Fluid Mechanics and the Joukowsky Airfoil

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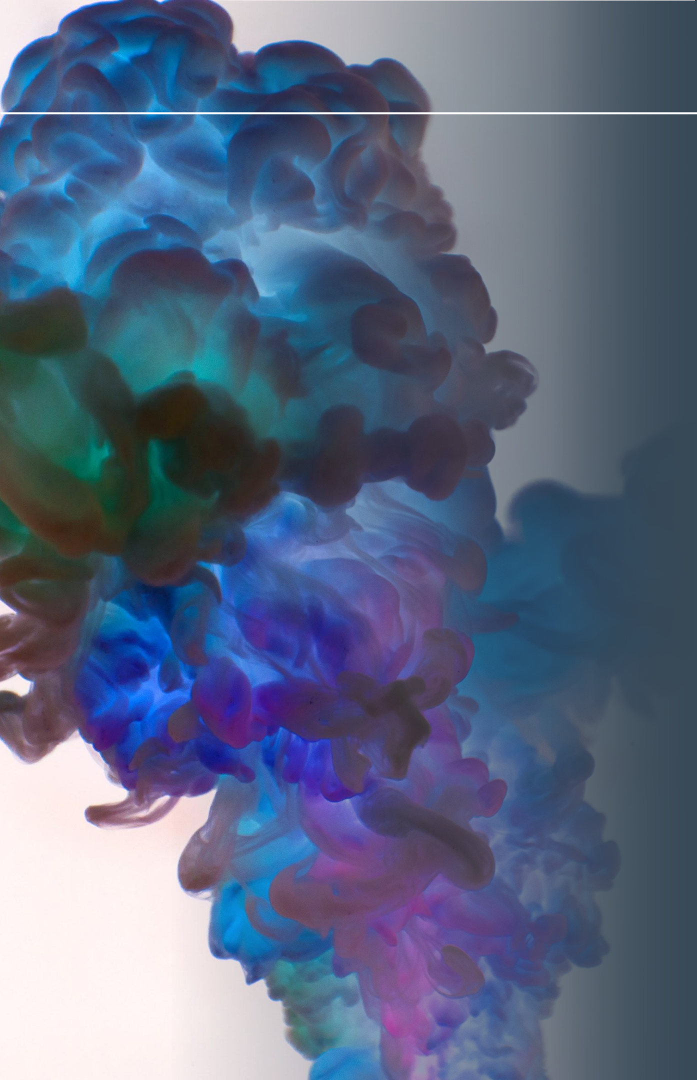


An introduction to fluid mechanics

Irrotational flows and the Laplace Equation

Fluid Velocity:

$$\vec{V} = \nabla \phi$$

Steady Flow:

$$\frac{\partial \vec{V}}{\partial t} = 0$$

Irrotational Flow:

$$\vec{w} = \nabla \times \vec{V} = 0$$

$$\nabla \times (\nabla \phi) = \vec{0}$$

Incompressible Flow:

$$\nabla \cdot \vec{V} = 0$$

Laplace Equation:

$$\nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0$$

Applying complex analysis to 2D flow

Combine the velocity potential and Lagrange's stream

By Cauchy-Riemann, the resulting complex velocity potential will be analytic and conformal

$$v_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$v_y = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

$$\Theta(z) = \phi + i\psi$$

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial y} = v_x - iv_y.$$

Types of fluid flow

- Uniform flow
- Source/Sink
- Doublet
- Vortex

$$w = U^* z,$$

$$w = \frac{m}{2\pi} \ln(z - z_0),$$

$$w = B/(z - z_0),$$

$$w = \frac{-i\Gamma}{2\pi} \ln(z - z_0),$$

TABLE 2.2.1 Velocity potentials and stream functions for irrotational flows

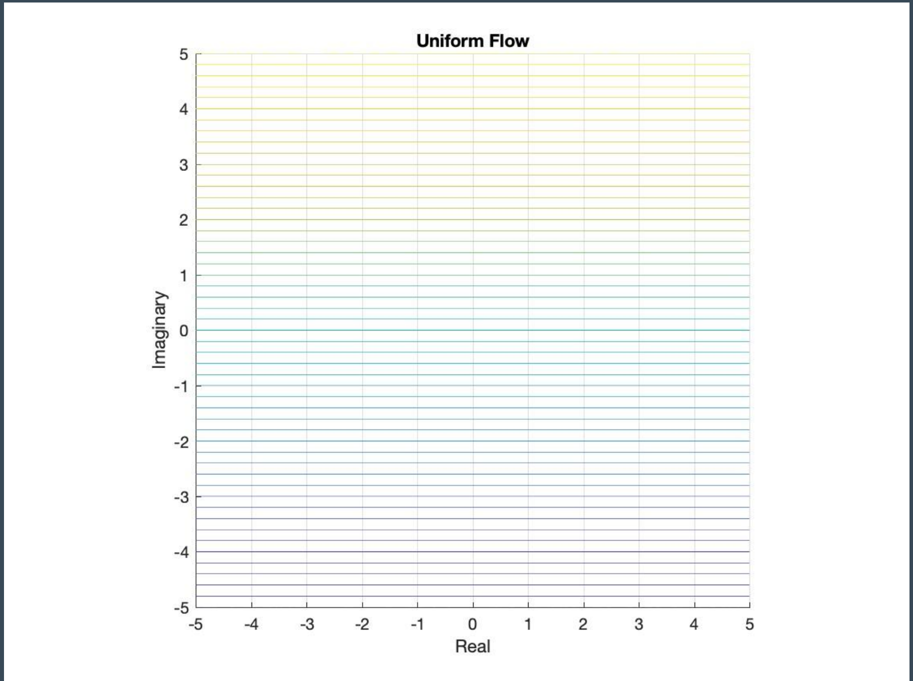
Flow Element	Two-dimensional	
	ϕ	ψ
Uniform stream	$xU_x + yU_{y'}$	$yU_x - xU_{y'}$
Source or sink	$\frac{m}{2\pi} \ln \sqrt{(x-x_0)^2 + (y-y_0)^2}$	$\frac{m}{2\pi} \tan^{-1} \frac{y-y_0}{x-x_0}$
Doublet	$\frac{B_x(x-x_0) + B_y(y-y_0)}{\lvert\mathbf{r}-\mathbf{r}_0\rvert^2}$	$\frac{B_y(x-x_0) - B_x(y-y_0)}{\lvert\mathbf{r}-\mathbf{r}_0\rvert^2}$
Line vortex	$\frac{\Gamma}{2\pi} \tan^{-1} \frac{y-y_0}{x-x_0}$	$-\frac{\Gamma}{2\pi} \tan^{-1} \frac{y-y_0}{x-x_0}$

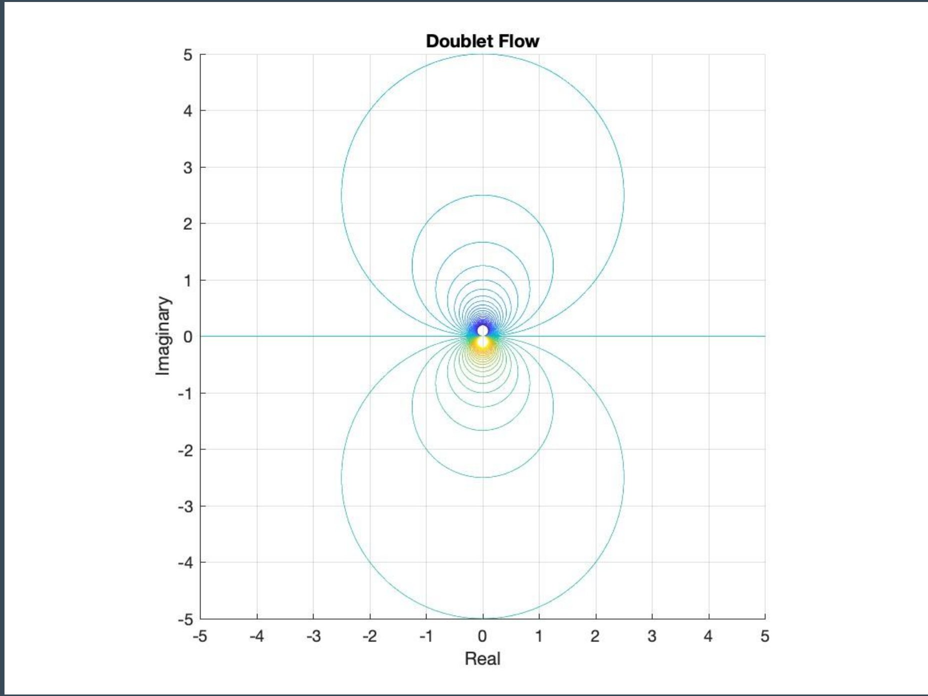
$$\phi = V_\infty r \cos\theta$$

$$\frac{\partial\phi}{\partial r} = V_\infty \cos\theta = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$\frac{\partial\psi}{\partial\theta} = V_\infty r \cos\theta$$

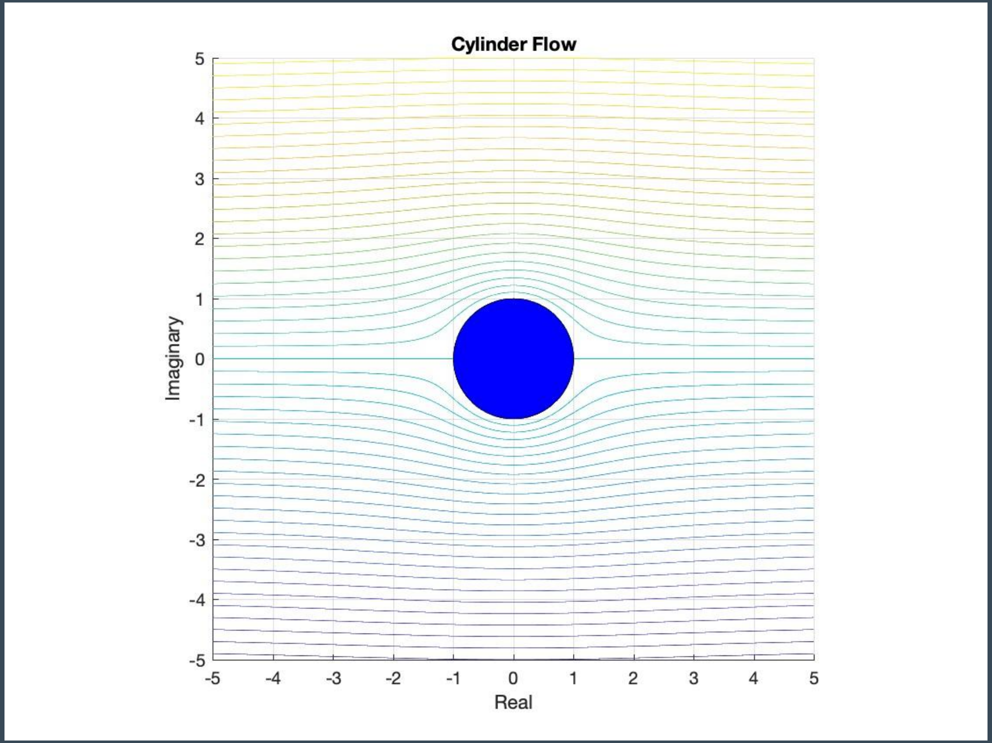
$$\psi = V_\infty r \sin\theta$$

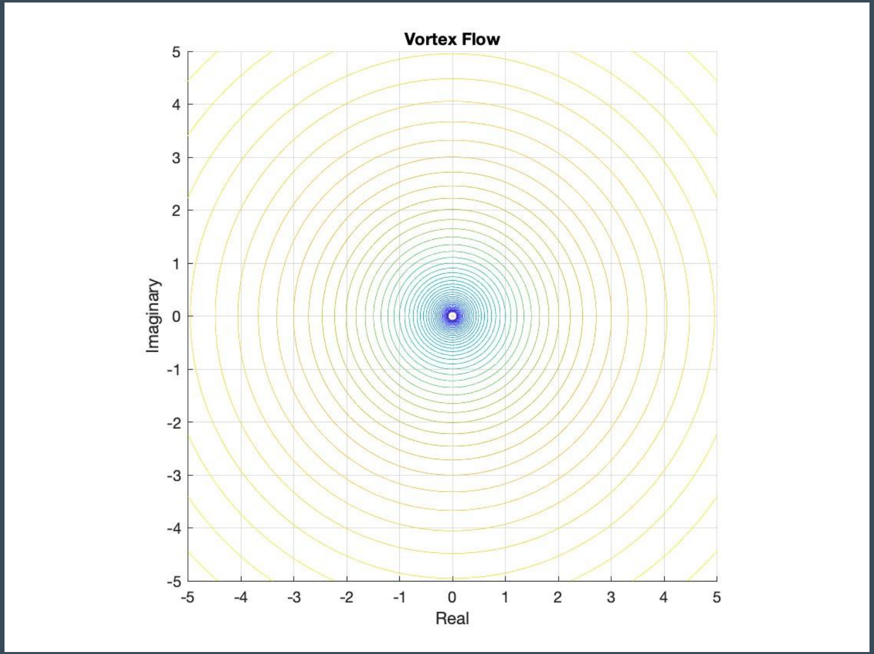




$$\phi = \frac{\kappa}{2\pi}\frac{\cos\theta}{r}$$

$$\psi = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r}$$


Cylinder Flow
Imaginary
Real
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5 -4 -3 -2 -1 0 1 2 3 4 5



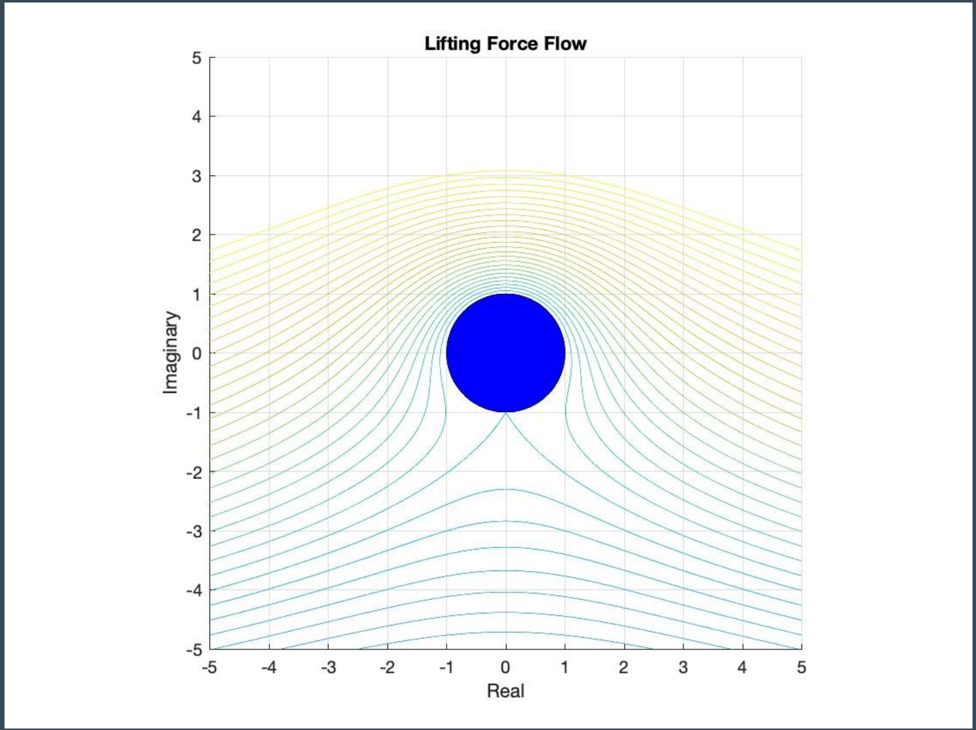
$$\phi = -\frac{\Gamma\theta}{2\pi}$$

$$\frac{\partial\phi}{\partial\theta} = -\frac{\Gamma}{2\pi} = -r\frac{\partial\psi}{\partial r}$$

$$\frac{\partial\psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

$$\psi = \frac{\Gamma}{2\pi}\ln r$$

$$\psi = V_\infty r \sin\theta + \frac{\Gamma}{2\pi}\ln r - \frac{\kappa}{2\pi}\frac{\sin\theta}{r}$$

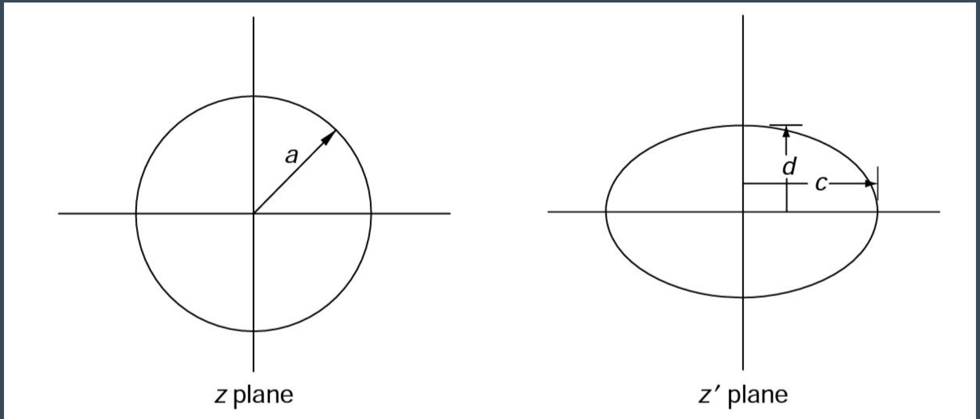


Conformal Mappings
and Joukowsky airfoils

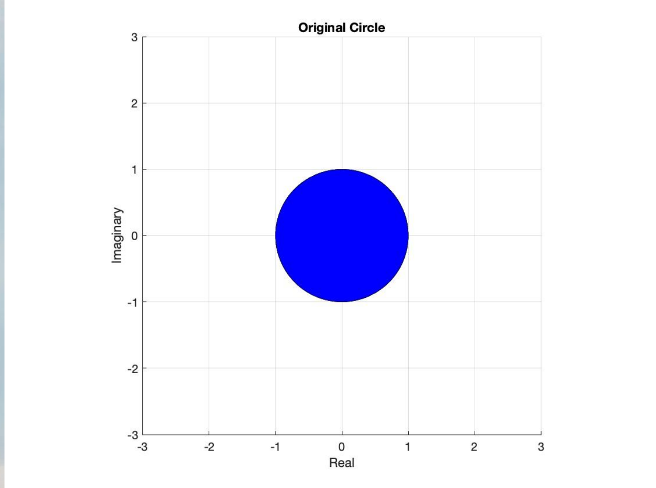
Finding the flow past an ellipse

The Joukowski transformation

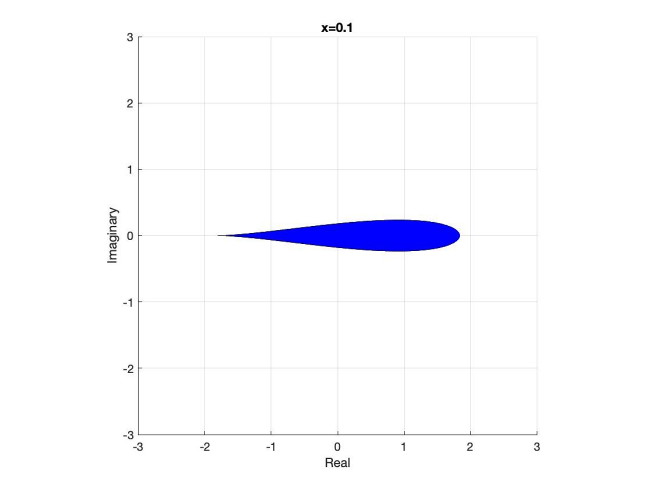
$$w(z) = z + \frac{\lambda^2}{z}$$



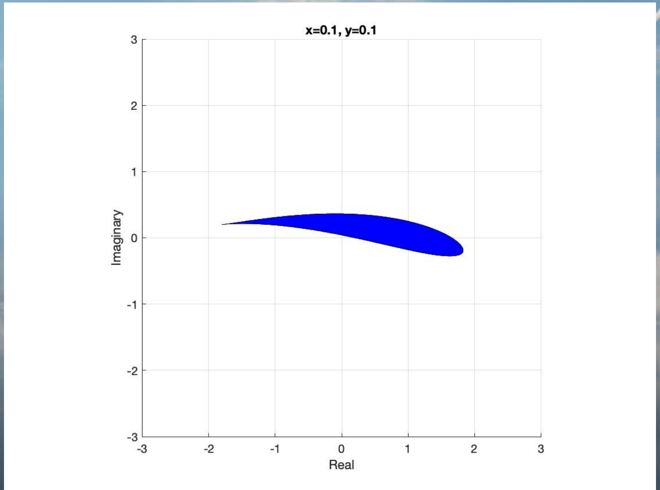
Transform a sphere into an ellipse with semimajor axis c = a + λ^2/a and semiminor axis d = a - λ^2/a



Original Circle
Imaginary
Real



x=0.1
Imaginary
Real



x=0.1, y=0.1
Imaginary
Real

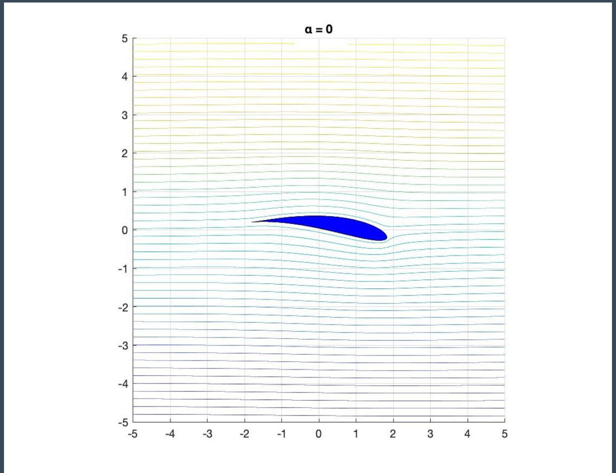
Complex Velocity Potential

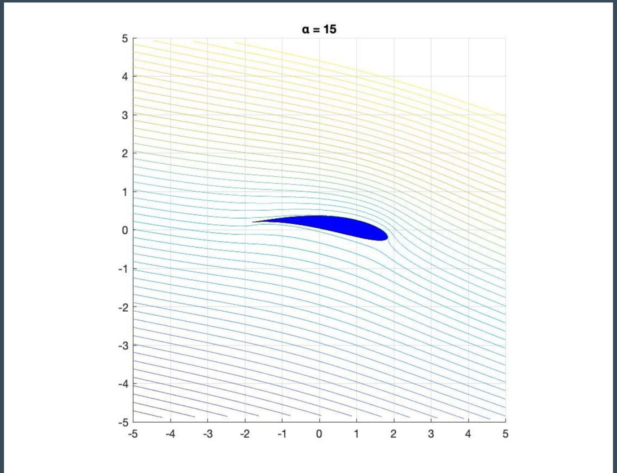
$$\Theta(z) = \phi + i\psi$$

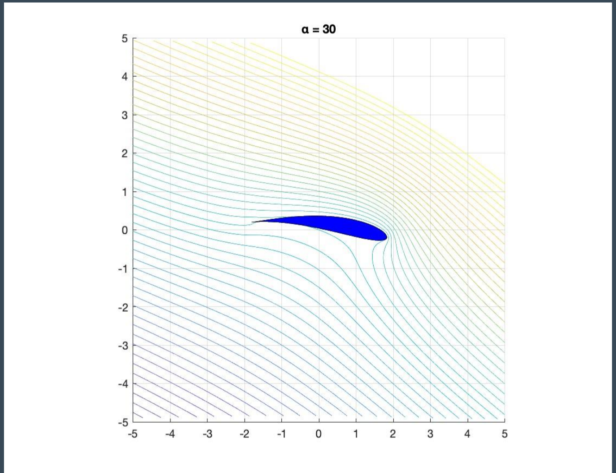
$$\Theta(z) = [V_\infty r \cos\theta + \frac{\kappa}{2\pi}\frac{\cos\theta}{r} - \frac{\Gamma\theta}{2\pi}] + i[V_\infty r \sin\theta - \frac{\kappa}{2\pi}\frac{\sin\theta}{r} + \frac{\Gamma}{2\pi}\ln r]$$

$$\Theta(z) = V_\infty\left(z + \frac{R^2}{z}\right) + i\frac{\Gamma}{2\pi}\ln z$$

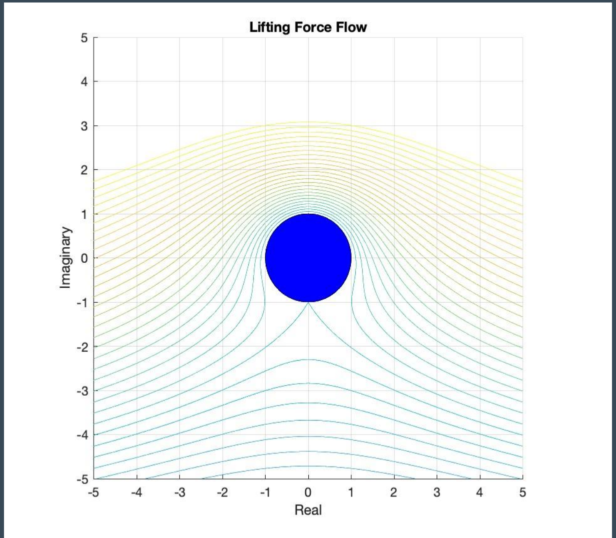
Angle of Attack (Vorticity)

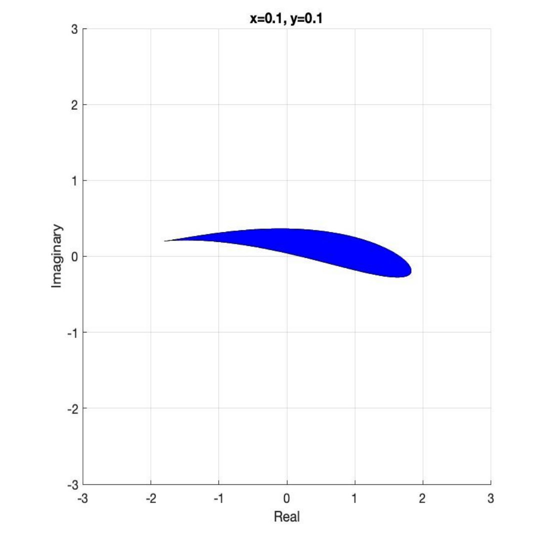






Kutta Condition





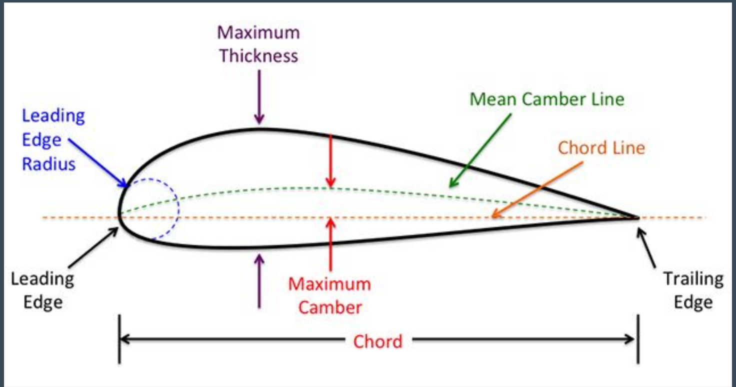
$$\Gamma = 4\pi V_\infty R \sin(\alpha + \beta)$$

$$L' = \rho_\infty V_\infty \Gamma$$

$$\Gamma = \oint_C \vec{V} \cdot ds = \iint_S (\nabla \times \vec{V}) ds$$

Calculation of lift around NACA airfoils

The NACA 4 digit airfoil



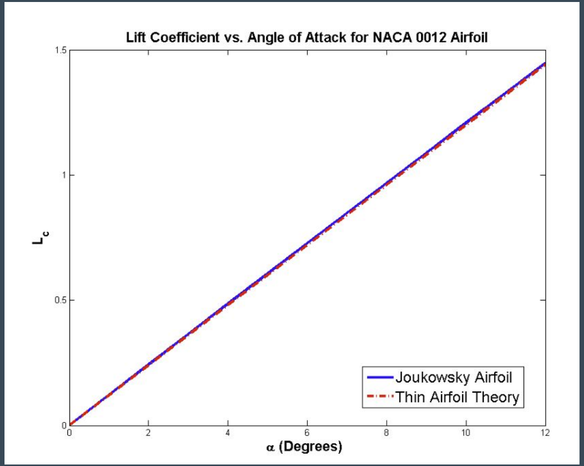
Joukowsky transform to NACA airfoils

Airfoil	x value	y value	radius
NACA 0012	-0.107	0	1.027
NACA 2215	-0.130	0.01	1.110
NACA 4412	-0.100	0.06	1.130

Calculation of lift around NACA airfoils

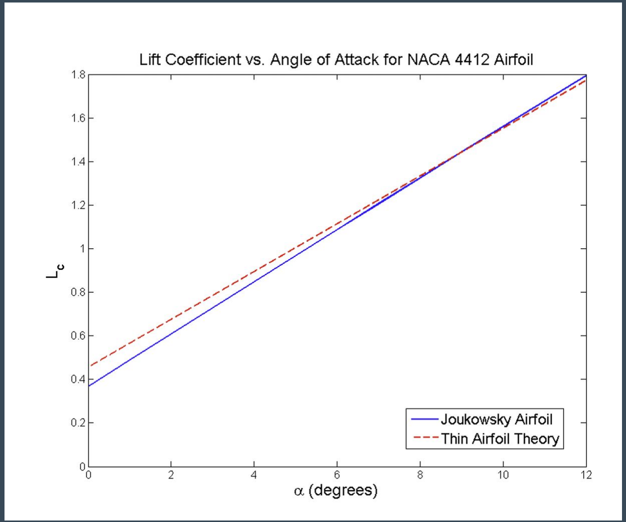
$$L' = \rho V_\infty \Gamma$$

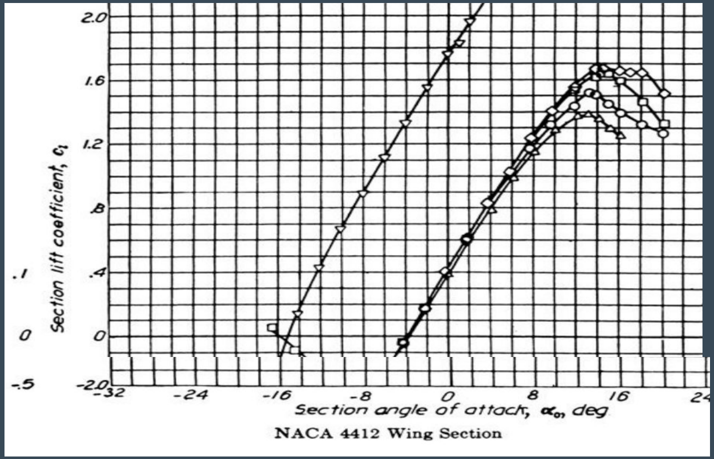
$$L_c = \frac{\Gamma}{2V_\infty^2 b}$$



Coefficient of lift vs Thin Airfoil
(Kapania, et al., 2008)

Calculation of lift around NACA airfoils





References

[1] Bosco, N., “APPLICATIONS OF CONFORMAL MAPPINGS TO THE FLUID FLOW,” Ph.D. thesis, 2018.

[2] Johnson, T., “Conformal Mapping in Wing Aerodynamics,” Ph.D. thesis, 2013.

[3] Graebel, W. P., Irrotational Two-Dimensional Flows, Elsevier Academic Press, 2007, p. 87–114.

[4] Burington, R. S., “On the use of conformal mapping in shaping wing profiles,” The American Mathematical Monthly , Vol. 47, No. 6, 1940, p. 362–373. https://doi.org/10.1080/00029890.1940.11990989.

[5] Kapania, N., Terracciano, K., and Taylor, S., “Modeling the fluid flow around airfoils using conformal mapping,” SIAM Undergraduate Research Online , Vol. 1, No. 2, 2008, p. 70–99. https://doi.org/10.1137/08s010104.