SUNY Polytechnic Institute

Gravitational field produced by elliptical distribution of mass & the resulting dynamics

Project submitted by

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PHY 490: Special Topics in Physics

Computational Methods in Classical Mechanics

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Abstract

The goal of this project is to model the gravitational field generated by an elliptical mass distribution, to allow for future analysis of the motion of objects within this field. We will first compute the gravitational potential in the plane of the ellipse, both inside and outside. This will be done using analytical methods as well as numerical techniques. This is a natural build off of the analytical computation of the gravitational potential generated by a ring shown in Chapter 6 of our Analytical Mechanics textbook. As will be shown, particular insights from the gravitational field generated by an ring mass distribution can help build intuition for the behavior of the gravitational field of an elliptical mass distribution. This computation proves useful astrophysics. Modeling the gravitational field created by galactic mass distributions and analyzing the kinematics within that field are important for a greater understanding of the universe. This project is a means to show the power of analytical mechanics.

1 Introduction

In our project, our goal is to model a gravitational field from a elliptical mass distribution. We study this notion to further our understanding of how gravity works as a force, as well as develop other aspects of physics like more complex mechanical systems that we would like to solve in the future. We attempt this by solving a series of equations to calculate a precise gravitational potential for an elliptical mass distribution. We will supplement our analytical solutions with numerical solutions to show how accurate our analysis is and to generalize and define our analysis.

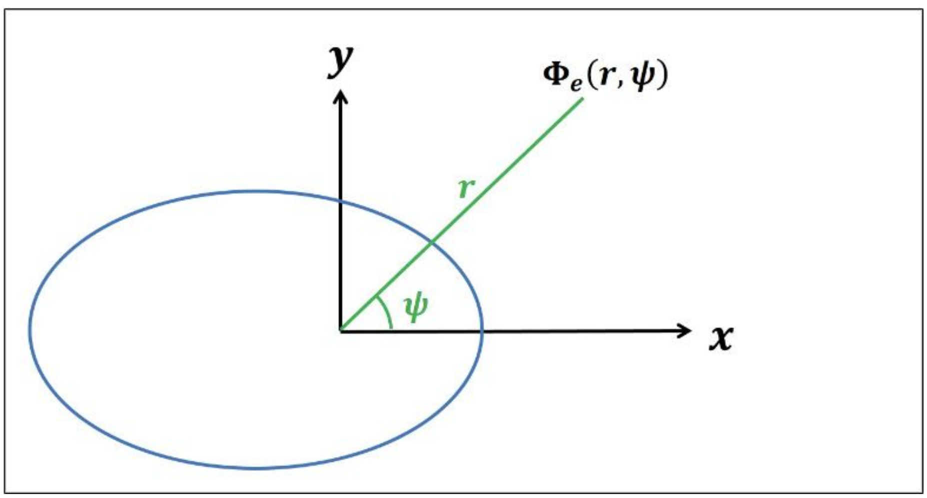

Figure 1: Illustration of the elliptical mass distribution where gravitational potential is expressed by distance and angle

2 Review of Theory

Before we get into the results, it is important that we go over the necessary theory that precedes the material covered in the results section. As well as establishing the fundamental geometry for what kind of system we are solving. The goal of this project is to find the gravitational field generated by a specific type of mass distribution. We must first know what a gravitational field is and how it is derived.

2.1 Gravity

A gravitational field is described by Newton's law of universal gravitation. Paraphrased from the textbook, this law states that every particle in the universe attracts every other particle with some force. This force is proportional to the product of the two masses and is inversely proportional the the square of the distance between the two particles. The direction of this force lies on the line that connects the two particles. The vector equation representing this phenomena is:

$$\vec{F_{ij}} = \frac{Gm_i m_j}{|r_{ij}^2|}(\frac{\vec{r_{ij}}}{|r_{ij}|}) \tag{1}$$

where $\vec{F_{ij}}$ is the gravitational force, magnitude and direction, that particle j exerts on particle i, G is the universal constant of gravitation, m_i and m_j are the masses of particle i and particle j, r_{ij} is the scalar distance between the two particles, and $\vec{r_{ij}}$ is the vector with the tail at particle i and the head at particle j. The accepted value of the universal law of gravitation is $G = (6.67259 \pm 0.00085) \times 10^{-11} \frac{Nm^2}{kg^2}$.

$\vec{F}$ produced by an object acting on some arbitrary point particle in space is what we refer to as the gravitational force field. Gravitational force fields are what we deem as a field derived from a central force. Central forces are forces whose lines of action either originate or terminate on a single point. So, in the case of two particles, i and j, the central gravitational force F_{ij} originates from particle j. Vice-versa for the force F_{ji}. Also, in the case of gravitation, the magnitude of the gravitational force is independent of any direction, thus, the force as defined is what we call isotropic. In other words, the magnitude of an isotropic force is constant on any point on the surface of any sphere that is centered by this force. Therefore, an isotropic force is independent of direction and dependent on distance between the two objects.

Note that the Newton's universal law of gravitation of classical mechanics is analogous to Coulomb's law of electrostatics. Coulomb's law describes the forces between two charged particles similarly to that of Newton's law of gravitation. Coulomb's law states that:

$$\vec{F_{ij}} = \frac{kq_i q_j}{|r_{ij}|^2}(\frac{\vec{r_{ij}}}{|r_{ij}|}) \tag{2}$$

where k is the Coulomb's law constant analogous to G and q_i and q_j are the charges of particle i and particle j which is analogous to m_i and m_j in the case of gravitational force. Why is this the case? It may be useful to dwell on the relationships between certain laws and theories in the study of physics. This relationship is not important for sake of this project, however, it is important to make this connection for a deeper understanding of concepts within physics.

2.2 Calculating the Gravitational Force Produces By a Mass Distribution

To proceed further, we must not only know how calculate the gravitational force generated by a point particle, we must also be able to calculate the gravitational force generated by a body or a mass distribution. We will go over an example of the gravitational force produced by an uniform spherical shell of mass M and radius R. Let r be the distance between the center of the spherical shell and some arbitrary point particle outside the spherical shell. Notice that we cannot simply plug in our known values into equation (1). We must break up the computation of the gravitational force into infinitesimal force vectors, then integrate over the surface of the spherical shell (there are many ways to do this so depending on how you break up the spherical shell, the integrand and bounds of the integral may be different). In this case, since there is spherical symmetry, it is easiest to break the sphere into rings of infinitesimal thickness $Rd\theta$ as shown in the figure (we need to add the figure for this problem). The magnitude of the infinitesimal gravitational force vector is of the form:

$$dF = \frac{GmdM}{s^2}\cos\phi \tag{3}$$

where m is the mass of the point particle, dM is the mass of the infinitesimal ring, s is the distance between the point particle and point Q, and ϕ is the angle between s and r where r is the line connecting the center of the spherical shell and the point particle. The reason $\cos\phi$ shows up is because, by symmetry, and the nature of central forces, the resultant gravitational force vector will lie on the line r. In other words, the components of each infinitesimal force vector that are not parallel with r will cancel out in the overall resultant force vector.

In order to find the total gravitational force, we must integrate over the entire spherical shell. i.e. we must add the infinitesimal forces generated by each infinitesimal ring of the spherical shell. First, since our integrand contains multiple variables that are dependent on each other, we must find the relations between each variable and express the integrand in terms of one of these variables. We know that $dM = \rho dV \approx \rho 2\pi R^2 \sin\theta d\theta$ where ρ is the density of the spherical shell, the radius of the infinitesimal ring is $R\sin\theta$ and the thickness is $Rd\theta$. This gives:

$$F = Gm2\pi\rho R^2 \int_0^\pi \frac{\sin\theta\cos\phi d\theta}{s^2} \tag{4}$$

We can express the relationship between s, θ, and ϕ using the law of cosines. Looking at figure we get [1]:

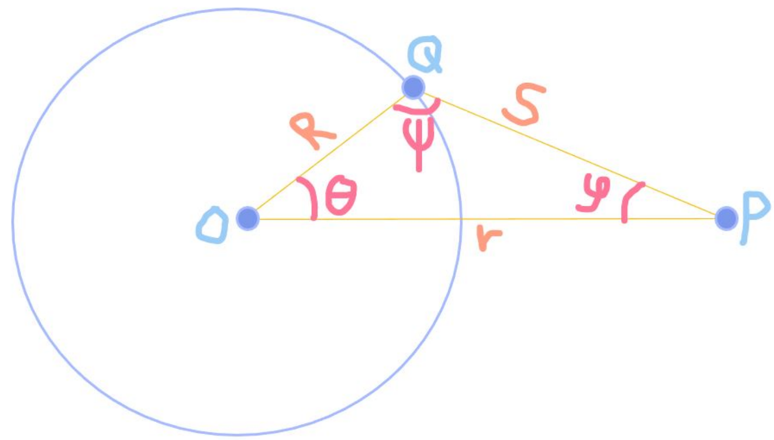

Figure 2: Coordinates for calculating the gravitational field of a ring

$$s^2 = r^2 + R^2 - 2rR\cos\theta \tag{5}$$

Now integrating:

$$sds = rR\sin\theta d\theta \implies \sin\theta d\theta = \frac{s}{rR}ds \tag{6}$$

Using the law of cosines with respect to ϕ:

$$\cos\phi = \frac{s^2 + r^2 - R^2}{2rs} \tag{7}$$

Now plugging (6) and (7) into equation (4) and changing the bounds with respect to s gives:

$$F = Gm2\pi\rho R^2 \int_{r+R}^{r-R} \frac{s^2 + r^2 - R^2}{2Rr^2s^2}ds \tag{8}$$

Simplifying the integrand and solving the integral gives us the gravitational force that a uniform spherical shell of mass M exerts on an arbitrary point particle of mass m:

$$F = \frac{GmM}{4Rr^2}\int_{r+R}^{r-R}(1 + \frac{r^2 - R^2}{s^2})ds = \frac{GmM}{r^2} \tag{9}$$

We know that this force will be directed towards the center of the spherical shell. So, given a coordinate system where the origin is at the center of the spherical shell gives the gravitational force vector:

$$\vec{F} = -\frac{GmM}{r^2}\hat{e}_r \tag{10}$$

This makes sense intuitively because of the spherical symmetry of the mass distribution. The gravitational field of a spherical shell that has uniform mass is the same of that of a point particle of mass M at the center of the spherical shell.

2.3 Kepler's First Law

Now we have the necessary background to compute the gravitational field produced by a mass distribution. In the case of an elliptical mass distribution, when using the same parameters as in the previous example, notice that R is dependent on the variable θ (i.e. R is not constant). So, when integrating over this mass distribution, we will have another variable to take into account when accessing the variable dependencies and expressing the integrand in terms of one of these variables. How can we represent R in a useful way?

One useful way of representing R in this case comes from Kepler's First Law: The Law of Ellipses. Kepler's First Law, which is derived from Newton's Second Law, states that the orbit of each planet is an ellipse, with the Sun located at one of its foci. The mere fact the orbit of each planet is an ellipse with the Sun at one of its foci is not important for this problem. However, the proof of this law gives rise to the equation of the planet's orbit with respect to the Sun, then giving us the relationship between R and θ that we need. We will derive this equation in this section and importance of this connection will come to fruition when it is implemented in the results section.

The first step to deriving the desired equation starts with deriving the general differential equation of the orbit. Starting with Newton's Second Law:

$$m\ddot{\vec{R}} = f(R)\hat{e}_r \tag{11}$$

where $f(R)$ is the central, isotropic gravitational force that the sun exerts on the planet. In chapter 1 from the textbook, we know that $\ddot{\vec{R}} = (\ddot{r} - R\dot{\theta}^2)\hat{e}_r + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$. Using this, we can split equation (11) into two component differential equations:

$$m(\ddot{R} - R\dot{\theta}^2) = f(R) \tag{12}$$

$$m(2\dot{R}\dot{\theta} + R\ddot{\theta}) = m\frac{1}{R}\frac{d}{dt}(R^2\dot{\theta}) = 0 \tag{13}$$

Simplifying equation (13) gives:

$$\frac{d}{dt}(R^2\dot{\theta}) = 0 \implies R^2\dot{\theta} = constant = l \tag{14}$$

where l is angular momentum per unit mass.

Now we will define a new variable u that will help us express equation (12) in a useful way:

$$R = \frac{1}{u} \tag{15}$$

Differentiating:

$$\dot{R} = -\frac{1}{u^2}\dot{u}\frac{d\theta}{d\theta} = -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta} = -l\frac{du}{d\theta} \tag{16}$$

Differentiating again:

$$\ddot{R} = -l\frac{d}{dt}\frac{du}{d\theta}\frac{d\theta}{d\theta} = -l\dot{\theta}\frac{d}{d\theta}\frac{du}{d\theta} = -l\dot{\theta}\frac{d^2u}{d\theta^2} \tag{17}$$

Rewriting equation (14):

$$\frac{\dot{\theta}}{u^2} = l \implies \dot{\theta} = lu^2 \tag{18}$$

Now subbing equation (18) into equation (17):

$$\ddot{R} = -l^2 u^2 \frac{d^2 u}{d\theta^2} \tag{19}$$

Finally, we can substitute equation (19), (18), and (15) into equation (12) gives us the differential equation of orbit some object moving in a central force field:

$$m(-l^2 u^2 \frac{d^2 u}{d\theta^2} - l^2 u^3) = f(u^{-1}) \implies \frac{d^2 u}{d\theta^2} + u = -\frac{1}{ml^2 u^2} f(u^{-1}) \tag{20}$$

With this differential equation, the explicit relationship between R and θ can be derived. As a consequence of Newton's Universal Law of Gravitation, we know that $f(R) = \frac{-GMm}{R^2} = \frac{-k}{R^2}$ where $k = GMm$. Now plugging this into equation (20) gives:

$$\frac{d^2 u}{d\theta^2} + u = \frac{k}{ml^2} \tag{21}$$

Notice that this differential equation is in the form of a simple harmonic oscillator with an additive constant! Therefore the general solution must be of the form:

$$u(\theta) = \frac{1}{R(\theta)} = A\cos(\theta - \theta_0) + \frac{k}{ml^2} \implies R(\theta) = \frac{1}{\frac{k}{ml^2} + A\cos(\theta - \theta_0)} \tag{22}$$

where A and θ_0 are the constants of integration that are determined by the initial conditions of the object in orbit. This is exactly what we are looking for!

Now, rotating the coordinate system so that $\theta_0 = 0$ and using the properties of an ellipse in Figure 6.5.1 from [1], we can represent $R(\theta)$ in terms more descriptive parameters

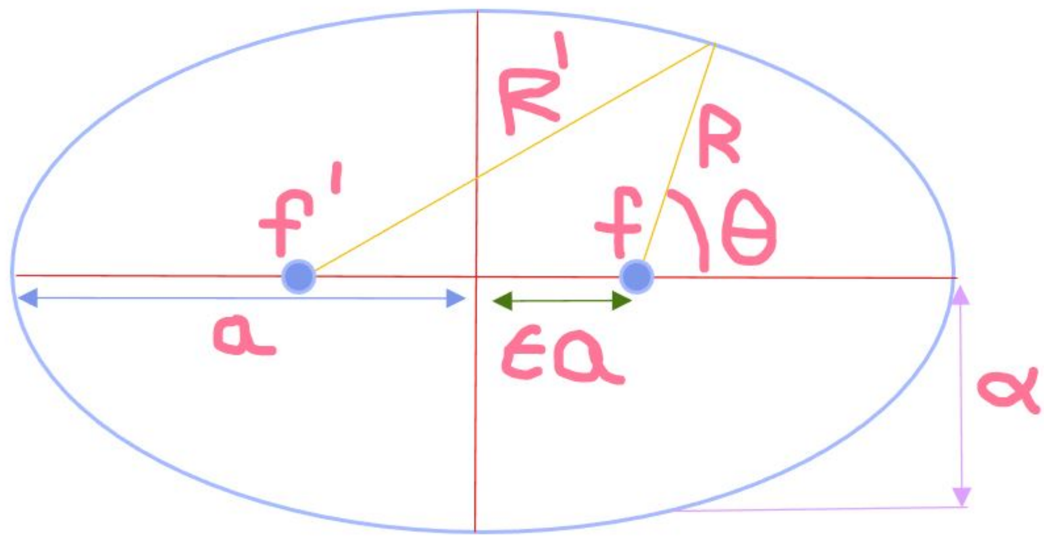

Figure 3: Breakdown of an Ellipse

$$R(\theta) = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\theta)} \tag{23}$$

where a is the semi-major axis of the ellipse and ϵ is the eccentricity of the ellipse. Eccentricity is the measure of how "circular" a conic section (cross-section of a cone) is. $\epsilon = 0$ characterizes a circle, $0 < \epsilon < 1$ characterises an ellipse, $\epsilon = 1$ characterizes a parabola, and $\epsilon > 1$ characterizes a hyperbola.

3 Calculating the Gravitational Potential of an Elliptical Mass Distribution

Now we begin our analysis of the gravitational potential of an elliptical mass distribution. First, we must define our axis. Since, in the previous section, we derived the equation of an ellipse relative to its foci, it only makes sense to center our axis at the foci.

Utilizing the first figure (figure of system) we can analyze and create the expressions for element of length, mass, and the radius of the ellipse as follows:

$$dl^2 = R^2 d\theta^2 + dR^2 \tag{24}$$

$$dM = \mu dl = \mu\sqrt{R^2 d\theta^2 + dR^2} \tag{25}$$

$$R = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\theta)} \tag{26}$$

Taking the derivative of Eq.(26) we can obtain the necessary components to express Eq.(24):

$$\frac{dR}{d\theta} = \frac{d}{d\theta}[a(1-\epsilon^2)(1+\epsilon\cos(\theta))^{-1}] = \frac{a\epsilon(1-\epsilon^2)\sin(\theta)}{(1+\epsilon\cos(\theta))^2}$$

From this we can obtain our expression for dR as:

$$dR = \frac{a\epsilon(1-\epsilon^2)\sin(\theta)}{(1+\epsilon\cos(\theta))^2}d\theta \tag{27}$$

then inputting this into dl we get:

$$dl^2 = \frac{a^2(1-\epsilon^2)^2}{(1+\epsilon\cos(\theta))^2}d\theta^2 + \frac{a^2\epsilon^2(1-\epsilon^2)^2\sin^2(\theta)}{(1+\epsilon cos(\theta))^4}d\theta^2$$

After simplifying, we end up with our final dl as:

$$dl = \frac{a(1-\epsilon^2)}{(1+\epsilon\cos(\theta))^2}d\theta\sqrt{1+2\epsilon\cos(\theta)+\epsilon^2} \tag{28}$$

Recalling Eq. (5) from earlier we must substitute in our new angle for our system so instead of θ we have $(\theta-\psi)$ which gives us:

$$s^2 = r^2 + R^2 - 2rR\cos(\theta-\psi) \tag{29}$$

inputting our previously established R value and simplifying we get s^2 to be:

$$s^2 = \frac{a^2(1-\epsilon^2)^2 + r^2(1+\epsilon\cos(\theta))^2 - 2a(1-\epsilon^2)r\cos(\theta-\psi)(1+\epsilon\cos(\theta))}{(1+\epsilon\cos(\theta))^2} \tag{30}$$

with all of this we can find $\frac{dl}{s}$ to begin expressing our integrand for the gravitational potential Φ

$$\frac{dl}{s} = \frac{a(1-\epsilon^2)\sqrt{1+2\epsilon\cos(\theta)+\epsilon^2}d\theta}{(1+\epsilon\cos(\theta))\sqrt{a^2(1-\epsilon^2)^2 + r^2(1+\epsilon\cos(\theta))^2 - 2a(1-\epsilon^2)r\cos(\theta-\psi)(1+\epsilon\cos(\theta))}} \tag{31}$$

This integral is not a trivial integral to solve even using different software packages so we will define constants to streamline the calculation:
substituting in

$C_0 = a^2(1-\epsilon^2)^2 + r^2,$
$C_1 = 4\epsilon r^2 + 2\epsilon a^2(1-\epsilon^2)^2 - 2a(1-\epsilon^2)r\cos(\psi),$
$C_2 = 6\epsilon^2 r^2 + a^2\epsilon^2(1-\epsilon^2)^2 - 6a(1-\epsilon^2)r\epsilon\cos(\psi),$
$C_3 = 4\epsilon^3 r^2 - 6\epsilon^2 ra(1-\epsilon^2)\cos(\psi),$
$C_4 = \epsilon^4 r^2 - 2ra\epsilon^3(1-\epsilon^2)\cos(\psi)$
$C_5 = -2a(1-\epsilon^2)r\sin(\psi)$
$C_6 = -6a(1-\epsilon^2)r\epsilon\sin(\psi)$
$C_7 = -6ra\epsilon^2(1-\epsilon^2)\sin(\psi)$
$C_8 = -2ra\epsilon^3(1-\epsilon^2)\sin(\psi)$

We can simplify our integrand as:

$\Phi = -G\mu a(1-\epsilon^2) * \int_0^{2\pi} \frac{\sqrt{D_0+D_1\cos(\theta)}d\theta}{\sqrt{C_0+C_1\cos(\theta)+C_2\cos^2(\theta)+C_3\cos^3(\theta)+C_4\cos^4(\theta)+C_5\sin(\theta)+C_6\sin(\theta)\cos(\theta)+C_7\sin(\theta)\cos^2(\theta)+C_8\sin(\theta)\cos^3(\theta)}}$

where we define
$D_0 = 1+\epsilon^2$,
$D_1 = 2\epsilon$

If this is the correct integrand for the gravitational potential produced by a elliptical mass distribution, then, if we take the limit as the ellipse approaches a circle, the integrand should match the integrand in equation (). As an ellipse approaches a circle, $R \to a$, $\epsilon \to 0$, and $\psi \to 0$. This yields, $C_0 = r^2 + a^2$
$C_1 = -2ra$
$C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = C_8 = 0$
$D_0 = 1$
$D_1 = 0$
Now plugging the constants into equation (32):

$$\Phi = -G\mu a \int_0^{2\pi} \frac{d\theta}{\sqrt{r^2+a^2-2ra\cos(\theta)}} \tag{32}$$

Which is expected.

Now we must solve the integral in equation (32). This integral cannot be solved analytically, so we will need to resort to other mathematical techniques.

Intuitively, the gravitational potential of an elliptical mass distribution should just be the gravitational potential of a circle mass distribution plus corrections of order 1, 2, 3, and so on. The only parameter that sufficiently characterizes an ellipse relative to a circle is the eccentricity, so it would make sense to do a series expansion of the integrand of equation (32) with respect to ϵ. Lets refer to this expression as I.

$$\begin{aligned} I &= \frac{a}{\sqrt{a^2+r^2-2ar\cos(\psi)\cos(\theta)-2ar\sin(\psi)\sin(\theta)}} \\ &\quad - \frac{ar^2\cos(\theta)-a^2r\cos(\psi)\cos^2(\theta)-a^2r\sin(\psi)\cos(\theta)\sin(\theta)}{(a^2+r^2-2ar\cos(\psi)\cos(\theta)-2ar\sin(\psi)\sin(\theta))^{3/2}}\epsilon + O(\epsilon^2) \end{aligned} \tag{33}$$

As you can see, for $\epsilon = 0$, I reduces to the integrand of the gravitational potential of a circular mass distribution. This integral cannot be solved analytically so we must resort to numerical techniques.

4 Numerical Results

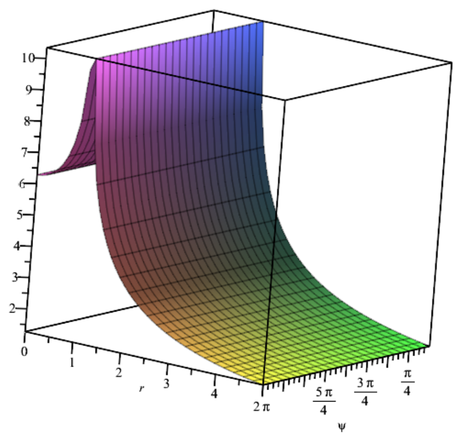

Figure 4: Graph of the potential calculation for a circular mass distribution with $R = 1$

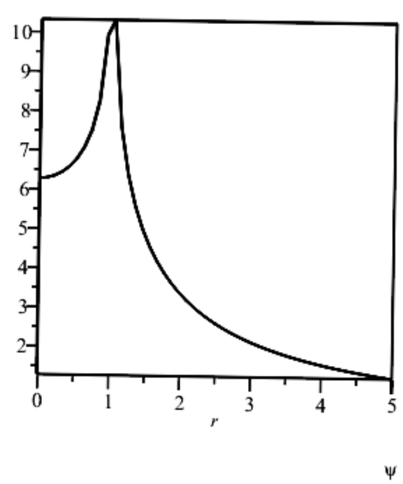

Figure 5: Graph of the potential calculation for a circular mass distribution with $R = 1$

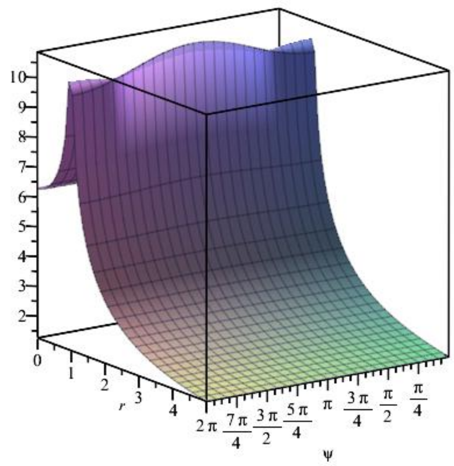

Figure 6: Graph of the integral of Eq. (33) over an ellipse with $\epsilon = .01$

You can visually see as we start making our mass distribution more akin to a ellipse from a circle we see a small deviation from the circle potential.
These next few plots were generated after we applied our correction term to turn the originally modeled circular system into a more elliptical system, we did this by constraining the parameters $\epsilon = 0.5$ and setting $a = 1$.

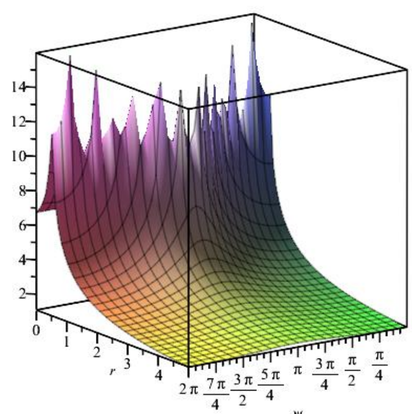

Figure 7: Graph of gravitational potential function of an elliptical mass distribution

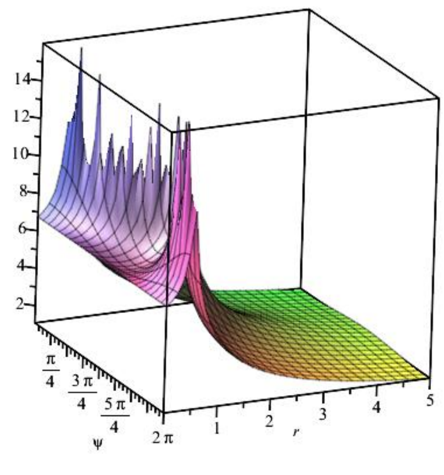

Figure 8: Graph of gravitational potential function of an elliptical mass distribution

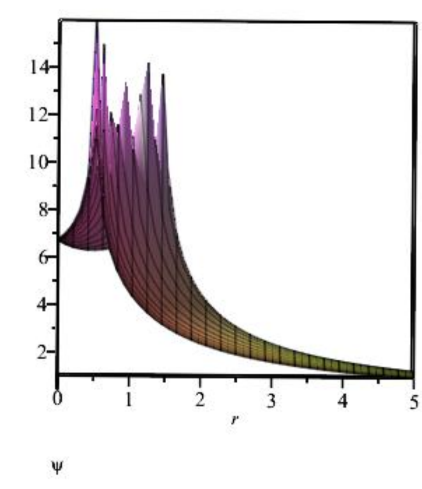

Figure 9: Side view of gravitational potential function

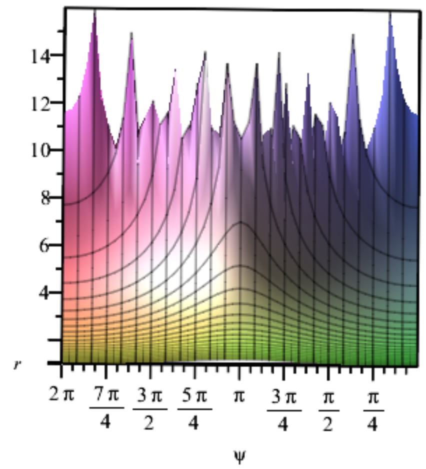

Figure 10: Side view of gravitational potential function

As you can see from this view, the spikes formed by the gravitational potential of an ellipse are symmetrical with respect to $\psi = \pi$. The symmetry can be easily understood by the horizontal symmetry of the ellipse with respect to the foci, but why the spikes?

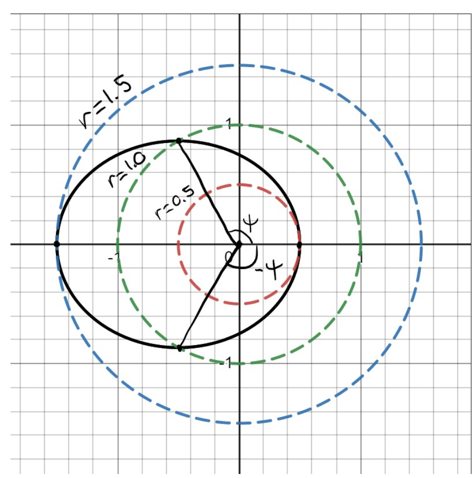

Figure 11: Illustration of the cause of the spikes for $a(1-\epsilon) < r < a(1+\epsilon)$

As you can see from figure 10, when $a(1-\epsilon) < r < a(1+\epsilon)$, the circular area of interest will intersect with the ellipse twice. This explains the symmetry of the spikes very well. Also, at $r = a(1-\epsilon)$ and $r = (1+\epsilon)$, this circle will intersect with the ellipse once. Ideally, there should be one continuous spike in the domain of $a(1-\epsilon) \leq r \leq a(1+\epsilon)$, however, it seems that our numerical precision is limited. In our case we have $a(1-\epsilon) = 0.5$ and $a(1+\epsilon) = 1.5$ which confines our spikes shown in figure 11 and 12.

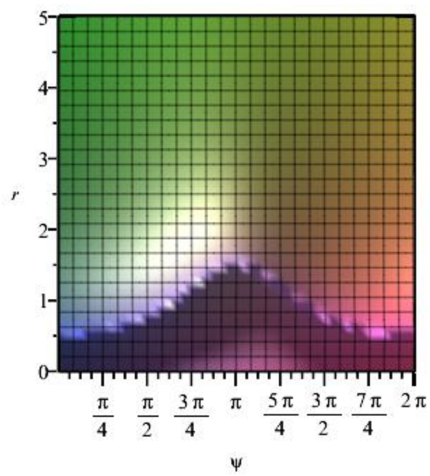

Figure 12: Top down view of Gravitational Potential function

This view shows spikes in the plot on the following curve:

$$r(\psi) = \frac{0.75}{1 + 0.5cos(\psi)} \tag{34}$$

This equation is analogous to equation (23).

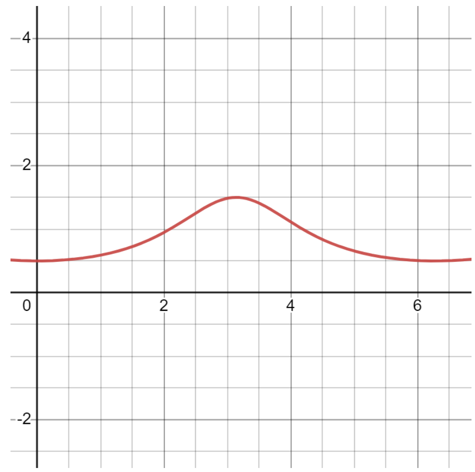

Figure 13: $r(\psi)$ Graph of distance from foci to field point in terms of angle ψ

5 Summary and Future Work

All in all, we have graphically modeled the gravitational potential of an elliptical mass distribution. We first analytically modeled this potential equation. This was done by picking the coordinate axis to measure the potential relative to, deriving relationships between the variables ψ, r, θ, R, dl, and plugged these relationships into the equation for gravitational potential. Solving this equation required the use of numerical integration since it is not solvable analytically. So to get an idea of how the gravitational potential of an ellipse should deviate from the gravitational potential of a circular mass distribution, we did a series expansion on the integrand of our potential with respect to ϵ. Then plotting the integral of this equation up until the first order of epsilon, with a small epsilon, we got a graph that was slightly deviated from the graph of a circular mass distribution. This deviation also gave us insight into what to expect for the elliptical mass distribution graph. We then graphed this potential and the results are intuitive. There is horizontal symmetry with respect to $\psi = \pi$ as expected . Also, the deviation of the "spike" of the potential function turned into the equation of distance between the ellipse and the foci. This is fascinating because we can see exactly how the geometry of a mass distribution affects the gravitational force around it. Going forward we would like to implement these methods on different types of mass distributions to get a better idea of the impact of gravitational force for different systems.

6 Maple Code

Different instances were tested by altering values of r, ψ, ϵ:
$a := 1;$
$mu := 1;$
$G := 1;$
$f := -G * mu * a * (-q^2 + 1) * sqrt(D0 + D1 * cos(theta))/sqrt(C0 + C1 * cos(theta) + C2 * cos(theta)^2 + C3 * cos(theta)^3 + C4 * cos(theta)^4 + C5 * sin(theta) + C6 * cos(theta) * sin(theta) + C7 * cos(theta)^2 * sin(theta) + C8 * cos(theta)^3 * sin(theta));$
$c0 := a^2 * (-q^2 + 1)^2 + r^2;$
$c1 := 4 * q * r^2 + 2 * q * a^2 * (-q^2 + 1)^2 - 2 * a * (-q^2 + 1) * r * cos(psi);$
$c2 := 6 * q^2 * r^2 + a^2 * q^2 * (-q^2 + 1)^2 - 6 * a * (-q^2 + 1) * r * q * cos(psi);$
$c3 := 4 * q^3 * r^2 - 6 * q^2 * r * a * (-q^2 + 1) * cos(psi);$
$c5 := -2 * a * (-q^2 + 1) * r * sin(psi);$
$c4 := q^4 * r^2 - 2 * r * a * q^3 * (-q^2 + 1) * cos(psi);$
$c6 := -6 * a * (-q^2 + 1) * r * q * sin(psi);$
$c7 := -6 * a * r * q^2 * (-q^2 + 1) * sin(psi);$
$d0 := q^2 + 1;$
$c8 := -2 * r * a * q^3(-q^2 + 1) * sin(psi);$
$d1 := 2 * q;$
$f_subs := subs(D0 = d0, D1 = d1, C0 = c0, C1 = c1, C2 = c2, C3 = c3, C4 = c4, C5 = c5, C6 = c6, C7 = c7, C8 = c8, f);$
$f_subs_expand := series(f_subs, q, 3);$
$f0_expand := limit(f_subs_expand, q = 0);$
$f1_expand := simplify(limit((f_subs_expand - f0_expand)/q, q = 0));$
$f0 := limit(f_subs, q = 0);$
$f1 := simplify(limit((f_subs - f0)/q, q = 0));$

Potential of circle mass distribution:
$circle := simplify(evalf(Int(limit(-f_subs, q = 0), theta = 0..2 * Pi)));$
$PlotBuilder(Int(1/(1. + r^2 - 2. * r * cos(psi) * cos(theta) - 2. * r * sin(psi) * sin(theta))^(1/2), theta = 0...6.283185308));$

Potential of circle + first order correction:
$small_epsilon := circle + correction;$
$PlotBuilder(Int(1/(1. + r^2 - 2. * r * cos(psi) * cos(theta) - 2. * r * sin(psi) * sin(theta))^(1/2), theta = 0...6.283185308) + 0.01 * r * Int(-(-sin(psi) * sin(theta) - cos(psi) * cos(theta) + r) * cos(theta)/(1. + r^2 - 2. * r * cos(psi) * cos(theta) - 2. * r * sin(psi) * sin(theta))^(3/2), theta = 0...6.283185308))$

Potential of an ellipse mass distribution:
$ellipse := -simplify(evalf(Int(f_subs, theta = 0..2 * Pi)))$
$PlotBuilder(-0.0201 * Int((2.0201 + 2.02 * cos(theta))^(1/2)/((0.0402 * r + 0.121806 * r * cos(theta) + 0.12302406 * r * cos(theta)^2 + 0.0414181002 * r * cos(theta)^3) * sin(theta) * sin(psi) + (0.0414181002 * r * cos(theta)^4 + 0.12302406 * r * cos(theta)^3 + 0.121806 * r * cos(theta)^2 + 0.0402 * r * cos(theta)) * cos(psi) + 1.04060401 * cos(theta)^4 * r^2 + 4.121204 * cos(theta)^3 * r^2 + (6.1206 * r^2 + 0.000412130601) * cos(theta)^2 + (4.04 * r^2 + 0.0008161002) * cos(theta) + 0.00040401 + r^2)^(1/2), theta = 0...6.283185308));$

References

[1] "Grant R. Fowles" and "George L. Cassidy". Analytical Mechanics 7th edition. book, 2004.