Cohomology of sheaves

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Introduction

This is an attempt, to rewrite B. Iversons book "Cohomology of Sheaves". Since the original book is very old, I will adapt certain passages to the modern language. This is also a project to experiment a little with Latex and the mathematics itself. If someone has tips to improve anything (it doesn't matter if its of mathematical, organisational or latex matter) he is welcome to propose those to me.

1 Homological Algebra

1.1 Exact categories

We consider a category with a **zero object** 0, that is, for every object A there is precisely one morphism $A \to 0$ and precisely one $0 \to A$.
A **zero morphism** $A \to B$ is a morphism which be factored $A \to 0 \to B$.
A **kernel**, $\ker(f)$ of a morphism $f : A \to B$ is a pair $(\ker(f), i)$, where $i : K \to A$ is a monomorphism with $f \circ i = 0$ and such that for any morphism $g : X \to A$ with $f \circ g = 0$ there is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, dotted, "\exists"'] \arrow[d, "g"] \arrow[dr, "0"] & \\ \ker(f) \arrow[r, "i"] & A \arrow[r, "f"] & B \end{tikzcd}$$

A **cokernel**, $\operatorname{coker}(f)$ of $f : A \to B$ is a pair $(\operatorname{coker}(f), p)$ where $p : B \to C$ is a epimorphism such that for any morphism $h : B \to Y$ with $h \circ f = 0$ there is a commutative diagram

$$\begin{tikzcd} & Y & \\ \operatorname{coker}(f) \arrow[ur, dotted, "\exists"] & B \arrow[l, "p"] \arrow[u, "h"] & A \arrow[l, "f"] \arrow[ul, "0"'] \end{tikzcd}$$

We shall assume that every morphism has a kernel and cokernel.
We define the image $\operatorname{Im}(f)$ of a morphism $f : A \to B$ to be the kernel of a cokernel.
We define the koimage $\operatorname{Coim}(f)$ of a morphism $f : A \to B$ to be the corkernel of the kernel. Every morphism $f : A \to B$ has a **canonical factorization**

$$\begin{tikzcd} A \arrow[rrr, "f"] \arrow[dr] & & & B \\ & \operatorname{Coim}(f) \arrow[r, "f'"] & \operatorname{Im}(f) \arrow[ur] & \end{tikzcd}$$

Definition 1.1.1. An **exact category** is a category with zero objects, kernels, cokernels, such that $\operatorname{Coim}(f) \xrightarrow{f'} \operatorname{Im}(f)$ is always an isomorphism.

In the remaining part of this section we shall work in an exact category.

Definition 1.1.2. A sequence of morphisms

$$\dots \xrightarrow{f^{n-2}} A^{n-1} \xrightarrow{f^{n-1}} A^n \xrightarrow{f^n} A^{n+1} \xrightarrow{f^{n+1}} \dots \tag{1.1.1}$$

is called **exact**, if $\operatorname{Im}(f^{n-1}) = \ker(f^n) \forall n$.

Proposition 1.1.3. *We consider commutative diagram with exact rows*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & D \arrow[d, "d"] \\ \overline{A} \arrow[r] \arrow[d] & \overline{B} \arrow[r] & \overline{C} \arrow[r] & \overline{D} \\ 0 & & & \end{tikzcd}$$

Then the induced sequence

$$\ker b \to \ker c \to \ker d \tag{1.1.2}$$

is exact.

Proof. We break the diagram into two pieces

$$
\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, "e"] & C \arrow[r] \arrow[d, "c"] & D \arrow[d, "d"] \\
0 \arrow[r] & \overline{E} \arrow[r] & \overline{C} \arrow[r] & \overline{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & E \arrow[r] \arrow[d, "e"] & 0 \\
\overline{A} \arrow[r] \arrow[d] & \overline{B} \arrow[r] & \overline{E} \arrow[r] & 0 \\
0 & & &
\end{tikzcd}
$$

We have to prove :

(i)
$$0 \to \ker e \to \ker c \to \ker d \tag{1.1.3}$$
is exact

(ii) $\ker b \to \ker e$ is surjective

(i) It is easy to check that $\ker e \to \ker c$ is a kernel for $\ker c \to D$.

(ii)
1. Check that $\operatorname{coker} b \cong \operatorname{coker} e$
2. The commutative diagram with exact rows

$$
\begin{tikzcd}
\overline{A} \arrow[r] \arrow[d] & \overline{B} \arrow[r] \arrow[d] & \overline{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{coker} b \arrow[r] & \operatorname{coker} e \arrow[r] & 0
\end{tikzcd}
$$

   shows that $\overline{A} \to \operatorname{Im} b \to \operatorname{Im} e$ is exact: we replace $\overline{A}$ by the kernel of $\overline{B} \to \overline{E}$ and use (i).
3. This gives a commutative diagram with exact rows

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "\overline{a}"] & B \arrow[r] \arrow[d, "\overline{b}"] & E \arrow[r] \arrow[d, "\overline{e}"] & 0 \\
\overline{A} \arrow[r] \arrow[d] & \operatorname{Im} b \arrow[r] \arrow[d] & \operatorname{Im} e \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}
$$

   Check that $\overline{e}$ is a cokernel for $\ker \overline{b} \to E$

□

The dual statement is

Proposition 1.1.4. *Consider the commutative diagram with exact rows*

$$
\begin{tikzcd}
 & & & 0 \arrow[d] \\
A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & D \arrow[d, "d"] \\
\overline{A} \arrow[r] & \overline{B} \arrow[r] & \overline{C} \arrow[r] & \overline{D}
\end{tikzcd}
$$

The induced sequence
$$\operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c \tag{1.1.4}$$
is exact

Proof. Dualize the proof above. □

Corollary 1.1.5. *Consider morphisms $f : X \to Y$ and $g : Y \to Z$. The following sequence*

$$0 \to \ker f \to \ker g \circ f \to \ker g \to \operatorname{coker} f \to \operatorname{coker} g \circ f \to \operatorname{coker} g \to 0 \tag{1.1.5}$$

is exact

Proof. Apply 1.3 and 1.4 to the two diagrams

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & \ker f \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, "g\circ f"] & Y \arrow[r] \arrow[d, "g"] & \operatorname{coker} f \arrow[d] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & Z \arrow[r] & Z \arrow[r] & 0 \\
0 & 0 & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & X \arrow[r, "id"] \arrow[d, "f"] & X \arrow[r] \arrow[d, "g\circ f"] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\ker g \arrow[r] & Y \arrow[r, "g"] & Z \arrow[r] & \operatorname{coker} g \arrow[r] & 0
\end{tikzcd}
$$

$\square$

Lemma 1.1.6. *(Snake Lemma) Consider the commutative diagram with exact rows*

$$
\begin{tikzcd}
& & & & 0 \arrow[d] \\
A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & D \arrow[r] \arrow[d, "d"] & E \arrow[d, "e"] \\
\overline{A} \arrow[r] \arrow[d] & \overline{B} \arrow[r] & \overline{C} \arrow[r] & \overline{D} \arrow[r] & \overline{E} \\
0 & & & &
\end{tikzcd}
$$

Then there is an exact sequence

$$\ker b \to \ker c \to \ker d \xrightarrow{\partial} \operatorname{coker} b \to \operatorname{coker} c \to \operatorname{coker} d \tag{1.1.6}$$

1.2 Homology of complexes

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