ON ESTIMATION AND PREDICTION FOR THE XLINDLEY DISTRIBUTION BASED ON RECORD DATA

F. Zanjiran, S.M.T.K. MirMostafaee

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Department of Statistics, University of Mazandaran, Iran
m.mirmostafaee@mail.umz.ac.ir

Abstract

This paper explores the estimation of the unknown parameter in the XLindley distribution using record values and inter-record times, both in classical and Bayesian frameworks. It also investigates Bayesian prediction of future record values. The study includes a simulation to compare the proposed estimators and approximate Bayes predictors. Real data on rainfall and COVID-19 records are analyzed, with four one-parameter lifetime distributions as base models for each dataset. The goodness-of-fit of these models is evaluated. Results show that record values and inter-record times from the XLindley distribution can reasonably predict future records.

Keywords: XLindley, lower record values, Bayes estimation, inter-record times, Prediction.

1. Introduction

The XLindley distribution was first proposed by [8] as an effective new distribution in modeling lifetime data. Suppose that X is a random variable following the one-parameter XLindley distribution. The probability density fucntion (PDF) and cumulative distribution function (CDF) of X are the given respectively by

$$f(x;\theta) = \frac{\theta^2}{(1+\theta)^2}(2+\theta+x)\mathrm{e}^{-\theta x},$$

$$F(x;\theta) = 1 - \left(1 + \frac{\theta x}{(1+\theta)^2}\right)\mathrm{e}^{-\theta x}.$$

The XLindley distribution enjoys an increasing hazard rate function. Chouia and Zeghdoudi [8] demontrated that the XLindley distribution can fit better than some other one-parameter distributions such as the exponential, xgamma and Lindley distributions. Due to the flexibility of the XLindley model, several inferential researches have accomplished by authors since its inception, for example Alotaibi et al. [2] addressed the estimation problem for the XLindley distribution using an adaptive Type-II progressively hybrid censored data, Nassar et al. [31] investigated the reliability estimation of the XLindley constant-stress partially accelerated life tests using progressively censored samples and Alotaibi et al. [3] worked on the reliability estimation under normal operating conditions for progressively Type-II XLindley censored data. Moreover, Metiri et al. [29] focued on the characterisation of XLindley distribution using the relation between the truncated moment and failure rate function or reverse failure rate function.

Suppose that $\{X_n, n = 1, 2, \cdots\}$ is a sequence of identical and independent random variables. Let $\{X_n, n = 1, 2, \cdots\}$ be a sequence of identically distributed and independent random variables. If an observation X_j is less than all its preceding observations, then it is termed a lower record value. Similarly, upper record values can be defined based on the comparisons with preceding observations in the sequence. The sequence of lower record values along with the inter-record times can be denoted by $(\mathbf{R}, \mathbf{T}) = \{R_1, T_1, R_2, T_2, \cdots, R_{m-1}, T_{m-1}, R_m\}$ where R_i represents the i-th record value and T_i is the i-th inter-record time, which is the number of observations needed after occurrence of R_i to obtain a new record value R_{i+1}. Record data play a crucial role in various practical scenarios, see for example [5]. Record values and the related subjects have been studied by many authors; see for example [1, 11, 12, 28]. For instance, Samaniego and Whitaker [38] explored the estimation problem of the mean parameter of the exponential distribution using records and inter-record times. Doostparast [9] delved into the Bayesian and non-Bayesian estimation of the two parameters of the exponential distribution based on records and inter-record times. In a similar study, Doostparast et al. [10] investigated the Bayesian estimation of the parameters of the Pareto distribution utilizing records and inter-record times. Kzlaslan and Nadar [21] estimated the parameter of the proportional reversed hazard rate model based on records and inter-record times. Nadar and Kzlaslan [30] discussed inferential methods for the Burr type XII distribution using record values and inter-record times. Additionally, Kzlaslan and Nadar [22, 23] centered their research on inferential procedures for the generalized exponential and Kumaraswamy distributions based on record values and inter-record time statistics, respectively. Amini and MirMostafaee [4] examined interval prediction of future order statistics from the exponential distribution based on records given the inter-record times. Pak and Dey [32] developed inferential procedures for the estimation of parameters and prediction of future record values for the power Lindley model using lower record values and inter-record times. Kumar et al. [24] directed their attention towards the estimation and prediction for the unit-Gompertz distribution based on records and inter-record times. Bastan and MirMostafaee [6] explored inferential problems for the Poisson-exponential distribution based on record values and inter-record times. Khoshkhoo Amiri and MirMostafaee [19] studied estimation and prediction issues for the xgamma distribution based on lower records and inter-record times. Most recently, Khoshkhoo Amiri and MirMostafaee [20] addressed the estimation and prediction problems for the Chen distribution, utilizing lower records and inter-record times.

In this paper, we intend to discuss estimation and prediction for the XLindley distribution based on lower records and inter-record times, as well as based on lower records alone. In what follows, first, we obtain maximum likelihood (ML) estimates and asymptotic confidence intervals for the parameter of the XLindley distribution in Section 2. In Section 3, we go through the Bayesian estimation method and find the Bayes estimates of the parameter under a symmetric loss function and an asymmetric loss function. The Bayes estimates do not seem to be expressible in closed forms, so we become inclined to use an approximation method such as the Metropolis-Hastings algorithm. Section 4 is devoted to the Bayesian prediction of a lower future record value. Numerical illustration, including a simulation study and a real data example, are given in Section 5. .... The paper is concluded with several remarks.

2. Maximum Likelihood Estimation

In this section, we proceed to compute the ML and Bayesian estimates for the unknown parameters θ of the XLindley model using record data. The record data is generated through an inverse sampling scheme, where units are sequentially selected until the mth minimum is observed. Additionally, for ease of computation, the mth inter-record time is designated as one.

2.1. Maximum Likelihood Estimation Based on Records and Inter-Record Times

In this section, our attention shifts towards the Maximum Likelihood (ML) estimate and an Associated Confidence Interval (ACI) for the parameter. Consider a sequence of record data $\{(r_1, t_1), (r_2, t_2), ..., (r_m, t_m)\}$ originating from the $XL(\theta)$ distribution. Subsequently, the likelihood function of θ, taking into account the observed lower records and inter-record times, is expressed as

$$L(\theta; \boldsymbol{r}, \boldsymbol{t}) = \prod_{i=1}^{m} f(r_i)[1 - F(r_i)]^{t_i - 1} = \left(\frac{\theta}{1+\theta}\right)^{2m} \mathrm{e}^{-\theta \sum_{i=1}^{m} r_i} \prod_{i=1}^{m} \left[(2 + \theta + r_i)[\xi(r_i, \theta)]^{t_i - 1}\right] \quad (1)$$

where

$$\xi(x, \theta) = \left(1 + \frac{\theta x}{(1+\theta)^2}\right)\mathrm{e}^{-\theta x}, \qquad \theta > 0. \quad (2)$$

$\boldsymbol{r} = \{r_1, \cdots, r_m\}$ and $\boldsymbol{t} = \{t_1, \cdots, t_{m-1}\}$ represent the observed sets of $\boldsymbol{R} = \{R_1, \cdots, R_{m-1}, R_m\}$ and $\boldsymbol{T} = \{T_1, \cdots, T_{m-1}\}$, respectively. It is important to note that t_m is consistently set to one for the sake of simplifying the equations. Therefore, the resulting log-likelihood function can be expressed as

$$l(\theta; \boldsymbol{r}, \boldsymbol{t}) = 2m \ln \theta - 2m \ln(1+\theta) - \theta \sum_{i=1}^{m} r_i + \sum_{i=1}^{m} \ln(2 + \theta + r_i) + \sum_{i=1}^{m} (t_i - 1) \ln \xi(r_i, \theta)$$

Next, we compute the initial partial derivative of the log-likelihood function with respect to θ and set it to zero. This yields

$$\frac{\partial l(\theta; \boldsymbol{r}, \boldsymbol{t})}{\partial \theta} = \frac{2m}{\theta(1+\theta)} - \sum_{i=1}^{m} r_i + \sum_{i=1}^{m} \frac{1}{2 + \theta + r_i} + \sum_{i=1}^{m} (t_i - 1) \frac{\psi(r_i, \theta)}{\xi(r_i, \theta)} = 0$$

where $\psi(x, \theta) = -x\mathrm{e}^{-\theta x}\left(1 + \frac{\theta x}{(1+\theta)^2} + \frac{\theta - 1}{(1+\theta)^3}\right)$ for $\theta > 0$. It appears that there is no explicit form for the equation presented above, which necessitates the use of numerical methods to determine the maximum likelihood (ML) estimate for θ. Subsequently, our focus shifts to constructing an associated confidence interval (ACI) for the parameter θ. In this context, Fisher's information is defined as follows $I(\theta) = -E\left(\frac{\partial^2 l \ln f_\theta(\boldsymbol{R}, \boldsymbol{T})}{\partial \theta^2}\right)$ Here, if the integral is valid, $f_\theta(\boldsymbol{R}, \boldsymbol{T})$ denotes the joint probability function of $R_1, T_1, R_2, T_2, \cdots, R_m - 1, T_m - 1, R_m$. This is expressed as

$$\frac{\partial^2 l(\theta; \boldsymbol{r}, \boldsymbol{t})}{\partial \theta^2} = -\frac{2m(1+2\theta)}{[\theta(1+\theta)]^2} - \sum_{i=1}^{m} \frac{1}{(2 + \theta + r_i)^2} + \sum_{i=1}^{m} (t_i - 1)\left(\frac{\psi'(r_i, \theta)\xi(r_i, \theta) - [\psi(r_i, \theta)]^2}{[\xi(r_i, \theta)]^2}\right)$$

where $\psi'(x, \theta) = x\mathrm{e}^{-\theta x}\left(x + \frac{\theta x^2}{(1+\theta)^2} + \frac{2x(\theta - 1)}{(1+\theta)^3} - \frac{2(2-\theta)}{(1+\theta)^4}\right)$ for $\theta > 0$. Denote the Maximum Likelihood Estimator (MLE) of θ as $\hat{\theta}_{ML}$ and z_γ as the $\gamma - th$ upper quantile of the standard normal distribution. Subsequently, the $100(1-\alpha)\%$ Modified Asymptotic Two-Sided Equi-Tailed Confidence Interval (MATE CI) for θ can be determined as per the methodology outlined in references such as [25].

$$\left(max\left\{0, \hat{\theta}_{ML} - \frac{z_{\frac{\alpha}{2}}}{\sqrt{\tilde{I}(\hat{\theta}_{ML})}}\right\}, \hat{\theta}_{ML} + \frac{z_{\frac{\alpha}{2}}}{\sqrt{\tilde{I}(\hat{\theta}_{ML})}}\right)$$

where $\tilde{I}(\hat{\theta}_{ML}) = -\frac{\partial^2 l(\theta|\boldsymbol{R}, \boldsymbol{T})}{\partial \theta^2}\Big|_{\theta = \hat{\theta}_{ML}}$

2.2. Maximum Likelihood Estimation Based on Record Values

$$L(\theta;\boldsymbol{r}) = f(r_m)\prod_{i=1}^{m-1}\frac{f(r_i)}{F(r_i)} = (\frac{\theta}{1+\theta})^{2m}\mathrm{e}^{-\theta\sum_{i=1}^{m} r_i}\frac{\prod_{i=1}^{m}(2+\theta+r_i)}{\prod_{i=1}^{m-1}(1-\xi(r_i,\theta))} \tag{3}$$

and the corresponding log-likelihood function is

$$l(\theta;\boldsymbol{r}) = 2m\ln\theta - 2m\ln(1+\theta) - \theta\sum_{i=1}^{m} r_i + \sum_{i=1}^{m}\ln(2+\theta+r_i) - \sum_{i=1}^{m-1}\ln[1-\xi(r_i,\theta)] \tag{4}$$

Taking the first partial derivatives of log-likelihood (4) with respect to θ and equating each to zero, we obtain

$$\frac{\partial l(\theta;\boldsymbol{r})}{\partial\theta} = \frac{2m}{\theta(1+\theta)} - \sum_{i=1}^{m} r_i + \sum_{i=1}^{m}\frac{1}{2+\theta+r_i} + \sum_{i=1}^{m-1}\frac{\psi(r_i,\theta)}{1-\xi(r_i,\theta)} = 0$$

$$\frac{\partial^2 l(\theta;\boldsymbol{r})}{\partial\theta^2} = -\frac{2m(1+2\theta)}{[\theta(1+\theta)]^2} - \sum_{i=1}^{m}\frac{1}{(2+\theta+r_i)^2} + \sum_{i=1}^{m-1}\left(\frac{\psi'(r_i,\theta)[1-\xi(r_i,\theta)] + [\psi(r_i,\theta)]^2}{[1-\xi(r_i,\theta)]^2}\right)$$

3. Bayesian Estimation

3.1. Bayesian Estimation Based on Records and Inter-Record Times

In the context of Bayesian estimation, the experimenter's information is incorporated through a probability distribution for the parameter, referred to as the prior distribution. Due to the constraint that the parameter of the XLindley distribution must be positive, a frequently employed prior is the gamma distribution for θ , which is defined by the following Probability Density Function (PDF) The posterior predictive density of R_m can be written as

$$\Pi(\theta) = \frac{b^a\theta^{a-1}\mathrm{e}^{-b\theta}}{\Gamma(a)} \tag{5}$$

Here, the positive hyperparameters a and b can be set based on the prior information available to the experimenter. By utilizing equations (1) and (5), we can derive the posterior density as

$$\Pi(\theta|\boldsymbol{r},\boldsymbol{t}) = \frac{\theta^{2m+a-1}}{D(1+\theta)^{2m}}\mathrm{e}^{-\theta(b+\sum_{i=1}^{m} r_i)}\prod_{i=1}^{m}\left[(2+\theta+r_i)[\xi(r_i,\theta)]^{t_i-1}\right]$$

in which $D = \int_0^\infty \frac{\theta^{2m+a-1}}{(1+\theta)^{2m}}\mathrm{e}^{-\theta(b+\sum_{i=1}^{m} r_i)}\prod_{i=1}^{m}\left[(2+\theta+r_i)[\xi(r_i,\theta)]^{t_i-1}\right]\mathrm{d}\theta$. The squared error loss function (SELF) is widely used as a quadratic loss function, although it may not be suitable for all scenarios since it treats underestimation and overestimation equally. An alternative asymmetric loss function is the linear-exponential loss function (LELF), proposed by [42], which is characterized by

$$L_{LE}(\theta,\hat{\theta}) = b[\exp\{c(\hat{\theta}-\theta)\} - c(\hat{\theta}-\theta) - 1], \qquad b>0, \qquad c\neq 0,$$

where $\hat{\theta}$ be an estimator of θ. Without loss of generality, we assume $b = 1$. The sign and magnitude of parameter c must be properly determined. If $c > 0$, then overestimation is more serious than underestimation and vice versa [43]. The Bayes estimates of θ under the SELF and LELF become

$$\hat{\theta}_{SE} = \int_0^\infty \theta\Pi(\theta|\boldsymbol{r},\boldsymbol{t})\mathrm{d}\theta = \frac{1}{D}\int_0^\infty \frac{\theta^{2m+a}}{(1+\theta)^{2m}}\mathrm{e}^{-\theta(b+\sum_{i=1}^{m} r_i)}\prod_{i=1}^{m}\left[(2+\theta+r_i)[\xi(r_i,\theta)]^{t_i-1}\right]\mathrm{d}\theta$$

and

$$\hat{\theta}_{LE} = -\frac{1}{c}\ln M(-c|\boldsymbol{r},\boldsymbol{t}) = -\frac{1}{c}\ln[\int_0^\infty \mathrm{e}^{-c\theta}\Pi(\theta|\boldsymbol{r},\boldsymbol{t})\mathrm{d}\theta]$$
$$= -\frac{1}{c}\ln\left(\frac{1}{D}\int_0^\infty \frac{\theta^{2m+a-1}}{(1+\theta)^{2m}}\mathrm{e}^{-\theta(c+b+\sum_{i=1}^m r_i)}\prod_{i=1}^m\left[(2+\theta+r_i)[\xi(r_i,\theta)]^{t_i-1}\right]\mathrm{d}\theta\right)$$

Assuming the integrals exist, the Bayes estimates of θ as described above may not be expressible in closed forms. Therefore, we resort to three methods to approximate the Bayes estimates of parameter θ.

The Metropolis-Hastings (M-H) technique was first introduced by [27] and later expanded upon by [15]. An M-H algorithm suitable for our scenario can be outlined as follows

Algorithm can be described as follows

Algorithm 1

Step1. Begin by initializing with an initial estimate $\theta_0 = \hat{\theta}_{ML}$ and setting $t = 1$.

Step2. With θ_{t-1} as input, generate θ^* from a truncated-normal distribution, $N(\theta_{t-1},\sigma^2)I_{\{\theta>0\}}$. Subsequently, assign $\theta_t = \theta^*$ with a certain probability.

$$P = \min\left\{\frac{\Pi(\theta^*|\boldsymbol{r},\boldsymbol{t})q(\theta_{t-1}|\theta^*)}{\Pi(\theta_{t-1}|\boldsymbol{r},\boldsymbol{t})q(\theta^*|\theta_{t-1})},1\right\}$$

where $q(x|b)$ represents the density of $N(b,\sigma^2)I_{\{x>0\}}$, otherwise keep $\theta_t = \theta_{t-1}$.

Step3. Proceed to increment t by 1 and repeat T times, where T is a considerably large value. Subsequently, $\{\theta_{M+1},\theta_{M+2},\cdots,\theta_T\}$ constitutes the generated sample, with M denoting the burn-in period. The estimated Bayes point approximations of θ under the SELF and LELF are now

$$\hat{\theta}_{SM} = \frac{1}{M^*}\sum_{t=M+1}^{T}\theta_t, \qquad \textit{and} \qquad \hat{\theta}_{LM} = -\frac{1}{c}\ln\left(\frac{1}{M^*}\sum_{t=M+1}^{T}\mathrm{e}^{-c\theta_t}\right)$$

Respectively, with $M^* = T - M$. In Section 5, we have taken $\sigma^2 = 1$. Let $\theta_{(1)}\cdots\theta_{(M^*)}$ denote the ordered values of $\theta_{M+1},\cdots,\theta_T$. Define the intervals $L_j(M^*) = [\theta_{(j)},\theta_{(j+\lceil(1-\alpha)M^*\rceil)}]$ for $j = 1,2,\cdots,M^* - \lceil(1-\alpha)M^*\rceil$. Consequently, the $100(1-\alpha)\%$ CSSW CrI for θ can be represented as $L_q(M^*)$, where q is determined such that [7]

$$\theta_{(q+[(1-\alpha)M^*])} - \theta_{(q)} = \min_{1\le j\le M^*-[(1-\alpha)M^*]}\theta_{(j+[(1-\alpha)M^*])} - \theta_{(j)}$$

3.2. Bayesian Estimation Based on record Values

The posterior predictive density of R_m can be written as

$$\Pi(\theta) = \frac{b^a\theta^{a-1}\mathrm{e}^{-b\theta}}{\Gamma(a)}$$

$$\Pi(\theta|\boldsymbol{r}) = \frac{\theta^{2m+a-1}}{D^*(1+\theta)^{2m}}\mathrm{e}^{-\theta(b+\sum_{i=1}^m r_i)}\frac{\prod_{i=1}^m(2+\theta+r_i)}{\prod_{i=1}^{m-1}[1-\xi(r_i,\theta)]}$$

in which

$$D^* = \int_0^\infty \frac{\theta^{2m+a-1}}{(1+\theta)^{2m}}\mathrm{e}^{-\theta(b+\sum_{i=1}^m r_i)}\frac{\prod_{i=1}^m(2+\theta+r_i)}{\prod_{i=1}^{m-1}[1-\xi(r_i,\theta)]}\mathrm{d}\theta$$

The Bayes estimates of θ under the SELF and LELF become

$$\hat{\theta}^*_{SE} = \frac{1}{D^*}\int_0^\infty \frac{\theta^{2m+a}}{(1+\theta)^{2m}}\mathrm{e}^{-\theta(b+\sum_{i=1}^m r_i)}\frac{\prod_{i=1}^m(2+\theta+r_i)}{\prod_{i=1}^{m-1}[1-\xi(r_i,\theta)]}\mathrm{d}\theta$$

and

$$\hat{\theta}^{*}_{LE} = -\frac{1}{c}\ln\left(\frac{1}{D^*}\int_0^\infty \frac{\theta^{2m+a-1}}{(1+\theta)^{2m}} e^{-\theta(c+b+\sum_{i=1}^m r_i)} \frac{\prod_{i=1}^m (2+\theta+r_i)}{\prod_{i=1}^{m-1}[1-\xi(r_i,\theta)]} d\theta\right)$$

4. Bayesian Prediction

Let $r = (r_1, \cdots, r_m)$ and $t = (t_1, \cdots, t_{m-1})$ be the observed sets of $R = (R_1, \cdots, R_m)$ and $T = (T_1, \cdots, T_{m-1})$, respectively. Here, we want to predict the s-th unobserved record value, denoted by $R_s (s > m)$. Note that the conditional distribution of R_s given r and t, denoted by $f(r_s|\theta, \boldsymbol{r}, \boldsymbol{t})$, is the same as the conditional distribution of R_s given $R_m = r_m$, therefore we have

$$\begin{aligned} f(r_s|\theta, \boldsymbol{r}, \boldsymbol{t}) \equiv f(r_s|\theta, r_m) &= \frac{f(r_s;\theta)[Q(r_s,\theta) - Q(r_m,\theta)]^{s-m-1}}{F(r_m,\theta)\Gamma(s-m)} \\ &= [Q(r_s,\theta) - Q(r_m,\theta)]^{s-m-1} \frac{\left(\frac{\theta}{1+\theta}\right)^2 (\theta+2+r_s)}{[1-\xi(r_m,\theta)]\Gamma(s-m)} e^{-\theta r_s} \end{aligned} \quad (6)$$

where $0 < r_s < r_m$ and $Q(x,\theta) = -\ln(F(x;\theta))$ and $\xi(x,\theta)$ is defined in (2). The posterior predictive density of R_s given the lower records and inter-record times is derived as

$$h(r_s|\boldsymbol{r}, \boldsymbol{t}) = \int_0^\infty f(r_s|\theta, r_m) \Pi(\theta|\boldsymbol{r}, \boldsymbol{t}) d\theta$$

It can be easily seen that the associated posterior predictive density cannot be obtained analytically. Thus we estimate $h(r_s|, \boldsymbol{r}, \boldsymbol{t})$ by means of a sample generated using the MHG algorithm. Let $\{\theta_v, v = 1, \cdots, M^*\}$ be the generated sample using Algorithm 1, where $M^* = T - M$. Then, an estimate of $h(r_s|\boldsymbol{r}, \boldsymbol{t})$ is given by

$$\tilde{h}(r_s|\boldsymbol{r}, \boldsymbol{t}) = \frac{1}{M^*}\sum_{v=1}^{M^*} f(r_s|\theta_v, r_m)$$

The approximate predictions of R_s under the SELF and LELF can be obtained as

$$\tilde{R}_s^{SEM} = \int_0^{r_m} r_s \tilde{h}(r_s|\boldsymbol{r}, \boldsymbol{t}) dr_s = \frac{1}{M^*}\sum_{v=1}^{M^*}\int_0^{r_m} r_s f(r_s|\theta_v, r_m) dr_s \quad (7)$$

$$\tilde{R}_s^{LEM} = \frac{-1}{c}\ln\left[\int_0^{r_m} e^{-cr_s}\tilde{h}(r_s|\boldsymbol{r}, \boldsymbol{t}) dr_s\right] = \frac{-1}{c}\ln[\frac{1}{M^*}\sum_{v=1}^{M^*}\int_0^{r_m} e^{-cr_s} f(r_s|\theta_v, r_m) dr_s] \quad (8)$$

A $100(1-\alpha)\%$ two-sided prediction interval (TPI) for R_s is given by $(L(\boldsymbol{r}, \boldsymbol{t}), U(\boldsymbol{r}, \boldsymbol{t}))$, where $L(\boldsymbol{r}, \boldsymbol{t})$ and $U(\boldsymbol{r}, \boldsymbol{t})$ are the simultaneous solutions of the following equations

$$\int_0^{L(\boldsymbol{r},\boldsymbol{t})} h(r_s \mid \boldsymbol{r}, \boldsymbol{t}) dr_s = \frac{\alpha}{2}, \quad \text{and} \quad \int_0^{U(\boldsymbol{r},\boldsymbol{t})} h(r_s \mid \boldsymbol{r}, \boldsymbol{t}) dr_s = 1 - \frac{\alpha}{2}.$$

A $100(1-\alpha)\%$ approximate two-sided prediction interval (ATPI) for R_s is given by (L, U), where L and U are the simultaneous solutions of the following equations

$$\frac{1}{M^*}\sum_{v=M+1}^{T}\int_0^L f(r_s \mid \theta_v, r_m) dr_s = \frac{\alpha}{2}, \quad \text{and} \quad \frac{1}{M^*}\sum_{v=M+1}^{T}\int_0^U f(r_s \mid \theta_v, r_m) dr_s = 1 - \frac{\alpha}{2}.$$

4.1. Special Case: $s = m + 1$

For the special case, when $s = m + 1$, then $Y = R_{m+1}$ given $R_m = r_m$ follows the truncated XLindley distribution on interval $(0, r_m)$. So, we have

$$f(r_{m+1}|\theta, \boldsymbol{r}, \boldsymbol{t}) \equiv f(r_{m+1}|\theta, r_m) = \frac{f(r_{m+1};\theta)}{F(r_m, \theta)} = \frac{\left(\frac{\theta}{1+\theta}\right)^2 (\theta + 2 + r_{m+1})}{1 - \xi(r_m, \theta)} \mathrm{e}^{-\theta r_{m+1}}, \quad 0 < r_{m+1} < r_m. \tag{9}$$

$$\begin{aligned}\int_0^{r_m} r_{m+1} f(r_{m+1}|\theta, r_m)\mathrm{d}r_{m+1} &= -\frac{(\theta+2)(\theta r_{m+1}+1) + r_{m+1}(\theta r_{m+1}+2) + 2/\theta}{(1+\theta)^2[1-\xi(r_m,\theta)]}\mathrm{e}^{-\theta r_{m+1}}\Bigg]_0^{r_m} \\ &= \frac{\theta + 2 + 2/\theta - [(\theta+2)(\theta r_m + 1) + r_m(\theta r_m + 2) + 2/\theta]\mathrm{e}^{-\theta r_m}}{(1+\theta)^2[1-\xi(r_m,\theta)]}\end{aligned}$$

$$\begin{aligned}\int_0^{r_m} \mathrm{e}^{-c r_{m+1}} f(r_{m+1}|\theta, r_m)\mathrm{d}r_{m+1} &= -\frac{\theta^2[1 + (\theta+c)(\theta+2+r_{m+1})]}{(\theta+c)^2(1+\theta)^2[1-\xi(r_m,\theta)]}\mathrm{e}^{-(\theta+c)r_{m+1}}\Bigg]_0^{r_m} \\ &= \frac{\theta^2\{1 + (\theta+c)(\theta+2) - [1 + (\theta+c)(\theta+2+r_m)]\mathrm{e}^{-(\theta+c)r_m}\}}{(\theta+c)^2(1+\theta)^2[1-\xi(r_m,\theta)]}\end{aligned}$$

Therefore, from (7) and (8), the approximate predictions of R_s under the SELF and LELF can be obtained as

$$\begin{aligned}\tilde{R}_{m+1}^{SEM} &= \frac{1}{M^*}\sum_{v=1}^{M^*}\int_0^{r_m} r_{m+1} f(r_{m+1}|\theta_v, r_m)\mathrm{d}r_{m+1} \\ &= \frac{1}{M^*}\sum_{v=1}^{M^*}\frac{\theta_v + 2 + 2/\theta_v - [(\theta_v+2)(\theta_v r_m + 1) + r_m(\theta_v r_m + 2) + 2/\theta_v]\mathrm{e}^{-\theta_v r_m}}{(1+\theta_v)^2[1-\xi(r_m,\theta_v)]}\end{aligned}$$

$$\begin{aligned}\tilde{R}_{m+1}^{LEM} &= \frac{-1}{c}\ln\left[\frac{1}{M^*}\sum_{v=1}^{M^*}\int_0^{r_m}\mathrm{e}^{-c r_s} f(r_s|\theta_v, r_m)\mathrm{d}r_s\right] \\ &= \frac{-1}{c}\ln\left[\frac{1}{M^*}\sum_{v=1}^{M^*}\frac{\theta_v^2\{1 + (\theta_v+c)(\theta_v+2) - [1 + (\theta_v+c)(\theta_v+2+r_m)]\mathrm{e}^{-(\theta_v+c)r_m}\}}{(\theta_v+c)^2(1+\theta_v)^2[1-\xi(r_m,\theta_v)]}\right]\end{aligned}$$

A $100(1-\alpha)\%$ approximate two-sided prediction interval (ATPI) for R_{m+1} is given by (L, U), where L and U are the simultaneous solutions of the following equations

$$\frac{1}{M^*}\sum_{v=M+1}^{T}\frac{1-\xi(L,\theta_v)}{1-\xi(r_m,\theta_v)} = \frac{\alpha}{2}, \qquad \text{and} \qquad \frac{1}{M^*}\sum_{v=M+1}^{T}\frac{1-\xi(U,\theta_v)}{1-\xi(r_m,\theta_v)} = 1 - \frac{\alpha}{2},$$

where $\xi(x, \theta)$ is defined in (2).

5. Numerical Illustration

5.1. A Simulation Study

In this study, we conduct a Monte Carlo simulation to evaluate the point estimators, interval estimators, and approximate predictors introduced in the paper. The simulation involves 1000 replications, denoted as $M^* = 1000$. For each replication, we create m+1 records along with their respective inter-record intervals using the XLindley (θ) distribution. We explore three scenarios for m, 3, 4 and 5, with the θ parameter set at 0.5, 1, and 2. Within the framework of Bayesian inference, two prior distributions are applied.

Table 1: *The ERs of the point estimators of θ when $\theta = 0.5$.*

	$m=4$				$m=5$			
$\theta = 0.5$	bias	ER_S	ER_L $c=0.5$	ER_L $c=-0.5$	bias	ER_S	ER_L $c=0.5$	ER_L $c=-0.5$
MLE	0.0868	0.0971	0.0156	0.0101	0.0669	0.0483	0.0066	0.0056
	1.6656	99.041	>100	0.6617	0.1215	>100	>100	0.8858
Bayes (SELF)	0.0943	0.0991	0.0156	0.0104	0.0729	0.0510	0.0070	0.0059
	0.6746	2.7953	1.7739	0.1732	0.6977	3.0185	2.6072	0.1802
Bayes (LELF)	0.0795	0.0821	0.0123	0.0089	0.0634	0.0457	0.0062	0.0053
$c=0.5$	0.3751	0.7616	0.1494	0.0684	0.3854	0.7803	0.1546	0.0697
Bayes (LELF)	0.1112	0.1246	0.0212	0.0126	0.0831	0.0573	0.0079	0.0066
$c=-0.5$	1.7028	25.912	>100	0.6366	1.7503	26.783	>100	0.6545
$\theta = 1$								
MLE	0.2293	0.5745	0.2497	0.0466	0.1564	0.3371	0.1164	0.0300
	7.4661	>100	>100	3.4686	4.8365	>100	>100	2.1844
Bayes (SELF)	0.2394	0.5045	0.1371	0.0443	0.1688	0.3186	0.0806	0.0297
	1.2180	6.3531	43.106	0.3665	1.0927	5.8714	9.0584	0.3326
Bayes (LELF)	0.1652	0.3243	0.0602	0.0317	0.1186	0.2277	0.0412	0.0230
$c=0.5$	0.4764	1.1666	0.2182	0.1077	0.4107	1.0520	0.1980	0.0973
Bayes (LELF)	0.3505	1.2209	14.838	0.0718	0.2316	0.5204	0.5563	0.0406
$c=-0.5$	3.9390	75.039	>100	1.6089	3.4741	67.636	>100	0.0481
$\theta = 2$								
MLE	0.5934	2.9392	2.0688	0.1996	0.4055	1.7905	0.9308	0.1353
	9.4667	>100	>100	4.4454	9.9441	>100	>100	4.6788
Bayes (SELF)	0.5169	2.1067	0.7134	0.1628	0.3695	1.4141	0.4473	0.1166
	1.5136	9.0327	7.3213	0.5359	1.5155	8.9470	6.8274	0.5322
Bayes (LELF)	0.2013	0.9833	0.1922	0.0943	0.1448	0.7758	0.1505	0.0759
$c=0.5$	0.1222	1.0090	0.1527	0.1139	0.1291	1.0102	0.1549	0.1128
Bayes (LELF)	1.0939	6.8180	59.371	0.3483	0.7305	3.6409	17.917	0.2150
$c=-0.5$	6.5314	>100	>100	2.7891	6.6037	>100	>100	2.8190

Table 2: *The EpRs of the approximate Bayes point predictors of θ for $m = 4$ and 5.*

	$m = 4$				$m = 5$			
$\theta = 0.5$	bias	ER_S	ER_L $c = 0.5$	ER_L $c = -0.5$	bias	ER_S	ER_L $c = 0.5$	ER_L $c = -0.5$
SELF	0.002842	0.017065	0.002167	0.002128	0.000548	0.006948	0.000912	0.000843
	0.005136	0.017106	0.002211	0.002098	0.001393	0.007154	0.000958	0.000854
LELF	-0.001371	0.017407	0.002159	0.002222	-0.000844	0.006759	0.000865	0.000837
$c = 0.5$	0.000895	0.017163	0.002166	0.002155	-0.000003	0.006893	0.000899	0.000841
LELF	0.007078	0.017195	0.002235	0.002097	0.001943	0.007233	0.000975	0.000858
$c = -0.5$	0.009345	0.017514	0.002317	0.002102	0.002780	0.007493	0.001031	0.000874
$\theta = 1$								
SELF	0.002136	0.002491	0.000312	0.000312	0.001369	0.000653	0.000082	0.000081
	0.002754	0.002453	0.000309	0.000305	0.001667	0.000663	0.000084	0.000082
LELF	0.001516	0.002487	0.000310	0.000313	0.001114	0.000645	0.000081	0.000080
$c = 0.5$	0.002130	0.002437	0.000305	0.000305	0.001411	0.000651	0.000082	0.000081
LELF	0.002760	0.002504	0.000315	0.000312	0.001625	0.000664	0.000084	0.000082
$c = -0.5$	0.003379	0.000248	0.000313	0.000307	0.001924	0.000678	0.000086	0.000084
$\theta = 2$								
SELF	0.000476	0.000378	0.000047	0.000047	-0.000272	0.000105	0.000013	0.000013
	0.000689	0.000397	0.000050	0.000049	-0.000208	0.000106	0.000013	0.000013
LELF	0.000381	0.000374	0.000047	0.000047	-0.000302	0.000105	0.000013	0.000013
$c = 0.5$	0.000594	0.000392	0.000049	0.000049	-0.000239	0.000106	0.000013	0.000013
LELF	0.000572	0.000382	0.000048	0.000048	-0.000242	0.000105	0.000013	0.000013
$c = -0.5$	0.000785	0.000401	0.000050	0.000050	-0.000177	0.000106	0.000013	0.000013

Table 3: *The AWs and CPs of ...*

	$m = 3$		$m = 4$		$m = 5$	
$\theta = 0.5$	AW	CP	AW	CP	AW	CP
ACI	1.09579	0.962	0.83556	0.963	0.70605	0.956
	6.92765	0.964	5.89956	0.959	7.21595	0.960
CSSW CrI	1.03103	0.955	0.80036	0.952	0.68460	0.951
	2.98580	0.957	2.93030	0.952	0.00196	0.957
PI	0.47067	0.945	0.23426	0.938	0.11708	0.948
	0.47077	0.942	0.23428	0.938	0.11708	0.949
$\theta = 1$						
ACI	2.55093	0.953	1.95860	0.954	1.61990	0.951
	11.8789	0.966	24.2695	0.977	16.4109	0.965
CSSW CrI	2.29057	0.954	1.82186	0.955	1.54003	0.946
	5.46901	0.962	5.77409	0.975	5.43254	0.960
PI	0.18213	0.955	0.08794	0.946	0.04852	0.950
	0.18224	0.955	0.08797	0.946	0.04853	0.950
$\theta = 2$						
ACI	5.81158	0.946	4.58167	0.956	3.75708	0.956
	29.2559	0.962	32.4821	0.955	33.8740	0.967
CSSW CrI	4.66783	0.952	4.00381	0.958	3.39805	0.953
	8.67913	0.959	9.29240	0.952	9.29188	0.962
PI	0.08007	0.935	0.03415	0.954	0.01698	0.950
	0.08012	0.937	0.03416	0.952	0.01699	0.950

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5.2. Real Data Example

Here, we consider the following data on the amount of rainfall (in inches) recorded at the Los Angeles Civic Center in February from 1999 to 2018; see the website of Los Angeles Almanac: www.laalmanac.com/weather/we08aa.php.

```
0.56, 5.54,  8.87, 0.29,  4.64, 4.89, 11.02, 2.37, 0.92,  1.64,
3.57, 4.27,  3.29, 0.16,  0.20, 3.58, 0.83,  0.79, 4.17,  0.03.
```

We use the Kolmogorov-Smirnov (K-S) test to check if the XLindley model fits the data. The K-S test statistic confirms that the XLindley distribution is quite suitable for fitting the above data (p-value greater than 0.5). We have extracted the lower records and the corresponding inter-record times as follows:

i	1	2	3	4
r_i	0.56	0.29	0.16	0.03
t_i	3	10	6	1

We have considered both the exponential and XLindley models, as we see that these models fit the data well. Here, we have used the approximate non-informative prior with $a = b = 0.1$We have calculated the ML and approximate Bayes point estimates, as well as the 95% interval estimates of the parameter for the XLindley distribution. The point predictions and 95% ABPIs for the next future record, namely R_5, have been obtained as well. We have used the M-H method for the Bayesian estimation and prediction for the XLindley distribution. The numerical results of this example have been given in Table ???. Here, we predict that the next lowest amount of rainfall (after the year 2018) would be approximately 0.015 inches, which is the predicted 5-th lower record value since 1999.

Table 4: *The numerical results of Example 1.*

Estimation	$\widehat{\theta}_{ML}$	$\widehat{\theta}_{SE}$	$\widehat{\theta}_{LE}$ $(c = 0.5)$	$\widehat{\theta}_{LE}$ $(c = -0.5)$	MATE CI	CSSW CrI
Times	0.9535	0.9729	0.9405	1.0087	(0.2470, 1.6601)	(0.3781, 1.7523)
Without Times	1.8809	1.8679	1.4782	2.9436	(0, 4.9336)	(0.0400, 4.6741)
Prediction		$\widehat{R}_5^S$	$\widehat{R}_5^L$ $(c = 0.5)$	$\widehat{R}_5^L$ $(c = -0.5)$	ABPI	
Times		0.01495	0.01493	0.01497	(0.00074, 0.02924)	
Without Times		0.01488	0.01486	0.01490	(0.00073, 0.02923)	

6. Concluding Remarks

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