On the construction of groupoid C^*-algebras

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Abstract

This article serves as an invitation to the field of research that was introduced by Renault in [1]. Generalizing topological groups, topological groupoids describe continuously parametrizable symmetries which may fail to be global. A natural construction of a group C^*-algebra $C^*(G)$, which captures the information about unitary representations of a group G on Hilbert spaces and naturally generalizes Fourier transform, may be extended to the groupoid setting using the notion of a Haar system – we provide examples thereof.

1 Group C^*-algebras

Groupoid C^*-algebras generalize the construction of a group C^*-algebra, so we begin our discussion with a relaxed motivation – a good reference for the results stated here is [2]. In this section, G will denote a locally compact Hausdorff, second-countable topological group.

A *Haar measure* on the Borel σ-algebra $\mathcal{B}(G)$ of G is a measure $\mu\colon \mathcal{B}(G) \to [0,\infty]$ that satisfies the following properties:

(i) *Regularity.* For any open set $U \subset G$:

$$\mu(U) = \sup\{\mu(K) \mid K \subset U,\ K \text{ is compact}\},$$

for any Borel set $S \in \mathcal{B}(G)$:

$$\mu(S) = \inf\{\mu(U) \mid S \subset U,\ U \text{ is open}\},$$

and for any compact set $K \subset G$, $\mu(K)$ is finite. If this holds, we say μ is a *Radon measure.*

(ii) *Left-invariance*: $\mu(gS) = \mu(S)$ for any $g \in G$ and $S \in \mathcal{B}(G)$.

By *Haar's theorem*, such a measure always exists on G and is unique up to a positive multiplicative constant – this is proved by constructing a linear functional I on the set $C_c(G)$ of compactly supported functions on G, such that $I(f) \geq 0$ if $f \geq 0$, and $I(f \circ L_g) = I(f)$ for any $g \in G$, and then invoking Riesz' representation theorem.* A *Haar measure* gives rise in the usual way to the integral $\int_G\colon C_c(G) \to \mathbb{C}$ – we will use the notation

$$\int_G f(x)\,\mathrm{d}x$$

*Note that in the important case when G is a Lie group, this functional is easily constructed by integrating f over a left-invariant differential n-form, where $n = \dim G$.

and keep the μ implicit. Note that left-invariance of μ implies that the integral is also left-invariant, i.e.

$$\int_G f(gx)\,\mathrm{d}x = \int_G f(x)\,\mathrm{d}x$$

for any $f \in C_c(G)$ and $g \in G$. It is important to note that in general, the Haar integral isn't right-invariant, but rather that uniqueness (up to a positive multiplicative constant) of the Haar measure implies that there exists a continuous homomorphism $\Delta\colon G \to (0,\infty)$, called the *modular function*, such that

$$\int_G f(xg)\,\mathrm{d}x = \Delta(g)\int_G f(x)\,\mathrm{d}x$$

for any $g \in G$. It is easy to see that if G is compact, then $\Delta \equiv 1$, so that the Haar integral is also right invariant.†

Defining the norm on $C_c(G)$ as $\|f\|_1 = \int_G |f(x)|\,\mathrm{d}x$, the product as the convolution:

$$(f * h)(x) = \int_G f(y)h(y^{-1}x)\,\mathrm{d}y,$$

and involution as:

$$f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1}),$$

it is straightforward to see that the space $C_c(G)$ becomes a normed $*$-algebra (here, the property $\int_G f(x)\,\mathrm{d}x = \int_G f(x^{-1})\Delta(x^{-1})\,\mathrm{d}x$ is relevant). The completion of $C_c(G)$ with respect to this norm is $*$-isomorphic to $L^1(G)$ – the importance of this space is stressed in the next example.

Example 1.1 (Abelian groups and Fourier transform)**.** In the case when G is abelian, it is not hard to show that there is a bijection from the set $\widehat{G} := \mathrm{Hom}(G, S^1)$ (continuous homomorphisms, called *characters*) to the set $\widehat{L^1(G)}$ of nonzero multiplicative functionals on $L^1(G)$; it is given by

$$\omega \mapsto h_\omega, \text{ where } h_\omega(f) = \int_G f(x)\omega(x)\,\mathrm{d}x.$$

If we equip the domain with the compact-open topology (i.e. the topology of uniform convergence on compacts) and the codomain with the weak-$*$ topology, it is possible to show that this map is a homeomorphism ([2, Lemma 1.78]).

On the other hand, the Gelfand transform on a (non-unital) commutative Banach algebra A maps $a \in A$ to $\widehat{a} \in C_0(\widehat{A})$, $\widehat{a}(h) = h(a)$, where $\widehat{A}$ denotes the set of nonzero multiplicative functionals on A, and $C_0(\widehat{A})$ denotes the set of all continuous (in the weak-$*$ topology on $\widehat{A}$) functions on $\widehat{A}$ which vanish at infinity (i.e. the set $\{x \in \widehat{A} \mid |f(x)| \geq \varepsilon\}$ is compact for any $\varepsilon > 0$). Picking $A = L^1(G)$ and keeping in mind the identification from the previous paragraph, the Gelfand transform of $f \in L^1(G)$ is a function $\widehat{f} \in C_0(\widehat{G})$, given by

$$\widehat{f}(\omega) = \int_G f(x)\omega(x)\,\mathrm{d}x.$$

In the case $G = \mathbb{R}$, it's easy to see that $\widehat{G} = \{x \mapsto \mathrm{e}^{ikx} \mid k \in \mathbb{R}\}$ and $\widehat{\mathbb{R}}$ is homeomorphic to $\mathbb{R}$, so that the Gelfand transform $L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is just the well-known Fourier transform. $\Diamond$

† $G = \mathbb{R}^n$ provides an example of a non-compact group with a right-invariant Haar integral. On a matrix Lie group, it is possible to show that $\Delta(g) = \left|\det(\mathrm{Ad}(g^{-1}))\right|$, where $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is given by $\mathrm{Ad}(g)(X) = gXg^{-1}$.

We now use the norm $\|.\|_1$ to construct another norm on $C_c(G)$ which captures information about its representations on Hilbert spaces. Recall that a representation $\pi\colon C_c(G) \to B(H)$ is said to be *cyclic*, if there exists a vector $\xi \in H$ such that $\overline{\pi(C_c(G))\xi} = H$, and *norm-decreasing*, if $\|\pi(f)\| \leq \|f\|_1$ for all $f \in C_c(G)$.

Definition 1.2. The *universal C^*-norm* $\|.\|_u$ on $C_c(G)$ is given by

$$\|f\|_u = \sup\{\|\pi(f)\| \mid \pi\colon C_c(G) \to B(H) \text{ is a cyclic and norm-decreasing representation}\}.$$

Notice that the norm-decreasing condition in the definition above ensures that $\|f\|_u \leq \|f\|_1$ for any $f \in C_c(G)$, so that $\|.\|_u$ is well-defined (note that we cannot rely here on the fact that every $*$-homomorphism between C^*-algebras is bounded, because $C_c(G)$ is not complete in the L^1-norm). Since every representation is a direct sum of cyclic representations, we also have $\|\pi(f)\| \leq \|f\|_u$ for any norm-decreasing representation π; the cyclic condition in the definition ensures that the supremum runs over a set, because any cyclic representation is unitarily equivalent to a subrepresentation of one fixed Hilbert space.

The only nontrivial thing needed to be checked is positive definiteness, which is done using regular representations. It follows straight from the definition of $\|.\|_u$ that $(C_c(G), \|.\|_u)$ satisfies all other axioms of a normed $*$-algebra.

Definition 1.3. The *group C^*-algebra* of G is the completion of $C_c(G)$ with respect to the universal norm $\|.\|_u$. We denote this completion by $C^*(G)$.

The mathematical importance of $C^*(G)$ is in the following:

Proposition 1.4. There is a bijective correspondence between unitary representations of G and nondegenerate representations of $C^*(G)$

Here, a *unitary representation* of G is a (strongly) continuous homomorphism $u\colon G \to \mathrm{U}(H)$ into the set of unitary operators on a Hilbert space H, with the latter endowed with the strong operator topology; a *nondegenerate representation* $\pi\colon A \to B(H)$ of a C^*-algebra A is such that $\overline{\pi(A)H} = H$.

The idea of proof of this proposition rests upon the notion of a vector-valued integral, which we briefly explain. If $\mathcal{D}$ is a Banach space, we consider the set of all compactly supported continuous functions $C_c(G;\mathcal{D})$ – there exists a linear map $\int_G\colon C_c(G;\mathcal{D}) \to \mathcal{D}$, characterized by the property

$$\varphi\left(\int_G v(x)\,\mathrm{d}x\right) = \int_G \varphi(v(x))\,\mathrm{d}x \quad \text{for all } v \in C_c(G;\mathcal{D}) \text{ and } \varphi \in \mathcal{D}^*.$$

(See e.g. [2, Lemma 1.91].) The bijection whose existence is asserted in the above proposition, is then given by mapping $u\colon G \to \mathrm{U}(H)$ to $\pi_u\colon C_c(G) \to B(H)$, called the *integrated form* of u, which is given by

$$\pi_u(f) = \int_G f(x)u_x\,\mathrm{d}x,$$

where $u_x = u \circ L_x \in C_c(G; B(H))$. For details of the proof, see [3, Section C.3]).

To conclude this motivating section, we note that in the case when G is abelian, $C^*(G)$ can be shown to be $*$-isomorphic to $C_0(\widehat{G})$, with the isomorphism given as the extension of the Gelfand transform $\widehat{\ }\colon C_c(G) \to C_0(\widehat{G})$ to the completion $C^*(G)$ (see [3, Example C.20]).

2 Topological groupoids

Definition 2.1. A *groupoid* is a small category (i.e. the classes of its objects and morphisms are sets) where every morphism is invertible. More precisely, a groupoid consists of:

(i) a set G of morphisms and a set M of objects,

(ii) two maps $s, t\colon G \to M$, called the *source* and *target* maps, which prescribe to any morphism its domain and codomain (respectively),

(iii) a *unit* map $1\colon M \to G$, which assigns to every object $x \in M$ the identity morphism $1_x = \mathrm{id}_x$, and an *inversion* map $\mathrm{inv}\colon G \to G$, which assigns to any morphism $g \in G$ its inverse g^{-1},

(iv) a *partial multiplication* map $m\colon G*G \to G$, where $G*G = \{(g,h) \in G\times G \mid s(g) = t(h)\}$ is the set of pairs of *composable* morphisms, which sends $(g,h) \in G*G$ to their composition gh,

so that the following properties hold for any $g, h, k \in G$ and $x \in M$:

(i) $s(hg) = s(g)$ and $t(hg) = t(h)$,

(ii) $s(1_x) = t(1_x) = x$ and $g1_{s(g)} = 1_{t(g)} = g$,

(iii) for any $g \in G$ there exists a unique $g^{-1} \in G$ such that $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$, $g^{-1}g = 1_{s(g)}$ and $gg^{-1} = 1_{t(g)}$,

(iv) $k(hg) = (kh)g$ whenever $s(k) = t(h)$ and $s(h) = t(g)$.

Given $x \in M$, we also define *s-fibre* over x as $G_x := s^{-1}(x)$, *t-fibre* over x as $G^x := t^{-1}(x)$, and the *vertex group* at x as $G_x^x := G_x \cap G^x$. Notice that the set of all morphisms from x to y is just $G_x^y := G_x \cap G^y$. We will sometimes denote a morphism $g \in G_x^y$ by $g\colon x \to y$.

A *topological groupoid* is a groupoid with G and M topological spaces, such that the maps $s, t, 1, \mathrm{inv}$ and m (with the relative topology on $G*G \subset G\times G$) are continuous – we say that the topology on the groupoid is *compatible* with the groupoid structure.

Remark 2.2. We write $G \rightrightarrows M$ for a groupoid, to mean the whole structure:

$$\begin{tikzcd} G*G \arrow[r, "m"] & G \arrow[loop above, "\mathrm{inv}"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & M \arrow[r, "1"] & G \end{tikzcd}$$

Furthermore, in a Hausdorff topological groupoid, we may view M as a closed subspace of G:

Lemma 2.3. Let $G \rightrightarrows M$ be a topological groupoid. Then $1\colon M \to G$ is a closed embedding if and only if G is Hausdorff.

Proof. Recall that a topological space is Hausdorff if and only if any converging net has a unique limit point. If G is Hausdorff, suppose that $(1_{x_\alpha})_{\alpha\in\Lambda}$ is a net in 1_M converging to $g \in G$. By continuity of s, we have $x_\alpha = s(1_{x_\alpha}) \xrightarrow{\alpha} s(g)$, and since G is Hausdorff, the limit point g is unique, hence $g = 1_{s(g)}$. Conversely, suppose that $(g_\alpha)_{\alpha\in\Lambda}$ is a converging net in G, with limit points h and k. By continuity of operations, $g_\alpha^{-1}g_\alpha \xrightarrow{\alpha} h^{-1}k$, and since $g_\alpha^{-1}g_\alpha = 1_{s(g_\alpha)}$ and 1_M is closed by assumption, this implies $h^{-1}k \in 1_M$, hence $h = k$. ■

Note that if G is Hausdorff, then by previous lemma, so is M. This implies also that $G*G$ is closed, as it is an incidence set of two continuous maps.

Example 2.4.

(i) *Base groupoid.* Let M be a topological space. Defining $G = \{1_x \mid x \in M\}$ and endowing it with the obvious groupoid structure and topology inherited from M, we obtain a topological groupoid.

(ii) *Trivial groupoid.* Let M be a topological space and G a topological group (G may not be acting on M). We define the trivial groupoid $M \times G \times M \rightrightarrows M$ with the product topology, and the groupoid structure given by:

- s and t are projections to the third and first factor (resp.),
- the unit map is given by $1_x = (x, e, x)$ and inverse map by $(y, g, x)^{-1} = (x, g^{-1}, y)$,
- the multiplication is given by $(z, h, y)(y, g, x) = (z, hg, x)$.

It is straightforward to check that the groupoid axioms are satisfied, and that the topology is compatible with the groupoid structure. Note that in the case when $M = \{*\}$ is a singleton, we can identify $\{*\} \times G \times \{*\} \rightrightarrows \{*\}$ with the group G, so that groupoids are a generalization of groups. In case $G = \{e\}$ is a trivial group, we call the obtained groupoid a *pair groupoid* and just write $M \times M \rightrightarrows M$.

(iii) *Action groupoid.* Let M be a topological space and G a topological group acting on it (from the left). We define a topological groupoid $G \times M \rightrightarrows M$ by endowing it with the product topology, and the compatible groupoid structure given by:

- $s(g, x) = x$, $t(g, x) = gx$,
- the unit map is given by $1_x = (e, x)$ and inverse map by $(g, x)^{-1} = (g^{-1}, gx)$,
- the multiplication is given by $(g_2, g_1x)(g_1, x) = (g_2g_1, x)$.

We will write $G \ltimes M$ to mean the obtained topological groupoid. It is easy to see that the s-fibres are $(G \ltimes M)_x = G \times \{x\}$, that the t-fibre of $G \ltimes M$ at x may be identified with the orbit $\mathrm{Orb}_G(x)$, and that the vertex group $(G \ltimes M)_x^x = \{(g, x) \mid g^{-1}x = x\}$ at x may be identified with the stabilizer group of x.

As a concrete example, the action of $\mathbb{R}$ on S^1 given by $(t, z) \mapsto \mathrm{e}^{2\pi it}z$, provides us with a groupoid structure on the cylinder, with the s-fibre at z given by a vertical line $(\mathbb{R} \ltimes S^1)_z = \mathbb{R} \times \{z\}$, the t-fibre at z given by $(\mathbb{R} \ltimes S^1)^z = \{(t, w) \mid \mathrm{e}^{2\pi it}w = z\}$ (visually depicted as a "spiral" on the cylinder), and $(\mathbb{R} \ltimes S^1)_z^z = \mathbb{Z} \times \{z\}$.

(iv) *The fundamental groupoid.* Let M be a connected topological manifold. A *relative homotopy* $H\colon [0, 1]^2 \to M$ between two paths $\gamma, \delta\colon [0, 1] \to M$ is a homotopy that preserves the basepoints, i.e. $H(\cdot, 0) = \gamma, H(\cdot, 1) = \delta$ and $H(0, \cdot) = \gamma(0) = \delta(0), H(1, \cdot) = \gamma(1) = \delta(1)$. Define $\Pi(M)$ as the set of all equivalence classes of relatively homotopic paths in M, and the groupoid structure on $\Pi(M) \rightrightarrows M$ as

- $s[\gamma] = \gamma(0)$, $t[\gamma] = \gamma(1)$,
- $1_x = [c_x]$, where c_x is the constant path to x, and $[\gamma]^{-1} = [\bar{\gamma}]$, where $[\bar{\gamma}]$ is the inverse path to γ,
- the multiplication is given by concatenation of paths, i.e. $[\delta][\gamma] = [\gamma * \delta]$.

As an interesting fact, we note that the topology on $\Pi(M)$ is given in the following way. If $p\colon \widetilde{M} \to M$ is a universal cover of M and $\mathrm{Aut}(\pi)$ denotes the deck transformation group, then the orbit space $(\widetilde{M} \times \widetilde{M})/\mathrm{Aut}(p)$ of the diagonal action[‡] can be shown to be in bijection with $\Pi(M)$, so we endow $\Pi(M)$ with the topology of the orbit space.

[‡]The diagonal action $\mathrm{Aut}(p) \times (\widetilde{M} \times \widetilde{M}) \to \widetilde{M} \times \widetilde{M}$ is given by $(\phi, (e_1, e_2)) = (\phi(e_1), \phi(e_2))$.

Furthermore, it follows from the theory of covering spaces that the s-fibre $\Pi(M)_x$ at x is homeomorphic to the universal cover of M with base point x, and that the vertex group $\Pi(M)_x^x$ is the fundamental group $\pi_1(M, x)$ of M (with base point x).

There are many more natural and interesting examples which arise from the theory of fibre bundles (e.g. the gauge groupoid or the frame groupoid), but we will omit them here; furthermore, it is possible to impose a differentiable structure on a groupoid $G \rightrightarrows M$, in which case the groupoid is called a *Lie groupoid* – most interesting examples of groupoids actually arise in the smooth category.

3 Haar systems

Throughout this section and in later sections, $G \rightrightarrows M$ will be a second countable, locally compact Hausdorff groupoid. In order to construct a convolution algebra on $C_c(G)$, we need an analogue of the Haar measure for locally compact groupoids. There are several possible definitions of such an analogue; we will use the following one.

Definition 3.1. A *left-invariant Haar system* on $G \rightrightarrows M$ is a family $\mu = \{\mu^x \mid x \in M\}$ of Radon measures on G, such that:

(i) *Full support.* For any $x \in M$, the support $\operatorname{supp}(\mu^x) := \{g \in G \mid \mu^x(U) > 0 \text{ for any nbd. } U \text{ of } g\}$ of μ^x equals the t-fibre G^x, i.e. $\operatorname{supp}(\mu^x) = G^x$.

(ii) *Continuity.* For any $f \in C_c(G)$, the map $\mu(f)\colon M \to \mathbb{C}$, given by $\mu(f)(x) = \int_G f(g)\,\mathrm{d}\mu^x(g)$, is continuous.

(iii) *Left-invariance.* For any Borel function $f\colon G \to \mathbb{C}$ and $g \in G$, there holds

$$\int_G f(h)\,\mathrm{d}\mu^{t(g)}(h) = \int_G f(gh)\,\mathrm{d}\mu^{s(g)}(h).$$

Remark 3.2. The continuity condition (ii) ensures that we have a map $\mu\colon C_c(G) \to C_c(M)$. Together with the condition (i), this means that μ may be seen as *integration on the t-fibres.* The product gh, which appears in condition (iii), is well-defined – indeed, since $\mu^{s(g)}$ is supported only on $t^{-1}(s(g))$, the integral runs over all $h \in t^{-1}(s(g))$, hence $t(h) = s(g)$. Furthermore, the left-invariance property (iii) is clearly equivalent to $\mu^y(S) = \mu^x(gS)$ for any $g\colon x \to y$ and $S \in \mathcal{B}(G)$, where we have written $gS = \{gh \mid t(h) = s(g), h \in S\}$

Remark 3.3 (Right-invariance)**.** Similarly as with groups where every left-invariant Haar measure μ gives rise to a right-invariant Haar measure (given by $\mu_{-1}(S) = \mu(S^{-1})$ for any $S \in \mathcal{B}(G)$), we can associate to every left-invariant Haar system on $G \rightrightarrows M$ a set of Radon measures $\{\mu_x \mid x \in M\}$ on G, by defining $\mu_x(S) = \mu^x(S^{-1})$, for all $S \in \mathcal{B}(G)$. It is easy to see that we then have $\operatorname{supp}(\mu_x) = G_x$ for all $x \in G$ (i.e. the set of measures is supported on the s-fibres), and

$$\int_G f(h)\,\mathrm{d}\mu_{s(g)}(h) = \int_G f(hg)\,\mathrm{d}\mu_{t(g)}(h)$$

for any $g \in G$. The continuity condition also holds, which comes as a consequence of the easily verified *pushforward formula*:

$$\int_G f\,\mathrm{d}\mu^x = \int_G (f \circ \mathrm{inv})\,\mathrm{d}\mu_x.$$

Such a system of measures on $G \rightrightarrows M$ is called a *right-invariant Haar system.* To sum up, left-invariant and right-invariant Haar systems are in a bijection.

Example 3.4.

(i) *Groups.* If G is a locally compact Hausdorff group, then as a groupoid, G admits a Haar system $\{\mu\}$, where μ is a Haar measure on the group G (cf. section 1).

(ii) *Action groupoids.* If G is a locally compact Hausdorff group, acting on a locally compact Hausdorff space M from the left, then $G \times M \rightrightarrows M$ admits a right-invariant Haar system $\{\mu \times \delta_x \mid x \in M\}$, where μ is a right-invariant Haar measure on G. It's easy to see that for every $f \in C_c(G \times M)$, $\mu(f)(x) = \int_G f(h,x)\,\mathrm{d}\mu(h)$ and that right-invariance of this Haar system is just $\int_G f(h,gx)\,\mathrm{d}\mu(h) = \int_G f(hg,x)\,\mathrm{d}\mu(h)$, for every $g \in G$.

(iii) *Pair groupoids.* Given a Radon measure μ on a locally compact Hausdorff space M with full support (i.e. $\operatorname{supp}(\mu) = X$), we have a right-invariant Haar system $\{\mu \times \delta_x \mid x \in M\}$ on the pair groupoid $M \times M \rightrightarrows M$.

Unlike with Haar measures on locally compact Hausdorff groups, a Haar system on a groupoid may not exist, and in case it does, it may not be unique (non-uniqueness is clear from example (iii) above). The following lemma gives a necessary condition for the existence of a Haar system.

Lemma 3.5. If a Haar system exists on $G \rightrightarrows M$, then the source and target maps are open.

Proof. Let U be an open subset of G and let $x \in t(U)$ and pick $g \in U$ with $t(g) = x$. Since G is locally compact Hausdorff, we may pick a nonnegative $f \in C_c(G)$ with compact support $\operatorname{supp}(f) \subset U$ and $f(g) = 1$, by Urysohn's lemma. Since $\operatorname{supp}(\mu^x) = G^x$, we have $\mu(f)(x) > 0$. But $\mu(f)$ is continuous and $\operatorname{supp}\mu(f) \subset t(U)$, so x is an interior point of U. Since $x \in t(U)$ was arbitrary, $t(U)$ is open. The same argument works for the right-invariant Haar system and the openness of source map. ■

Openness of the source and target maps are not a very exciting feature of a topological groupoid – in fact, when dealing with Lie groupoids, we always assume that the source and target maps are submersions, so that their fibres and the set $M * M$ are embedded submanifolds of G (otherwise we cannot make sense of smoothness of partial multiplication).

It turns out (see [4]) that the sufficient condition for existence of a Haar system is that $G \rightrightarrows M$ is *local transitivity* (in addition to being locally compact Hausdorff), i.e. the map $t|_{G_x}$ is open for any $x \in X$. Instead of pursuing a proof of this statement, we will now focus on a special kind of groupoids.

4 Étale groupoids

We will now focus on topological groupoids that admit particularly well-behaved Haar systems. These are groupoids with the space of morphisms and the space of objects having "the same dimension". Of course, since we're not in the category of manifolds, we mean the following.

Definition 4.1. A topological groupoid $G \rightrightarrows M$ is *étale*, if the source and target maps are local homeomorphisms.

Remark 4.2. The word étale means *spread-out* or *flat* in French.

It is easy to see that as soon as one of the source or target maps is a local homeomorphism, so is the other; just recall the relation $s = t \circ \mathrm{inv}$ and note that inv is a homeomorphism. Furthermore, if $G \rightrightarrows M$ is étale, it follows immediately that s and t are open maps. The next proposition shows that any étale groupoid admits a Haar measure.

Proposition 4.3. For a second-countable, locally compact Hausdorff groupoid, the following are equivalent.

(i) $G \rightrightarrows M$ is étale.

(ii) The partial multiplication map $m \colon G * G \to G$ is a local homeomorphism.

(iii) G has a basis of open bisections.

(iv) $1_M \subset G$ is open and $G \rightrightarrows M$ admits a Haar system.

Remark 4.4. A subset $A \subset G$ is said to be a *bisection*, if $s|_A$ and $t|_A$ are injective (i.e. no two morphisms in A have the same source or target). It is an easy exercise to show that $A \subset G$ is a bisection if and only if $AA^{-1}, A^{-1}A \subset 1_M$, where we have defined

$$AB = \{ab \mid s(a) = t(b), a \in A, b \in B\}.$$

Proof. (i) $\to$ (ii): If $(g, h) \in G * G$, by assumption there exist compact neighborhoods U of g and V of h, such that $s|_U$ and $t|_V$ are homeomorphisms onto their images. Then $U * V := (G * G) \cap (U \times V)$ is a compact neighborhood of (g, h) in $G * G$, on which the restriction of m is injective. Indeed, if $g_1h_1 = g_2h_2$, applying the map t to the both sides of this identity yields $g_1 = g_2$, and applying s yields $h_1 = h_2$. This restriction is closed as a map from a compact to a Hausdorff space, hence a homeomorphism.

(ii) $\to$ (iii): If U is a neighborhood of a given point $g \in G$, by assumption there exist open neighborhoods $V \subset U$ of g and $W \subset U^{-1}$ of g^{-1}, such that the restriction of m to $V * W$ is injective. Then $V \cap W^{-1} \subset U$ is the wanted neighborhood of g.

(iii) $\to$ (iv): To show that $1_M \subset G$ is open, note that for any 1_x, by assumption there exists an open bisection S that is a neighborhood of 1_x with $SS^{-1} \subset 1_M$. The set SS^{-1} is open in G (this is true because clearly (iii) implies (i), and we have already proved that (i) implies (ii)).

Since 1_M is open, the spaces G_x and G^x are discrete (and thus by second countability, countable), for any $x \in M$. Indeed, observe that any $g \colon x \to y$ defines a homeomorphism $G^x \to G^y$, given by $h \mapsto gh$, and that in the induced topology on G^x, the subset $\{1_x\} = 1_M \cap G^x \subset G^x$ is open. This implies that the singleton $\{g\}$ is open in G^y as the image of $\{1_x\}$ via the mentioned homeomorphism. Similarly, $g \colon x \to y$ defines a homeomorphism $G_y \to G_x$, given by $h \mapsto hg$, and $\{1_x\}$ is open in G_x, so G_x is discrete.

To construct the Haar system, we define μ^x as the counting measure on G^x, for any $x \in M$. Using a partition of unity subordinate to a countable open cover by bisections of G, we may write a given $f \in C_c(G)$ as a finite sum of functions supported on bisections. Therefore it is enough to consider functions $f \in C_c(G)$ which are supported in an open bisection S. For such a function, we have that for any $x \in M$ with $f|_{G^x} \neq 0$, there holds

$$\mu(f)(x) = \sum_{h \in S \cap G^x} f(h) = f(h_x),$$

where $h_x \in S \cap G^x$ denotes the unique element at which $f|_{S \cap G^x}$ is nonzero. In other words, there holds $\mu(f) \circ t = f$, and by using the fact that t is a local homeomorphism, this proves continuity of f (note that we are again using that (iii) implies (i)). Finally, to show left-invariance of this Haar system, just note again that if $g \colon x \to y$ is a morphism, it induces a bijection $G^x \to G^y$, $h \mapsto gh$, and so

$$\sum_{h \in G^y} f(h) = \sum_{h \in G^x} f(gh).$$

(iv) $\to$ (i): We may assume that for any $x \in M$, the measure μ^x in the given Haar system is a counting measure on G^x. Indeed, since 1_M is open, G^x is discrete and since $\mathrm{supp}(\mu^x) = G^x$, every point in G^x has a positive μ^x-measure, so we can define a positive (continuous) function $\alpha = \mu(\chi_{1_M})$ where χ_{1_M} is the characteristic function on 1_M. This means that the measure $\nu^x := \alpha(x)^{-1}\mu^x$ has the property $\nu^x(\{1_x\}) = 1$, and thus by left-invariance $\nu^x(\{g\}) = 1$, for any $g \in G^x$.

Let's prove that r is locally injective (we will do so similarly as in Lemma 3.5, which showed that r is open). Let $g \in G$ and $x = t(g)$. Since G is locally compact Hausdorff, we can find a compact neighborhood U of g which intersects G^x only in g. Then $\nu^x(U) = 1$ and by continuity of the Haar system, we may assume (by shrinking U if necessary) that $\nu^y(U) = 1$ for any $y \in t(U)$, which shows that $r|_U$ is injective. The proof for the map s is similar. ■

Remark 4.5. Topological groupoids with the property that 1_M is open in G are sometimes called r-*discrete*. The implication (iv) $\to$ (i) in the proof above shows that on such a groupoid, the s- and t-fibres are discrete, and that *if* a Haar system exists, it is uniquely determined up to a positive multiplicative function on the space M of objects, and may hence be chosen as the counting measure on each G^x. As we have shown, the only feature that distinguishes r-discrete from etale groupoids, is precisely the existence of a Haar measure.

Example 4.6.

(i) The base groupoid of a locally compact Hausdorff space is étale.

(ii) A discrete group is an étale groupoid over a one point space.

(iii) If a discrete group G acts on a locally compact Hausdorff space M, the action groupoid $G \times M \rightrightarrows M$ is étale.

There are other important examples of étale groupoids, including the groupoid of local bisections of $G \rightrightarrows M$, the Haeflinger groupoid, and the étale monodromy groupoid (see [5, p. 114 and 134] for more details).

5 Groupoid C^*-algebras

To produce a C^*-algebra from a second-countable, locally compact Hausdorff groupoid $G \rightrightarrows M$ with a Haar system μ, we must give $C_c(G)$ a $*$-algebra structure. Although this structure is dependent upon the chosen Haar measure μ, we will just write $C_c(G)$ instead of e.g. $C_c(G, \mu)$. Naturally, we define for $f_1, f_2 \in C_c(G)$ their convolution by[§]

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g)\, \mathrm{d}\mu^{t(g)}(h), \tag{1}$$

and for any $f \in C_c(G)$, its involution by

$$f^*(g) = \overline{f(g^{-1})}. \tag{2}$$

It is clear that if $f \in C_c(G)$, then $f^* \in C_c(G)$ with $\mathrm{supp}(f^*) = \mathrm{supp}(f)^{-1}$. However, it is not so immediate that $f_1 * f_2$ is continuous if $f_1, f_2 \in C_c(G)$; the latter follows from the following.

[§]In terms of the associated right-invariant action, $(f_1 * f_2)(g) = \int_G f_1(gh^{-1}) f_2(h)\, \mathrm{d}\mu_{s(g)}(h)$.

Lemma 5.1. Let $G \rightrightarrows M$ be a second-countable, locally compact Hausdorff groupoid with a Haar system μ. If $F \in C_c(G * G)$, then $\Lambda(F) \in C_c(G)$, where

$$\Lambda(F)(g) = \int_G F(h,g)\,\mathrm{d}\mu^{t(g)}(h).$$

Proof. Since G is second-countable, $G \times G$ is normal. By Tieze extension theorem and the closedness of $G*G$ in $G\times G$, we extend F to a bounded continuous function on $G\times G$, and since $G\times G$ is paracompact, we may use a partition of unity to obtain a compactly supported function which equals F on $G*G$. Hence we may assume without loss of generality that $F \in C_c(G\times G)$.

Now the function $G \to C_c(G)$, $g \mapsto (h \mapsto F(h,g))$ is continuous with respect to the compact-open topology on $C_c(G)$, hence the function $\Phi\colon G\times M \to \mathbb{C}$, given by $(g,x) \mapsto \int_G F(h,g)\,\mathrm{d}\mu^x(h)$ is also continuous, and so is $\Lambda(F) = \Phi\circ\iota_t$, where $\iota_t\colon G \to G\times M$ is defined as $g \mapsto (g,t(g))$. ∎

Proposition 5.2. The vector space $C_c(G)$, endowed with operations (1) and (2), is a $*$-algebra.

Proof. By the previous lemma, the convolution $f_1 * f_2$ is continuous (just take $F(h,g) = f_1(h)f_2(h^{-1}g)$). Since $(f_1*f_2)(g)$ is clearly nonzero only if there exists an arrow $h \in G$ such that $f_1(h)$ and $f_2(h^{-1}g)$ are both nonzero, we have $\operatorname{supp}(f_1*f_2) \subset \operatorname{supp}(f_1)\operatorname{supp}(f_2)$, so that f_1*f_2 is indeed compactly supported.[¶] We check the property $(f_1*f_2)^* = {f_2}^* * {f_1}^*$: for any $g\in G$, we have

$$\begin{aligned}
(f_1*f_2)^*(g) &= \int_G \bar f_1(h)\bar f_2\left(h^{-1}g^{-1}\right)\mathrm{d}\mu^{\overbrace{t(g^{-1})}^{s(g)}}(h)\\
&= \int_G \bar f_1\left(g^{-1}h\right)\bar f_2\left(h^{-1}\right)\mathrm{d}\mu^{t(g)}(h)\\
&= ({f_2}^* * {f_1}^*)(g),
\end{aligned}$$

where we have used left-invariance of μ in the second line. For associativity of convolution, just note that we have to use left-invariance and Fubini's theorem in a straightforward manner (or see [6, Proposition 1.34]). ∎

Example 5.3.

(i) *Groups.* If G is a locally compact Hausdorff group with a Haar measure μ, and we observe it as a groupoid with a Haar system $\{\mu\}$, then the convolution coincides with the definition in section 1. However, the involution in the case of a groupoid is missing a modular function; the main reason for the factor $\Delta(g^{-1})$ in the group case is that it makes involution an L^1-isometry. At the first glance, this means that our definitions of group and groupoid C^*-algebras do not coincide. However, it may be shown that the groupoid C^*-algebra $C^*(G \rightrightarrows \{\star\})$ of G (cf. Definition 5.9) is $*$-isomorphic to the group C^*-algebra $C^*(G)$ as defined in the first section, with the isomorphism induced by $\varphi\colon C_c^{\mathrm{Group}}(G) \to C_c(G)$, $\varphi(f)(g) = \Delta(g)^{\frac{1}{2}}f(g)$. For more details, see [6, Example 1.50].

(ii) *Action groupoids.* If $G\times M \rightrightarrows M$ is an action groupoid with the right-invariant Haar system $\{\mu\times\delta_x \mid x\in M\}$ from example 3.4 (ii), the convolution and involution formulas read:

$$(f_1*f_2)(g,x) = \int_G f_1(gh^{-1},hx)f_2(h,x)\,\mathrm{d}\mu(h), \quad f^*(g,x) = \overline{f(g^{-1},gx)}.$$

[¶]Note that $K_1K_2 = m(G*G\cap(K_1\times K_2))$ is compact, if K_1 and K_2 are.

(iii) *Pair groupoids.* If $M \times M \rightrightarrows M$ is a pair groupoid from example 3.4 (iii), the convolution and involution formulas read:

$$(f_1 * f_2)(y,x) = \int_X f_1(y,z) f_2(z,x)\, \mathrm{d}\mu(z), \quad f^*(y,x) = \overline{f(x,y)}.$$

(iv) *Étale groupoids.* Let $G \rightrightarrows M$ be étale. As we have shown in section 4, we may pick a Haar system which consists of counting measures on t-fibres, so that

$$(f_1 * f_2)(g) = \sum_{t(h)=t(g)} f_1(h) f_2(h^{-1}g) = \sum_{s(h)=s(g)} f_1(gh^{-1}) f_2(h) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2).$$

In order for $C_c(G)$ to become a *normed* $*$-algebra, we must equip it with a submultiplicative norm $\|.\|$, so that the involution becomes an isometry. A natural choice for the norm is

$$\|f\|_I := \max\{\|f\|_{I,t}\,, \|f\|_{I,s}\},$$

where

$$\|f\|_{I,t} = \sup_{x \in M} \lambda(|f|)(x) = \sup_{x \in M} \int_G |f|\, \mathrm{d}\mu^x \quad \text{and} \quad \|f\|_{I,s} = \sup_{x \in M} \int_G |f|\, \mathrm{d}\mu_x.$$

In the literature, $\|.\|_I$ is known as the I-norm; note that a suggestive notation could also be $\|.\|_{\infty,t}$ and $\|.\|_{\infty,s}$ since these are just the supremum norms of a fibre-integrated function. We will stick to $\|.\|_I$ as to avoid potential confusion.

Proposition 5.4. $\|.\|_I$ is a norm on the $*$-algebra $C_c(G)$.

Proof. It is a routine check that $\|.\|_I$ is a norm on the vector space $C_c(G)$; for example, positive definitness follows from the observation that $\|f\|_{I,t} = 0$ implies $\int_G |f|\, \mathrm{d}\mu^x = 0$ for all $x \in M$, which further implies $f|_{t^{-1}(x)} \equiv 0$ μ^x-a.e., so that $f \equiv 0$ by continuity.

From the definition of involution, it is clear that $\|f^*\|_I = \|f\|_I$ for any $f \in C_c(G)$. To check submultiplicativity, let $f_1, f_2 \in C_c(G)$. We have:

$$\begin{aligned}
\|f_1 * f_2\|_{I,t} &= \sup_{x \in M} \int_G \left| \int_G f_1(h) f_2(h^{-1}g)\, \mathrm{d}\mu^{\overbrace{t(g)}^{x}}(h) \right| \mathrm{d}\mu^x(g) \\
&\leq \sup_{x \in M} \int_G \left(\int_G |f_1(h)| \left|f_2(h^{-1}g)\right| \mathrm{d}\mu^x(h) \right) \mathrm{d}\mu^x(g) \\
&= \sup_{x \in M} \int_G |f_1(h)| \left(\int_G \left|f_2(h^{-1}g)\right| \mathrm{d}\mu^x(g) \right) \mathrm{d}\mu^x(h) \\
&= \sup_{x \in M} \int_G |f_1(h)| \underbrace{\left(\int_G |f_2(g)|\, \mathrm{d}\mu^x(g) \right)}_{\leq \|f_2\|_{I,t}} \mathrm{d}\mu^x(h) \\
&\leq \|f_1\|_{I,t} \|f_2\|_{I,t}\,,
\end{aligned}$$

where we've used Fubini's theorem in the third line and left-invariance in the fourth. We similarly prove $\|f_1 * f_2\|_{I,s} \leq \|f_1\|_{I,s} \|f_2\|_{I,s}$, so it is clear that $\|f_1 * f_2\|_I \leq \|f_1\|_I \|f_2\|_I$. ■

We sometimes denote the completion of $C_c(G)$ with respect to the norm $\|.\|_I$ as $L^1(G \rightrightarrows M)$.

Definition 5.5. Let H be a Hilbert space. A $*$-homomorphism $\pi\colon C_c(G) \to B(H)$ is called a *representation* of $C_c(G)$. It is *norm-decreasing* if there holds $\|\pi(f)\| \leq \|f\|_I$ for all $f \in C_c(G)$.

Example 5.6. *Regular representations.* Important nontrivial examples of representations of a groupoid $G \rightrightarrows M$ with a Haar system μ may be constructed in the following way. Let θ be any Radon measure on M, and define two positive linear functionals on $C_c(G)$: $\nu = \theta \circ \mu$ and $\nu^{-1} = \theta \circ \mu^{-1}$, i.e.

$$\nu(f) = \int_M \left(\int_G f(g) \, \mathrm{d}\mu^x(g) \right) \mathrm{d}\theta(x) \quad \text{and} \quad \nu^{-1}(f) = \int_M \left(\int_G f(g) \, \mathrm{d}\mu_x(g) \right) \mathrm{d}\theta(x).$$

We now take $H = L^2(G, \nu^{-1})$ as our Hilbert space and define $\mathrm{Ind}_\theta\colon C_c(G) \to B(H)$ as the convolution of $f \in C_c(G)$ with $\rho \in L^2(G, \nu^{-1})$, i.e.

$$\mathrm{Ind}_\theta(f)(\rho)(g) = \int_G f(h)\rho(h^{-1}g) \, \mathrm{d}\mu^{t(g)}(h). \tag{3}$$

We omit the proof of the fact that Ind_θ is indeed norm-decreasing (see [6, Remark 1.40]). To see that it is a $*$-representation, we let $\rho, \sigma \in L^2(G, \nu^{-1})$ and straightforwardly compute:

$$\begin{aligned}
\langle \mathrm{Ind}_\theta(f^*)\rho, \sigma \rangle &= \int_M \int_G \left(\int_G \bar{f}(h^{-1})\rho(h^{-1}g) \, \mathrm{d}\mu^{t(g)}(h) \right) \bar{\sigma}(g) \, \mathrm{d}\mu_x(g) \, \mathrm{d}\theta(x) \\
&= \int_M \int_G \left(\int_G \bar{f}(kg^{-1})\rho(k) \, \mathrm{d}\mu_{s(g)}(k) \right) \bar{\sigma}(g) \, \mathrm{d}\mu_x(g) \, \mathrm{d}\theta(x) \\
&= \int_M \int_G \rho(k) \left(\int_G \bar{f}(kg^{-1})\bar{\sigma}(g) \, \mathrm{d}\mu_x(g) \right) \mathrm{d}\mu_x(k) \, \mathrm{d}\theta(x) \\
&= \int_M \int_G \rho(k) \left(\int_G \bar{f}(h)\bar{\sigma}(h^{-1}k) \, \mathrm{d}\mu^{t(k)}(h) \right) \mathrm{d}\mu_x(k) \, \mathrm{d}\theta(x) \\
&= \langle \rho, \mathrm{Ind}_\theta(f)\sigma \rangle ,
\end{aligned}$$

where we have used the pushforward formula together with right-invariance in the second line ($k = h^{-1}g$), Fubini's theorem in the third line, and pushforward formula together with left-invariance in the fourth line ($h = kg^{-1}$).

Remark 5.7. In particular, we have the important case of the regular representation of $G \rightrightarrows M$ which arises from a Dirac measure δ_x on M. In that case, Ind_{δ_x} is a representation on $L^2(G_x, \mu_x)$, and is given for any $f \in C_c(G)$ and $\rho \in C_c(G_x)$ by

$$\mathrm{Ind}_{\delta_x}(f)(\rho)(g) = \int_G f(h)\rho(h^{-1}g) \, \mathrm{d}\mu^{t(g)}(h).$$

Proposition 5.8. Let $f \in C_c(G)$. If we define

$$\|f\|_u = \sup\{\|\pi(f)\| \mid \pi\colon C_c(G) \to B(H) \text{ is a norm-decreasing representation}\},$$

we obtain a norm on $C_c(G)$, called the *universal norm.*

Proof. The only nontrivial thing to check is positive definiteness. Suppose $f \neq 0$, so that there exists an element $g \in G$ with $f(g) \neq 0$. By continuity, there exists a neighborhood U of g such that $f|_U \neq 0$. Now let $x = s(g)$ and choose the Dirac measure δ_x on M to obtain a regular representation $\mathrm{Ind}_{\delta_x}\colon C_c(G) \to B(L^2(G_x, \mu_x))$, so that if we pick $\rho \in C_c(G_x)$ as a nonnegative function with $\operatorname{supp}(\rho) \subset (G^{t(g)} \cap U)^{-1}g$ (note that this is an open subset of G_x containing 1_x), $\rho(1_x) = 1$ and $\int_G \rho \, \mathrm{d}\mu_x = 1$, we get

$$\operatorname{Ind}_{\delta_x}(f)(\rho)(h) \neq 0,$$

for all h in some neighborhood of g. This proves $\operatorname{Ind}_{\delta_x}(f) \neq 0$. ■

Definition 5.9. The *groupoid C^*-algebra $C^*(G \rightrightarrows M)$* is defined as the completion of the space $C_c(G)$ with respect to the universal norm.

5.1 The case of an étale groupoid

To conclude, we show that the construction of the groupoid C^*-algebra simplifies in the case of étale groupoids. Throughout this section, we will continue assuming that $G \rightrightarrows M$ is a second-countable, locally compact Hausdorff étale groupoid with a counting measure on each t-fibre, as made possible by Proposition 4.3. For any $f \in C_c(G)$, the I-norm on an étale groupoid reads

$$\|f\|_I = \sup_{x \in M} \max \Bigg\{ \sum_{s(g)=x} |f(g)| \, , \sum_{t(g)=x} |f(g)| \Bigg\}.$$

We claim that every $*$-representation of $C_c(G)$ is automatically continuous:

Proposition 5.10. On an étale groupoid, every $*$-representation is norm-decreasing. The universal norm on $C_c(G)$ is hence given for any $f \in C_c(G)$ by

$$\|f\|_u = \sup\{\|\pi(f)\| \mid \pi \colon C_c(G) \to B(H) \text{ is a representation}\}.$$

Lemma 5.11. For any $f \in C_c(G)$, there is a constant $K_f \geq 0$ such that $\|\pi(f)\| \leq K_f$ for any representation $\pi \colon C_c(G) \to B(H)$. If f is supported in a bisection, we can take $K_f = \|f\|_\infty$.

Proof. Since G is étale and f is compactly supported, we may write f as a finite sum $f = \sum_i f_i$, where each f_i is supported on a bisection. We define $K_f = \sum_i \|f_i\|_\infty$.

Since $\pi|_{C_c(1_M)}$ is a representation of a commutative $*$-algebra, we must have $\|\pi(f)\| \leq \|f\|_\infty$ for any $f \in C_c(1_M)$. Since f_i is supported on a bisection, so is f_i^*, and we must have that $f_i^* * f_i$ is supported on 1_M, and also $\|f_i^* * f_i\|_\infty = \|f_i\|_\infty^2$. (To see this, just note that in general, if f_1, f_2 are functions on G supported on bisections U_1, U_2, then we must have that for any $g = g_1 g_2 \in U_1 U_2$, there holds $(f_1 * f_2)(g) = f_1(g_1) f_2(g_2)$.) Altogether,

$$\|\pi(f_i)\|^2 = \|\pi(f_i^* * f_i)\| \leq \|f_i^* * f_i\|_\infty = \|f_i\|_\infty^2 \, ,$$

and now by triangle inequality, $\|\pi(f)\| \leq K_f$. The conclusion for when f is supported in a bisection is clear since there is only one term in the sum. ■

Proof of proposition. Let $\pi \colon C_c(G) \to B(H)$ be a $*$-representation. To see that π is a norm-decreasing representation, we only need to check that π is continuous with respect to the I-norm, because then the (unique) extension of π to the completion $(\overline{C_c(G)}, \|.\|_I)$ of the $*$-algebra $C_c(G)$, is continuous – it is a well-known fact that any $*$-homomorphism of C^*-algebras is automatically norm-decreasing.

We first show that π is continuous in the final topology on $C_c(G)$ induced by the inclusions of subsets $C_c^K(G) = \{f \in C(G) \mid \operatorname{supp}(f) \subset K\}$ indexed by compact subsets $K \subset G$, where each $C_c^K(G)$ carries the $\|.\|_\infty$-norm. By the characterization of final topology, it is enough to check that $\pi|_{C_c^K(G)}$ is continuous for any compact K. By Proposition 4.3, we may cover K by finitely many open bisections $(U_i)_{i=1}^n$, and write $f = \sum_i f_i$ where $\mathrm{supp}(f_i) \in U_i$. Then

$$\|\pi(f)\| = \left\| \sum_i \pi(f_i) \right\| \leq \sum_i \|\pi(f_i)\| \leq \sum_i \|f_i\|_\infty \leq n\,\|f\|_\infty$$

where the second inequality follows from the previous lemma. This means that π is Lipschitz, hence continuous in the final topology.

Now just note that for any $f \in C_c(G)$, there holds $\|f\|_\infty \leq \|f\|_I$ since the fibrewise integral is just summation, so that the final topology is coarser than the I-norm topology, implying that π is continuous in the I-norm topology. ∎

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