UNIVERSITY OF COLORADO - BOULDER

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COMPLEX VARIABLES AND APPLICATIONS | SPRING 2023

Complex Variables and Fluid Mechanics

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Conformal Mappings have a wide arrange of applications in order to transform complicated domain shapes into simplified regions while still maintaining angular geometry between functions within the domain. In the study of fluid mechanics, a special type of mapping called the Joukowsky transform can be applied to a simple fluid flow in order to model the flow past an airfoil cross-section. This is accomplished by using the velocity potential function of known fluid flows, solving for their harmonic conjugates that are known as the stream function, and using the analytic nature of these functions to form a complex function in the complex plane. Conformal mappings can then be used to transform these functions into a more complicated shape that still has the same circulation and lift properties but can model the real-world shape of airfoils.

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I. Introduction

An important area of study in many branches of engineering is fluid mechanics and the way that fluids flow and exert force on rigid bodies. This is especially relevant in the branch of aerospace engineering related to the lift that is exerted on the wing of a plane. However, this is also a complicated problem to understand as it involves an understanding of fluid dynamics as well as the ability to apply that to complex shapes such as an airfoil. The purpose of this paper is to introduce a way to model the flow of air past a specific type shape and calculate the resulting lift from the atmosphere.

Fluid flow is considered inviscid if it involves no friction, thermal conductivity, or diffusion. In reality, no actual fluids exhibit the characteristics of inviscid flow, they are all viscous in some manner. However, many of these effects are often small in comparison to the actual calculations being done and so in the study of aerodynamic flows, fluids can often be modeled as inviscid with reasonable accuracy[1].

Fluid flow can be modeled as a collection of fluid particles moving together. As such, the velocity of a fluid particle can be measured using simple kinematic equations:

$$\vec{V} = \frac{ds}{dt}$$

The motion of a single particle can then be mapped using *path lines*, whose components are made up of the motion of the particle in each direction. Another useful tool in studying fluid flow is the mapping of *stream lines*, which show the velocity of the entire fluid at a point in time [2]. As the velocity of a fluid may change over time, path lines and stream lines often do not coincide except in one important case, when the fluid is steady. This refers to a fluid when it has a constant flow with respect to time, and therefore all particles will follow the same path that coincides with the stream lines of a flow.

An incompressible fluid is one where the density is constant with respect to time and is shown by the following representation:

$$\nabla \cdot \vec{V} = 0$$

Vorticity is defined as the curl of the velocity field of a fluid, and measures the amount of rotation in a fluid. A fluid is considered irrotational if its vorticity is equal to 0:

$$\vec{w} = \nabla \times \vec{V} = 0$$

If we now define a function ϕ such that $\vec{V} = \nabla\phi$, meaning that the velocity field is the gradient of some function, the following relation holds:

$$\nabla \times (\nabla\phi) = \vec{0}$$

The above relation is true because the curl of the gradient of a scalar function is always 0. This shows that for an irrotational field, the velocity can be written as the gradient of a function ϕ, which is referred to as the velocity potential. By further plugging this into the above equation for divergence:

$$\nabla \cdot (\nabla\phi) = \nabla^2\phi = 0$$

This equation is also known as Laplace's equation. Because Laplace's equation is a second order linear homogeneous partial differential equation, the property of superposition holds where the sum of any particular solutions to the equation is also a solution itself. This has important consequences for the rest of this paper. Any function that satisfies the Laplace equation at all points, like ϕ above, is called a harmonic function, and therefore also has a harmonic conjugate that satisfies the Laplace equation as well. In this case, we define the stream function ψ as the harmonic function [3]. The relation between these two functions is then given by the Cauchy-Riemann equations:

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

$$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{\partial\phi}{\partial\theta} = -\frac{1}{r}\frac{\partial\psi}{\partial r}$$

(written in polar form)

And because these functions satisfy the Cauchy-Riemann equations, they can be written as a single analytic function in the form $\Theta = \phi(x, y) + i\psi(x, y)$ called the complex velocity potential [1]. Additionally, because these functions are harmonic conjugates, they will be orthogonal to each other at intersection points which is important for conformal mappings.

The Kutta-Joukowsky Theorem relates the lift force on a cylindrical body to the fluid density, fluid velocity, and circulation around the body. This theorem holds as long as the Kutta condition applies, which states that the fluid flowing over the upper and lower surfaces of an airfoil must meet at the trailing edge. Practically this is true if the airfoil is moving at subsonic speeds, otherwise shockwaves in the air disrupt fluid flow, and as long as the angle of attack is small enough to not disrupt the flow at the trailing edge of the airflow[2]. If all of these conditions are met, the Kutta-Joukowsky theorem states that:

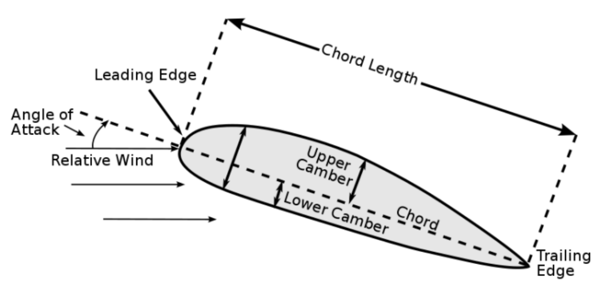
$$L' = \rho_\infty V_\infty \Gamma$$

Where ρ_∞ is the fluid density far away from the airfoil, V_∞ is the fluid velocity far way from the airfoil, and Γ is the circulation which is defined by the following contour integral, which can be related to a fluid's vorticity using Stoke's Theorem:

$$\Gamma = \oint_C \vec{V} \cdot ds = \iint_S (\nabla \times \vec{V}) ds$$

Therefore, if a flow is irrotational everywhere inside of a domain, the boundary of that domain will have a circulation of 0.

The shape of an airfoil is complicated, with many different characteristics and terms to define. An airfoil is the shape of an airplane wing as seen in a cross-section, as shown below. The chord line is the straight line connecting the leading edge of the airfoil to the trailing edge. The angle of attack, commonly denoted by α, is the angle between the chord line and the relative wind. The mean camber line is the locus of points between the top and bottom surfaces of the airfoil. Most airfoils tend to have to have a slightly curved body[4].



The technique of conformal mapping and the principles described above can be used to model the fluid flow around an airfoil and therefore calculate the lift force acting upon it. Due to the angle-preserving nature of conformal mappings the velocity potential and stream functions will still be orthogonal to each other in the new plane and therefore will still satisfy Laplace's equation and the circulation will still be the same as in the original plane. This significantly simplifies the calculations that need to be done in order to calculate the lift on an airfoil.

The most widely used transformation to approximate an airfoil shape is called the Joukowsky transformation[5]:

$$w = z + \frac{\lambda^2}{z}$$

Different values of lambda will be explored in this paper and how they relate to the shape of an airfoil. Throughout this investigation, all fluids will have the same 4 assumptions unless otherwise stated: all fluids will be considered steady, incompressible, irrotational, and inviscid.

II. Mathematical Formulation

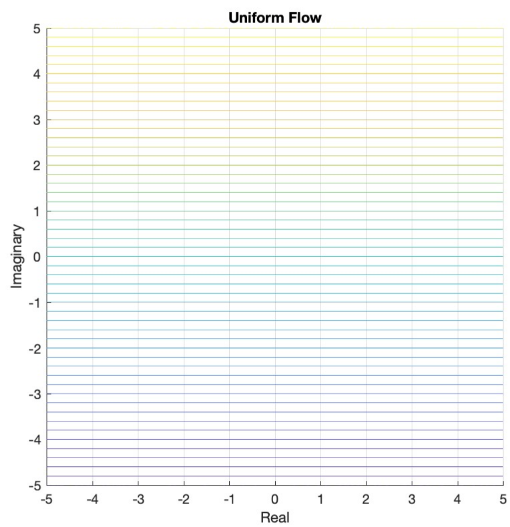
A. Relevant Types of Fluid Flow

Uniform Flow

The simplest type of fluid flow is the case of uniform flow. Defining the velocity potential function as $\phi = V_\infty r \cos\theta$ for some velocity oriented in the positive direction, it is easy to see that this function has a constant velocity for all values of y. Note that the function was written using polar form to simplify later calculations. Finding the correlated stream function involves solving for its harmonic conjugate using the Cauchy-Riemann equations[3]:

$$\begin{aligned}
\phi &= V_\infty r \cos\theta \\
\frac{\partial \phi}{\partial r} &= V_\infty \cos\theta = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \\
\frac{\partial \psi}{\partial \theta} &= V_\infty r \cos\theta \\
\psi &= V_\infty r \sin\theta
\end{aligned}$$

By setting the stream function equal to various constants, the stream lines for this flow can be visualized:



Source and Sink Flow

A flow where all streamlines are straight lines converging or diverging from a central point is known as a source or a sink. The velocity field will only have a radial component to its velocity and is given by the following velocity potential with strength Λ (if Λ is positive it is a source, and if it is negative it is a sink). The harmonic conjugate stream function can be found in a similar way[5]:

$$\begin{aligned}
\phi &= \frac{\Lambda}{2\pi} \ln r \\
\frac{\partial \phi}{\partial r} &= \frac{\Lambda}{2\pi r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \\
\frac{\partial \psi}{\partial \theta} &= \frac{\Lambda}{2\pi} \\
\psi &= \frac{\Lambda}{2\pi}\theta
\end{aligned}$$

A source or sink satisfies the incompressible and irrotational conditions everywhere except for the source, at which point it is singular.

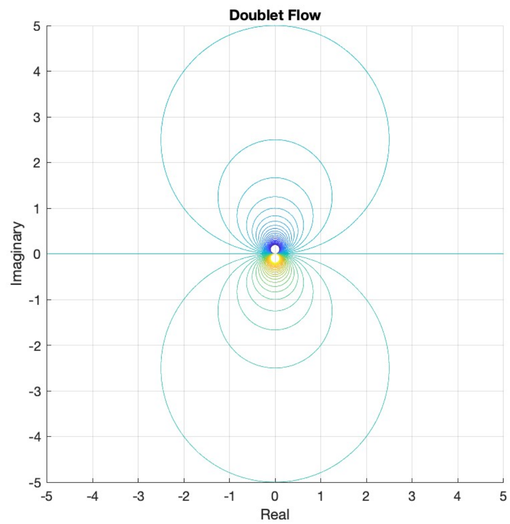
Doublet Flow

If you consider a source and a sink with the same strength, as the limit of the distance between them goes to zero, a doublet is formed with the resulting velocity potential and stream function with strength κ [1]:

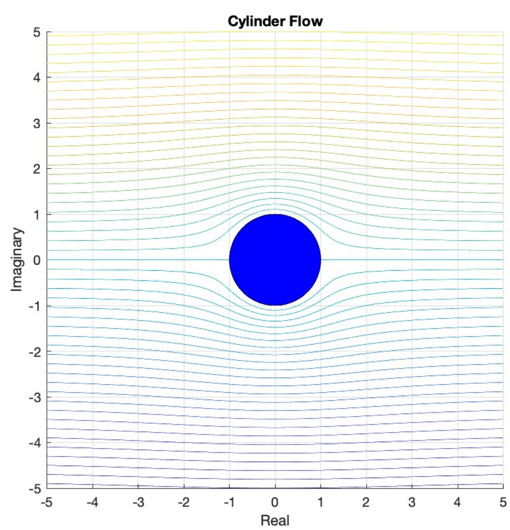
$$\phi = \frac{\kappa}{2\pi}\frac{\cos\theta}{r}$$

$$\psi = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r}$$

Once again, these functions satisfy the requirements of an incompressible and irrotational field.



Due to the linear nature of Laplace's equation, as described in the introduction, these flows can be superimposed to create a model of flow around a cylinder:

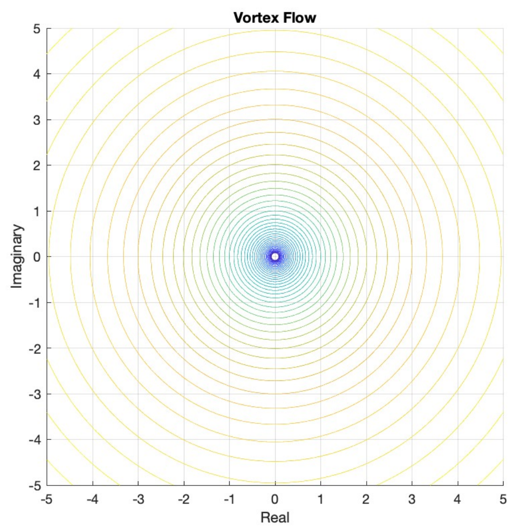


From a qualitative standpoint it is easy to see that this type of fluid flow would create no upward lift force on the cylinder however. The flow on top of the cylinder is equal to the flow on the bottom, and therefore no pressure difference is present and by Bernoulli's principle there would be no upward lift[2]. This is verified by the fact that there is no circulation on the inside of the circular domain, and therefore the circulation around the cylinder must be zero. The Kutta-Joukowsky theorem would then show that there is zero lift.

Vortex flow

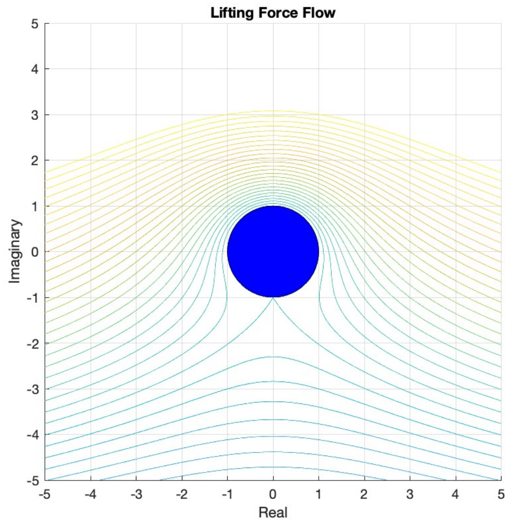
In order to add circulation around the cylinder to create a lifting force, a vortex can be superimposed onto the center of the circle[1]. A vortex is the rotational counterpart to a source/sink flow where its radial velocity is 0 everywhere, but its angular velocity is inversely proportional to the radial distance. The strength of the vortex is represented by Γ, which is also referred to as the circulation because it happens to be the value of the circulation around a contour when that contour encloses the vortex point. The velocity potential is given below, and the harmonic conjugate stream function is found as before:

$$
\begin{aligned}
\phi &= -\frac{\Gamma\theta}{2\pi} \\
\frac{\partial\phi}{\partial\theta} &= -\frac{\Gamma}{2\pi} = -r\frac{\partial\psi}{\partial r} \\
\frac{\partial\psi}{\partial r} &= \frac{\Gamma}{2\pi r} \\
\psi &= \frac{\Gamma}{2\pi}\ln r
\end{aligned}
$$



Note that this fluid flow is incompressible and irrotational everywhere except for the origin, at which the curl of the field is non-zero. This non-zero curl is the source of circulation as described in the introduction. All of these stream functions can then be superimposed to create a model of fluid flow that generates lift on a cylinder centered at the origin:

$$\psi = V_\infty r\sin\theta + \tfrac{\Gamma}{2\pi}\ln r - \tfrac{\kappa}{2\pi}\tfrac{\sin\theta}{r}$$



As stated above, the circulation around the cylinder is given by Γ, and so according to the Kutta-Joukowsky Theorem, the lift acting on this cylinder is equal to:

$$L' = \rho_\infty V_\infty \Gamma$$

B. Conformal Mappings

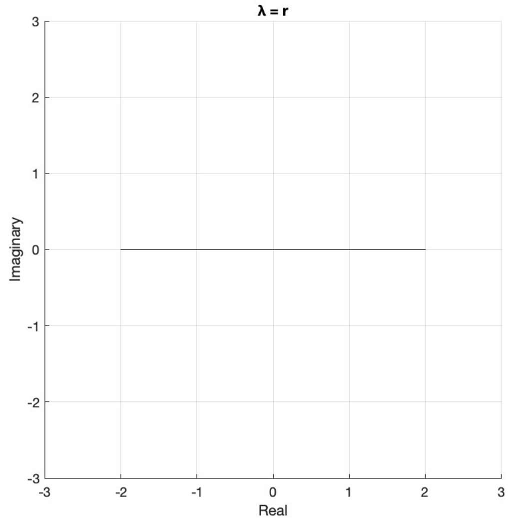
Now that we have found a way to create a circle in the complex plane with a lifting force acting on it by the fluid flow, we can use conformal mappings to map that circle into the shape of an airfoil. This is accomplished by using the Joukoswki transformation:

$$w(z) = z + \frac{\lambda^2}{z}$$

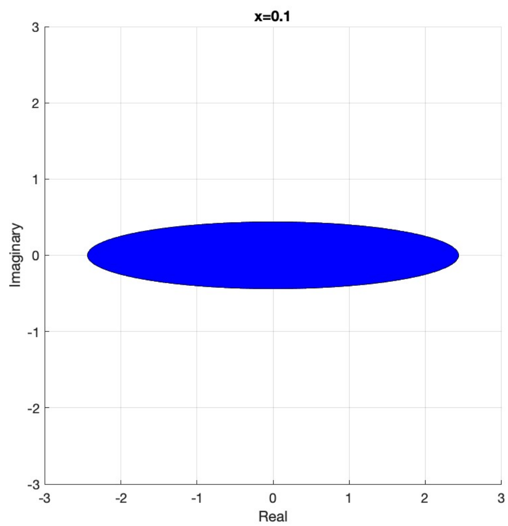
Where λ is a parameter used to determine the shape of the resulting transformation. By taking a circle with radius r, which can be shown in the complex plane by $z = re^{i\theta}$, we can analyze the mapping to determine how to create the shape of a traditional airfoil. In setting $\lambda = r$:

$$\begin{aligned}
w &= z + \frac{\lambda^2}{z} \\
&= re^{i\theta} + \frac{r^2}{re^{i\theta}} \\
&= re^{i\theta} + re^{-i\theta} \\
&= r(\cos\theta + i\sin\theta =) + r(\cos\theta - i\sin\theta =) \\
&= 2r\cos\theta
\end{aligned}$$

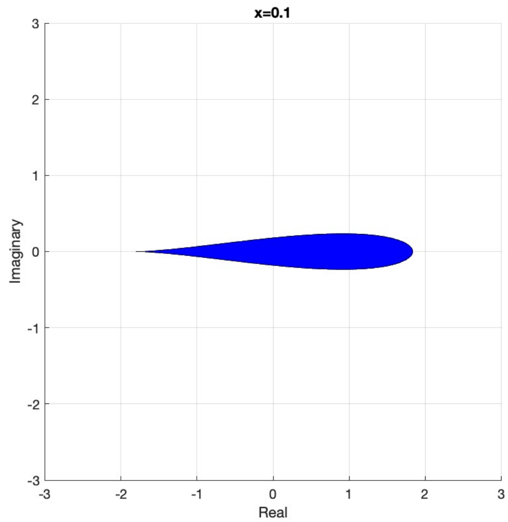
Which lies entirely on the real axis and oscillates between $2r$ and $-2r$. This is shown below:

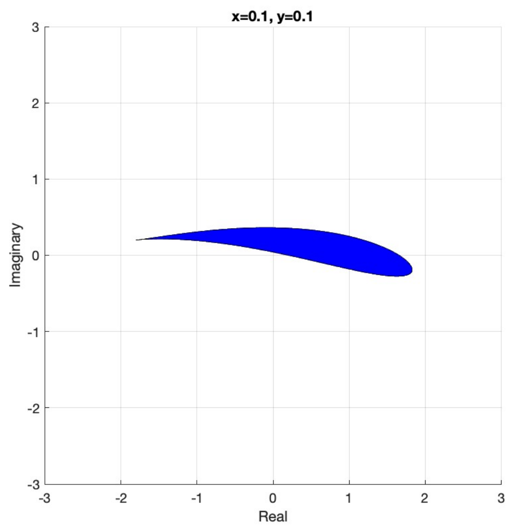


If however, $\lambda > r$ then the circle gets mapped to an ellipse:



While an ellipse is closer to the shape of an airfoil, it still is missing the camber and chord components that define a characteristic wing shape. Instead, by shifting the coordinates of the center of the original circle, the defining characteristics begin to take shape. The desired parameter is then given by $\lambda = r - s$ Where r is the radius of the circle and s is the center of the circle:





By applying this transformation, we have successfully transformed a circle in the complex plane into a Joukowsky airfoil, however this does not account for the fluid flow discussed in the previous section. This is where the harmonic conjugate nature of the velocity potential and stream functions plays an essential role. Owing to their analytic nature, the functions can be combined into a single complex velocity potential function that is analytic, and therefore any conformal mappings would be analytic as well, barring any points of singularity or stagnation described later:

$$\Theta(z) = \phi + i\psi$$

We can use the potential velocity and stream functions from the previous section to find the complex potential velocity of a lifting flow on a cylinder:

uniform flow:

$$\begin{aligned}\Theta(z) &= \phi + i\psi \\ &= V_\infty r\cos\theta + i(V_\infty r\sin\theta) \\ &= V_\infty r(\cos\theta + i\sin\theta) \\ &= V_\infty r e^{i\theta} \\ &= V_\infty z\end{aligned}$$

doublet flow:

$$\begin{aligned}\Theta(z) &= \phi + i\psi \\ &= \frac{\kappa}{2\pi}\frac{\cos\theta}{r} - i(\frac{\kappa}{2\pi}\frac{\sin\theta}{r}) \\ &= \frac{\kappa}{2\pi r}(\cos\theta - i\sin\theta) \\ &= \frac{\kappa}{2\pi r e^{i\theta}} \\ &= \frac{\kappa}{2\pi z}\end{aligned}$$

vortex flow:

$$\begin{aligned}\Theta(z) &= \phi + i\psi \\ &= -\frac{\Gamma\theta}{2\pi} + i(\frac{\Gamma}{2\pi}\ln r) \\ &= \frac{\Gamma}{2\pi}(-\theta + i\ln r) \\ &= \frac{\Gamma}{2\pi}\cdot i(\ln r + i\theta) \\ &= i\frac{\Gamma}{2\pi}(\ln re^{i\theta}) \\ &= i\frac{\Gamma}{2\pi}\ln z\end{aligned}$$

Therefore, by combining the complex velocity potentials of the superimposed flows, the full complex velocity potential is shown to be:

$$\Theta(z) = [V_\infty r\cos\theta + \tfrac{\kappa}{2\pi}\tfrac{\cos\theta}{r} - \tfrac{\Gamma\theta}{2\pi}] + i[V_\infty r\sin\theta - \tfrac{\kappa}{2\pi}\tfrac{\sin\theta}{r} + \tfrac{\Gamma}{2\pi}\ln r]$$

$$\Theta(z) = V_\infty\left(z + \tfrac{R^2}{z}\right) + i\tfrac{\Gamma}{2\pi}\ln z$$

When the complex function shown above is transformed by a conformal mapping, the circulation Γ is the same as in the original z-plane, and therefore the lift is the same in both mappings. This allows the shape of an airfoil to be inversely transformed to a simple cylinder with a lifting flow in which it is much simpler to calculate the circulation and resulting lift acting on the airfoil[3].

As stated above, the only thing that the circulation around the cylinder relies on is the vorticity Γ. However, different values of vorticity affect the stagnation points on the surface of the airfoil. Stagnation points are locations in the fluid flow where the velocity is disappears. This means that $\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = 0$, and by the Cauchy-Riemann equations we therefore know that $\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = 0$ must hold as well. This implies that stagnation points occur when $\Theta'(z) = 0$. In order to satisfy the Kutta conditions to make this a valid model, these stagnation points must map to the leading and trailing points of the resulting Joukowsky airfoil, and their can only be velocity in the θ component at the cylinder's boundary (because no flow can exit a rigid body) [2]. This occurs when:

$$\Gamma = 4\pi V_\infty R\sin(\alpha + \beta)$$

Where α is the angle of attack of the fluid on the airfoil and β is the angle between the x axis and the center of the original cylinder (before the Joukoswki transformation). In this way, circulation is directly related to the angle of attack of the fluid, and the shape of an airfoil (which is uniquely determined by the center coordinates of the transformed circle) as expected. In turn, this means that lift is determined by those factors as well as the velocity and density of the surrounding air (by the Kutta-Joukowsky theorem). Shown below is the transformed Joukowsky airfoil with a vareity of different α values. In reality, the vorticity of fluid flow as described above is determined by the angle of attack, and was just represented in a rotated form above.

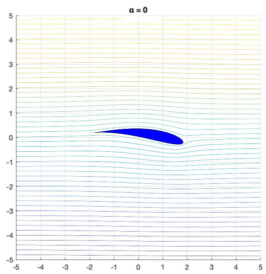


Fig. 1 0 degrees

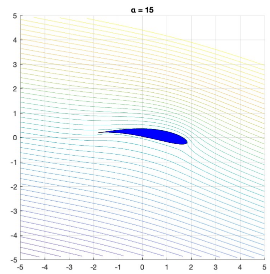


Fig. 2 15 degrees

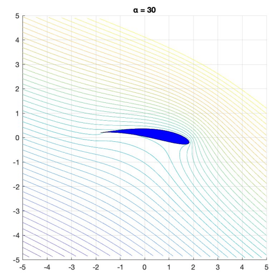


Fig. 3 30 degrees

III. Calculation of lift around NACA airfoils

The NACA airfoils are airfoil shapes that were developed by the National Advisory Committee for Aeronautics in the 1920s and 1930s. As there has been significant testing with the NACA family of airfoils, they provide a good opportunity to compare the computational techniques developed in this paper to experimental data. In the four-digit NACA series, the thickness of the of the airfoil can be modeled by:

$$y_t = 5y_{max}\left[0.2969\sqrt{\frac{x}{c}} - 0.126\frac{x}{c} + 0.3537\left(\frac{x}{c}\right)^2 + 0.28431\left(\frac{x}{c}\right)^3 - 0.1015\left(\frac{x}{c}\right)^4\right] \tag{1}$$

[3]

The numbers in the series control the foil shape. The first is $100\frac{m}{c}$, the second is $10\frac{p}{c}$, and the last two digits are $100\frac{y_{max}}{c}$. The difficulty then is utilizing an inverse Joukowsky transform to use conformal mapping to shift from these airfoils to a circle. We will be analizing three matched NACA/Joukowsky airfoils developed by Kapania, et al. NACA 0012, NACA 2215 and NACA 4412 [5]. With the airfoils described below, the Kutta-Joukowsky theorem can be used to determine the lift coefficient of each airfoil at a given angle of attack. This will then be compared to the known experimental data for each of these airfoils to characterize the accuracy of this simple model.

Airfoil	x value	y value	radius
NACA 0012	-0.107	0	1.027
NACA 2215	-0.130	0.01	1.110
NACA 4412	-0.100	0.06	1.130

With the assistance of conformal mapping, the lift and lift coefficient of these airfoils can be easily deduced with a couple of simple formulas. The lift coefficient can be determined by

$$L_c = \frac{\Gamma}{2V_{\text{inf}}^2 R} \tag{2}$$

Where Gamma is a function of α and the airfoil characteristics (through β). By analyzing over a range of angles, the lift coefficient of each of these airfoils can be calculated.

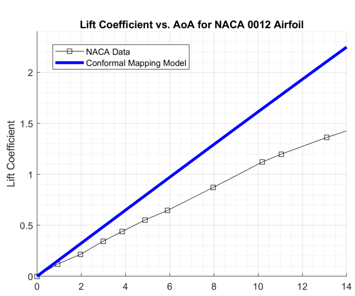


Fig. 4 NACA 0012

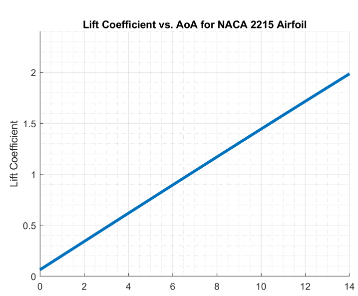


Fig. 5 NACA 2215

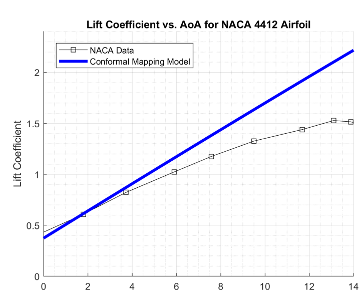


Fig. 6 NACA 4412

In the comparison to the NACA data, the Joukowsky airfoil method does not prove to be an effective approximation of the coefficient of lift, particularly as the angle of attack approaches stall. However, the model is at least a good starting point, and does appear to provide a good small angle approximation of the lift.

IV. Conclusions

Despite the assumptions made concerning ideal fluid flow and the approximated shape of an airfoil using the Joukowsky conformal mapping, it has been shown that any calculations regarding lift and circulation are sufficiently accurate to model the flight of an aircraft. This greatly simplifies the calculations necessary to study the lift force acting on airplane wings and demonstrates another application of complex variables to the real world despite the subject of study being grounded entirely in the world of real numbers. The properties of solutions to the Laplace Equation align well with the applications of analytic functions and harmonic conjugates as well, making it possible to express fluid flow in the complex plane and perform operations that are normally not possible in the real plane. Overall, the extension of fluid mechanics using complex variables and their applications benefits the study of fluid flow greatly.

References

[1] Bosco, N., "APPLICATIONS OF CONFORMAL MAPPINGS TO THE FLUID FLOW," Ph.D. thesis, 2018.

[2] Johnson, T., "Conformal Mapping in Wing Aerodynamics," Ph.D. thesis, 2013.

[3] Graebel, W. P., *Irrotational Two-Dimensional Flows*, Elsevier Academic Press, 2007, p. 87–114.

[4] Burington, R. S., "On the use of conformal mapping in shaping wing profiles," *The American Mathematical Monthly*, Vol. 47, No. 6, 1940, p. 362–373. https://doi.org/10.1080/00029890.1940.11990989.

[5] Kapania, N., Terracciano, K., and Taylor, S., "Modeling the fluid flow around airfoils using conformal mapping," *SIAM Undergraduate Research Online*, Vol. 1, No. 2, 2008, p. 70–99. https://doi.org/10.1137/08s010104.

V. Appendix

```
clear all
```

Code to create angle of attack diagrams

```
v_inf = 200; % Velocity far from the airfoil
v = v_inf / v_inf;
theta = 30 * pi / 180; % Angle of attack

s_x = 0.1; % x coordinate of circle center
s_y = 0.1; % y coordinate of circle center
s = s_x + i*s_y; % z coordinates of circle center

r = 1; % radius size of circle

lambda = r - s; % Joukowski parameter


beta = (theta);
k = 2*r*v*sin(beta); % kappa value for doublet

w = v * exp(i*theta);


x = meshgrid(-5:0.02:5);
y = x';


z = x + i*y; % z complex plane values


f = w*z + ((v*exp(-i*theta)*r^2) ./ (z-s)) + i*k*log(z); % function to make
 streamlines
J = z + lambda^2./z; % Joukowski transformation

angles = 0:0.01:2*pi; % full values for a circle
z_circle = r*(cos(angles) + i*sin(angles)) + s; % circle in the z plane
z_airfoil = z_circle + lambda^2./z_circle; % circle in the transformed w plane



%{
figure(2)
hold on
grid on
contour(real(J),imag(J),imag(f),[-5:.2:5])
fill(real(z_airfoil),imag(z_airfoil), 'b')
axis equal
axis([-5 5 -5 5])
title("# = 30")
%}
```

Code to create flow diagrams and circle transformations:

```
s_x = 0; % x coordinate of circle center
s_y = 0; % y coordinate of circle center
s = s_x + i*s_y; % z coordinates of circle center
r = 1; % radius of circle

v_inf = 1; % fluid velocity


kappa = 2*pi*v_inf*r^2; % kappa value for doublet
Gamma = 4*pi; %Gamma value for vortex

x = meshgrid(-5:0.02:5);
y = x';
z = x + i*y; % z plane values



Psi_U0 = v_inf .* abs(z) .* sin(angle(z)); % stream function for uniform flow

Psi_K = -kappa/(2*pi)*sin(angle(z))./(abs(z)); % stream function for doublet
 flow

Psi_Gamma = Gamma/(2*pi)*log(abs(z)); % stream function for vortex flow

Cylinder = Psi_U0 + Psi_K; % stream function for cylindrical flow

StreamLine = Psi_U0 + Psi_K + Psi_Gamma; % stream function for lifting flow on
 cylinder



angles = 0:0.01:2*pi; % full range of circle values
z_circle = r*(cos(angles) + i*sin(angles)) + s; % circle in z plane


%{
figure(1)
hold on
grid on
contour(real(z),imag(z),StreamLine, [-5:.2:5])
fill(real(z_circle),imag(z_circle), 'b')
axis equal
axis([-5 5 -5 5])
xlabel('Real')
ylabel("Imaginary")
title("Lifting Force Flow")
%}
```

```
clear all
close all
clc

%Data0012 = importdata("0012.abbottdata.cl.dat");
AoA =
 [-0.256880734,1.798165138,3.724770642,5.908256881,7.577981651,9.504587156,11.68807339,13.
 15.79816514];%Data0012.data(15:28,1);
Cl =
 [0.407692308,0.609090909,0.823076923,1.024475524,1.175524476,1.326573427,1.43986014,1.527

N = 100;
thetad = zeros(1,N);
L_c = zeros(1,N);

for idx = 1:N
v_inf = 100;
v = v_inf/v_inf;
thetad(idx) = (idx-1)*0.142;
theta = thetad(idx) * pi/180;
s_x = -0.10;
s_y = .06;
s = s_x + i*s_y;
r = 1.13;
rho = 1.225;
mu = 1.79e-5;
Re = rho*v_inf*2*r/mu;
lambda = r-s;
beta = acos((r^2+(r-s_x)^2-s_x^2)/(2*r*sqrt((r-s_x)^2+s_y^2)));
k = 2*r*v*sin(beta+theta);
Gamma = 4*pi*r*v_inf*sin(beta+theta);

w = v*exp(i*theta);


toll = +7e-2;


x = meshgrid(-5:0.1:5);
y = x';
z = x + i*y;

for a = 1:length(x)
    for b = 1:length(y)
        if abs(z(a,b)-s) <= r-toll
            z(a,b) = NaN;
        end
    end
end

f = w*(z) + (v*exp(-i*theta)*r^2)./(z-s) + i*k*log(z);
```

```
J = z + lambda^2./z;

angle = 0:0.1:2*pi;
z_circle = r*(cos(angle)+i*sin(angle))+s;
z_airfoil = z_circle + (lambda^2./z_circle);

L = v_inf*rho*Gamma;
L_c(idx) = L/(pi/2*v_inf^2*r^3);
L_str = num2str(L);

end



figure(1)
hold on
grid on
contour(real(z),imag(z),imag(f),[-5:.2:5])
fill(real(z_circle),imag(z_circle),'b')
axis equal
axis([-5 5 -5 5])

figure(2)
hold on
grid on
contour(real(J),imag(J),imag(f),[-5:.2:5])
fill(real(z_airfoil),imag(z_airfoil),'b')
axis equal
axis([-5 5 -5 5])

figure(3)
hold on
grid on
grid minor
plot(AoA,Cl,'s-k')
plot(thetad,L_c,'LineWidth',3,'Color','b')
title("Lift Coefficient vs. AoA for NACA 4412 Airfoil")
axis([0 14 0 2.4])
xlabel("Angle of Attack (degrees)")
ylabel("Lift Coefficient")
legend("NACA Data", "Conformal Mapping Model")
```