = 1.$$

Exercise

What is the state space of the smallest possible quantum register? **Answer:** $\mathbb{C}$, when $n = 0$.

Definition

Given an n-qubit state $|\psi\rangle$ and an m-qubit state $|\phi\rangle$, then the *composed system* containing $|\psi\rangle$ and $|\phi\rangle$ is described by the $n+m$-qubit state $|\psi\phi\rangle \stackrel{\text{def}}{=} |\psi\rangle \otimes |\phi\rangle$, where $(- \otimes -)$ denotes the Kronecker product.

Remark

Recall that the Kronecker product of an $n \times m$ matrix $A = (a_{i,j})$ and $p \times r$ matrix B is the $(np \times mr)$ matrix

$$\begin{pmatrix} a_{1,1}B & \cdots & a_{1,m}B \\ a_{2,1}B & \cdots & a_{2,m}B \\ \vdots & \cdots & \vdots \\ a_{n,1}B & \cdots & a_{n,m}B \end{pmatrix}.$$

Properties of the Tensor/Kronecker Product

From linear algebra we know that:

- Tensor product of (finite-dimensional) Hilbert spaces: $\mathbb{C}^n \otimes \mathbb{C}^m \cong \mathbb{C}^{nm}$.
- The tensor product is a bilinear operation. In particular:
  - $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$.
  - $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$.
  - $(zA) \otimes B = A \otimes (zB) = z(A \otimes B)$.
- The tensor product is associative: $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.
- $\mathbf{0} \otimes B = \mathbf{0} = A \otimes \mathbf{0}$.
- Interchange law: $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.
- Adjoints: $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

where $z \in \mathbb{C}$, $\mathbf{0}$ is a zero matrix and where A, B, C, D are complex matrices (of appropriate dimensions in some of the above equations).

Exercise

Compute $(H \otimes H) |00\rangle$ using the above properties.

Exercise

Simplify the following expression: $(T \otimes H^\dagger)(I \otimes H)(T^\dagger \otimes I)$.

Exercise

Rewrite $A \otimes (\sum_{i=1}^n z_i B_i)$ in another form. Do the same for $(\sum_{j=1}^m y_j A_j) \otimes (\sum_{i=1}^n z_i B_i)$.

Composite quantum systems

Exercise

Write down the following states in vector notation:

- $|00\rangle$.
- $|11\rangle$.
- $|0+\rangle$.
- $|1-\rangle$.
- $|+1\rangle$.
- $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$.

Definition

An *n*-qubit state $|\psi\rangle$ is *entangled* when there exists no non-trivial quantum states $|\phi\rangle$ and $|\tau\rangle$, such that $|\psi\rangle = |\phi\rangle \otimes |\tau\rangle$ (non-trivial means that the two states contain at least one qubit).

Exercise

The *Bell state* is the state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Is this state entangled?

Composite quantum systems

Exercise

Write down the following states in vector notation:

- $|00\rangle$.
- $|11\rangle$.
- $|0+\rangle$.
- $|1-\rangle$.
- $|+1\rangle$.
- $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$.

Definition

An n-qubit state $|\psi\rangle$ is *entangled* when there exists no non-trivial quantum states $|\phi\rangle$ and $|\tau\rangle$, such that $|\psi\rangle = |\phi\rangle \otimes |\tau\rangle$ (non-trivial means that the two states contain at least one qubit).

Exercise

The *Bell state* is the state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Is this state entangled? **Answer:** Yes, it is. A simple algebraic argument shows that it is not the Kronecker product of any two vectors in $\mathbb{C}^2$. The Bell state is the most important example of quantum entanglement.

Composite quantum system dynamics

Definition

Deterministic operations on an n-qubit system are described by unitary matrices acting on $\mathbb{C}^{2^n}$.

Exercise

How do the following quantum states evolve when we apply the $H \otimes X$ operation on them (recall that $(- \otimes -)$ is bilinear)?

- $|01\rangle$.
- $|{+}0\rangle$.
- $|0{+}\rangle$.
- $|0{-}\rangle$.
- $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$.

How can we manipulate entanglement?

Exercise

Assume that $|\psi\rangle = |\phi\rangle \otimes |\tau\rangle$ is a non-entangled state, where $|\phi\rangle$ is an n-qubit state and $|\tau\rangle$ is an m-qubit state. Assume further that $U_1 : \mathbb{C}^{2^m} \to \mathbb{C}^{2^m}$ and $U_2 : \mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$ are unitary maps. Is the state $(U_1 \otimes U_2)(|\psi\rangle \otimes |\phi\rangle)$ entangled?

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Answer: No, because $(U_1 \otimes U_2)(|\phi\rangle \otimes |\tau\rangle) = (U_1 |\phi\rangle) \otimes (U_2 |\tau\rangle)$ due to bilinearity of the Kronecker product.

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Assume that $|\psi\rangle$ is an entangled 2-qubit state. Assume further that $U_1 : \mathbb{C}^2 \to \mathbb{C}^2$ and $U_2 : \mathbb{C}^2 \to \mathbb{C}^2$ are unitary maps. Is the state $(U_1 \otimes U_2) |\psi\rangle$ entangled?

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Assume that $|\psi\rangle$ is an entangled 2-qubit state. Assume further that $U_1 : \mathbb{C}^2 \to \mathbb{C}^2$ and $U_2 : \mathbb{C}^2 \to \mathbb{C}^2$ are unitary maps. Is the state $(U_1 \otimes U_2) |\psi\rangle$ entangled?
Answer: Yes. Assume for contradiction that it is not entangled. Then by the above exercise it follows that applying $(U_1^\dagger \otimes U_2^\dagger)$ to the non-entangled state would still result in a non-entangled state. But this means

$$(U_1^\dagger \otimes U_2^\dagger)(U_1 \otimes U_2) |\psi\rangle = (U_1^\dagger U_1 \otimes U_2^\dagger U_2) |\psi\rangle = (I \otimes I) |\psi\rangle = |\psi\rangle$$

must be non-entangled which is a contradiction.

- So, how can we change introduce/eliminate entanglement in a quantum system?
- For this we need to consider some additional unitary operations that we have not seen so far.

The CNOT gate

Definition

The CNOT operation is a 2-qubit unitary map defined by

$$\mathrm{CNOT} \stackrel{\mathrm{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Exercise

What is $\mathrm{CNOT}^\dagger$? What is the action of CNOT on the computational basis states:

- $|00\rangle$.

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- $|00\rangle$. **Answer:** $\mathrm{CNOT}\,|00\rangle = |00\rangle$.
- $|01\rangle$. **Answer:** $\mathrm{CNOT}\,|01\rangle = |01\rangle$.
- $|10\rangle$.

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- $|10\rangle$. **Answer:** $\mathrm{CNOT}\,|10\rangle = |11\rangle$.
- $|11\rangle$.

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- $|00\rangle$. **Answer:** $\mathrm{CNOT}\,|00\rangle = |00\rangle$.
- $|01\rangle$. **Answer:** $\mathrm{CNOT}\,|01\rangle = |01\rangle$.
- $|10\rangle$. **Answer:** $\mathrm{CNOT}\,|10\rangle = |11\rangle$.
- $|11\rangle$. **Answer:** $\mathrm{CNOT}\,|11\rangle = |10\rangle$.

Creating Entanglement

Exercise

Consider the 2-qubit state $|00\rangle$. Find two unitary gates which can be applied to $|00\rangle$ resulting in the Bell state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$. Hint: the second one should be CNOT. The first one should create superposition on one of the qubits.

Creating Entanglement

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Answer: Consider the map $\mathrm{CNOT}(H \otimes I)$. Then, we get:

$$\mathrm{CNOT}(H \otimes I)\,|00\rangle = \mathrm{CNOT}\,|{+}0\rangle = \mathrm{CNOT}\frac{|00\rangle + |10\rangle}{\sqrt{2}} = \frac{\mathrm{CNOT}\,|00\rangle + \mathrm{CNOT}\,|10\rangle}{\sqrt{2}} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

This shows that we can use the *combination* of a CNOT and H gates to create entanglement.

Measurement in composite systems

Remark

Every non-zero vector $v \in \mathbb{C}^n$ can be normalised by setting $v' = \frac{v}{||v||}$. Why is this true?

Remark

In this course we only consider measurements in the computational basis and they are given in the following way.

Definition

Assume we are given an n-qubit quantum system $|\psi\rangle \in \mathbb{C}^{2^n}$. A measurement on qubit $1 \leq i \leq n$ is determined by the following process. Let $P_0^i = I \otimes \cdots \otimes I \otimes |0\rangle \langle 0| \otimes I \otimes \cdots \otimes I$ and $P_1^i = I \otimes \cdots \otimes I \otimes |1\rangle \langle 1| \otimes I \otimes \cdots \otimes I$. That is in P_0^i we apply $|0\rangle \langle 0|$ at the i-th position and we tensor with the identity matrix on all other positions. Similarly for P_1^i.

After performing the measurement:

- the state of the system collapses to $\frac{P_0^i|\psi\rangle}{||P_0^i|\psi\rangle||}$ with probability $||P_0^i |\psi\rangle ||^2$.
- the state of the system collapses to $\frac{P_1^i|\psi\rangle}{||P_1^i|\psi\rangle||}$ with probability $||P_1^i |\psi\rangle ||^2$.

Exercise

Describe the probability distributions that result from measuring the first qubit of the following states:

- $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$.
- $|{+}{+}\rangle$, $|{+}{-}\rangle$, $|{-}{+}\rangle$, $|{-}{-}\rangle$.
- $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$. After we measure the first qubit here, what happens if you measure the second one?

Measurement in composite systems

- Measuring several qubits of a composite system simultaneously is the same as measuring individual qubits one after the other (in any order).
- We describe the special case where we measure all qubits simultaneously. This is usually what is done.

Definition

Assume we are given an n-qubit quantum system $|\psi\rangle \in \mathbb{C}^{2^n}$. Measuring all qubits of $|\psi\rangle$ in the computational basis is determined by the following process. Let

$$P_{i_1,\ldots,i_n} = |i_1\rangle\langle i_1| \otimes \cdots \otimes |i_n\rangle\langle i_n| \in \mathbb{C}^{2^n \times 2^n}$$

where $i_j \in \{0,1\}$. After performing the measurement:

- the state of the system collapses to $|i_1 i_2 \cdots i_n\rangle$ with probability $||P_{i_1 i_2 \cdots i_n} |\psi\rangle ||^2$.

Exercise

Describe the probability distributions that result from measuring all qubits of the following states:

- $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$.
- $|{+}{+}\rangle$, $|{+}{-}\rangle$, $|{-}{+}\rangle$, $|{-}{-}\rangle$.
- $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$.

Exercise

The above formula for the probability computation can be simplified. How?

Measurements of entangled states

- Consider the Bell state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$. This is arguably the most important entangled state.
- We just determined that measuring any one qubit would produce measurement outcome 0 or 1 with equal probability.
- However, we also determined that measuring both qubits produces outcomes 00 or 11 with equal probability.
- That is, if we measure one qubit first and consider the outcome, then with probability 100% we know the outcome of the measurement on the second qubit.
- These correlations *cannot* be explained by classical statistical mechanics.

No cloning

- Quantum information cannot be copied, in general.

Proposition

There exists no unitary operation $C : \mathbb{C}^4 \to \mathbb{C}^4$, such that for an arbitrary qubit $|\psi\rangle$:

$$C(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle .$$

Proof.

Exercise:

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- Quantum information cannot be copied, in general.

Proposition

There exists no unitary operation $C : \mathbb{C}^4 \to \mathbb{C}^4$*, such that for an arbitrary qubit* $|\psi\rangle$ *:*

$$C(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle .$$

Proof.

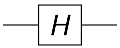
Exercise: Assume that such C exists. Let $|\psi\rangle$ and $|\phi\rangle$ be arbitrary qubits. Then:

$$\begin{aligned}\langle\psi|\phi\rangle &= \langle\psi|\phi\rangle \cdot 1 = \langle\psi|\phi\rangle \cdot \langle 0|0\rangle = (\langle\psi| \otimes \langle 0|)(|\phi\rangle \otimes |0\rangle) = \\ &= (\langle\psi| \otimes \langle 0|)I(|\phi\rangle \otimes |0\rangle) = (\langle\psi| \otimes \langle 0|)C^\dagger C(|\phi\rangle \otimes |0\rangle) = (\langle\psi| \otimes \langle\psi|)(|\phi\rangle \otimes |\phi\rangle) \\ &= \langle\psi|\phi\rangle \cdot \langle\psi|\phi\rangle\end{aligned}$$

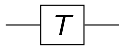
With this, we can now easily reach a contradiction by choosing appropriate $|\psi\rangle$ and $|\phi\rangle$. Example: choose $|\psi\rangle = |0\rangle$ and $|\phi\rangle = |+\rangle$. □

Quantum circuits

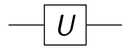
Quantum operations admit a diagrammatic representation in the form of *quantum circuit diagrams*.



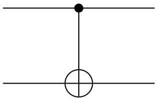
H unitary operator



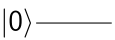
T unitary operator



U unitary operator

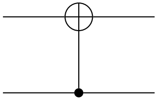


$CNOT$ unitary operator

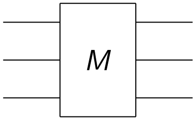


Prepare a new qubit in state $|0\rangle$

Measure a qubit in the computational basis



$CNOT$ unitary operator with swapped inputs



A three qubit unitary operator called M

Remark

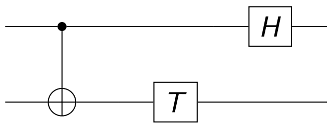
In the literature, authors often use other notations for measurement.

Quantum circuits

- Circuits should be read left-to-right and top-to-bottom.
- Left-to-right direction corresponds to sequential composition (matrix multiplication).
- Topt-to-bottom corresponds to spatial composition (kronecker product/tensor product).

Example

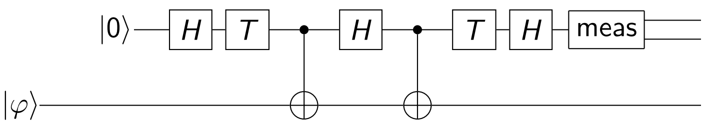
The following circuit:



describes the unitary operator $(H \otimes I)(I \otimes T)CNOT$.

Example

The following circuit:



describes the following quantum algorithm:

1. Input: an arbitrary qubit (abstracted to state $|\varphi\rangle$ above).
2. Prepare a new qubit in state $|0\rangle$. The new state is now $|0\rangle \otimes |\varphi\rangle$.
3. Apply the unitary operator $(H \otimes I)(T \otimes I)CNOT(H \otimes I)CNOT(T \otimes I)(H \otimes I)$.
4. Measure the first (auxiliary, i.e., non-input) qubit.

Quantum teleportation

- Quantum teleportation is an interesting protocol which allows (possibly separated) parties to move quantum information from one place to another.
- In the protocol, there are two parties – Alice and Bob.
- Alice has some qubit $|\psi\rangle$ which she wishes to send to Bob.
- How can this be done? Remember, we cannot copy quantum information.

Quantum teleportation

The protocol is described as follows:

1. Alice has an input qubit $|\psi\rangle$ in her possession.
2. Alice and Bob prepare the Bell state together.
3. After preparing the Bell state, Alice controls one qubit and Bob controls the other.
4. Alice applies a CNOT operation on her two qubits with $|\psi\rangle$ being the control qubit.
5. Alice applies a Hadamard operation on her first qubit.
6. Alice measures her two qubits in the computational basis and reads the measurement outcome (b_1, b_2), which indicates to what state her subsystem has collapsed.
7. Alice sends the two classical bits (b_1, b_2) to Bob.
8. Bob now applies the unitary operation $Z^{b_1} \circ X^{b_2}$ on his qubit. This means, he applies X iff $b_2 = 1$ and he applies Z iff $b_1 = 1$.
9. Bob's qubit is now in state $|\psi\rangle$.

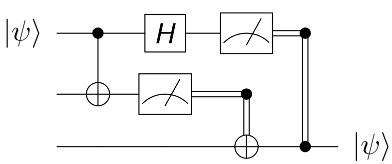


Figure: Quantum circuit representation of quantum teleportation (as seen in the literature).

Quantum teleportation

- **Exercise:** verify the quantum teleportation protocol.

Quantum teleportation

- **Exercise:** verify the quantum teleportation protocol.
- So what can we learn from this?
- 2 bits of classsical information + entanglement = quantum teleportation.
- But a qubit can be in uncountably many states and the shared entangled state is always the Bell state.
- Does it seems counter-intuitive?
- Experimentally confirmed many times, so this does indeed work.

Shor's algorithm

- Problem: Given an integer N, find a non-trivial integer divisor of N.
- Classical results from number theory: it suffices to solve the period finding problem.
- Period finding: Given a function $f(x) = a^x \mod N$, where a and N are positive integers, $a < N$ and such that a and N have no common factors, find the smallest integer $r > 0$, such that $a^r \mod N = 1$.
- Shor's algorithm can solve this problem in polynomial time on a quantum computer. The best known classical algorithms need exponential time.
- For the setup of the algorithm, let us assume that $N^2 \leq 2^q = Q$.
- In the description of the algorithm, for an integer $k < N$, we shall write $|k\rangle = |k_1\rangle \otimes \cdots \otimes |k_n\rangle$, where $k_1 \cdots k_n$ is the bit representation of k.

Shor's algorithm

1. Initialise the state to $|0^{2q}\rangle$.
2. Apply H^q to the first q qubits. The new state is now

$$\frac{1}{\sqrt{Q}}\sum_{x=0}^{Q-1}|x\rangle \otimes |0^q\rangle$$

3. Implement the quantum oracle U_f which realises the classical function f. It has action $U_f(|x\rangle \otimes |0^q\rangle) = |x\rangle \otimes |f(x)\rangle$.
4. Apply the quantum oracle to the current state. The new state is then

$$U_f\left(\frac{1}{\sqrt{Q}}\sum_{x=0}^{Q-1}|x\rangle \otimes |0^q\rangle\right) = \frac{1}{\sqrt{Q}}\sum_{x=0}^{Q-1}|x\rangle \otimes |f(x)\rangle$$

5. Apply the quantum Fourier transform to the first q qubits. The QFT unitary is given by

$$\mathrm{QFT}\,|x\rangle = \frac{1}{\sqrt{Q}}\sum_{y=0}^{Q-1}\omega^{xy}\,|y\rangle$$

where $\omega = e^{2\pi i/Q}$ is the Q-th root of unity. After applying QFT to the previous state, we get

$$(QFT \otimes I)\left(\frac{1}{\sqrt{Q}}\sum_{x=0}^{Q-1}|x\rangle \otimes |f(x)\rangle\right) = \frac{1}{Q}\sum_{x=0}^{Q-1}\sum_{y=0}^{Q-1}\omega^{xy}\,|y\rangle \otimes |f(x)\rangle$$

6. Measure in the computational basis. Now, with high probability and some classical computation, the period is found. If it is not found, then repeat the process until we find it.