Math 314 Final Exam **NAME:**

Due by 5:00pm on Thursday 5/13 UNM Learn.

Instructions:

You may use your notes and textbook, but you are expected to work on this without the help of any other person. You can use MATLAB or Octave to check your calculations, *but you must write every step of your work unless instructed otherwise.*

To complete this exam and submit your work properly:

I. Print this document and solve all problems (writing solutions by hand), or if you plan to write your work electronically (typing or using a stylus/tablet) download this document and solve the problems by writing that way.

II. If you printed and wrote your work by hand, scan or take a clear picture of each page of your work. All exams must be saved and submitted as a single .pdf file, with pages in the correct order.

III. Upload the single .pdf file of your work in the Final Exam Submission page, under "exams" in the left-menu of UNM Learn, by 5pm on 5/13.

IMPORTANT:

Start with the problems that you are the most comfortable with. If you get stuck on a problem, skip it and come back to it later. Many parts of these problems can be solved without computation.

There is a total of 50 points to be earned.

Problem 1. (9pts $= 18\%$)

Solve each system or show that it is inconsistent. If there are infinitely many solutions, express your answer parametrically.

a. $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

Problem 2. (14pts= 28%) (short answer)

a. Solve the system using an inverse matrix

$$3x_1 + x_2 = 1$$
$$2x_1 - 2x_2 = 13$$

b. Write down a linearly independent set in P_1 which does not span P_1 (recall that P_1 is the vector space of polynomials degree one or less).

c. Write down a linearly dependent set which spans P_1.

d. If $proj_v\boldsymbol{u} = \boldsymbol{0}$, what can you say about vectors $\boldsymbol{u}$ and $\boldsymbol{v}$? Assume that neither $\boldsymbol{u}$ nor $\boldsymbol{v}$ is the zero vector.

e. Let $T: P_1 \to R^3$ be defined by $T(p) = (2a_1, 3a_0 + a_1, 2a_0)$ for all functions $p = a_0 + a_1x$ in P_1.
Is T an isomorphism? Briefly explain your reasoning.

f. Let $S = span\{1 + x, 1 - 2x\}$ be a subspace of inner product space P_2 with the inner product defined by $\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2$ for all functions $p = a_0 + a_1x + a_2x^2$ and $q = b_0 + b_1x + b_2x^2$ in P_2.
Find a basis for $S^{\perp}$, the orthogonal complement of S in P_2.

g. Assume $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $A = P^{-1}BP$. Let $\boldsymbol{y} = P\boldsymbol{x}$. Show that $B\boldsymbol{y} = \lambda\boldsymbol{y}$. In other words, show that similar matrices A and B have the same eigenvalues λ.

Problem 3. (3pts$=6\%$)

Use the Gram-Schmidt orthonormalization process to find an orthonormal basis for the column space of A.

$$A = \begin{bmatrix} -3 & 1 \\ 0 & -2 \\ 4 & 2 \end{bmatrix}$$

Problem 4. (3pts$=6\%$)

Find the equation of the least squares regression line for the four points:

$$(-1,-2),(-2,3),(3,3),(0,1)$$

Show all work by hand.

Problem 5. (9pts= 18%)

Let $T: R^4 \rightarrow R^3$ be defined by

$T\big((x_1, x_2, x_3, x_4)\big) = (x_1 - x_2 - x_3 + x_4,\ x_1 + x_2 - x_3 - x_4,\ x_1 - x_2 + x_3 - x_4).$

You can use the "rref" command in MATLAB/Octave, but do not use any other commands.

a. Find the standard matrix of T.

b. Find a basis for the range of T, and find the rank of T.

c. Find a basis for the kernel of T, and find the nullity of T.

Problem 6. (6pts= 12%)

Let

$$A = \begin{bmatrix} 7 & -4 \\ 12 & -7 \end{bmatrix}$$

a. Diagonalize matrix A.

b. Write down the general solution of the linear system of differential equations $\boldsymbol{y}' = A\boldsymbol{y}$.

Problem 7. (6pts= 12%)

P_n, the set of all polynomials of degree n or less, is a subspace of $C[0,1]$, with inner product defined by $\langle f,g\rangle = \int_0^1 f(x)g(x)dx$.

a. The set $\{\boldsymbol{u_1}, \boldsymbol{u_2}\} = \{\sqrt{3}x,\ 2-3x\}$ forms an orthonormal basis for P_1. Use this to help you find $\boldsymbol{u_3}$ such that the set $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is an orthonormal basis for P_2.

b. Let $f = \ln(x+1)$ be a function in inner product space $C[0,1]$. The orthogonal projection of f on the subspace P_2 is denoted $proj_{P_2}f$ (no need to compute this).
Answer "True" or "False" for each statement and explain your answers briefly:

i. $proj_{P_2}f$ is orthogonal to every function p in P_2.

ii. The function $f - proj_{P_2}f$ is in P_2.

iii. $\|f-p\| \geq \|f - proj_{P_2}f\|$ for all functions p in P_2 (with inner product as stated above).