

Homework 4

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1 Multilayer Perceptron

There are 2 units on input layer, 6 units on hidden layer and 2 units on output layer. The W, b for each layer edges are as following:

$$W^{(1)} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad b^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$W^{(2)} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad b^{(2)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

How does it work?

$$h1 = ReLU(x_1 - x_2)$$

$$h2 = ReLU(x_2 - x_1)$$

$$h3 = ReLU(x_1)$$

$$h4 = ReLU(x_2)$$

$$h5 = ReLU(-x_1)$$

$$h6 = ReLU(-x_2)$$

We can get y_1, y_2

$$\begin{aligned} y1 &= -\frac{1}{2}h1 + -\frac{1}{2}h2 + \frac{1}{2}h3 + \frac{1}{2}h4 + -\frac{1}{2}h5 + -\frac{1}{2}h6 \\ &= -\frac{1}{2}h1 + -\frac{1}{2}h2 + \frac{1}{2}(h3 - h5) + \frac{1}{2}(h4 - h6) \\ &= -\frac{1}{2}h1 + -\frac{1}{2}h2 + \frac{1}{2}(max(0, x_1) - max(0, -x_1)) + \frac{1}{2}(max(0, x_2) - max(0, -x_2)) \end{aligned}$$

If $x_1 \geq 0$, $max(0, x_1) - max(0, -x_1) = x_1 - 0 = x_1$;

If $x_1 < 0$, $max(0, x_1) - max(0, -x_1) = 0 - (-x_1) = x_1$; So is x_2 . Then,

$$\begin{aligned} y1 &= -\frac{1}{2}h1 + -\frac{1}{2}h2 + \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ y2 &= \frac{1}{2}h1 + \frac{1}{2}h2 + \frac{1}{2}x_1 + \frac{1}{2}x_2 \end{aligned}$$

If $x_1 \geq x_2$,

$$h1 = x_1 - x_2$$

$$h2 = 0$$

Then,

$$\begin{aligned} y1 &= -\frac{1}{2}(x_1 - x_2) + -\frac{1}{2} * 0 + \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ &= -\frac{1}{2}x_1 + -\frac{1}{2}(-x_2) + \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ &= x_2 \end{aligned}$$

$$\begin{aligned} y2 &= \frac{1}{2}(x_1 - x_2) + \frac{1}{2} * 0 + \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ &= x_1 \end{aligned}$$

If $x_1 < x_2$,

$$h1 = 0$$

$$h2 = x_2 - x_1$$

Then,

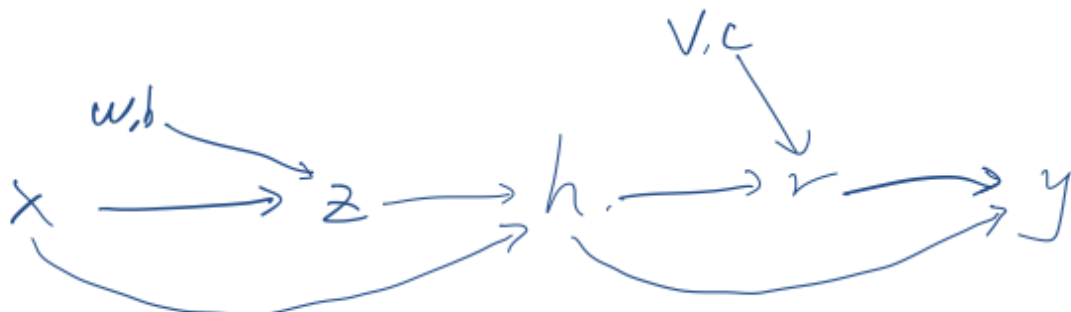
$$\begin{aligned} y1 &= -\frac{1}{2}(x_2 - x_1) + \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ &= x_1 \end{aligned}$$

$$\begin{aligned} y2 &= \frac{1}{2}(x_2 - x_1) + \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ &= x_2 \end{aligned}$$

They are sorted in order.

2 Backprop

(a)



(b)

Assume we know $\bar{y} = \frac{\partial L}{\partial y}$,

$$\begin{aligned}\frac{\partial L}{\partial V} &= \bar{y} \frac{\partial y}{\partial V} \\ &= \bar{y} \frac{\partial \phi(r)}{\partial V} + \bar{y} \frac{\partial h}{\partial V} \\ &= \bar{y} \phi'(r) \frac{\partial Vh + c}{\partial V} + 0 \\ &= \bar{y} \phi'(r) h\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial c} &= \bar{y} \frac{\partial y}{\partial c} \\ &= \bar{y} \frac{\partial \phi(r)}{\partial c} + \bar{y} \frac{\partial h}{\partial c} \\ &= \bar{y} \phi'(r) \frac{\partial Vh + c}{\partial c} + 0 \\ &= \bar{y} \phi'(r)\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial W} &= \bar{y} \frac{\partial y}{\partial W} \\ &= \bar{y} \frac{\partial \phi(r)}{\partial W} + \bar{y} \frac{\partial h}{\partial W} \\ &= \bar{y} \phi'(r) \frac{\partial Vh + c}{\partial W} + \bar{y} \frac{\partial \phi(z) + x}{\partial W} \\ &= \bar{y} \phi'(r) \frac{\partial Vh}{\partial W} + \bar{y} \phi'(z) \frac{\partial Wx + b}{\partial W} \\ &= \bar{y} \phi'(r) V \frac{\partial \phi(z) + x}{\partial W} + \bar{y} \phi'(z) x \\ &= \bar{y} \phi'(r) V \phi'(z) \frac{\partial Wx + b}{\partial W} + \bar{y} \phi'(z) x \\ &= \bar{y} \phi'(r) V \phi'(z) x + \bar{y} \phi'(z) x\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial b} &= \bar{y} \frac{\partial y}{\partial b} \\ &= \bar{y} \frac{\partial \phi(r)}{\partial b} + \bar{y} \frac{\partial h}{\partial b} \\ &= \bar{y} \phi'(r) \frac{\partial Vh + c}{\partial b} + \bar{y} \frac{\partial \phi(z) + x}{\partial b} \\ &= \bar{y} \phi'(r) \frac{\partial Vh}{\partial b} + \bar{y} \phi'(z) \frac{\partial Wx + b}{\partial b} \\ &= \bar{y} \phi'(r) V \frac{\partial \phi(z) + x}{\partial b} + \bar{y} \phi'(z) \\ &= \bar{y} \phi'(r) V \phi'(z) \frac{\partial Wx + b}{\partial b} + \bar{y} \phi'(z) \\ &= \bar{y} \phi'(r) V \phi'(z) + \bar{y} \phi'(z)\end{aligned}$$

3 EM for Probabilistic PCA

(a)

See appendix equations of $p(z)$ and $p(x|z)$, In this problem, we know $p(z) = N(z|0, 1)$ and $p(x|z) = N(x|zu, \sigma^2 I)$. Plug into the equations in appendix. We got:

$$\begin{aligned}\mu &= 0 \\ \Sigma &= 1 \\ A &= u \\ b &= 0 \\ S &= \sigma^2 I\end{aligned}$$

Then:

$$\begin{aligned}C &= (1 + u^T(\sigma^2 I)^{-1}u)^{-1} \\ &= (1 + \frac{1}{\sigma^2}u^T u)^{-1} \\ &= \frac{\sigma^2}{\sigma^2 + u^T u}\end{aligned}$$

Then:

$$\begin{aligned}mean = m &= C(A^T S^{-1}(x - b) + \Sigma^{-1}\mu) \\ &= \frac{\sigma^2}{\sigma^2 + u^T u}(u^T \sigma^{-2} I x) \\ &= \frac{u^T x}{\sigma^2 + u^T u} \\ Var = C &= \frac{\sigma^2}{\sigma^2 + u^T u}\end{aligned}$$

Then:

$$\begin{aligned}s = Var + m^2 &= \frac{\sigma^2}{\sigma^2 + u^T u} + \frac{(u^T x)^2}{(\sigma^2 + u^T u)^2} \\ &= \frac{\sigma^2(\sigma^2 + u^T u) + (u^T x)^2}{(\sigma^2 + u^T u)^2} \\ &= \frac{\sigma^4 + \sigma^2 u^T u + (u^T x)^2}{(\sigma^2 + u^T u)^2}\end{aligned}$$

To conclude:

$$\begin{aligned}m &= \frac{u^T x}{\sigma^2 + u^T u} \\ s &= \frac{\sigma^4 + \sigma^2 u^T u + (u^T x)^2}{(\sigma^2 + u^T u)^2}\end{aligned}$$

(b)

Find $\log p(z, x)$ first:

$$\begin{aligned}
\log p(z, x) &= \log(p(x|z)) + \log(p(z)) \\
&= -\frac{\frac{1}{\sigma^2}(x - zu)^T(x - zu)}{2} \log\left(\frac{1}{\sqrt{((2\pi)^d \sigma^{2d})}}\right) + \text{const} \\
&= \frac{\frac{1}{\sigma^2}(x - zu)^T(x - zu)}{2} \log(\sqrt{((2\pi)^d \sigma^{2d})}) + \text{const} \\
&= \frac{\frac{1}{\sigma^2}(x - zu)^T(x - zu)}{4} \log((2\pi)^d \sigma^{2d}) + \text{const}
\end{aligned}$$

Define a const $D = \frac{\log((2\pi)^d \sigma^{2d})}{4\sigma^2}$

$$\begin{aligned}
&= D(x - zu)^T(x - zu) + \text{const} \\
&= D(x^T x - x^T zu - zu^T x + zu^T zu) + \text{const} \\
&= D(-2zu^T x + z^2 u^T u) + \text{const}'
\end{aligned}$$

Then:

$$\begin{aligned}
E[\log p(z, x)] &= -2DE[z]u^T x + DE[z^2]u^T u + \text{const}' \\
&= -2Dmu^T x + Dsu^T u + \text{const}'
\end{aligned}$$

Drop const' , then the constant-dropped expected log-likelihood L is:

$$\begin{aligned}
L &= \frac{1}{N} \sum_{i=1}^N (-2Dm^{(i)}u^T x^{(i)} + Ds^{(i)}u^T u) \\
\frac{\partial L}{\partial u} &= \frac{1}{N} \sum_{i=1}^N (-2Dm^{(i)}x^{(i)} + 2Ds^{(i)}u)
\end{aligned}$$

Let $\frac{\partial L}{\partial u} = 0$, then:

$$\begin{aligned}
\frac{1}{N} \sum_{i=1}^N (-2Dm^{(i)}x^{(i)} + 2Ds^{(i)}u) &= 0 \\
u_{new} &= \frac{1}{N} \sum_{i=1}^N \frac{m^{(i)}x^{(i)}}{s^{(i)}}
\end{aligned}$$