On Unfoldings of Stretched Polyhedra

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Abstract

We give a short proof of a result obtained by Mohammad Ghomi concerning existence of nets of a convex polyhedron after a suitable linear transformation.

Introduction

A *net of a polyhedron* is an arrangement of edge-joined polygons in the plane which can be folded along its edges to become the faces of the polyhedron. The first known record of this procedure is the renaissance book *The Painter's Manual* by Albert Durer [1]. In this book, Durer shows how to cut and develop some figures, including all five regular polyhedra.

Figure 1: Albert Durer self portrait.

In 1975 Geoffrey Shephard [5] posed the problem to determine whether all convex polyhedra have a net. Let us discuss this problem in more detail.

To obtain a net one has to cut a convex polyhedron P along a spanning tree T of its 1-skeleton. This way we obtain a flat surface P_T, which is homeomorphic to a closed disc. The surface P_T can be mapped isometrically face by face into the plane in an essentially unique way; denote this map by $f_T\colon P_T \to \mathbb{R}^2$.

If f_T is injective then the image $f_T(P_T)$ is a net of P. However f_T might not be injective as figure 2 shows. Therefore Shephard's problem is asking if for any convex polyhedron P there is a spanning tree T such that f_T is injective.

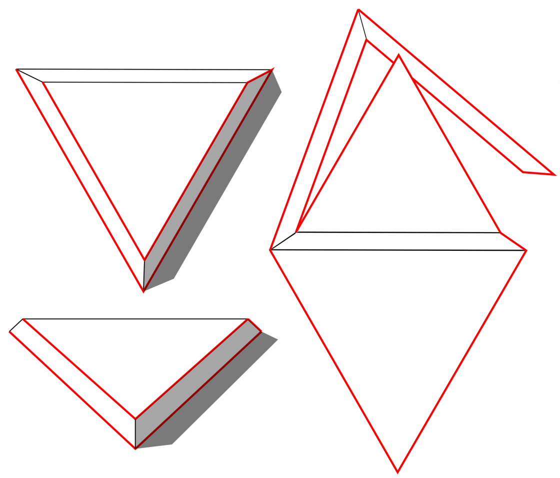

Figure 2: Cutting a thin truncated equilateral pyramid along the red line (left) generates an unfolding with self overlaps (right).

Recently Mohammad Ghomi [2] proved the existence of a net for any polyhedron after a suitable stretching in one direction (theorem 1 below). In particular, every polyhedron is affinely equivalent to one with a net, and having a net does not depend on the combinatorial structure of the polyhedron.

Theorem 1. Let P be a convex polyhedron, then there is a linear stretching L, such that $L(P)$ has a net.

Let ξ be a direction not orthogonal to any line determined by two vertices of P. Applying a rotation sending ξ to e_1, we obtain a polyhedron such that no edge is orthogonal to e_1. Therefore, after a suitable stretching in the direction of the x axis, all the edges of the obtained polyhedron Q form a sufficiently small angle with e_1 (less than $\frac{\pi}{20N}$ will do, where N is the number of edges of P).

We now define an ordering on the set of vertices of Q. We say that $v \leq v'$ if the first coordinate of v is less or equal than the first coordinate of v'. We will denote by y and z the minimal and maximal vertices respectively.

We define an *ascending sequence* as a set of vertices $\{p_0, p_1, \ldots, p_n\}$ such that $p_i \leq p_{i+1}$ and $p_i p_{i+1}$ are connected by an edge for all $i \in \{0, 1, \ldots, n-1\}$. We say that a spanning tree T with root z is *increasing* if for any vertex $v \in Q$ there is a (unique) ascending sequence from v to z contained in T.

Note that all terminal edges of an increasing tree form an angle close to π with e_1. Otherwise, the path in T joining the corresponding leaf to z wouldn't be an ascending sequence (see figure 3). Also, all the ends of the surface $Q \backslash T$ point in the direction of the x axis.

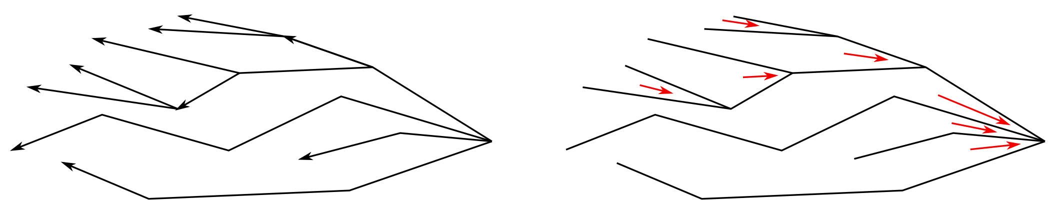

Figure 3: Terminal edges of an increasing tree T point leftwards and ends of $Q \backslash T$ point rightwards.

Theorem 2. (Ghomi) If Q is cut along any increasing tree T, then the unfolding map f_T is injective.

In this note we give a short and elementary proof this theorem, which clearly implies theorem 1.

Preliminaries

Let a, b, c be vertices such that ab and ac are connected by edges. We will denote by $\measuredangle bac$ the intrinsic angle at a from ab to ac measured counterclockwise. Note that by convexity

$$\measuredangle bac + \measuredangle cab < 2\pi. \tag{1}$$

For distinct $x, y, z \in \mathbb{R}^2$, $arg(x)$ will denote the argument from $-\pi$ to π of x as a complex number, and $\angle yxz$ will denote the angle at x measured counterclockwise. We will say that two sequences of points in $\mathbb{R}^2$ *cross* if the broken line's determined by them do so.

Lemma 3. **(Arm Lemma)** Suppose $\{u_0, u_1, \ldots, u_m\}, \{v_0, v_1, \ldots, v_m\} \subset \mathbb{R}^2$ satisfy

- $u_0 = v_0$.
- $|u_j - u_{j-1}| = |v_j - v_{j-1}|$ for $j = 1, 2, \ldots, m$.
- $u_j - u_{j-1}$ and $v_j - v_{j-1}$ are almost horizontal for $j = 1, 2, \ldots, m$ (their argument is in $\left(-\frac{\pi}{10}, \frac{\pi}{10}\right)$).
- $arg(v_i - v_{i-1}) \geq arg(u_i - u_{i-1})$ for $i \in \{1, 2, \ldots, m\}$.

Then $\{u_0, u_1, \ldots, u_m\}$ and $\{v_0, v_1, \ldots, v_m\}$ do not cross and $v_m - u_m$ is almost vertical (its argument lies in the interval $\left(\frac{\pi}{2} - \frac{\pi}{10}, \frac{\pi}{2} + \frac{\pi}{10}\right)$)

Remark 4. One can observe from figure 2 that this lemma does not hold if we remove the condition of $u_j - u_{j-1}$ and $v_j - v_{j-1}$ being almost horizontal. The stretching is applied to meet this condition.

Proof of the lemma: We will prove this lemma by induction on m. The case $m = 1$ is elementary plane geometry.

Suppose it holds for $m = \nu$ and take $m = \nu + 1$. Construct another sequence $\{w_1, w_2, \ldots, w_m\}$ such that $w_1 = v_1$ and $u_{j+1}u_jw_jw_{j+1}$ is a parallelogram for $j \in \{1, 2, \ldots, m-1\}$. Applying the induction hypothesis to $\{w_1, w_2, \ldots w_m\}$ and $\{v_1, v_2, \ldots, v_m\}$, we see that they do not cross and $arg(w_m - v_m) \in \left(\frac{4\pi}{10}, \frac{6\pi}{10}\right)$. Also, $arg(u_m - w_m) = arg(u_1 - w_1) = arg(u_1 - v_1) \in \left(\frac{4\pi}{10}, \frac{6\pi}{10}\right)$, which implies $arg(u_m - v_m) \in \left(\frac{4\pi}{10}, \frac{6\pi}{10}\right)$.

$\square$

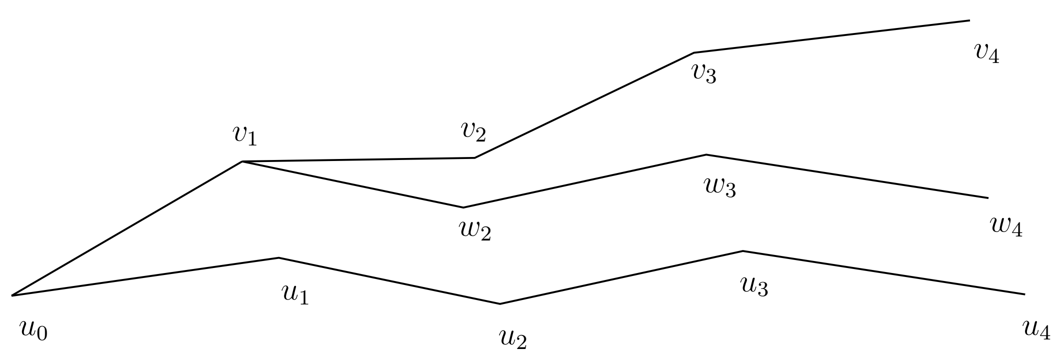

Figure 4: Proof of arm lemma.

Lemma 5. Let S be a flat surface homeomorphic to a closed disc. Consider a map $f : S \to \mathbb{R}^2$ such that restricted to the interior of S is an isometric immersion. Then f is injective if and only if $f(\partial S)$ is a simple closed curve.

Proof: One implication is trivial. The other one follows from the fact that for conformal maps $f : S \to \mathbb{R}^2$, the number of preimages $f^{-1}(x)$ equals the winding number $I(f(\partial S), x)$ for all $x \in \mathbb{R}^2 \backslash f(\partial S)$, which is a standard fact in complex analysis ([4], p. 384). If $f(\partial S)$ is a simple closed curve, by Jordan Curve Theorem [3] the winding number $I(f(\partial S), x)$ equals 0 or 1, then the function is injective.

$\square$

Proof of Theorem 2

First, construct an unfolding f_T of Q_T in which the images of all the edges are almost horizontal (they form an angle less than $\frac{\pi}{10}$ with $e_1 \in \mathbb{R}^2$). Next, we are going to prove that $f_T(\partial Q_T)$ does not self intersect.

Consider $y' = f_T(y) \in \mathbb{R}^2$. Then, starting at y', the counterclockwise image of ∂Q_T is a sequence of piecewise linear curves (broken lines) going almost horizontally rightwards and leftwards, alternately. We are going to denote the first broken line going right as R_1, the first one going left as L_1, and so on. Since T is ascending, when we switch from going rightwards to leftwards, we rotate counterclockwise and when we switch from going leftwards to rightwards, we rotate clockwise (see figure 5).

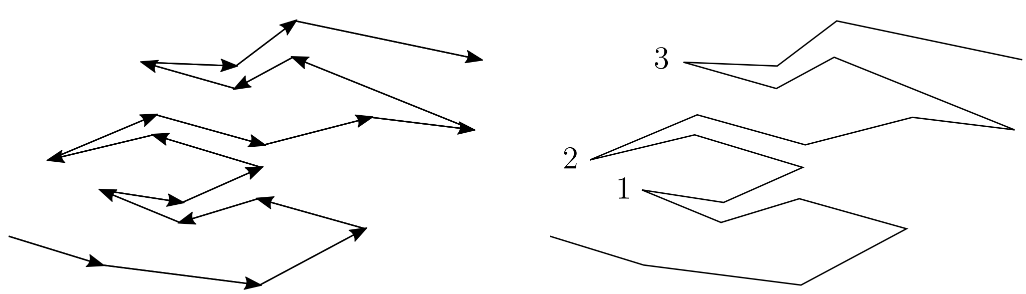

Figure 5: The image of the boundary is a sequence of broken lines going rightwards and leftwards alternately.

Proposition 6. As defined above, if the sequence of broken lines $R_1L_1R_2\ldots L_n$ doesn't self intersect, then $R_1L_1R_2\ldots L_nR_{n+1}$ doesn't self intersect either.

Proof: The proof goes by induction on the number n of times it has gone leftwards (in figure 5, $n=3$). For the case $n=1$, observe that the condition $arg(v_i - v_{i-1}) \geq arg(u_i - u_{i-1})$ for $i = 1, 2, \ldots, m$ in the arm lemma is implied by $arg(v_1 - v_0) \geq arg(u_1 - u_0)$ and $\angle v_{i+1}v_iv_{i-1} + \angle u_{i-1}u_iu_{i+1} \leq 2\pi$ for $i \in \{1, 2, \ldots, m-1\}$. Therefore the arm lemma completes the base of induction.

Suppose the assertion is true for $n \leq k$ and consider the case $n = k+1$. Note that the edges of ∂Q_T are paired in such a way that we glue paired edges together to obtain Q from Q_T. Each one will be called the *dual* of the other.

Observe that when we start R_2, we are traveling the dual edges of the leftmost part of L_1. If the length of R_2 is greater than or equal to the length of L_1, we can apply the induction hypothesis to $R_2L_2R_3\ldots L_nR_{n+1}$ and the result will follow.

If the length of R_2 is less than the length of L_1, we extend R_2 with a broken line S parallel to L_1 minus the dual of R_2. By the arm lemma, S will be above L_1 (see figure 6), and by the induction hypothesis S does not touch $L_2R_3\ldots L_nR_{n+1}$ so this finishes the proof of proposition 6.

□

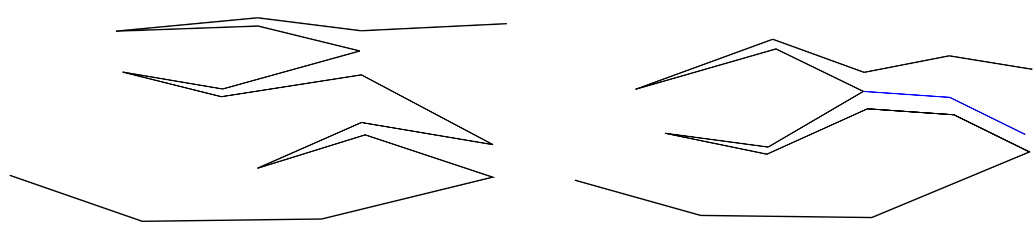

Figure 6: On the left R_2 and L_1 have the same length, on the right we construct S in blue.

Now, the image of ∂Q_T will self intersect for the first time in a point y'' while going leftwards. Because T is increasing, $f_T(\partial Q_T)$ contains a simple closed curve γ starting at y'' consisting of a sequence of broken lines turning clockwise when changing from going leftwards to rightwards and counter-clockwise in the other case.

Since all ends point rightwards, we can contract the curve γ to the point y'' moving leftwards all the time. Such contraction can be performed in the

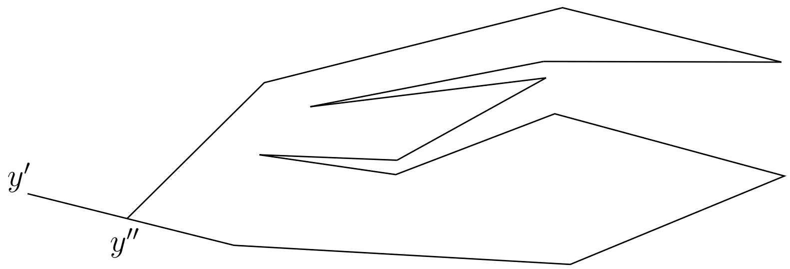

Figure 7: The image of ∂Q_T first self intersects at y''.

same way in Q_T. Therefore only one point of ∂Q_T is sent to y''. This is only possible if $y' = y''$ and f_T restricted to ∂Q_T is injective.

Applying lemma 5 with $S = Q_T$ we get that f_T is a net, completing the proof.

References

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[5] G. C. Shephard, *Convex polytopes with convex nets.* Math. Proc. Cambridge Philos. Soc., 78(3):389-403, 1975.