

# Algebraic Geometry

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THIS IS A DRAFT. Expect inaccuracies and incomplete information.

## Introduction

Algebraic Geometry is a field of mathematics with roots as far back as at least the 19<sup>th</sup> century, and with major advancement and applications in recent years. At its core, Algebraic Geometry is concerned with studying the sets of zeroes of a system of polynomials, in particular their algebraic structure. Though real polynomials are surely the first that come to mind, much of the power in Algebraic Geometry comes from the generalization to polynomials with coefficients in arbitrary fields.

The undergraduate mathematics student would hopefully find this paper accessible, but experience with algebra and geometry would help. A “Basics” section is included, but will not be exhaustive. As unhelpful as it seems, it’s recommended to do independent research as necessary using Wikipedia, YouTube (videotaped lectures) and the referenced textbooks.

## 1 Basics

Here’s where I’ll put all the theorems and definitions that are simply necessary. I’ll definitely talk about projective space here, and homogenous equations.

**Definition 1.1.** *Given a Field  $k$ , the Projective Space of Degree  $n$  over  $k$ , denoted  $\mathbb{P}_k^n$  is the quotient  $(k^{n+1} \setminus \{0\}) / \sim$ , where  $\sim$  is the equivalence relation defined as*

$$(x_0, \dots, x_n) \sim (y_0, \dots, y_n) \text{ if } \exists \lambda \in k \text{ s.t. } \lambda \neq 0, (x_0, \dots, x_n) = (\lambda y_0, \dots, \lambda y_n)$$

*Given this, points in projective space are often denoted  $P = [x_0 : \dots : x_n]$ , with the colons and brackets indicating this is an equivalence class. So then,  $[y_0 : \dots : y_n] = P = [x_0 : \dots : x_n]$  if  $(y_0, \dots, y_n) \sim (x_0, \dots, x_n)$ .*

The definition is abstract to purposely highlight the fact that, given a field  $k$  we can create a projective space. Effectively, the projective space is the set of all lines in  $k^n$  through the origin. This is also why the projective space has one fewer dimension than its corresponding vector space.

So, the “points” in projective space are lines in the corresponding vector space. Another interesting feature of projective space is that parallel lines intersect at a “point at infinity”. This can be thought of as making partitions of lines in the projective space based on parallelism, as if we were defining an equivalence relation. Now, each equivalence class is assigned a unique “point at infinity” that each line in the equivalence class intersects. Thus, if two lines are parallel, they intersect at some point at infinity. Likewise, if two lines aren’t parallel, they’ll have their own points at infinity, but they won’t be the same, thus they will only have the usual intersection normally expected of non-parallel lines.

This may seem very strange, but can be illustrated with a stereographic projection. Suppose we’re in  $\mathbb{R}^3$ , and picture two things: the plane  $z = 1$  and a sphere of radius 1 centered at the origin. Now, we can map any point  $p$  on  $z = 1$  to a unique point on the upper hemisphere of our sphere; simply note that the line passing through  $p$  and the origin  $(0, 0, 0)$  intersects the specified hemisphere precisely once at some point  $q$ . Knowing this, we can see that lines on our plane  $z = 1$  are mapped to great circles on our sphere. In the case of two parallel lines, they will never intersect in  $z = 1$ , but the great circles they’re mapped to will intersect on the equator of the hemisphere.

**Definition 1.2.** A function  $f(x)$  is Homogenous of Degree  $n$  if, for some constant  $\lambda$ , we have  $f(\lambda x) = \lambda^n f(x)$ . If  $f$  is a homogenous polynomial of degree  $n$ , it is known as a form of degree  $n$ .

Homogenous functions are particularly useful when considering their sets of zeroes in projective space. If  $f$  is homogenous of degree  $n$ , and some point  $x = (x_0, \dots, x_n)$  is a zero of  $f$ , then  $f(\lambda x) = \lambda^n f(x) = \lambda^n 0 = 0$ . Thus, we can say that, if  $f$  is homogenous, then can indeed have zeroes of  $f$  in  $\mathbb{P}_k^n$ , assuming  $k$  is algebraically closed.

## 2 Conic Sections

Use conic sections to illustrate some basics here and provide necessary theorems/tools for elliptic curve section. Will draw heavily from the book

## 3 Bezout’s Theorem

Bezout’s theorem will be important later, but we probably won’t be able to fully prove it here. However, the book does prove the limited cases that we’ll most rely on.

## 4 Elliptic Curves

One of the more interesting sections. Definitely need to cover proving the given construction on the non-singular elliptic curve creates a group. Should also

cover how in practice for crypto the identity is chosen as the point at infinity, allowing for much cheaper computation.

Current strategy is to develop this section, then determine dependencies for understanding, and stuff those in the earlier sections.

**Definition 4.1.** Group Law on a Plane Cubic: Suppose  $k \subset \mathbb{C}$  is a subfield of  $\mathbb{C}$ , and  $F \in k[X, Y, Z]$  is a cubic form defining a non-empty plane curve  $C : (F = 0) \subset \mathbb{P}_k^2$ . Additionally, assume that  $F$  satisfies the following:

$$F \text{ is irreducible, (} C \text{ won't contain a line or conic)} \quad (\text{a})$$

$$\forall P \in C, \exists! L \subset \mathbb{P}_k^2 \text{ s.t. } P \text{ is a repeated zero of } F|_L \quad (\text{b})$$

Now, fix any point  $O \in C$  and make the following construction:

$$\text{for } A \in C, \text{ let } \bar{A} \text{ be the third point of intersection of } C \text{ with } \overleftrightarrow{OA} \quad (\text{i})$$

$$\begin{aligned} &\text{for } A, B \in C \text{ let } R \text{ be the third point of intersection of } C \text{ with } \overleftrightarrow{AB} \\ &\text{and define } A + B = \bar{R} \end{aligned} \quad (\text{ii})$$

Geometrically, the uniqueness in (b) says that  $C$  should be non-singular, and we can see that this line  $L$  is in fact the tangent line to  $C$  at  $P$ . Likewise, (a) prevents somewhat degenerate cases.

As you may have noticed, we've defined an operation on points of this cubic curve  $C$ , so naturally we're curious what sort of structure then arises. It turns out that this actually defines an Abelian Group on  $C$ , as we will now prove.

**Theorem 4.1.** The above construction in Definition 4.1 defines an Abelian Group  $(C, +)$  with  $O$  as the identity element.

*Proof.* Well, to prove this is an abelian group, we have a few conditions we must satisfy:

1. Well-definedness of the operation and inverses
2. Existence of an identity element
3. Existence of unique inverses
4. Associativity

*Well-Definedness:*

Let  $P, Q \in C$ . Then, we have two simple cases:

case 1:  $P \neq Q$ . Then  $\exists! L \subset \mathbb{P}_k^2$  such that  $L = \overleftrightarrow{PQ}$ .

case 2:  $P = Q$ . Then, part (b) of the definition gives that  $\exists! L \subset \mathbb{P}_k^2$  such that  $P$  is a repeated root of  $F|_L$ .

Note, however, that in both cases  $F|_L$  is a cubic form in 2 variables.  $L$  is a line, and thus defined as the set of zeroes of some linear form  $l \in k[X, Y, Z]$ . As is common, we've held  $Z$  constant and non-zero to simplify our work, and use the common substitution  $x = X/Z$ ,  $y = Y/Z$ . Thus, as  $F|_L = F \circ l$ ,  $F|_L$  can only be of 3rd degree, and is still homogenous. ( $F(l(\lambda x)) = F(\lambda l(x)) = \lambda^3 F(l(x))$ ).

So, as  $F|_L$  is a 3rd degree form, it has precisely 3 roots, 2 of which are known to be  $P$  and  $Q$ . Well,  $P \in \mathbb{P}_k^2$ ,  $Q \in \mathbb{P}_k^2$ , so  $F|_L$  is the product of 3 linear terms from  $k[X, Y, Z]$ . Thus, the third root of  $F|_L$  must also be from  $\mathbb{P}_k^2$ , and thus from  $C$ .

So, given any two points on  $C$ , the line they determine intersects  $C$  at some unique third point. Thus our construction, which relies on such a third point existing, is well defined. That is, Given any  $A \in C$ ,  $\bar{A}$  is a unique element of  $C$ , and given any  $A, B \in C$ ,  $A + B$  is a unique point of  $C$ .

*Identity:*

Let  $A \in C$  be some arbitrary point of  $C$ , and consider  $O + A$ . Well, by our construction, we first find the third point of intersection of  $\overleftrightarrow{OA}$  with  $C$ , which by (i) is  $\bar{A}$ . From here, to find  $O + A$  we must now find the third point of intersection of  $\overleftrightarrow{O\bar{A}}$  with  $C$ . However, note that  $O$ ,  $A$ ,  $\bar{A}$  are all collinear. Thus, this third point of intersection of  $\overleftrightarrow{O\bar{A}}$  with  $C$  is  $A$ . Hence,  $O + A = A$ ,  $\forall A \in C$ .  $\square$

## 5 Regular Polyhedra??

If we even get here, this should be a neat topic to talk about for a bit to offset the earlier material.