Boundary Integral Equations

Zane Perry

A Motivating Example

Consider the Boundary Value Problem given by:

$$-\frac{d^2u}{dx^2} = f(x)$$
$$u(0) = u(1) = 0$$

How can we avoid the error produced by approximating the derivative operators and discretizing on a mesh grid?

What if we tried integrating it?

$$-\frac{du}{dx} = \int_0^x f(y)dy + C_1$$
$$-u(x) = \int_0^x \int_0^y f(s)ds + C_1 x + C_2$$

With a little help from the following theorem:

$$\int_a^x \int_a^s f(y)dy = \int_a^x f(y)(x-y)dy$$

And applying the boundary conditions:

$$-u(x) = \int_0^x (x-s)f(s)ds + x\int_0^1 (s-1)f(s)ds$$

And changing the bounds of integration:

$$= \int_0^x s(1-x)f(s)ds + \int_x^1 x(1-s)f(s)ds$$

The Green's Function!!

The previous solution is often rewritten as

$$u(x) = \int_0^1 G(x,s) f(s) ds$$

$$G(x,s) = \begin{cases} x(1-s) & x < s \\ s(1-x) & x > s \end{cases}$$

(Note: this is an exact formula)

How is this evaluated numerically?

$$= \int_0^x s(1-x) f(s) ds + \int_x^1 x(1-s) f(s) ds$$

Apply quadrature for each desired point

Benefits:

- Integration is more stable than derivatives with numerical methods
- Error is only dependent on the quadrature method being used
- Error is localized and does not rely on previous steps

Unfortunately this method requires more overhead to set up, and requires a Green's function to exist for the given problem

How do you even find the Green's function?

Linear Algebra Recap

Any linear differential equation can be written in the form

$$L[u] = f$$

How can we invert a differential operator?

Some form of integration!

$$u(x) = L^{-1}[f] = \int_a^b G(x,s)f(s)ds$$

But where does this come from?

Introducing: PHYSICS!!!

$$L[u] = \delta(x)$$

What if we moved it around and introduced the boundary conditions?

$$L[u] = \delta(x - x_0)$$

$$u(a) = u(b) = 0$$

Solving this problem would result in the Green's Function

$$u(x) = G(x, x_0)$$

(Note: Dependent on the operator L and the B.C.s)

Convolving this with the forcing function allows the point charge to be "moved around" and capture the behavior of the solution

Sturm-Liouville Problems

Consider the well-known problem:

$$\frac{d}{dx}\left(p(x)\frac{du(x)}{dx}\right) + q(x)u(x) = f(x) \quad a < x < b$$

Using variation of parameters:

$$G(x,s) = \begin{cases} \frac{u_1(s)u_2(x)}{pW}, & a \le x \le s \\ \frac{u_1(x)u_2(s)}{pW}, & s \le x \le b \end{cases}$$

Where u1 and u2 are linearly independent solutions of the associated homogeneous problem with the respective homogeneous boundary conditions

Notes:

- The Wronskian is never zero on the interval because they are solutions to the ODE
- pW will always be constant for ODEs of this form

But what about non-homogeneous boundary conditions?

$$u(x) = u_h(x) + u_p(x)$$

$$L[u_h] = 0 \quad u_h(a) = \alpha \;\; u_h(b) = \beta$$

$$L[u_p] = f(x) \quad u_p(a) = 0 \;\; u_p(b) = 0$$

Numerical Examples!!!!

(Pretty graphs & actual numbers)

Trapezoidal Quadratures use 100 nodes, Gaussian Quadratures use 10

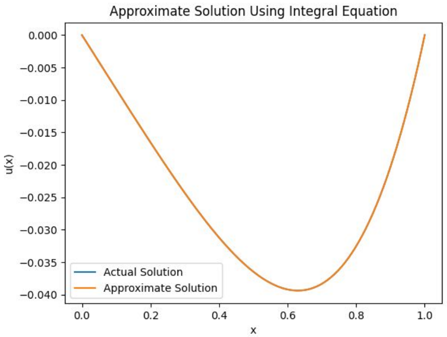
Starting Easy

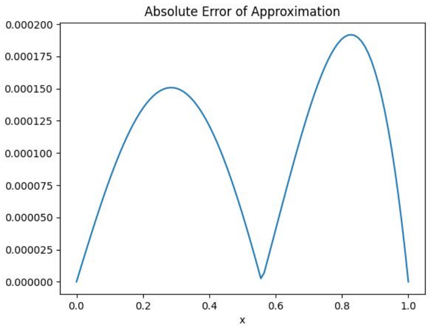
$$u''(x) = x^2$$
$$u(0) = u(1) = 0$$

Green's Function:

$$G(x,s) = \begin{cases} s(x-1) & 0 \le x \le s \\ x(s-1) & s \le x \le 1 \end{cases}$$

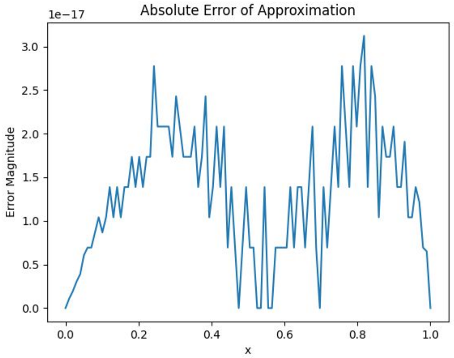
Solution:





Trapezoidal Error:

Gauss-Legendre Error:



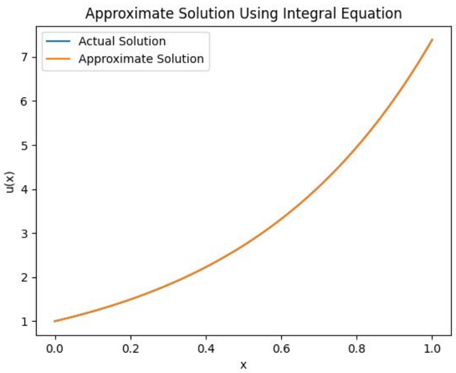
Nonhomogeneous B.C.s

$$u''(x) = 4e^{2x}$$
$$u(0) = 1 \quad u(1) = e^2$$

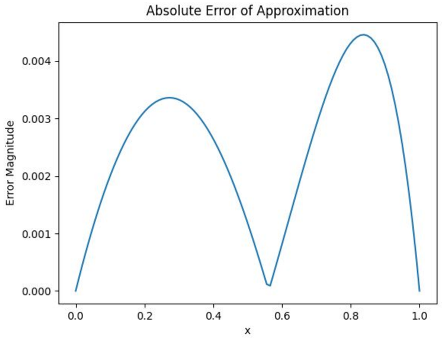
Note: Same Green's function

Particular Solution:

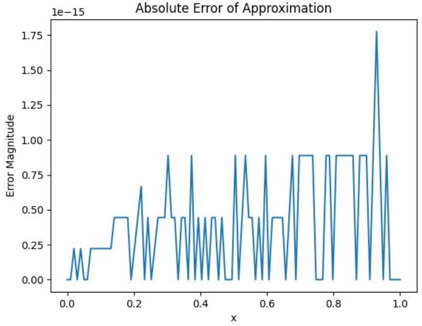
$$u_p(x) = x(e^2 + 1) + 1$$



Solution



Trapezoidal Error



Gauss-Legendre Error

Meaner Problem

$$\frac{d}{dx}\left(x^2\frac{du}{dx}\right) - 2u(x) = 4x^2$$
$$u(1) = 2$$
$$u(2) = -1$$

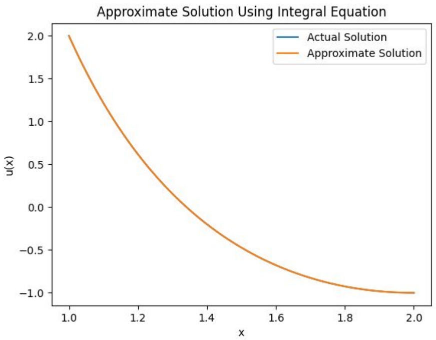
Green's Function:

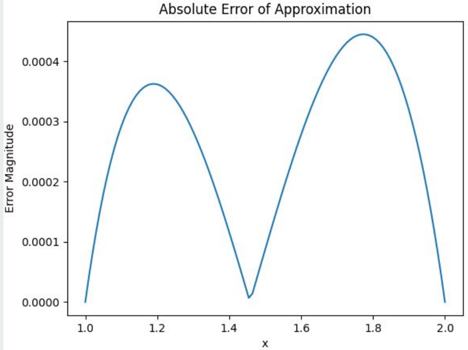
$$G(x,s) = \begin{cases} \frac{1}{21}(s - s^{-2})(x - 8x^{-2}), & 1 \le x \le s \\ \frac{1}{21}(x - x^{-2})(s - 8s^{-2}), & s \le x \le b \end{cases}$$

Particular Solution:

$$u_p(x) = -\frac{6}{7}x + \frac{20}{7}x^{-2}$$

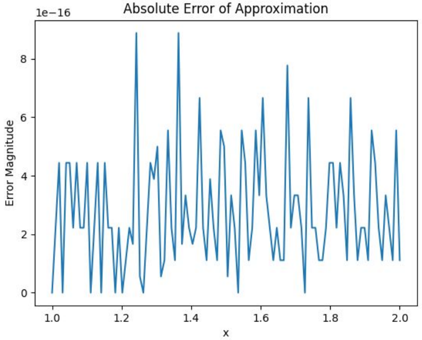
Solution





Trapezoidal Error

Gauss-Legendre Error



Independent Extension

Two dimensional PDEs (spooky scary)

Green's Functions vs Fundamental Solutions

- In multiple dimensions, solutions still take the same form as before
- However boundary conditions become much harder to account for, especially on weird domain
- Instead we transfer the conditions onto an unknown density function

$$h(x) = \int_\Gamma \phi(x,y)\sigma(y)dl(y) \quad \text{where } u(x) = h(x) \; x \in \Gamma$$

Let's look at Laplace's Equation

$$u(\mathbf{x}) = -\frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0|.$$

$$\begin{aligned} -\Delta u(x) &= 0 \quad x \in \Omega \\ u(x) &= g(x) \quad x \in \partial\Omega = \Gamma \end{aligned}$$

Solving for the Density

- Will result in a dense linear system
- Becomes well conditioned when a double layer potential (dipole) is used

$$u(x) = \int_\Gamma \frac{n(y)\cdot(x-y)}{2\pi(x-y)^2}\sigma(y)dl(y)$$

When evaluated on the boundary, the jump discontinuity results in an extra term

$$g(x) = \frac{1}{2}\sigma(x) + \int_\Gamma \frac{n(y)\cdot(x-y)}{2\pi(x-y)^2}\sigma(y)dl(y)$$

(Fredholm Equation of the Second Kind)

Using any desired quadrature method (and parametrizing the boundary):

$$g(x_i) = \frac{1}{2}\sigma(x_i) + \sum_{j=1}^{N}\partial_{x_j}\phi(x_i,x_j)\sigma(x_j)w_j$$

$$(\frac{1}{2}I + D)\sigma = g$$

So we found the density, but what about the actual solution?

(Hint: It's already on this slide)

Use the same quadrature to evaluate the solution

More Numerical Examples!!!

(Even prettier graphs, but less numbers)

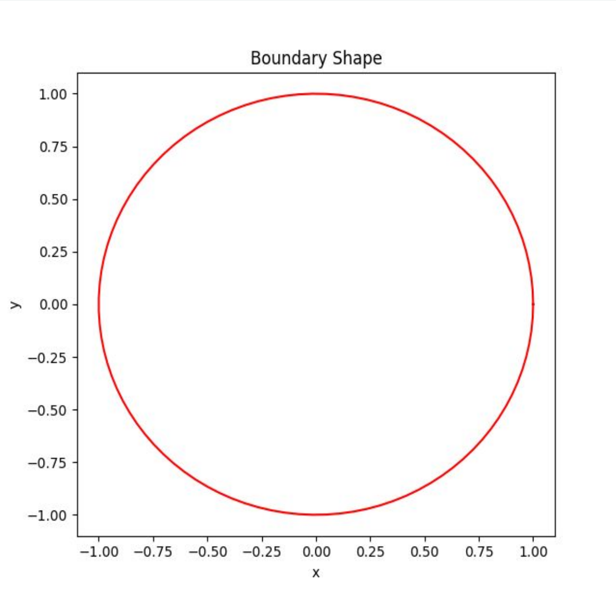
Note that all examples approximate Laplace's Equation with exact solution given by

$$\begin{aligned} -\Delta u(x) &= 0 \quad x \in \Omega \\ u(x) &= g(x) \quad x \in \partial\Omega = \Gamma \end{aligned}$$

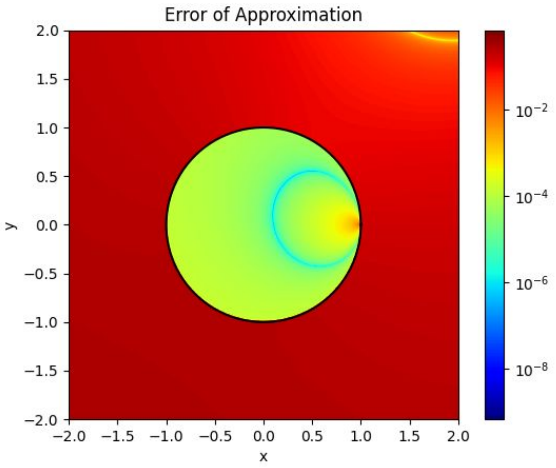
$$u(x) = -\frac{1}{2\pi} \log |x - (2,3)|$$

Trapezoidal Quadratures use 1000 nodes, Gaussian Quadratures use 160 (10 panels of 16)

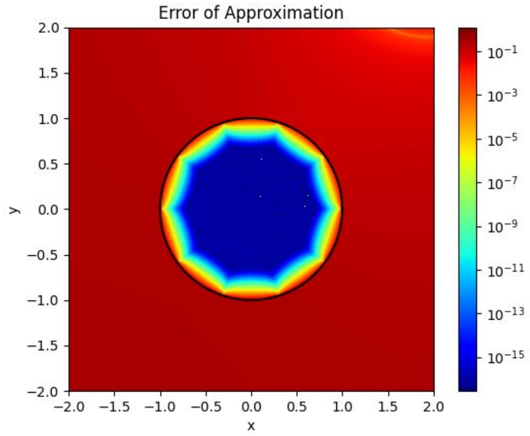
Circle Boundary



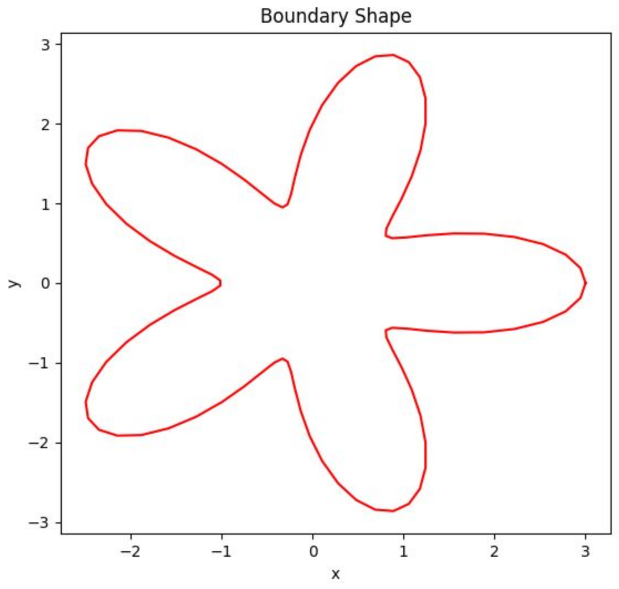
Trapezoidal



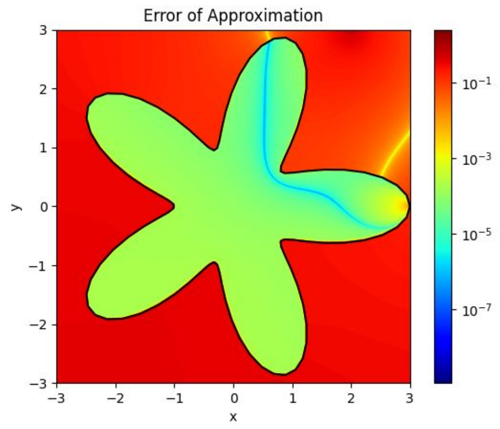
Gauss-Legendre



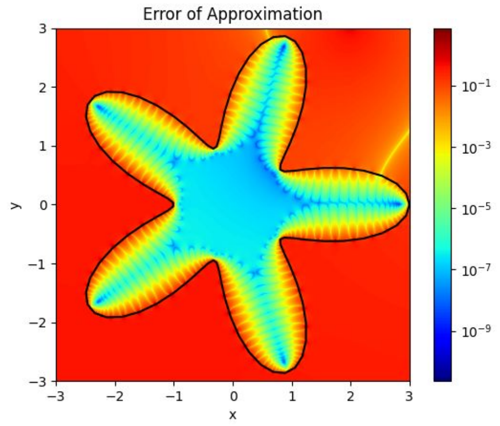
Clover Boundary



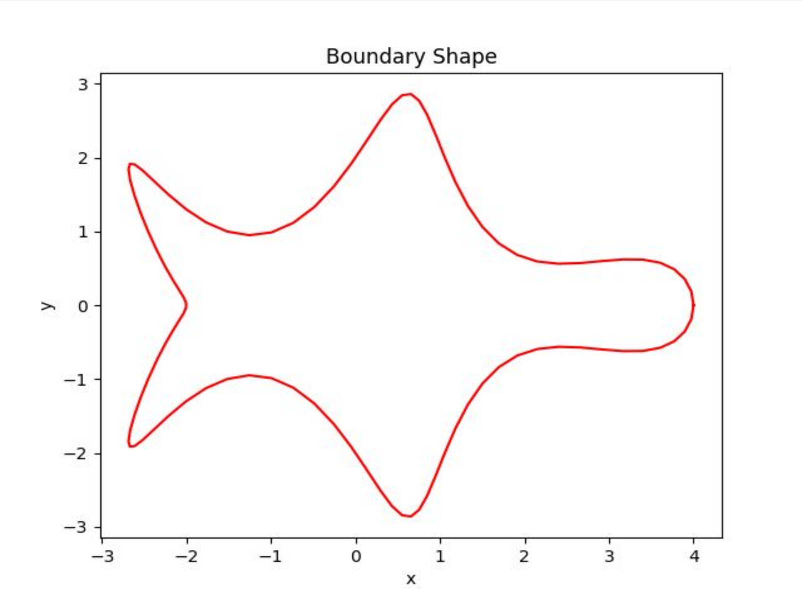
Trapezoidal



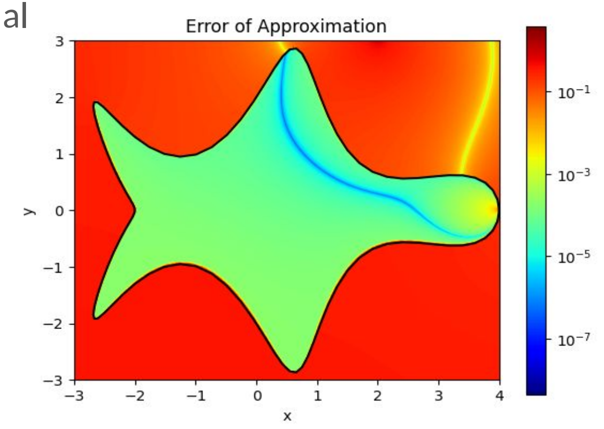
Gauss-Legendre



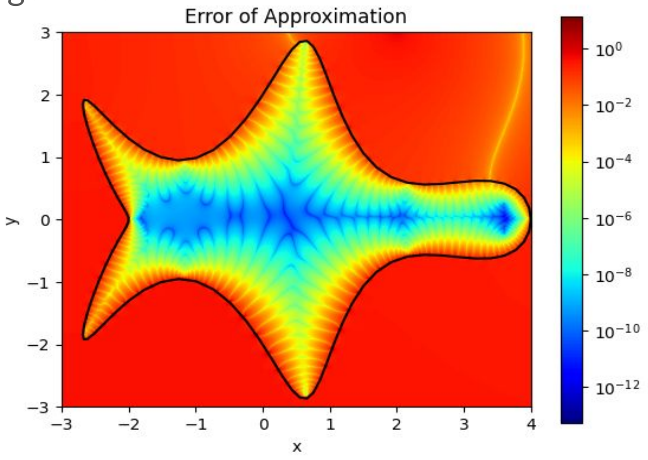
Little Guy Boundary



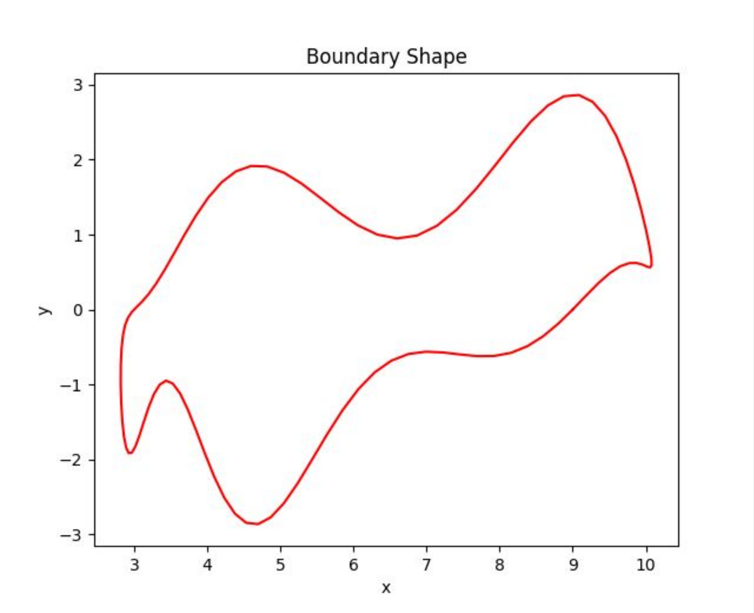
Trapezoidal



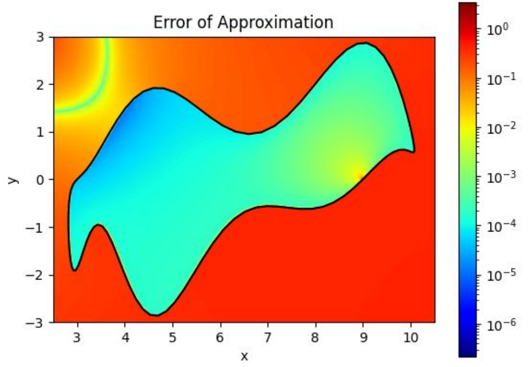
Gauss-Legendre



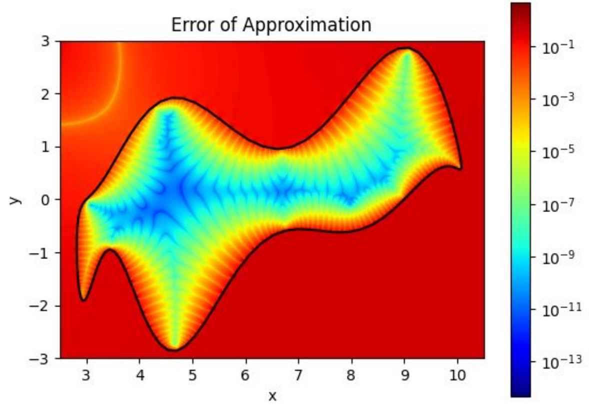
The Blob



Trapezoidal



Gauss-Legendre



Even More???

External Domain Problems!!! (WOOOO)

Just Kidding, but you can use this method to solve those as well

Other areas of extension:

- Non-smooth kernels (Helmholtz)
- Neumann Boundary Conditions (Single Layer)
- Evaluation close to the boundary
- Different quadrature methods
- Iterative solvers for the density
- Interpolation of the density nodes

Sources

[1] "Chapter 10: Integral equation formulations". In: *Fast Direct Solvers for Elliptic PDEs*, pp. 105–114. DOI: `10.1137/1.9781611976045.ch10`. eprint: `https://epubs.siam.org/doi/pdf/10.1137/1.9781611976045.ch10`. URL: `https://epubs.siam.org/doi/abs/10.1137/1.9781611976045.ch10`.

[2] Jerry Farlow et al. *Differential equations & Linear Algebra*. 2nd ed. Pearson, 2018.

[3] Dominique Habault. "Chapter 6 - Boundary Integral Equation Methods - Numerical Techniques". In: *Acoustics*. Ed. by Paul Filippi et al. London: Academic Press, 1999, pp. 189–202. ISBN: 978-0-12-256190-0. DOI: `https://doi.org/10.1016/B978-012256190-0/50007-5`. URL: `https://www.sciencedirect.com/science/article/pii/B9780122561900500075`.

[4] Richard Haberman. *Applied partial differential equations with Fourier series and boundary value problems*. 5th ed. Pearson, 2019.

[5] John David Logan. *Applied mathematics*. 4th ed. Wiley-Interscience, 2006.

[6] Gunnar Martinsson. "CBMS Conference on Fast Direct Solvers". In: *Fast direct solvers for elliptic pdes*. Society for Industrial and Applied Mathematics, 2020.

Questions?