Homework 4

Zihan Zhao

1001103708

1 Multilayer Perceptron

There are 2 units on input layer, 6 units on hidden layer and 2 units on output layer. The W, b for each layer edges are as following:

$$W^{(1)} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} b^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$W^{(2)} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} b^{(2)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

How does it work?

$$\begin{aligned} h1 &= ReLU(x_1 - x_2) \\ h2 &= ReLU(x_2 - x_1) \\ h3 &= ReLU(x_1) \\ h4 &= ReLU(x_2) \\ h5 &= ReLU(-x_1) \\ h6 &= ReLU(-x_2) \end{aligned}$$

We can get y_1, y_2

$$\begin{aligned} y1 &= -\frac{1}{2}h1 + -\frac{1}{2}h2 + \frac{1}{2}h3 + \frac{1}{2}h4 + -\frac{1}{2}h5 + -\frac{1}{2}h6 \\ &= -\frac{1}{2}h1 + -\frac{1}{2}h2 + \frac{1}{2}(h3 - h5) + \frac{1}{2}(h4 - h6) \\ &= -\frac{1}{2}h1 + -\frac{1}{2}h2 + \frac{1}{2}(max(0, x_1) - max(0, -x_1)) + \frac{1}{2}(max(0, x_2) - max(0, -x_2)) \end{aligned}$$

If $x_1 \geq 0$, $max(0, x_1) - max(0, -x_1) = x_1 - 0 = x_1$;

If $x_1 < 0$, $max(0, x_1) - max(0, -x_1) = 0 - (-x_1) = x_1$; So is x_2. Then,

$$\begin{aligned} y1 &= -\frac{1}{2}h1 + -\frac{1}{2}h2 + \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ y2 &= \frac{1}{2}h1 + \frac{1}{2}h2 + +\frac{1}{2}x_1 + \frac{1}{2}x_2 \end{aligned}$$

If $x_1 \geq x_2$,

$$h1 = x_1 - x_2$$
$$h2 = 0$$

Then,

$$\begin{aligned}
y1 &= -\frac{1}{2}(x_1 - x_2) + -\frac{1}{2} * 0 + \frac{1}{2}x_1 + \frac{1}{2}x_2 \\
&= -\frac{1}{2}x_1 + -\frac{1}{2}(-x_2) + \frac{1}{2}x_1 + \frac{1}{2}x_2 \\
&= x_2 \\
y2 &= \frac{1}{2}(x_1 - x_2) + \frac{1}{2} * 0 + \frac{1}{2}x_1 + \frac{1}{2}x_2 \\
&= x_1
\end{aligned}$$

If $x_1 < x_2$,

$$h1 = 0$$
$$h2 = x_2 - x_1$$

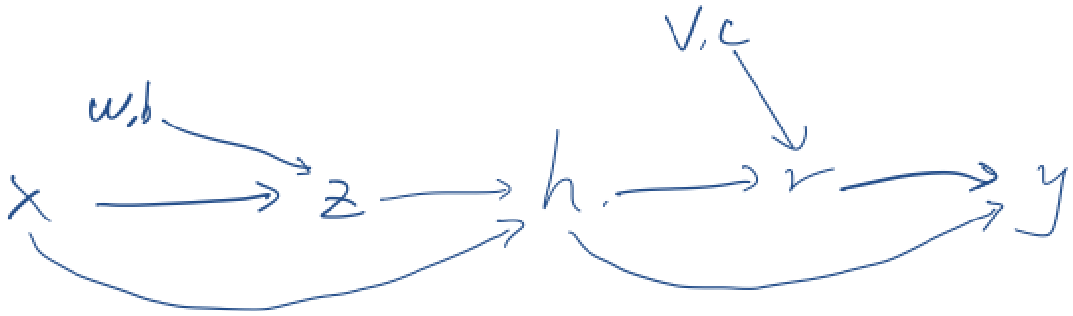
Then,

$$\begin{aligned}
y1 &= -\frac{1}{2}(x_2 - x_1) + \frac{1}{2}x_1 + \frac{1}{2}x_2 \\
&= x_1 \\
y2 &= \frac{1}{2}(x_2 - x_1) + \frac{1}{2}x_1 + \frac{1}{2}x_2 \\
&= x_2
\end{aligned}$$

They are sorted in order.

2 Backprop

(a)



(b)

Assume we know $\overline{y} = \frac{\partial L}{\partial y}$,

$$
\begin{aligned}
\frac{\partial L}{\partial V} &= \overline{y}\frac{\partial y}{\partial V} \\
&= \overline{y}\frac{\partial \phi(r)}{\partial V} + \overline{y}\frac{\partial h}{\partial V} \\
&= \overline{y}\phi'(r)\frac{\partial Vh + c}{\partial V} + 0 \\
&= \overline{y}\phi'(r)h \\
\frac{\partial L}{\partial c} &= \overline{y}\frac{\partial y}{\partial c} \\
&= \overline{y}\frac{\partial \phi(r)}{\partial c} + \overline{y}\frac{\partial h}{\partial c} \\
&= \overline{y}\phi'(r)\frac{\partial Vh + c}{\partial c} + 0 \\
&= \overline{y}\phi'(r) \\
\frac{\partial L}{\partial W} &= \overline{y}\frac{\partial y}{\partial W} \\
&= \overline{y}\frac{\partial \phi(r)}{\partial W} + \overline{y}\frac{\partial h}{\partial W} \\
&= \overline{y}\phi'(r)\frac{\partial Vh + c}{\partial W} + \overline{y}\frac{\partial \phi(z) + x}{\partial W} \\
&= \overline{y}\phi'(r)\frac{\partial Vh}{\partial W} + \overline{y}\phi'(z)\frac{\partial Wx + b}{\partial W} \\
&= \overline{y}\phi'(r)V\frac{\partial \phi(z) + x}{\partial W} + \overline{y}\phi'(z)x \\
&= \overline{y}\phi'(r)V\phi'(z)\frac{\partial Wx + b}{\partial W} + \overline{y}\phi'(z)x \\
&= \overline{y}\phi'(r)V\phi'(z)x + \overline{y}\phi'(z)x \\
\frac{\partial L}{\partial b} &= \overline{y}\frac{\partial y}{\partial b} \\
&= \overline{y}\frac{\partial \phi(r)}{\partial b} + \overline{y}\frac{\partial h}{\partial b} \\
&= \overline{y}\phi'(r)\frac{\partial Vh + c}{\partial b} + \overline{y}\frac{\partial \phi(z) + x}{\partial b} \\
&= \overline{y}\phi'(r)\frac{\partial Vh}{\partial b} + \overline{y}\phi'(z)\frac{\partial Wx + b}{\partial b} \\
&= \overline{y}\phi'(r)V\frac{\partial \phi(z) + x}{\partial b} + \overline{y}\phi'(z) \\
&= \overline{y}\phi'(r)V\phi'(z)\frac{\partial Wx + b}{\partial b} + \overline{y}\phi'(z) \\
&= \overline{y}\phi'(r)V\phi'(z) + \overline{y}\phi'(z)
\end{aligned}
$$

3 EM for Probabilistic PCA

(a)

See appendix equations of $p(z)$ and $p(x|z)$, In this problem, we know $p(z) = N(z|0,1)$ and $p(x|z) = N(x|zu, \sigma^2 I)$. Plug into the equations in appendix. We got:

$$\begin{aligned}
\mu &= 0 \\
\Sigma &= 1 \\
A &= u \\
b &= 0 \\
S &= \sigma^2 I
\end{aligned}$$

Then:

$$\begin{aligned}
C &= (1 + u^T(\sigma^2 I)^{-1}u)^{-1} \\
&= (1 + \frac{1}{\sigma^2}u^T u)^{-1} \\
&= \frac{\sigma^2}{\sigma^2 + u^T u}
\end{aligned}$$

Then:

$$\begin{aligned}
mean = m &= C(A^T S^{-1}(x - b) + \Sigma^{-1}\mu) \\
&= \frac{\sigma^2}{\sigma^2 + u^T u}(u^T \sigma^{-2} I x) \\
&= \frac{u^T x}{\sigma^2 + u^T u} \\
Var = C &= \frac{\sigma^2}{\sigma^2 + u^T u}
\end{aligned}$$

Then:

$$\begin{aligned}
s = Var + m^2 &= \frac{\sigma^2}{\sigma^2 + u^T u} + \frac{(u^T x)^2}{(\sigma^2 + u^T u)^2} \\
&= \frac{\sigma^2(\sigma^2 + u^T u) + (u^T x)^2}{(\sigma^2 + u^T u)^2} \\
&= \frac{\sigma^4 + \sigma^2 u^T u + (u^T x)^2}{(\sigma^2 + u^T u)^2}
\end{aligned}$$

To conclude:

$$\begin{aligned}
m &= \frac{u^T x}{\sigma^2 + u^T u} \\
s &= \frac{\sigma^4 + \sigma^2 u^T u + (u^T x)^2}{(\sigma^2 + u^T u)^2}
\end{aligned}$$

(b)

Assume the dimension of $x^{(i)}$ is dx1. Find $logp(z,x)$ first:

$$\begin{aligned} log(p(z,x)) &= log(p(x|z)) + log(p(z)) \\ &= -\frac{\frac{1}{\sigma^2}(x-zu)^T(x-zu)}{2} log(\frac{1}{\sqrt{((2\pi)^d\sigma^{2d})}}) + const \\ &= \frac{\frac{1}{\sigma^2}(x-zu)^T(x-zu)}{2} log(\sqrt{((2\pi)^d\sigma^{2d})}) + const \\ &= \frac{\frac{1}{\sigma^2}(x-zu)^T(x-zu)}{4} log((2\pi)^d\sigma^{2d}) + const \end{aligned}$$

Define a const $D = \frac{log((2\pi)^d\sigma^{2d})}{4\sigma^2}$

$$\begin{aligned} &= D(x-zu)^T(x-zu) + const \\ &= D(x^Tx - x^Tzu - zu^Tx + zu^Tzu) + const \\ &= D(-2zu^Tx + z^2u^Tu) + const' \end{aligned}$$

Then:

$$\begin{aligned} E[log(p(z,x))] &= -2DE[z]u^Tx + DE[z^2]u^Tu + const' \\ &= -2Dmu^Tx + Dsu^Tu + const' \end{aligned}$$

Drop $const'$, then the constant-dropped expected log-likelihood L is:

$$L = \frac{1}{N}\sum_{i=1}^{N}(-2Dm^{(i)}u^Tx^{(i)} + Ds^{(i)}u^Tu)$$

$$\frac{\partial L}{\partial u} = \frac{1}{N}\sum_{i=1}^{N}(-2Dm^{(i)}x^{(i)} + 2Ds^{(i)}u)$$

Let $\frac{\partial L}{\partial u} = 0$, then:

$$\frac{1}{N}\sum_{i=1}^{N}(-2Dm^{(i)}x^{(i)} + 2Ds^{(i)}u) = 0$$

$$u_{new} = \frac{\sum_{i=1}^{N}(m^{(i)}x^{(i)})}{\sum_{i=1}^{N}s^{(i)}}$$