LAPLACE EQ. IN SPHERICAL COORDINATES

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\phi\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\phi\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\phi=0$$

WE USE AGAIN SEP. OF VARIABLES, BUT FIRST FOR SIMPLICITY

WE ASSUME

$$\phi=\phi(r,\theta)=R(r)\Theta(\theta)$$

$$\Rightarrow\quad \Theta\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R\right)=-R\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\Theta\right)$$

WE MULTIPLY BY

$$\frac{r^2}{\Theta R}$$

$$\Rightarrow\quad \frac{1}{R}\partial_r\left(r^2\partial_r R\right)=-\frac{1}{\Theta}\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\Theta\right)=K$$

RADIAL

$$\partial_r\left(r^2\partial_r R\right)=kR$$

LET US TRY

$$R=r^l$$

$$\partial_r r^l=l\,r^{l-1}\qquad r^2\left(\partial_r r^l\right)=l\,r^{l+1}$$

$$\partial_r\left(r^2\left(\partial_r r^l\right)\right)=l(l+1)\,r^l\quad\Rightarrow$$

$\Rightarrow$ r^l IS SOL. OR

$$\partial_r\left(r^2\partial_r R\right) = k R$$

IF $\boxed{l(l+1) = k}$

ANGULAR

$$\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\Theta\right) = -l(l+1)\Theta$$

LET US DEFINE $\mu = \cos\theta$

$\Rightarrow$ $\mu^2 = \cos^2\theta$ $\Rightarrow$ $\sin^2\theta = 1-\cos^2\theta = 1-\mu^2$

$$\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\mu}\frac{\partial\cos\theta}{\partial\theta} = -\sin\theta\,\partial_\mu$$

$$\Rightarrow \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta(-\sin\theta)\partial_\mu\Theta(\mu)\right) = -l(l+1)\Theta$$

$$\Rightarrow \frac{1}{\sin\theta}(-\sin\theta)\partial_\mu\left((\mu^2-1)\partial_\mu\Theta(\mu)\right) = -l(l+1)\Theta$$

$$\boxed{\partial_\mu\left[(\mu^2-1)\partial_\mu\Theta(\mu)\right] = l(l+1)\Theta(\mu)}$$

THE SOLUTIONS OF THIS EQ. ARE CALLED LEGENDRE POLYNOMIALS

WE LOOK FOR A SOLUTION IN THE FORM μ^n

$$\partial_\mu\mu^n = n\mu^{n-1}$$

$$(\mu^2-1)\partial_\mu\mu^n = n\mu^{n+1} - n\mu^{n-1} = n\mu^{n-1}\left[\mu^2-1\right]$$

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$$\partial_\mu\left[(\mu^2-1)\,\partial_\mu(\mu^n)\right] = n(n+1)\mu^n - n(n-1)\mu^{n-2}$$

$\Rightarrow$

$$n(n+1)\mu^n - n(n-1)\mu^{n-2} = \ell(\ell+1)\mu^n$$

DIFFERENT POWERS OF μ ARE INVOLVED

WE MAY TRY

$$\Theta(\mu) = \sum_{n=0}^{\infty} a_n \mu^n$$

$\Rightarrow$

$$\sum_{n=0}^{\infty} a_n\, n(n+1)\mu^n - \sum_{n=0}^{\infty} a_n\, n(n-1)\mu^{n-2} = \sum_{n=0}^{\infty} a_n\, \ell(\ell+1)\mu^n$$

$\Rightarrow$

$$\sum_{n=0}^{\infty} a_n \mu^n\left[\ell(\ell+1) - n(n+1)\right] + \sum_{n=0}^{\infty} a_n\, n(n-1)\mu^{n-2} = 0$$

NOTE

$n(n-1) = 0$ IF $n = 0$

$n(n-1) = 0$ IF $n = 1$

$\Rightarrow$

$$\sum_{n=0}^{\infty} a_n \mu^n\left[\ell(\ell+1) - n(n+1)\right] + \sum_{n=2}^{\infty} a_n\, n(n-1)\mu^{n-2} = 0$$

$\boxed{n' = n-2} \Rightarrow n = n'+2$

$$\sum_{n=0}^{\infty} a_n \mu^n \left[l(l+1) - n(n+1) \right] + \sum_{n'=0}^{\infty} a_{n'+2} (n'+2)(n'+1) \mu^{n'} = 0$$

TRUE IF

$$a_{n+2} = - a_n \frac{l(l+1) - n(n+1)}{(n+2)(n+1)}$$

IF $l \notin \mathbb{Z}$, $l(l+1) \neq n(n+1)$

$a_0 \Rightarrow a_2 \Rightarrow a_4 \Rightarrow$ ALL EVEN $\neq 0$ IF $a_0 \neq 0$

$a_1 \Rightarrow a_3$ - - ALL ODD $\neq 0$ IF $a_1 \neq 0$

BUT $\theta = 0 \Rightarrow \cos\theta = 1$

$$\sum_{n=0}^{\infty} a_n \sim \sum_{n=0}^{\infty} \text{CONST DIVERGES} \sim \frac{n(n+1)}{(n+2)(n+1)}$$ FOR HIGH n

$\Rightarrow$ $\boxed{l \text{ INTEGER}}$

FOR THE SAME REASON THE SERIES ~~INCLUDES~~ SHOULD INCLUDE ONLY EVEN OR ODD

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FOR EACH l THE SOLUTIONS ARE CALLED

$P_l(\mu)$

NOTE

$l = 0 \Rightarrow l(l+1) = 0$ $\qquad$ $l = 1 \Rightarrow l(l+1) = 2$

$l = -1 \Rightarrow l(l+1) = 0$ $\qquad$ $l = -2 \Rightarrow l(l+1) = 2$

$\ell = 2 \Rightarrow \ell(\ell+1) = 6$ $\qquad$ $\ell = -3 \Rightarrow \ell(\ell+1) = 6$

P_ℓ DOES NOT SEE IF $\ell \geq 0$ OR $\ell < 0$

WE DEFINE IT FOR $\ell \geq 0$

BUT r^ℓ SEES IT!

$$\Phi = \sum_{\ell=0}^{\infty} A_\ell \underbrace{r^\ell}_{\text{DIVERGE FOR } r \to \infty} P_\ell(\cos\theta) + \sum_{\ell=0}^{\infty} B_\ell \, r^{-\ell-1} P_\ell(\cos\theta)$$

DIVERGE FOR $\ell \to 0$

LEGENDRE POLYNOMIALS

$P_0(\mu) =$ CONST

WE DEFINE THEM SUCH THAT $P_\ell(1) = 1$

$\Rightarrow$ NOT NORMALISED!

$$\int_{-1}^{1} 1 = 2$$

$P_2(\mu)$ IF $a_0 = 1 \Rightarrow a_2 = -\dfrac{\ell(\ell+1) - 0}{1 \cdot 2} = -3$

$P_2(\mu) \propto 1 - 3\mu^2$ $\qquad$ $P_2(1) = 1$ IF $P_2(\mu) = \frac{3}{2}\mu^2 - \frac{1}{2}$

$P_1(\mu) = \mu$ $\qquad$ etc.

NOTE :

$$\int_{-1}^{1} P_0(\mu) P_1(\mu) = \int_{-1}^{1} \mu = 0$$

$$\int_{-1}^{1} P_2(\mu) P_1(\mu) \propto \int_{-1}^{1} \mu - 3\mu^3 = 0$$

$$\int_{-1}^{1} P_0(\mu) P_2(\mu) \propto \int_{-1}^{1} (1 - 3\mu^2) = 2 - 3\left[\frac{\mu^3}{3}\right]_{-1}^{1} = 2 - 3\,\frac{2}{3} = 0$$

AS WE WILL SEE LATER, THIS IS NOT BY CHANCE

WE ARE SOLVING

$$L\, P_\ell(\mu) = \ell(\ell+1) P_\ell(\mu)$$

$$L = \partial_\mu \left((\mu^2 - 1)\, \partial_\mu \right)$$

$$\hookrightarrow L^+ = L$$

(SEE LATER, S.L. THEORY)

$$\Rightarrow \int_{-1}^{1} P_n(\mu) P_m(\mu)\, d\mu \propto \delta_{nm}$$

IF ϕ DEPENDS ON φ

$$\phi = R\,\Theta\,\Phi$$

WE MULTIPLY BY

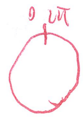
$$\frac{r^2 \sin^2\theta}{R\,\Theta} \implies \frac{1}{\Phi}\,\partial_\varphi^2 \Phi = f(r,\theta)$$

$$\implies \partial_\varphi^2 \Phi = k\,\Phi$$

$$\implies \Phi = e^{im\varphi} \quad \text{WITH} \quad m \in \mathbb{Z}$$

TO HAVE $\Phi(0) = \Phi(2\pi)$

THE USUAL SQC OR



SUBSTITUTING, WE GET

$$\frac{1}{r^2}\,\partial_r\left(r^2 \partial_r \phi\right) + \frac{1}{r^2 \sin\theta}\,\partial_\theta\left(\sin\theta\,\partial_\theta\right)\phi - \frac{m^2}{r^2 \sin^2\theta}\,\phi = 0$$

$$\implies \partial_r\left(r^2 \partial_r \phi\right) = \text{CONST}\;\phi$$

$$\implies \text{CONST} = \ell(\ell+1)$$

$$\implies \partial_\mu\left[(\mu^2-1)\,\partial_\mu \Theta(\mu)\right] + \frac{m^2}{(1-\mu^2)}\,\Theta = \ell(\ell+1)\,\Theta$$

THE SOLUTIONS ARE CALLED

$P_{\ell m}$

ℓ INTEGER $(0, 1, 2, \ldots)$

m INTEGER $-\ell, \ldots, +\ell$

BYRON FULLER PAGE 256 $\quad P_{\ell m} = (1-\mu)^{\frac{m}{2}} \frac{d^m P_\ell}{d^m \mu}$

THE FUNCTIONS

$$Y_{\ell m}(\theta, \varphi) \propto P_{\ell m}(\cos\theta)\, e^{im\phi}$$

ARE CALLED SPHERICAL HARMONICS

THEY SATISFY (SEE THE DISCUSSION IN Q.M.)

$$\int_0^\pi d\theta \int_0^{2\pi} d\varphi\, Y_{\ell m}(\theta,\varphi)\, Y_{nq}(\theta,\varphi) \sin\theta\, d\theta = \delta_{\ell n} \delta_{mq}$$

IN ANALOGY WITH FOURIER (WE ASSUME SOL. $r \leq R$)

$$\Phi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} r^\ell\, Y_{\ell m}(\theta,\varphi)\, A_{\ell m}$$

$$\Phi(R,\theta,\varphi) = f(\theta,\varphi)$$

$$\Rightarrow \quad A_{\ell m} = \frac{1}{R^\ell} \int f(\theta,\varphi)\, Y^*_{\ell m}(\theta,\varphi)$$