Radiation Hydrodynamics in Python

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Background

Hydrodynamics

conserved $\boxed{\mathcal{U} = (\rho, p_x, p_y, E, \rho_{\rm ion})}$

primitive $\boxed{\mathcal{W} = (\rho, v_x, v_y, P, \rho_{\rm ion})}$

Euler Equations

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \rho + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \quad \text{mass conservation}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} + \frac{\boldsymbol{\nabla} P}{\rho} = 0 \quad \text{momentum conservation}$$

$$\frac{\partial P}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} P + \gamma P \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \quad \text{energy conservation}$$

$$\frac{\partial}{\partial t} \rho_{\rm ion} + \boldsymbol{\nabla} \cdot (\rho_{\rm ion} \mathbf{u}) = 0 \quad \text{passive scalar}$$

$$\rho_{\rm ion} \equiv \rho x_{\rm HII}$$

Conservative Form

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \mathbf{p} \\ E \end{pmatrix} + \boldsymbol{\nabla} \cdot \begin{pmatrix} \mathbf{p} \\ \mathbf{p} \otimes \mathbf{u} + P \\ (E + P)\mathbf{u} \end{pmatrix} = 0$$

Quasi-linear Form

$$\frac{\partial \mathcal{W}}{\partial t} + \mathbb{A} \frac{\partial \mathcal{W}}{\partial x} + \mathbb{B} \frac{\partial \mathcal{W}}{\partial y} = 0$$

Radiative Transfer

Moment equations

$$\frac{\partial N_{\rm phot}}{\partial t} + \nabla \cdot \mathbf{F} = -(1-x) n_{\rm H} \sigma + \dot{N}_{\rm inj} + \dot{N}_{\rm rec}$$
$$\frac{\partial \mathbf{F}}{\partial t} + c^2 \nabla \cdot \mathbb{P} = -(1-x) n_{\rm H} \sigma c \mathbf{F}$$
$$\mathbb{P} = \mathbb{D} N_{\rm phot}$$

Eddington tensor closure

$$\mathbb{D} = \frac{1-\chi}{2}\mathbf{I} + \frac{3\chi - 1}{2}\mathbf{n} \otimes \mathbf{n}$$

$$\mathbf{n} = \frac{\mathbf{F}}{|\mathbf{F}|}, \quad \chi = \frac{3 + 4f^2}{5 + 2\sqrt{4 - 3f^2}}, \quad f = \frac{|\mathbf{F}|}{c N_{\rm phot}}$$

Conservative Form

$$\frac{\partial}{\partial t}\begin{pmatrix} N_{\rm phot} \\ \mathbf{F} \end{pmatrix} + \boldsymbol{\nabla} \cdot \begin{pmatrix} \mathbf{F} \\ c^2 \mathbb{P} \end{pmatrix} = 0$$

$$\boxed{\mathcal{U}_{\rm rad} = (N_{\rm phot}, F_x, F_y)}$$

Non-Equilibrium Thermochemistry

$$\boxed{\mathcal{U}_T = (\epsilon, x_{\mathrm{HII}}, N_{\mathrm{phot}}, \mathbf{F})}$$

Photon Number Density $$\frac{\partial N_{\mathrm{phot}}}{\partial t} = -n_{\mathrm{HI}} c \sigma_{\mathrm{HI}}^{N} N_{\mathrm{phot}}$$

Photon Flux $$\frac{\partial \mathbf{F}}{\partial t} = -n_{\mathrm{HI}} c \sigma_{\mathrm{HI}}^{N} \mathbf{F}$$

Energy Density $$\frac{\partial \epsilon}{\partial t} = \mathcal{H} + \mathcal{L}$$

Ionization Fraction $$n_{\mathrm{H}} \frac{\partial x_{\mathrm{HII}}}{\partial t} = n_{\mathrm{HI}} \left(\beta_{\mathrm{HI}} n_e + \sigma_{\mathrm{HI}}^{N} c N_{\mathrm{phot}}\right) - n_{\mathrm{HII}} \alpha_{\mathrm{HII}}^{B} n_e$$

Note: see final report for all definitions

Photoheating Rate

$$\mathcal{H} = n_{\mathrm{HI}} c_r N_{\mathrm{phot}} \left(\bar{\epsilon}_\gamma \sigma_{\mathrm{HI}}^{E} - \epsilon_{\mathrm{HI}} \sigma_{\mathrm{HI}}^{N}\right)$$

Primordial Cooling Rate

$$\begin{aligned} \mathcal{L} = & \left[\zeta_{\mathrm{HI}}(T) + \psi_{\mathrm{HI}}(T)\right] n_e n_{\mathrm{HI}} \\ & + \eta_{\mathrm{HII}}^{B}(T) n_e n_{\mathrm{HII}} \\ & + \theta(T) n_e n_{\mathrm{HII}} \\ & + \varpi(T) n_e \; . \end{aligned}$$

Collisional Ionizations ζ
Collisional Excitations ψ
Recombinations η
Bremsstrahlung θ
Compton cooling ϖ

Methods

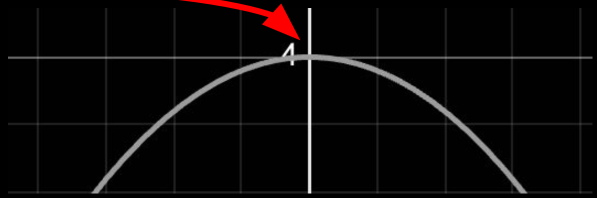
Hydrodynamics

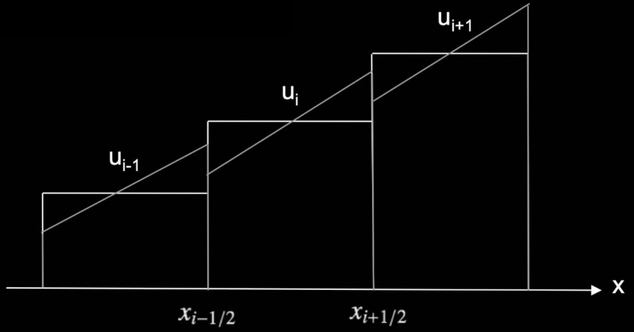
Second-Order Finite Volume Godunov Scheme

MonCen slope limiter

Smooth extrema detection

No slope limiter





Credit: Romain Teyssier, lecture notes

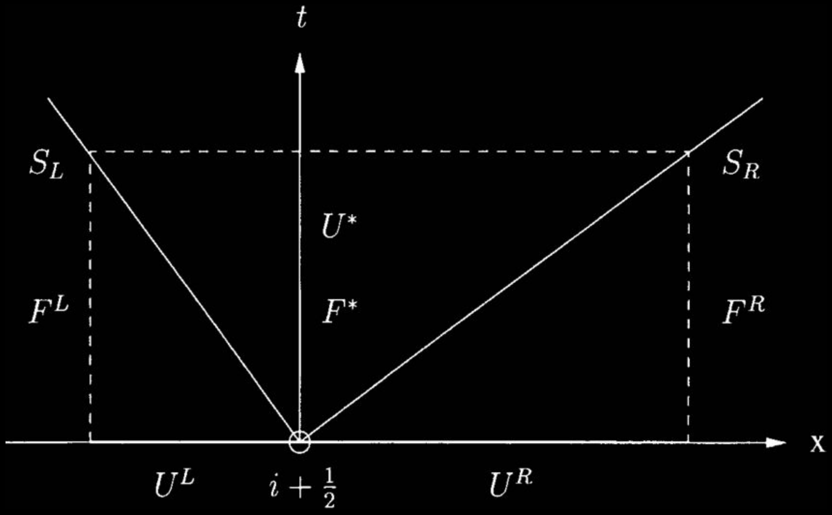
MUSCL+Hancock scheme

Half step $$\mathcal{W}^{n+1/2} = \mathcal{W}^n - \frac{\Delta t}{2\Delta x}\left(\mathbb{A}\Delta_x \mathcal{W}^n + \mathbb{B}\Delta_y \mathcal{W}^n\right)$$

Full step $$\mathcal{U}^{n+1}_{i,j} = \mathcal{U}_{i,j} + \frac{\Delta t}{\Delta x}\left(\mathcal{F}^{n+1/2}_{i-1/2,j} - \mathcal{F}^{n+1/2}_{i+1/2,j} + \mathcal{F}^{n+1/2}_{i,j-1/2} - \mathcal{F}^{n+1/2}_{i,j-1/2}\right)$$

Hydrodynamics

HLL Riemann solver



Credit: Zhou+2002

$$S_{\rm L} = \min\{v_{\rm L}, v_{\rm R}\} - \max\{c_{s,\rm L}, c_{s,\rm R}\}$$
$$S_{\rm R} = \max\{v_{\rm L}, v_{\rm R}\} + \max\{c_{s,\rm L}, c_{s,\rm R}\}$$

Three cases

1. $S_{\rm R} > 0$ $S_{\rm L} > 0$ supersonic (right)
2. $S_{\rm R} < 0$ $S_{\rm L} < 0$ supersonic (left)
3. $S_{\rm R} > 0$ $S_{\rm L} < 0$ subsonic

Radiative Transfer

First-Order Finite Volume Godunov Scheme

$$(\mathcal{U}_{\rm rad})^{n+1}_{i,j} = (\mathcal{U}_{\rm rad})^{n}_{i,j} + \frac{\Delta t}{\Delta x}\left((\mathcal{F}_{\rm rad})^{n}_{i-1/2,j} - (\mathcal{F}_{\rm rad})^{n}_{i+1/2,j} + (\mathcal{F}_{\rm rad})^{n}_{i,j-1/2} - (\mathcal{F}_{\rm rad})^{n}_{i,j+1/2}\right)$$

Global Lax-Friedman Riemann Solver

$$\mathcal{F}_{\rm rad} = \frac{\mathcal{F}_{\rm rad,L} + \mathcal{F}_{\rm rad,R}}{2} - \frac{c}{2}(\mathcal{U}_{\rm rad,R} - \mathcal{U}_{\rm rad,L})$$

Thermochemistry

Photon Update

$$N_i^{t+\Delta t} = \frac{N_i^t}{1+\Delta t D}$$

$$\mathbf{F}_i^{t+\Delta t} = \frac{\mathbf{F}_i^t}{1+\Delta t D}$$

10 % Rule
→ Δt = Δt / 2

Thermal Update

$$T_\mu^{t+\Delta t} = T_\mu^t + \frac{\Lambda K \Delta t}{1-\Lambda' K \Delta t}$$

10 % Rule
→ Δt = Δt / 2

Ionization Fraction Update

$$x_{\rm HII}^{t+\Delta t} = x_{\rm HII}^t + \Delta t \frac{C - x_{\rm HII}^t (C+D)}{1 - J\Delta t}$$

Semi-Implicit Scheme: Each update step uses the previous update's state variables to take the next time step

Adaptive Time Stepping with 10% Rule: if any step fails to meet the 10% rule, restart the entire thermochemical step with a halved time step and take 2^n steps (n = number of halvings) in the total thermochemical step

Photon Update Definitions

$$D = c_r \sigma_{\rm HI}^N n_{\rm HI}$$

Thermal Update Definitions

$$T_\mu \equiv \frac{T}{\mu} \qquad \mu = [X(1+x_{\rm HII})]^{-1}$$

$$\Lambda \equiv \mathcal{H} + \mathcal{L} \qquad \Lambda' \equiv \mu \frac{\partial \mathcal{L}}{\partial T}$$

$$K \equiv \frac{(\gamma - 1) m_{\rm H}}{\rho k_{\rm B}}$$

Ionization Fraction Definitions

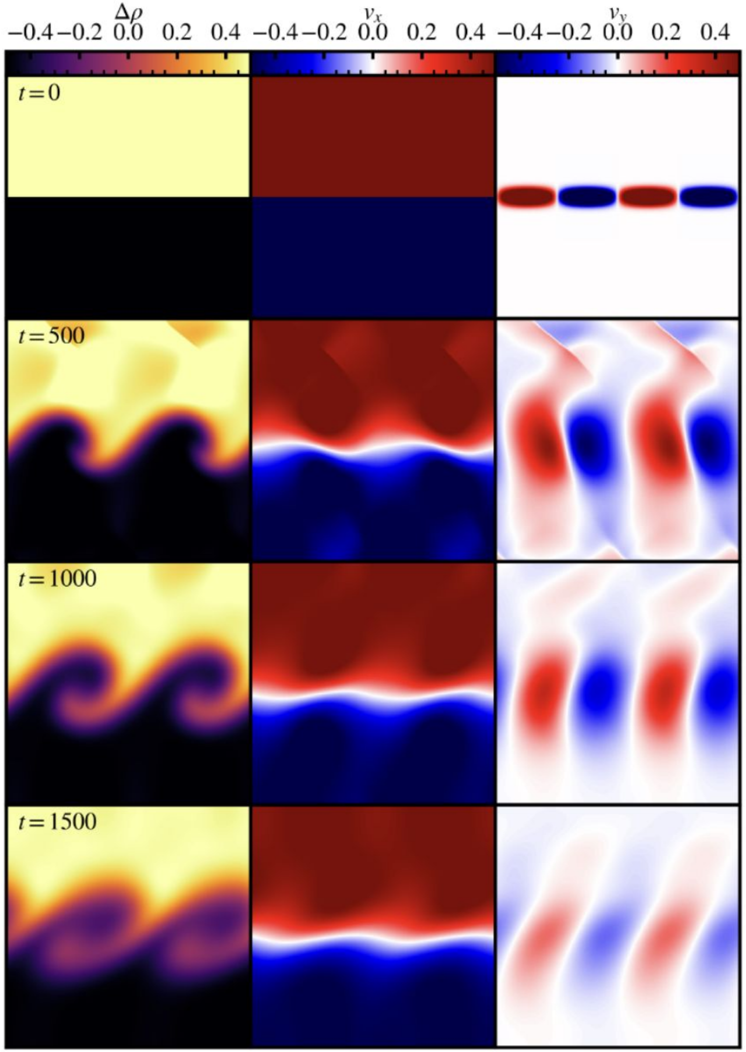
$$\begin{aligned}\frac{\partial x_{\rm HII}}{\partial t} &= (1-x_{\rm HII})\left[\beta_{\rm HII} n_e + \sigma_{\rm HI}^N c_r N\right] - x_{\rm HII}\alpha_{\rm HII}^B n_e \\ &= (1-x_{\rm HII})C - x_{\rm HII} D \\ &= C - x_{\rm HII}(C+D)\ ,\end{aligned}$$

$$J \equiv \frac{\partial \dot{x}_{\rm HII}}{\partial x_{\rm HII}} = \frac{\partial C}{\partial x_{\rm HII}} - (C+D) - x_{\rm HII}\left(\frac{\partial C}{\partial x_{\rm HII}} + \frac{\partial D}{\partial x_{\rm HII}}\right)$$

Results

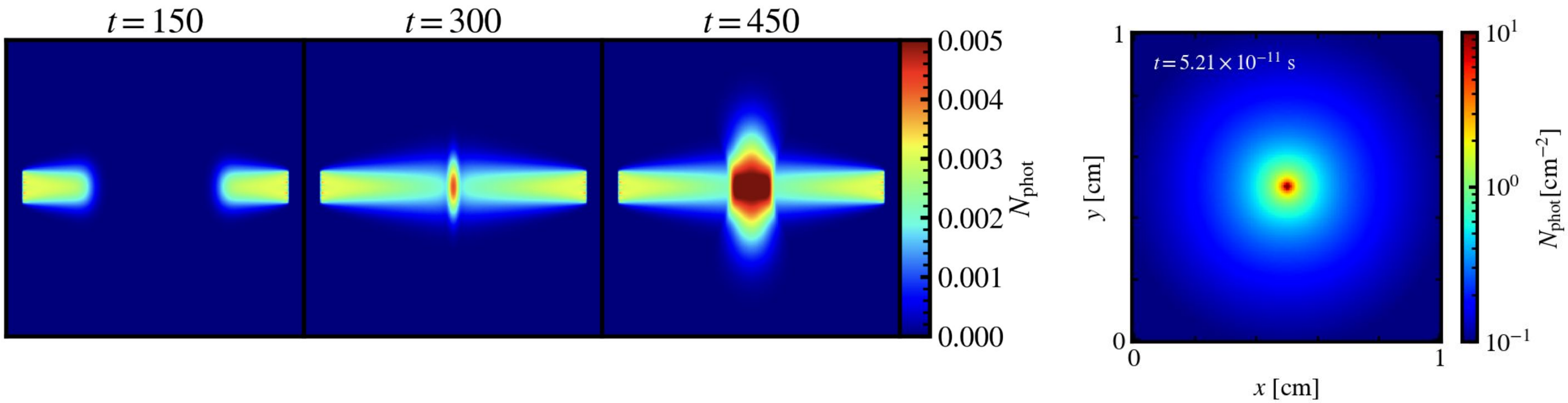
Hydrodynamics Solver

- 2nd Order Finite Volume Godunov Method (MUSCL-Hancock)
- MonCen Slope Limiter
- Smooth Extrema Detection
- HLL Riemann Solver
- Test: Kelvin-Helmholtz instability



Radiative Transfer Solver

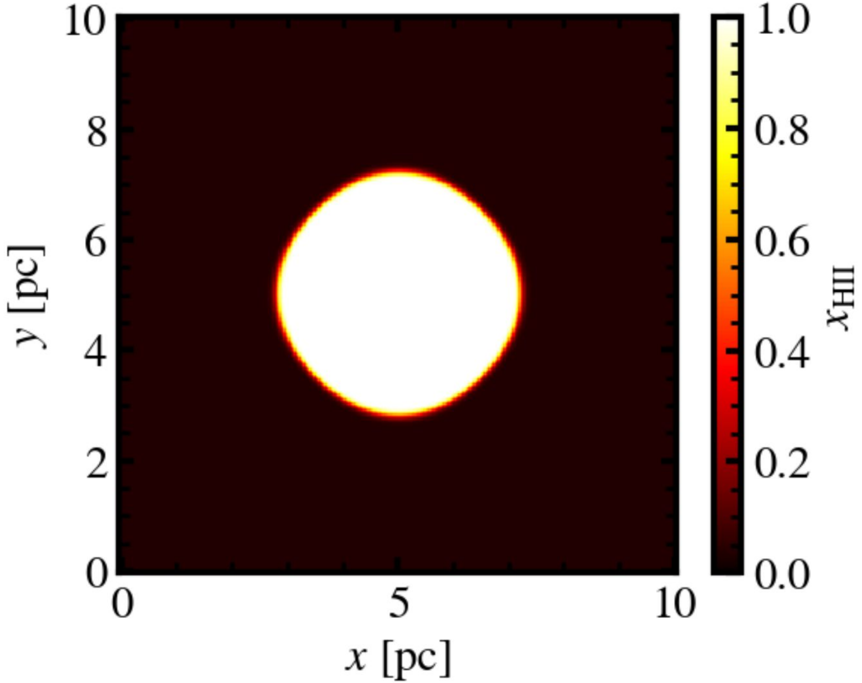
- First-Order Finite Volume Godunov Scheme
- GLF Riemann Solver



Radiation Hydrodynamics

Initial conditions for simulating an OB-type star in a MW cloud and problem setup

$$n_{\mathrm{H}} = 10^2\ \mathrm{cm}^{-3}$$
$$\dot{N} = 2 \times 10^{48}\ \mathrm{s}^{-1}$$
$$r_{\mathrm{S}} = 2\ \mathrm{pc}$$
$$t_{\mathrm{sim}} = 0.5\ \mathrm{Myr}$$
$$T = 100\ \mathrm{K}$$
$$c_{\mathrm{r}} = 6 \times 10^{-4} c$$
$$x_{\mathrm{HII}} = 0$$



Strömgen Sphere Test Problem

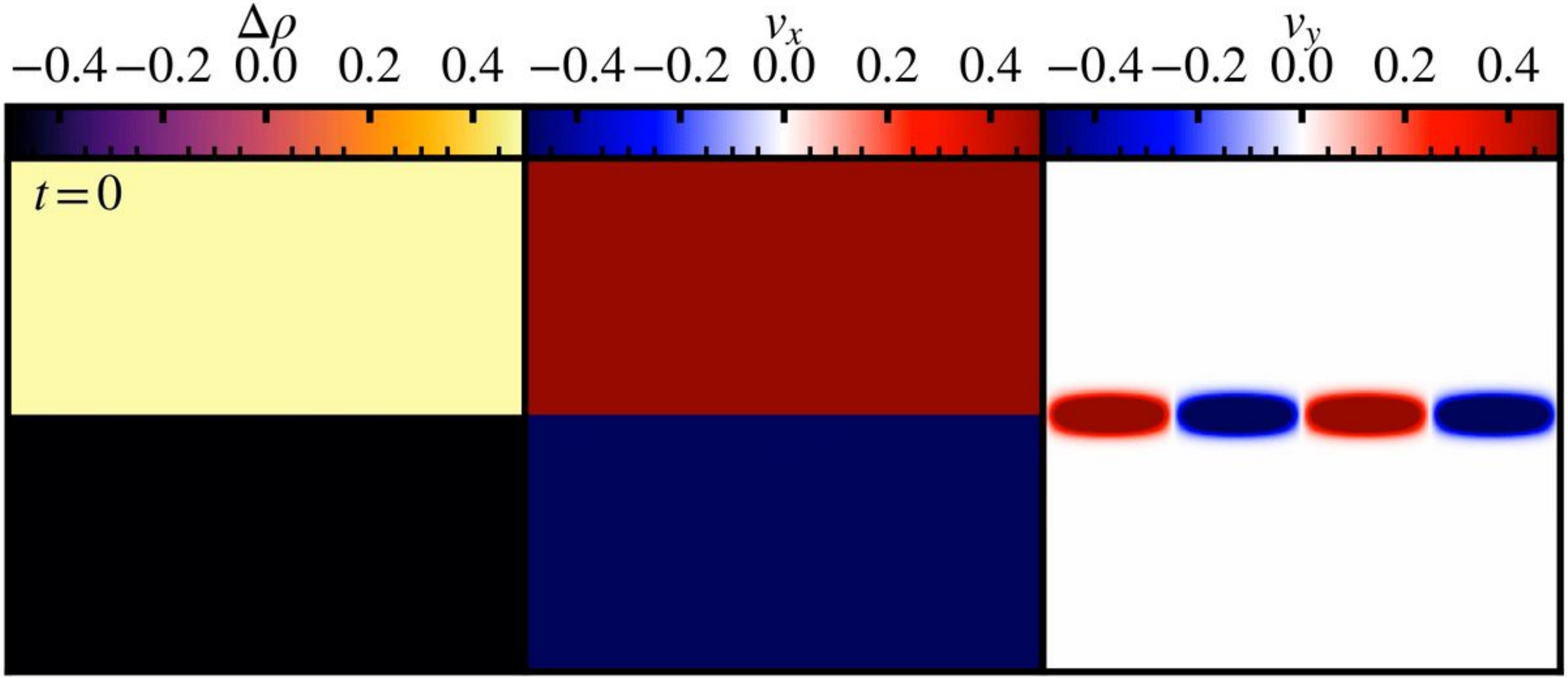
Conclusions

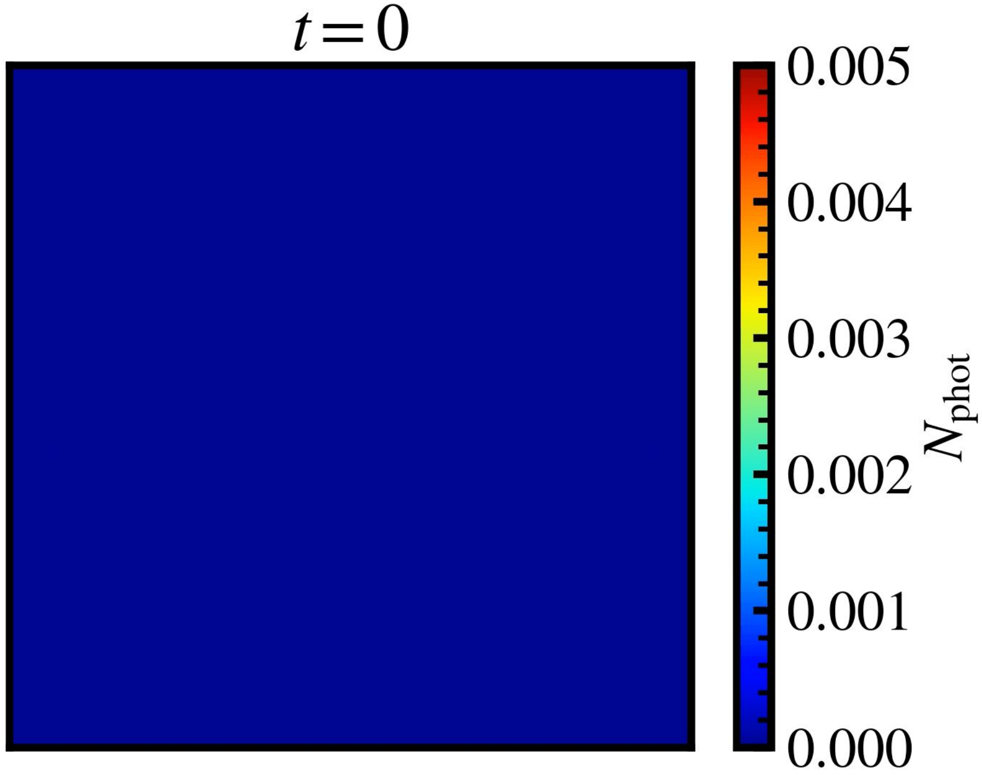
- ➔ **Euler equations**
  - ◆ MUSCL-Hancock + MonCen slope limiter + smooth extrema detection + HLL Riemann solver
- ➔ **RT equations**
  - ◆ M1 closure
  - ◆ 1st-order FE + GLF Riemann solver
- ➔ **Thermochemistry coupling**
  - ◆ Only hydrogen + one group of photons with 13.6 eV
  - ◆ On-the-spot Approximation (OTSA)
  - ◆ Semi-implicit scheme
  - ◆ Adaptive Time Stepping

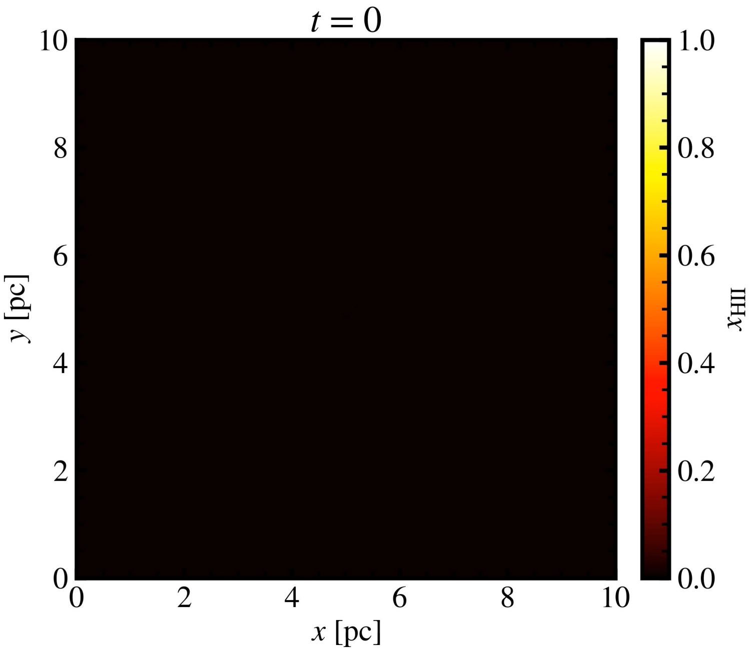
References

1. Ramses-RT : https://arxiv.org/pdf/1304.7126
2. Foundations of Radiation Hydrodynamics - D. Mihalas & B. Weibel Mihalas

Movies Appendix


Δρ
v_x
v_y
−0.4 −0.2 0.0 0.2 0.4
−0.4 −0.2 0.0 0.2 0.4
−0.4 −0.2 0.0 0.2 0.4
$t = 0$


$t = 0$
0.005
0.004
0.003
0.002
0.001
0.000
N_{phot}


$t = 0$
10
8
6
4
2
0
y [pc]
0
2
4
6
8
10
x [pc]
1.0
0.8
0.6
0.4
0.2
0.0
$x_{\rm HII}$