On Unfoldings of Stretched Polyhedra

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Abstract

Here we give a short proof of a result obtained by Mohammad Ghomi concerning existence of nets of a convex polyhedron after a suitable linear transformation.

Introduction

Once we cut a convex polyhedron P along a spanning tree T of its 1-skeleton, we obtain a compact surface P_T, which is homeomorphic to a closed disc. Since P_T does not have any cycles, once we map isometrically one face into $\mathbb{R}^2$, there is a unique way of mapping (unfolding) its complement face by face in such a way that when two faces share an edge not in T, then their images share the corresponding edge, and consecutive faces do not overlap. We are going to denote this map as f_T. This map restricted to the interior of P_T is an isometric immersion into the plane. However, f_T does not have to be injective (see figure 1). If it is, then we say it is a *net* of P.

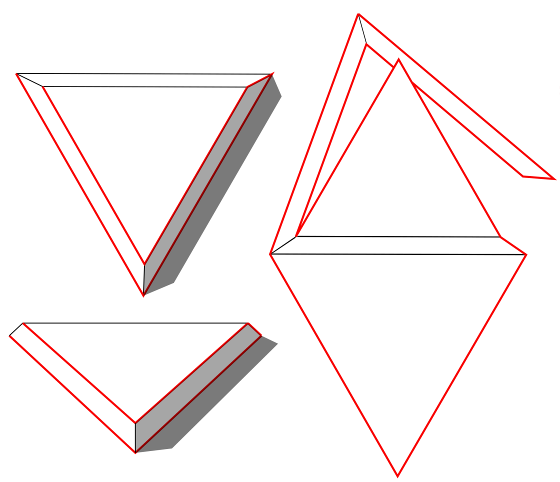

Figure 1: Cutting a thin truncated equilateral pyramid along the red line (left) generates an unfolding with self overlaps (right).

Figure 2: Albert Durer self portrait.

G.C. Shephard [5] posed in 1975 the problem to determine whether all convex polyhedra have a net. Experimental results do not give us enough information to expect a positive or negative answer to the problem. However, Mohammad Ghomi solved a similar problem, which is to develop any polyhedron after a suitable stretching in one direction (theorem 1 below). In particular, every polyhedron is affinely equivalent to one with a net, and having a net does not depend on the combinatorial structure of the polyhedron.

Theorem 1. Let P be a convex polyhedron, then there is a linear stretching L, such that $L(P)$ has a net.

Let ξ be a direction not orthogonal to any line determined by two vertices of P. Applying a rotation sending ξ to e_1, we obtain a polyhedron such that no edge is orthogonal to e_1. Therefore, after a suitable stretching in the direction of the x axis, all the edges of the obtained polyhedron Q form an angle of less than $\frac{\pi}{20N}$ with e_1, where N is the number of edges of P.

We now define an ordering on the set of vertices of Q. We say that $v \leq v'$ if the first coordinate of v is less or equal than the first coordinate of v'. We will denote by y and z the minimal and maximal vertices respectively.

We define an *ascending sequence* as a set of vertices $\{p_0, p_1, \ldots, p_n\}$ such that $p_i \leq p_{i+1}$ and $p_i p_{i+1}$ are connected by an edge for all $i \in \{0, 1, \ldots, n-1\}$.

We say that a spanning tree T with root z is *increasing* if for any vertex $v \in Q$ there is a (unique) ascending sequence from v to z contained in T.

Note that all leaves of an increasing tree form an angle close to π with e_1. Otherwise, the path joining the corresponding leaf to z wouldn't be an ascending sequence (see figure 3). Also, all the ends of the surface $Q \backslash T$ point in the direction of the x axis.

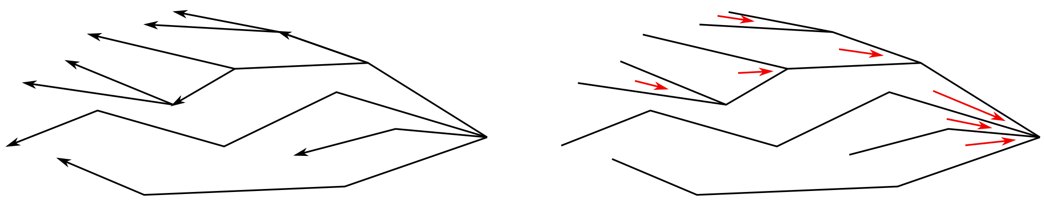

Figure 3: Leaves of an increasing tree point leftwards and ends of its complement point rightwards.

In what follows, we give a proof of theorem 2 below, which clearly implies theorem 1.

Theorem 2. (Ghomi) If Q is cut along any increasing tree T, then it can be unfolded without overlapping in $\mathbb{R}^2$.

Preliminaries

Let a, b, c be vertices such that ab and ac are connected by edges. We will denote by $\measuredangle bac$ the intrinsic angle at a from ab to ac measured counterclockwise. Note that by convexity

$$\measuredangle bac + \measuredangle cab < 2\pi. \tag{1}$$

For distinct $x, y, z \in \mathbb{R}^2$, $arg(x)$ will denote the argument from $-\pi$ to π of x as a complex number, and $\angle yxz$ will denote the angle at x measured counterclockwise. We will say that two sequences of points in $\mathbb{R}^2$ *cross* if the broken line's determined by them do so.

Lemma 3. (Arm Lemma) Suppose $\{u_0, u_1, \dots, u_m\}, \{v_0, v_1, \dots, v_m\} \subset \mathbb{R}^2$ satisfy

- $u_0 = v_0$.

- $|u_j - u_{j-1}| = |v_j - v_{j-1}|$ for $j = 1, 2, \ldots, m$.
- $u_j - u_{j-1}$ and $v_j - v_{j-1}$ are almost horizontal for $j = 1, 2, \ldots, m$ (their argument is in $\left(-\frac{\pi}{10}, \frac{\pi}{10}\right)$).
- $arg(v_i - v_{i-1}) \geq arg(u_i - u_{i-1})$ for $i \in \{1, 2, \ldots, m\}$.

Then $\{u_0, u_1, \ldots, u_m\}$ and $\{v_0, v_1, \ldots, v_m\}$ do not cross and $v_m - u_m$ is almost vertical (its argument lies in the interval $\left(\frac{\pi}{2} - \frac{\pi}{10}, \frac{\pi}{2} + \frac{\pi}{10}\right)$)

Remark 4. One can observe from figure 1 that this lemma does not hold if we remove the condition of $u_j - u_{j-1}$ and $v_j - v_{j-1}$ being almost horizontal. Here is where we use the stretching.

Proof of the lemma: We will prove this lemma by induction on m. The case $m = 1$ is elementary plane geometry.

Suppose it holds for $m = \nu$ and take $m = \nu + 1$. Construct another sequence $\{w_1, w_2, \ldots, w_m\}$ such that $w_1 = v_1$ and $u_{j+1}u_jw_jw_{j+1}$ is a parallelogram for $j \in \{1, 2, \ldots, m-1\}$. Applying the induction hypothesis to $\{w_1, w_2, \ldots w_m\}$ and $\{v_1, v_2, \ldots, v_m\}$, we see that they do not cross and $arg(w_m - v_m) \in \left(-\frac{6\pi}{10}, -\frac{4\pi}{10}\right)$. Also, $arg(u_m - w_m) = arg(u_1 - w_1) = arg(u_1 - v_1) \in \left(-\frac{6\pi}{10}, -\frac{4\pi}{10}\right)$, which implies $arg(u_m - v_m) \in \left(-\frac{6\pi}{10}, -\frac{4\pi}{10}\right)$.

□

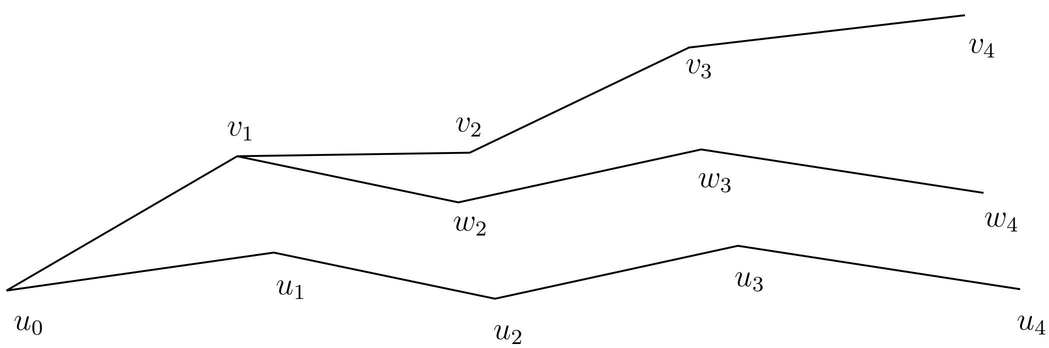

Figure 4: Proof of arm lemma.

Lemma 5. Let S be a flat surface homeomorphic to a closed disc. Consider a map $f : S \to \mathbb{R}^2$ such that restricted to the interior of S is an isometric immersion. Then f is injective if and only if $f(\partial S)$ is a simple closed curve.

Proof: One implication is trivial. The other one follows from the fact that for conformal maps $f : S \to \mathbb{R}^2$, the number of preimages $f^{-1}(x)$ equals the winding number $I(f(\partial S), x)$ for all x in $\mathbb{R}^2 \backslash f(\partial S)$, which is a standard fact in complex analysis ([4], p. 384). If $f(\partial S)$ is a simple closed curve, by Jordan's Curve Theorem [3] the winding number $I(f(\partial S), x)$ equals 0 or 1, then the function is injective.

□

Proof of Theorem 2

First, construct an unfolding f_T of Q_T in which the images of all the edges are almost horizontal (they form an angle less than $\frac{\pi}{10}$ with $e_1 \in \mathbb{R}^2$). Next, we are going to prove that $f_T(\partial Q_T)$ does not self intersect.

Consider $y' = f_T(y) \in \mathbb{R}^2$. Then, starting at y', the counterclockwise image of ∂Q_T is a sequence of piecewise linear curves (broken lines) going in the direction of the x axis and backwards, alternately. We are going to denote the first broken line going right as R_1, the first one going left as L_1, and so on. Since T is ascending, when we switch from going rightwards to leftwards, we rotate counterclockwise and when we switch from going leftwards to rightwards, we rotate clockwise (see figure 5).

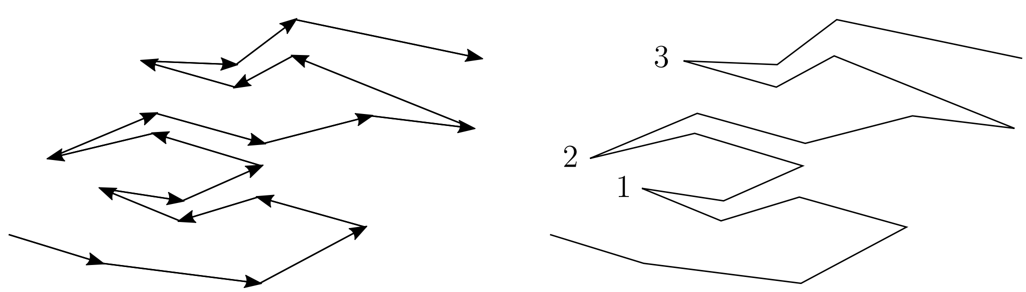

Figure 5: The image of the boundary is a sequence of broken lines going rightwards and leftwards alternately.

Proposition 6. As defined above, if the sequence of broken lines $R_1 L_1 R_2 \ldots L_n$ does not self intersect, then $R_1 L_1 R_2 \ldots L_n R_{n+1}$ won't self intersect either.

Proof: The proof goes by induction on the number n of times it has gone leftwards (in figure 5, $n = 3$). For the case $n = 1$, observe that the condition

$arg(v_i - v_{i-1}) \geq arg(u_i - u_{i-1})$ for $i = 1, 2, \ldots, m$ in the arm lemma is implied by $arg(v_1 - v_0) \geq arg(u_1 - u_0)$ and $\angle v_{i+1} v_i v_{i-1} + \angle u_{i-1} u_i u_{i+1} \leq 2\pi$ for $i \in \{1, 2, \ldots, m-1\}$. Therefore the arm lemma completes the base of induction.

Suppose the assertion is true for $n \leq k$ and consider the case $n = k+1$. Note that the edges of ∂Q_T are paired in such a way that we glue paired edges together to obtain Q from Q_T. Each one will be called the *dual* of the other.

Observe that when we start R_2, we are traveling the dual edges of the leftmost part of L_1. If the length of R_2 is greater than or equal to the length of L_1, we can apply the induction hypothesis to $R_2 L_2 R_3 \ldots L_n R_{n+1}$ and the result will follow.

If the length of R_2 is less than the length of L_1, we extend R_2 with a broken line S parallel to L_1 minus the dual of R_2. By the arm lemma, S will be above L_1 (see figure 6), and by the induction hypothesis S does not touch $L_2 R_3 \ldots L_n R_{n+1}$ so this finishes the proof of proposition 6.

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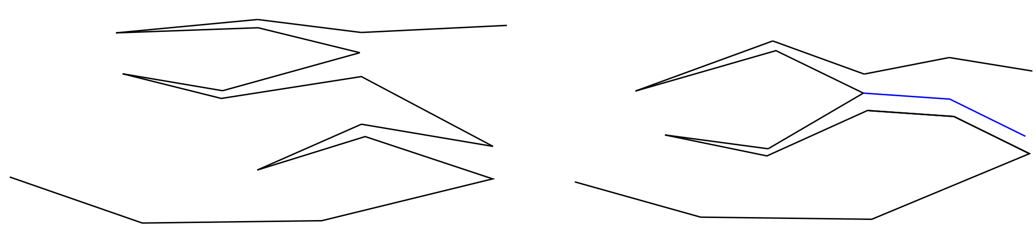

Figure 6: On the left R_2 and L_1 have the same length, on the right we construct S in blue.

Now, the image of ∂Q_T will self intersect for the first time in a point y'' while going leftwards. Because T is increasing, $f_T(\partial Q_T)$ contains a simple closed curve γ starting at y'' consisting of a sequence of broken lines turning clockwise when changing from going leftwards to rightwards and counter-clockwise in the other case.

Since all ends point rightwards, we can contract the curve γ to the point y'' moving leftwards all the time. Such contraction can be performed in the same way in Q_T. Therefore only one point of ∂Q_T is sent to y''. This is only possible if $y' = y''$ and f_T restricted to ∂Q_T is injective.

Applying lemma 5 with $S = Q_T$ we get that f_T is a net, completing the proof.

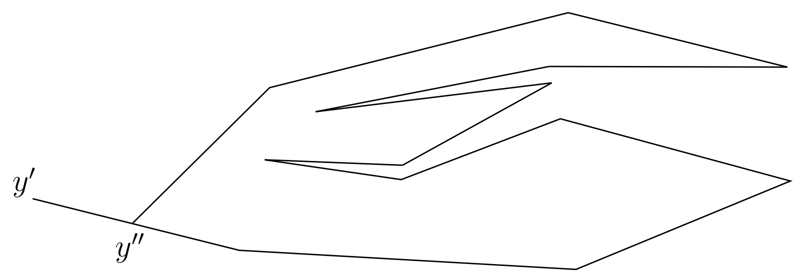

Figure 7: The image of ∂Q_T first self intersects at y''.

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