Cohomology of Sheaves

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1 Homological Algebra

1.1 Exact categories

We consider a category with a **zero object** 0, that is, for every object A there is precisely one morphism $A \to 0$ and precisely one $0 \to A$.
A **zero morphism** $A \to B$ is a morphism which be factored $A \to 0 \to B$.
A **kernel**, $ker(f)$ of a morphism $f : A \to B$ is a pair $(ker(f), i)$, where $i : K \to A$ is a monomorphism with $f \circ i = 0$ and such that for any morphism $g : X \to A$ with $f \circ g = 0$ there is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, dotted, "\exists"'] \arrow[d, "g"] \arrow[dr, "0"] & \\ ker(f) \arrow[r, "i"] & A \arrow[r, "f"] & B \end{tikzcd}$$

A **cokernel**, $coker(f)$ of $f : A \to B$ is a pair $(coker(f), p)$ where $p : B \to C$ is a epimorphism such that for any morphism $h : B \to Y$ with $h \circ f = 0$ there is a commutative diagram

$$\begin{tikzcd} & Y & \\ coker(f) \arrow[ur, dotted, "\exists"] & B \arrow[l, "p"] \arrow[u, "h"] & A \arrow[l, "f"] \arrow[ul, "0"'] \end{tikzcd}$$

We shall assume that every morphism has a kernel and cokernel.
We define the image $im(f)$ of a morphism $f : A \to B$ to be the kernel of a cokernel.
We define the koimage $coim(f)$ of a morphism $f : A \to B$ to be the corkernel of the kernel. Every morphism $f : A \to B$ has a **canonical factorization**

$$\begin{tikzcd} A \arrow[rrr, "f"] \arrow[dr] & & & B \\ & coim(f) \arrow[r, "f'"] & im(f) \arrow[ur] & \end{tikzcd}$$

Definition 1.1.1. An **exact category** is a category with zero objects, kernels, cokernels, such that $coim(f) \xrightarrow{f'} im(f)$ is always an isomorphism.

In the remaining part of this section we shall work in an exact category.

Definition 1.1.2. A sequence of morphisms

$$\ldots \xrightarrow{f^{n-2}} A^{n-1} \xrightarrow{f^{n-1}} A^n \xrightarrow{f^n} A^{n+1} \xrightarrow{f^{n+1}} \ldots \tag{1.1.1}$$

is called **exact**, if $im(f^{n-1}) = ker(f^n) \forall n$.

Proposition 1.1.3. *s*