Exact BER Computation for the Cross 32-QAM Constellation *

Pavan Kumar Vitthaladevuni and Mohamed-Slim Alouini
Department of Electrical and Computer Engineering
University of Minnesota
Minneapolis, MN 55455, USA.
E-mail: <{pavan,alouini}@ece.umn.edu > .

Abstract— **When the number of bits per symbol is odd, the peak and average power of transmission can be reduced by using cross quadrature amplitude modulations (QAMs) instead of rectangular QAMs. Since perfect Gray coding is not possible for cross QAMs, using Smith-style Gray coding, we derive in this paper the exact bit error rate (BER) for the cross 32-QAM constellation over additive white Gaussian noise (AWGN) and Rayleigh fading channels.**

I. Introduction

It is well known that square quadrature amplitude modulations (QAM) are the typically used constellations when the number of bits in a symbol is even (4, 16, 64, 256-QAMs). However, when the number of bits per symbol is odd, the rectangular constellation is not a good choice. In [1], the author shows how both the peak and average power can be reduced by using a cross QAM constellation. Also, the author shows that the average signal to noise ratio (SNR) is reduced by atleast 1 dB by using cross QAMs. A 32 QAM is shown in Fig. 2 in both the rectangular and cross shaped constellations.

More recently, cross QAMs have been found useful in adaptive modulation schemes wherein the constellation size is adjusted depending on the channel quality. As the channel quality improves, the constellation size ($M = 2^k$) is increased by incrementing k to $k+1$. If one were to use just square QAMs, the increments should be from k to $k+2$ (for instance, we need to go from 16 to 64 to 256-QAM $\cdots$). Using cross QAMs however, the increment is smoother (16-QAM to 32 cross QAM to 64-QAM $\cdots$).

The remainder of this paper is organized as follows. A systematic way to pseudo-Gray-map cross QAM constellations will be briefly discussed in the next section. Section III describes the main differences between the nature of decision boundaries for cross QAMs and those for rectangular QAMs. In section IV, we derive the bit error rate (BER) expressions for each of the 5 bits under additive white Gaussian noise (AWGN), and show how these can be extended to Rayleigh fading channels. In section VI we verify our results with simulations and compare the accuracy of our equations with the approximate expressions given in [1]. Finally, section VII concludes the paper.

II. Impure Gray Coding for Cross QAMs

The cross 32-QAM constellation is constructed as shown in Fig 2. The rectangular 32 QAM constellation is first Gray coded. Then, the last columns of symbols on the far left and the far right are moved to the top and bottom. Perfect Gray mapping is not possible for cross QAMs.

Consider the rectangular QAM constellation. The Gray code and the binary code (in both x and y-directions) are defined by [2]

$$\begin{aligned} g_j^1 &= b_j^1 \\ g_j^l &= b_j^l \oplus b_j^{l-1}, 1 \le j \le (2^{k+1} \text{or } 2^k), 1 < l \le (k \text{ or } k+1), \end{aligned} \quad (1)$$

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where g_j^l and b_j^l are the l^{th} bits in the gray and binary codes of the j^{th} symbol. This relation thereby gives a neat transformation for the amplitude of the symbol.

However, when the symbol constellation points are moved to form cross QAM constellation, this relation breaks down. Also, perfect Gray coding is no longer possible for the cross-QAM constellation. Fortunately, Smith [1] came up with a new labeling scheme that adds a sixth bit as the least significant bit and thereby creates a new Gray code. The 6-bit code will hereafter be referred to as the pseudo-code. The 5-bit one will be referred to as the impure Gray code. The transformation between the two sets of codes is quite simple and is easily invertible. Since 5-bit Gray codes are impure, Smith introduced the concept of Gray penalty, G_p, as the average number of bits by which two adjacent symbols differ in the constellation [1]. Constellations for which perfect Gray coding is possible, have a Gray penalty of 1. The Gray penalty of the cross 32-QAM constellation is $\frac{7}{6}$ (there are 4 symbols at with a Gray penalty of $\frac{3}{2}$, 4 with a Gray penalty of $\frac{4}{3}$, 8 with a Gray penalty of $\frac{5}{4}$ and the remaining 16 with a Gray penalty of 1, giving an average of $\frac{7}{6}$). Any other type of labeling for this constellation will result in a higher Gray penalty and as a result, a higher BER.

III. Decision Making at the receiver

In the case of rectangular QAM, decision boundaries were quite simple [2]. The in-phase bits ($i_1 i_2 i_3 \cdots$) had vertical decision boundaries, while the quadrature phase bits ($q_1 q_2 q_3 \cdots$) had horizontal ones. The same is the case for cross 32-QAM if we consider the 6-bit gray codes. We could find out the error probability for each of the 6-bits in the pseudo-code and deduce the error for each of the 5-bits in the impure code (which is what we are interested in), as suggested in [1].

Equivalently, for analysis purposes, we could directly analyze the BER of the 5-bit impure Gray code instead of deducing it from the 6-bit pseudo-code. However, for the 5-bit cross QAM constellation, due to the fact that symbols are moved from the end columns to the top and the bottom of the constellation, the decision boundaries for each bit will not be just horizontal or vertical lines. As we shall see, they will be a combination of these for most of the bits.

The decision boundaries for the 5-bit impure Gray code are shown in Fig. 3. Not that for bits i_2, q_2 and i_3, the decision boundaries are a combination of vertical, horizontal and 45° lines.

IV. BER Expressions for Cross 32-QAM

For the sake of convenience, we use the following notation:

$$\mathrm{e}_l(i) = \operatorname{erfc}\left(\frac{id}{\sqrt{l N_0}}\right), l = 1, 2, \quad (2)$$

where $\operatorname{erfc}(\cdot)$ is the error complimentary function, $2d$ is the minimum distance between adjacent symbols in the constellation and $\frac{N_0}{2}$ is the two-sided noise power spectral density.

A. Bits i_1 and q_1

When writing the BER of bit i_1 for a particular symbol, (say for symbol "A" in Fig. 3), all we need to look for is the distance between "A" and its decision boundary (y-axis in this case), which is $3d$ in this case. The BER for bit i_1 for symbol "A" can be written as

$$P_b(32, i_1|A) = \frac{1}{2}\mathrm{e}_1(3), \tag{3}$$

where erfc$(\cdot)$ is the error complimentary function, $2d$ is the minimum distance between adjacent symbols in the constellation and $\frac{N_0}{2}$ is the two-sided noise power spectral density. Noting that E_s, the average symbol energy is given by $E_s = 15d^2$, $\frac{d}{\sqrt{N_0}}$ can be rewritten as $\sqrt{\frac{\gamma}{15}}$, where $\gamma = \frac{E_s}{N_0}$ is the carrier to noise ratio (CNR) at the receiver.

Averaging over all the 32 symbols (averaging over 8 symbols of one quadrant is enough due to symmetry) in the cross-QAM constellation, the BER of bit i_1, $P_b(32, i_1)$ can be written as

$$P_b(32, i_1) = \frac{1}{8}\left(2\left(\frac{1}{2}\mathrm{e}_1(5)\right) + 3\left(\frac{1}{2}\mathrm{e}_1(3)\right) + 3\left(\frac{1}{2}\mathrm{e}(1)\right)\right) \tag{4}$$

Similarly, for bit q_1, the x-axis is the decision boundary, and hence, by symmetry,

$$P_b(32, q_1) = P_b(32, i_1). \tag{5}$$

B. Bit i_2

The decision boundary for bit i_2 is a square centered at the origin and of side length $8d$. For this expression, we divide the constellation symbols into two parts: those outside the rectangle (A, B, C and F), and those inside the rectangle (D, E, G and H). For the symbols outside the rectangle, the BER is given by

$$\frac{1}{2}\left(\frac{1}{2}\mathrm{e}_1(1) - \frac{1}{2}\mathrm{e}_1(9)\right)\left(2-\frac{1}{2}\mathrm{e}_1(1)-\frac{1}{2}\mathrm{e}_1(3)-\frac{1}{2}\mathrm{e}_1(5)-\frac{1}{2}\mathrm{e}_1(7)\right). \tag{6}$$

For the symbols inside the rectangle, the average error expression is given by

$$\begin{aligned}&\frac{1}{4}\left(\frac{1}{2}\mathrm{e}_1(1)+\frac{1}{2}\mathrm{e}_1(3)+\frac{1}{2}\mathrm{e}_1(5)+\frac{1}{2}\mathrm{e}_1(7)\right)\\ \times\ &\left(4-\frac{1}{2}\mathrm{e}_1(1)-\frac{1}{2}\mathrm{e}_1(3)-\frac{1}{2}\mathrm{e}_1(5)-\frac{1}{2}\mathrm{e}_1(7)\right).\end{aligned} \tag{7}$$

The expression for average BER for bit i_2, $P_b(32, i_2)$ is therefore given by

$$\begin{aligned}&P_b(32, i_2)\\ =\ &\frac{1}{4}\left(\frac{1}{2}\mathrm{e}_1(1) - \frac{1}{2}\mathrm{e}_1(9)\right)\left(2-\frac{1}{2}\mathrm{e}_1(1)-\frac{1}{2}\mathrm{e}_1(3)-\frac{1}{2}\mathrm{e}_1(5)-\frac{1}{2}\mathrm{e}_1(7)\right)\\ +\ &\frac{1}{8}\left(\frac{1}{2}\mathrm{e}_1(1)+\frac{1}{2}\mathrm{e}_1(3)+\frac{1}{2}\mathrm{e}_1(5)+\frac{1}{2}\mathrm{e}_1(7)\right)\\ \times\ &\left(4-\frac{1}{2}\mathrm{e}_1(1)-\frac{1}{2}\mathrm{e}_1(3)-\frac{1}{2}\mathrm{e}_1(5)-\frac{1}{2}\mathrm{e}_1(7)\right).\end{aligned} \tag{8}$$

C. Bit q_2

Using the procedures similar to those used for bits i_1, q_1 and i_2, we divide the constellation points into three parts: The row closest to the x-axis (consisting of symbols F, G and H), the next row (consisting of symbols D, E and F) and the top row (consisting of symbols A and B).

For the first part (symbols F, G and H), the average BER is given by

$$\frac{1}{2}\mathrm{e}_1(1)-\frac{1}{2}\mathrm{e}_1(5)+\frac{1}{3}\left(\frac{1}{2}\mathrm{e}_1(3)+\frac{1}{2}\mathrm{e}_1(5)\right)\left(2+\frac{1}{2}\mathrm{e}_1(7d)+\frac{1}{2}\mathrm{e}_1(5)\right). \tag{9}$$

For the second part (symbols D, E and F), the average BER is given by

$$\frac{1}{2}\mathrm{e}_1(1)-\frac{1}{2}\mathrm{e}_1(5)+\frac{1}{3}\left(\frac{1}{2}\mathrm{e}_1(1)+\frac{1}{2}\mathrm{e}_1(7)\right)\left(1-\frac{1}{2}\mathrm{e}_1(7)-\frac{1}{2}\mathrm{e}_1(5)\right). \tag{10}$$

For the third part (symbols A and B), the average BER is given by

$$\begin{aligned}&\frac{1}{2}\left(\frac{1}{2}\mathrm{e}_1(1) - \frac{1}{2}\mathrm{e}_1(9)\right)\\ +\ &\frac{1}{2}\left(\frac{1}{2}2\mathrm{e}_1(1)+\frac{1}{2}\mathrm{e}_1(3)-\frac{1}{2}\mathrm{e}_1(5)\right)\left(1-\frac{1}{2}\mathrm{e}_1(1)+\frac{1}{2}\mathrm{e}_1(9)\right).\end{aligned} \tag{11}$$

Putting the three parts together, we can write $P_b(32, q_2)$ as

$$\begin{aligned}&P_b(32, q_2)\\ =\ &\frac{1}{8}\Big[3\mathrm{e}_1(1) - 3\mathrm{e}_1(5) + \left(\frac{1}{2}\mathrm{e}_1(3) + \frac{1}{2}\mathrm{e}_1(5)\right)\left(2 + \frac{1}{2}\mathrm{e}_1(7) + \frac{1}{2}\mathrm{e}_1(5)\right)\\ +\ &\left(\frac{1}{2}\mathrm{e}_1(1) + \frac{1}{2}\mathrm{e}_1(7)\right)\left(1 - \frac{1}{2}\mathrm{e}_1(7) - \frac{1}{2}\mathrm{e}_1(5)\right)\\ +\ &\left(\frac{1}{2}\mathrm{e}_1(1)-\frac{1}{2}\mathrm{e}_1(9)\right)+\left(\mathrm{e}_1(1)+\frac{1}{2}\mathrm{e}_1(3)-\frac{1}{2}\mathrm{e}_1(5)\right)\left(1-\frac{1}{2}\mathrm{e}_1(1)+\frac{1}{2}\mathrm{e}_1(9)\right)\end{aligned} \tag{12}$$

D. Bit i_3

For bit i_3, the decision regions are slightly more complicated as shown in Fig. 3. The main difference here is to find the probability that a symbol might fall on the wrong side of the 45° lines. Consider the line $x - y = 0$. Symbol "H" with coordinates $(-d, d)$ is on the left side of this line, which basically is the side where $x - y < 0$. If n_x and n_y represent independent and identically distributed (iid) AWGN values in the x and y directions, then one of the error regions, say region X_2, for bit i_3 in symbol "H" is defined by the region, $n_y > 3d$ and $-d + n_x - d - n_y > 0$. This is the same as $n_y > 3d$ and $n_x - n_y > 2d$. Since n_x and n_y are zero mean iid Gaussian random variables, $n_x - n_y$ is also Gaussian with zero mean and a variance twice that of n_x and n_y. Hence, the probability that symbol "H" falls in the error region X_2 is given by

$$\frac{1}{2}\mathrm{e}_1(3)\frac{1}{2}\mathrm{e}_2(2). \tag{13}$$

The average probability of error $P_b(32, i_3)$ can therefore be written as

$$\begin{aligned}&P_b(32, i_3)\\ =&\frac{1}{8}\Big[\left(\frac{1}{2}\mathrm{e}_1(1) - \frac{1}{2}\mathrm{e}_1(9)\right)\left(1 + \frac{1}{2}\mathrm{e}_1(3) + \frac{1}{2}\mathrm{e}_1(5)\right)\\ &+(\mathrm{e}_1(1)+\mathrm{e}_1(3)+\mathrm{e}_1(5)+\mathrm{e}_1(7))\\ &+\left(\mathrm{e}_1(1)+\mathrm{e}_1(3)-\frac{1}{2}\mathrm{e}_1(5)-\frac{1}{2}\mathrm{e}_1(7)\right)\left(2-\frac{1}{2}\mathrm{e}_1(1)-\frac{1}{2}\mathrm{e}_1(3)-\frac{1}{2}\mathrm{e}_1(5)-\frac{1}{2}\mathrm{e}_1(7)\right)\Big]\\ &+\frac{1}{8}\Big[\left(1 - \frac{1}{2}\mathrm{e}_1(1)\right)\left(\frac{1}{2}\mathrm{e}_2(2)+\frac{1}{2}\mathrm{e}_2(4)+\frac{1}{2}\mathrm{e}_2(6)+\frac{1}{2}\mathrm{e}_2(8)\right)\\ &+\frac{1}{2}\mathrm{e}_1(9)\left(4-\frac{1}{2}\mathrm{e}_2(2)-\frac{1}{2}\mathrm{e}_2(4)-\frac{1}{2}\mathrm{e}_2(6)+\frac{1}{2}\mathrm{e}_2(8)\right)\\ &+\frac{1}{2}\mathrm{e}_1(1)\left(-\frac{3}{2}+\mathrm{e}_2(2)+\frac{1}{2}\mathrm{e}_2(4)-\frac{1}{2}\mathrm{e}_2(6)-\frac{1}{2}\mathrm{e}_2(8)\right)\\ &+\frac{1}{2}\mathrm{e}_1(7)\left(-\frac{1}{2}-\mathrm{e}_2(2)-\frac{1}{2}\mathrm{e}_2(4)+\frac{1}{2}\mathrm{e}_2(6)+\frac{1}{2}\mathrm{e}_2(8)\right)\\ &+\frac{1}{2}\mathrm{e}_1(3)\left(-\frac{3}{2}+\mathrm{e}_2(2)-\frac{1}{2}\mathrm{e}_2(6)\right) + \frac{1}{2}\mathrm{e}_1(5)\left(-\frac{1}{2}-\mathrm{e}_2(2)+\frac{1}{2}\mathrm{e}_2(6)\right)\Big]\end{aligned} \tag{14}$$

The first part of the sum contains the contribution of rectangular error regions to the BER and the second part is the contribution of regions X_1, X_2, X_3 and X_4.

E. Average BER

Using the exact BER of bit i_1, q_1, i_2, q_2 and i_3 given in (4), (5), (8), (12) and (14), the exact average BER for the 32 cross QAM constellation, $P_b(32)$ can be written as

$$\begin{aligned} P_b(32) &= \frac{1}{5}\left(P_b(32,i_1)+P_b(32,q_1)+P_b(32,i_2)\right. \\ &+ \left. P_b(32,q_2)+P_b(32,i_3)\right). \end{aligned} \tag{15}$$

V. Extension to Rayleigh Fading

Note that all the BER expressions are linear combinations of terms of the form $F(\alpha,\beta;\gamma)=\mathrm{erfc}(\sqrt{\alpha\gamma})\mathrm{erfc}(\sqrt{\beta\gamma})$. Terms that are not products of erfc$(\cdot)$ can be seen as a special case by setting $\beta=0$. To obtain the exact average BER over Rayleigh fading channels, we need to average these terms over the probability density function of the CNR [3, Chapter 2], which is defined by

$$p_\gamma(\gamma)=\frac{1}{\bar{\gamma}}e^{-\frac{\gamma}{\bar{\gamma}}}, \gamma>0. \tag{16}$$

The resultant BER over fading channels is going to be the same linear combination, but of function $\bar{F}(\alpha,\beta;\bar{\gamma})$ in the place of $F(\alpha,\beta;\gamma)$. $\bar{F}(\alpha,\beta;\bar{\gamma})$ can be shown to be given by

$$\begin{aligned} \bar{F}(\alpha,\beta;\bar{\gamma}) &= 1-\frac{2}{\pi}\sqrt{\frac{\alpha\bar{\gamma}}{1+\alpha\bar{\gamma}}}\tan^{-1}\left(\sqrt{\frac{\alpha(1+\alpha\bar{\gamma})}{\beta\alpha\bar{\gamma}}}\right) \\ &- \frac{2}{\pi}\sqrt{\frac{\beta\bar{\gamma}}{1+\beta\bar{\gamma}}}\tan^{-1}\left(\sqrt{\frac{\beta(1+\beta\bar{\gamma})}{\alpha\beta\bar{\gamma}}}\right). \end{aligned} \tag{17}$$

When $\beta=0$, $\bar{F}(\alpha,\beta;\bar{\gamma})$ has to be evaluated in the limit, giving

$$\bar{F}(\alpha,0;\bar{\gamma})=1-\sqrt{\frac{\alpha\bar{\gamma}}{1+\alpha\bar{\gamma}}} \tag{18}$$

VI. Numerical Examples

As a double check, we have verified our exact BER expressions for all the 5 bits through Monte Carlo simulations. Fig. 4 shows the excellent agreement between these equations and the simulation results. Fig. 5 compares the exact average BER for the cross 32-QAM constellation with (i) the exact average BER for the rectangular 32-QAM and (ii) the approximate BER given in [1, Eq. (7)] which we reproduce below for reader convenience:

$$P_b(M)\simeq G_p\frac{N}{\log_2 M}\frac{1}{2}\mathrm{erfc}\left(\frac{d}{\sqrt{N_0}}\right). \tag{19}$$

In this equation, G_p is the Gray penalty (7/6 for cross 32-QAM), N is the average number of nearest neighbors for a symbol in the constellation (3.25[1] for cross 32-QAM). Note that we gain around 1 dB at a BER of 10^{-2} by using cross QAM constellations instead of rectangular ones. This SNR gain remains almost the same as the SNR increases. Also note the small difference between the approximate bound and the exact BER, especially when the SNR is below 0 dB. We believe that this difference will grow as we move to larger constellations (such as 128 cross QAMs), which is the subject of our ongoing efforts. Also, the discrepancy between the exact and approximate expressions increases once we start considering channel fading. For example, Fig. 6 shows the performance of the cross 32-QAM constellation under Rayleigh fading. Note the relatively large difference between the exact and approximate expressions when the SNR per bit falls below 0dB.

[1] In [1], N appears erroneously as 3.38

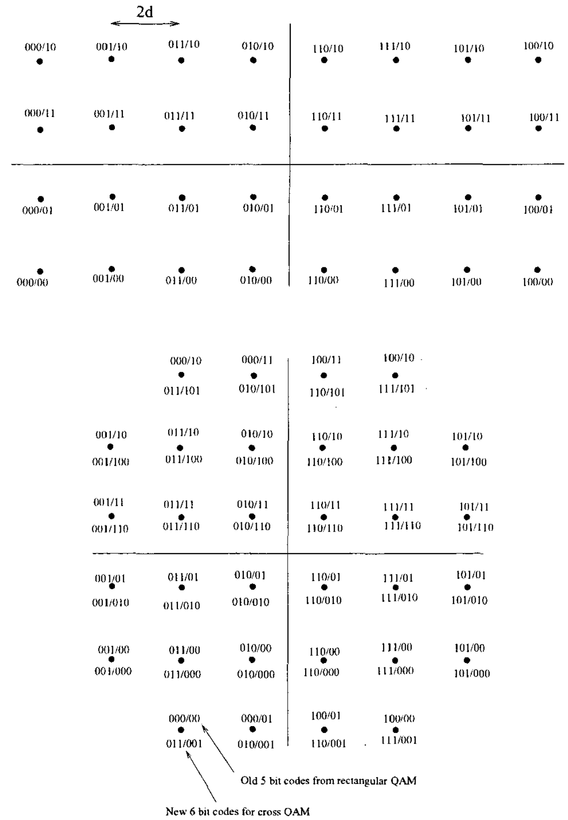

Fig. 1. Rectangular (above) and Cross 32-QAM (below) with Gray coding. The 5-bit Gray codes for rectangular and cross QAMs has been written in the $i_1i_2i_3/q_1q_2$ fashion. The 6-bit pseudo-code generation has also been shown for the cross QAM constellation in the $i_1i_2i_3/q_1q_2q_3$ fashion.

VII. Conclusion

In this paper, we have derived exact expressions for the average BER of a cross 32-QAM constellation. The expressions are a simple linear combination of erfc$(\cdot)$ and product of two erfc$(\cdot)$ functions and can easily be extended to the case when the channel experiences fading.

References

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[3] M. K. Simon and M. -S. Alouini, "Digital communications over fading channels: A unified approach to performance analysis." New York, NY: John Wiley & Sons, July 2000.

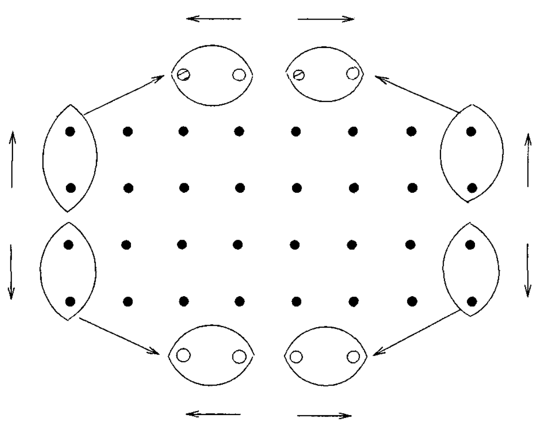

Fig. 2. Construction of a Cross 32-QAM from a rectangular 32-QAM. Note that the last two columns are reflected over to the top and bottom of the constellation as shown by the arrows.

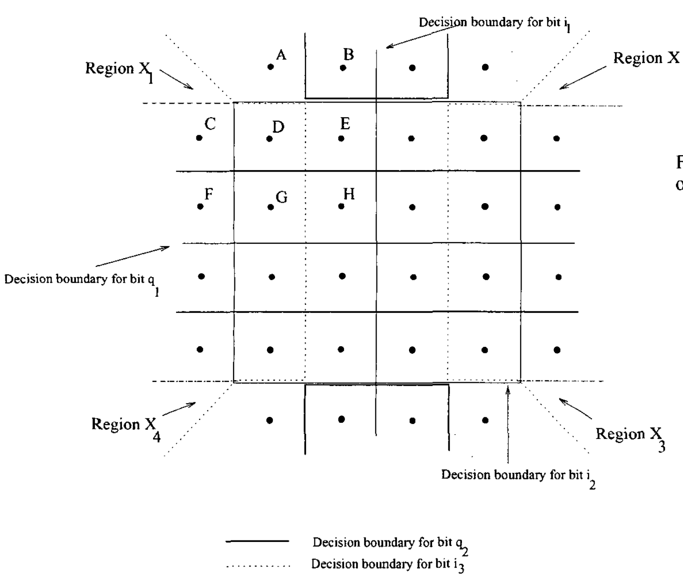

Fig. 3. Decision boundaries for the various bits in cross 32-QAM.

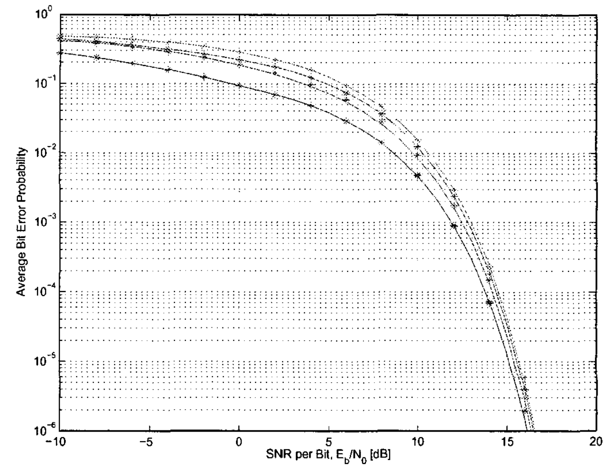

Fig. 4. Comparison of the analytical and simulation results for the average BER of various bits in cross 32-QAM.

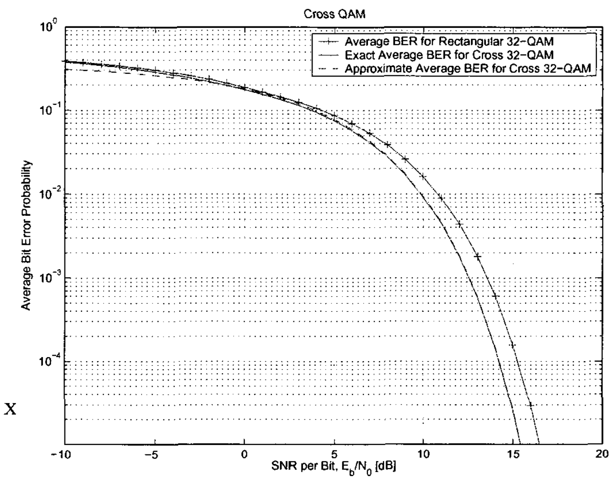

Fig. 5. Comparison of the average BER of rectangular 32-QAM with that of cross 32-QAM

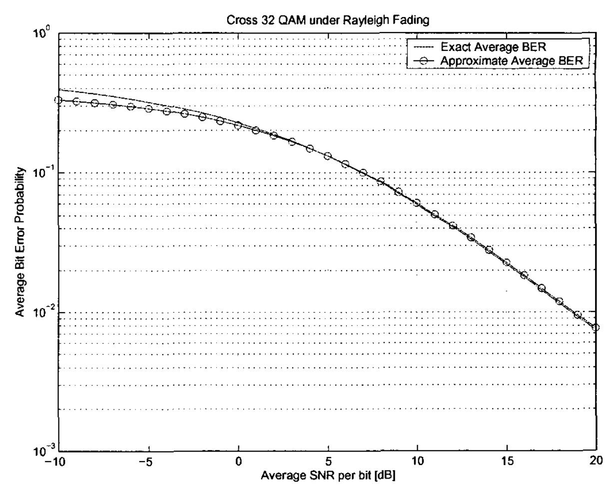

Fig. 6. Average BER of cross 32-QAM under Rayleigh Fading.