Homework 1

Zihan Zhao

1001103708

1

(a)

First determine $E[Z]$

$$\begin{aligned}
E[Z] &= E[(X-Y)^2] \\
&= E[(X^2 - 2XY + Y^2] \\
&= E[X^2] - E[2XY] + E[Y^2] \\
&= E[X^2]2E[X]E[Y] + E[Y^2]
\end{aligned}$$

Since X and Y both are variables from uniform distribution in $[0,1]$, $E[x] = E[Y]$ and $E[X^2] = E[Y^2]$. Therefore,

$$\begin{aligned}
&continue: \\
&E[Z] = 2E[X^2] + 2E[X]^2 \\
&\quad = 2\int_0^1 x^2 f(x)dx - 2(\int_0^1 xf(x)dx)^2
\end{aligned}$$

Note that $f(x)$ is the pdf of X. So $f(x) = \frac{1}{1-0} = 1$. Therefore,

$$\begin{aligned}
&continue: \\
&E[Z] = 2\int_0^1 x^2 dx - 2(\int_0^1 xdx)^2 \\
&\quad = 2 * \frac{1}{3} - 2 * (\frac{1}{2})^2 \\
&\quad = \frac{1}{6} = 0.16667
\end{aligned}$$

Second determine $Var[Z]$

$$\begin{aligned}
Var[Z] &= E[Z^2] - E[Z]^2 \\
&= E[(X-Y)^4] - (\frac{1}{6})^2 \\
&= E[X^4 - 4X^3Y + 6X^2Y^2 - 4XY^3 + Y^4] - (\frac{1}{6})^2 \\
&= E[X^4] - 4E[X^3]E[Y] + 6E[X^2]E[Y^2] - 4E[X]E[Y^3] + E[Y^4] - (\frac{1}{6})^2
\end{aligned}$$

Since X and Y both are variables from uniform distribution in $[0,1]$, $E[x] = E[Y]$, $E[X^2] = E[Y^2]$, $E[X^3] = E[Y^3]$, and $E[X^4] = E[Y^4]$ Therefore,

$$
\begin{aligned}
Var[Z] &= 2E[X^4] - 8E[X^3]E[X] + 6E[X^2]^2 - (\frac{1}{6})^2 \\
&= 2\int_0^1 x^4 f(x)dx - 8(\int_0^1 x^3 f(x)dx)(\int_0^1 xf(x)dx) + 6(\int_0^1 x^2 f(x)dx)^2 - (\frac{1}{6})^2 \\
&= 2 * \frac{1}{5} - 8 * \frac{1}{4} * \frac{1}{2} + 6(\frac{1}{3})^2 - (\frac{1}{6})^2 = \frac{7}{180} = 0.38889
\end{aligned}
$$

(b)

First determine $E[R]$. Since $X_1, .., Y_d$ are i.i.d, $(X_1 - Y_1)^2, ..., (X_d - Y_d)^2$ are independent. Therefore,

$$
\begin{aligned}
E[R] &= E[\sum_{i=1}^{d} Z_i] \\
&= \sum_{i=1}^{d} E[Z_i] \\
&= dE[Z] = \frac{d}{6}
\end{aligned}
$$

Second determine $Var[R]$. Since $Z_1, .., Z_d$ are independent, $Cov(Z_i, Z_j) = 0$ where i and j are arbitrary picks from 1 to d. Therefore,

$$
\begin{aligned}
Var[R] &= Var[\sum_{i=1}^{d} Z_i] \\
&= \sum_{i=1}^{d} Var[Z_i] + 2\sum_{i<j}^{d} Cov[Z_i, Z_j] \\
&= dVar[Z] = \frac{7d}{180}
\end{aligned}
$$

2

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

(i)

3

(a)

Since $p(x)$ is possibility mass, $0 \leq p(x) \leq 1$. Therefore,

$$p(x) \leq 1$$
$$\frac{1}{p(x)} \leq 1$$
$$\log(\frac{1}{p(x)}) \geq 1 \geq 0$$

since $p(x) \geq 0$,

$$p(x)\log(\frac{1}{p(x)}) \geq 0$$
$$H(X) = \sum_x p(x)\log(\frac{1}{p(x)}) \geq 0$$

(b)

In order to prove $KL(p||q) \geq 0$, it is equivalent to prove $\sum_x^X p(x)\log(\frac{p(x)}{q(x)}) \geq 0$. Let's define new variable Y which is $y = f(x) = \frac{q(x)}{p(x)}$. Then let's define a new function $\phi(y) = -\log y$, which is convex function. According to Jensen's Inequality,

$$
\begin{aligned}
E[\phi(Y)] &\geq \phi(E[Y]) \\
\sum_x^X p(x)\phi(f(x)) &\geq \phi(E[Y]) \\
-\sum_x^X p(x)\log(\frac{q(x)}{p(x)}) &\geq \phi(E[Y]) \\
\sum_x^X p(x)\log(\frac{p(x)}{q(x)}) &\geq \phi(E[Y]) \\
KL(p||q) &\geq \phi(E[Y]) \\
KL(p||q) &\geq \phi(E_x[f(x)]) \\
KL(p||q) &\geq \phi(\sum_x^X p(x)\frac{q(x)}{p(x)}) \\
KL(p||q) &\geq \phi(\sum_x^X q(x)) \\
KL(p||q) &\geq \phi(1) \\
KL(p||q) &\geq -\log(1) \\
KL(p||q) &\geq 0
\end{aligned}
$$

(c)

In order to show that

$$I[Y;X] = KL(p(x,y)||p(x)p(y))$$

On left side,

$$
\begin{aligned}
I[Y;X] &= H(Y) - H(Y|X) \\
&= \sum_y p(y)\log(\frac{1}{p(y)}) - (-\sum_y\sum_x p(x,y)\log(p(y|x))) \\
&= \sum_y p(y)\log(\frac{1}{p(y)}) + \sum_y\sum_x p(x,y)\log(p(y|x))
\end{aligned}
$$

On right side,

$$KL(p(x,y)||p(x)p(y)) = \sum_y\sum_x p(x,y)\log(\frac{p(x,y)}{p(x)p(y)})$$

note that $p(y|x) = \frac{p(x,y)}{p(x)}$

$$
\begin{aligned}
KL(p(x,y)||p(x)p(y)) &= \sum_y\sum_x p(x,y)\log(\frac{p(y|x)}{p(y)}) \\
&= \sum_y\sum_x (p(x,y)(\log(p(y|x)) + \log(\frac{1}{p(y)})))
\end{aligned}
$$