HYDROGEN ATOM : SOLUTION

2 BODY PROBLEM

WE ACTUALLY HAVE ELECTRON AND PROTON, 2 BODIES, BUT

BODY 1 (PROTON) $\quad \vec{r}_1, \quad m_1, \quad \vec{P}_1 = m_1 \dot{\vec{r}}_1$

BODY 2 (ELECTRON) $\quad \vec{r}_2, \quad m_2, \quad \vec{P}_2 = m_2 \dot{\vec{r}}_2$

THE RELATIVE DISTANCE IS $\quad \vec{r} = \vec{r}_2 - \vec{r}_1$

THE RELATIVE VELOCITY IS $\quad \vec{v} = \vec{V}_2 - \vec{V}_1 = \dfrac{\vec{P}_2}{m_2} - \dfrac{\vec{P}_1}{m_1}$

THE CENTER OF MASS IS
$$\vec{R} = \frac{m_2 \vec{r}_2 + m_1 \vec{r}_1}{m_2 + m_1}$$

IF WE DEFINE $\quad M = m_1 + m_2 \quad \Rightarrow \quad \vec{R} = \dfrac{m_2 \vec{r}_2 + m_1 \vec{r}_1}{M}$

ITS VELOCITY IS
$$\vec{V} = \frac{m_2 \vec{V}_2 + m_1 \vec{V}_1}{M}$$

THE TOTAL MOMENTUM IS $\quad \vec{P} = M\vec{V} = m_2 \vec{V}_2 + m_1 \vec{V}_1 = \vec{P}_2 + \vec{P}_1$

WE DEFINE $\quad m = \dfrac{m_1 m_2}{M}$

AND $\quad \vec{p} = m \vec{v} = m\left(\dfrac{\vec{P}_2}{m_2} - \dfrac{P_1}{m_1}\right) \quad$ IS THE RELATIVE MOMENTUM

THE ENERGY OF THE TWO BODIES IS

$$E = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(r_2 - r_1)$$

WE STATE THAT THIS IS EQUAL TO

$$E = \frac{P^2}{2M} + \frac{p^2}{2m} + U(r)$$

INDEED

$$E = \frac{1}{2}\left[\frac{P_2^2 + P_1^2 + 2\vec{P}_1\cdot\vec{P}_2}{M} + \frac{1}{m}\cdot m^2 \, \frac{P_2^2 m_1^2 + P_1^2 m_2^2 - 2\vec{P}_1\cdot\vec{P}_2 m_1 m_2}{m_1 m_2} \right]$$

$$+ U(r) = \frac{1}{2}\left[\frac{P_2^2 + P_1^2 + 2\vec{P}_1\cdot\vec{P}_2}{m_1+m_2} + \frac{m_1 m_2}{m_1+m_2}\left[\frac{P_2^2 m_1^2 + P_1^2 m_2^2 - 2\vec{P}_1\cdot\vec{P}_2 m_1 m_2}{m_1^2 m_2^2} \right] \right]$$

$$+ U(r) =$$

$$= \frac{1}{2}\left[P_2^2 \left[\frac{1}{m_1+m_2}\left(1 + \frac{m_1}{m_2}\right)\right] + P_1^2 \left[\frac{1}{m_1+m_2}\left(1+\frac{m_2}{m_1}\right)\right] + \frac{2\vec{P}_1\cdot\vec{P}_2}{m_1+m_2} - \frac{2\vec{P}_1\cdot\vec{P}_2}{m_1+m_2} \right] + U(r) =$$

$$= \frac{1}{2}\left[P_2^2\left(\frac{1}{m_2}\right) + P_1^2\left(\frac{1}{m_1}\right) \right] + U(r_2 - r_1)$$

THE DYNAMICS SEPARATES IN THE ONE OF P

(NO FORCE, NO POTENTIAL, MOVES FREELY)

AND p, r

IF $m_1 >> m_2 \Rightarrow m \simeq m_1$

$\vec{R} \simeq \vec{r}_1$

WE MAY TREAT THE 2-BODY PROBLEM AS THE PROBLEM OF A SINGLE BODY WITH MASS $m \simeq m_1$ AT DISTANCE $\simeq \vec{r}$ FROM THE CENTRE OF MASS

$\Rightarrow$ $\quad -\hbar^2\left(\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta)+\frac{1}{\sin^2\theta}\partial_\varphi^2\right)$

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\psi+\frac{\hat{L}^2}{2mr^2}\psi-\frac{e^2}{r}\psi=E\psi$$

IF WE WRITE $\psi = R\,Y_{em}(\theta,\varphi)$ WE HAVE

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\partial_r(r^2\partial_r)R+\frac{\hbar^2 \ell(\ell+1)}{2m\,r^2}R-\frac{e^2}{r}R=ER$$

LET US DEFINE

$$\boxed{\xi(r)=r\,R(r)} \Rightarrow R=\frac{\xi}{r}$$

WE HAVE

$$r^2\left(\frac{\partial}{\partial r}\left(\frac{\xi(r)}{r}\right)\right)=r^2\left(\frac{1}{r}\partial_r\xi(r)-\frac{1}{r^2}\xi(r)\right)=r\,\partial_r\xi(r)-\xi(r)$$

$$\Rightarrow \frac{1}{r^2}\partial_r\left(r^2\partial_r(R(r))\right)=\frac{1}{r^2}\partial_r\left[r\,\partial_r\xi(r)-\xi(r)\right]=$$

$$=\cancel{\frac{1}{r^2}\partial_r\xi(r)}+\frac{1}{r}\partial_r^2\xi(r)-\cancel{\frac{1}{r^2}\partial_r\xi(r)}$$

$\Rightarrow$

$$-\frac{\hbar^2}{2m}\frac{1}{r}\partial_r^2\xi+\frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^3}\xi-\frac{e^2}{r^2}\xi-E\frac{\xi}{r}=0$$

$\Rightarrow$ (MULTIPLY BY r)

$$\frac{\hbar^2}{2m}\partial_r^2\xi(r)-\frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\xi(r)+\frac{e^2}{r}\xi(r)+E\,\xi(r)=0$$

$$\Rightarrow \boxed{\partial_r^2\xi(r)+\frac{2m}{\hbar^2}\left[-\frac{\hbar^2\ell(\ell+1)}{2m\,r^2}+\frac{e^2}{r}+E\right]\xi(r)=0}$$

DIMENSIONAL ANALYSIS

WE ARE USING

$$M\frac{L}{T^2} = F = \frac{e^2}{r^2} = \frac{e^2}{L^2} \Rightarrow [e^2] = \frac{[M][L^3]}{[T^2]}$$

THE QUANTITY

$\frac{\hbar^2}{e^2 m}$ HAS DIMENSIONS $\quad \hbar = E\cdot T = M\frac{L^2}{\cancel{T^2}}\cdot\cancel{T}$

$$\Rightarrow \frac{M^2 L^4}{T^2}\cdot\frac{1}{M}\cdot\frac{T^2}{M L^3} = L$$

WE DEFINE

$$\frac{\hbar^2}{e^2 m} = r_0 \qquad \text{(BOHR RADIUS)}$$

AND MEASURE IN UNITS OF r_0, $\boxed{\mu = \frac{r}{r_0}} \Rightarrow \boxed{r = r_0\mu}$

$$\partial_r = \partial_\mu \frac{\partial\mu}{\partial r} = \frac{1}{r_0}\partial_\mu \Rightarrow \boxed{\partial_r^2 = \frac{1}{r_0^2}\partial_\mu^2}$$

$\Rightarrow$ WE DIVIDE THE EQUATION BY e^2

$$\frac{1}{r_0^2}\partial^2_\mu f(\mu) + \frac{2}{r_0}\left[-\frac{\ell(\ell+1)}{2r_0^2\mu^2}r_0 + \frac{1}{r_0\mu} + \frac{E}{e^2}\right]f(\mu) = 0$$

$$\Uparrow$$

$$\partial_r^2 f + 2\frac{m e^2}{\hbar^2}\left[-\frac{1}{2}\frac{\ell(\ell+1)}{r^2}\frac{\hbar^2}{m e^2} + \frac{1}{r} + \frac{E}{e^2}\right]f(r) = 0$$

NOW WE MULTIPLY BY r_0^2

$\Rightarrow$

$$\partial_\mu^2 f(\mu) + \left[-\frac{\ell(\ell+1)}{\mu^2} + \frac{2}{\mu} + \frac{r_0}{e^2} E \right] f(\mu) = 0$$

NOW WE NOTE THAT

$$\left[\frac{e^2}{r_0} \right] = \frac{M L^3}{T^2 L} = \frac{M L^2}{T^2} = [E]$$

$\Rightarrow$ WE DEFINE

$$\boxed{\frac{e^2}{r_0} \equiv E_0}$$

$$\partial_\mu^2 f(\mu) + \left[-\frac{\ell(\ell+1)}{\mu^2} + \frac{2}{\mu} + \frac{E}{E_0} \right] f(\mu) = 0$$

WE ARE INTERESTED IN ELECTRONS IN THE ATOM, MEANING THOSE WITH NEGATIVE ENERGY $E = \frac{p^2}{2m} - \frac{e}{r}$, $r \to \infty$ $E > 0$ MEANS POSITIVE E MAY GO ANYWHERE

$$\frac{E}{E_0} = -\frac{1}{n^2} \qquad \left(\text{SINCE } E_0 = \frac{e^2}{r_0} > 0 \right)$$

$(n > 0)$

$\Rightarrow$

$$\partial_\mu^2 f(\mu) + \left[-\frac{\ell(\ell+1)}{\mu^2} + \frac{2}{\mu} - \frac{1}{n^2} \right] f(\mu) = 0$$

NOW WE TRY TO UNDERSTAND HOW $f(\mu)$ BEHAVES FOR

$\mu \to \infty$ $(r \to \infty)$ $\qquad \frac{1}{\mu^2} \to 0$, $\frac{1}{\mu} \to 0$

$$\partial_\mu^2 f(\mu) \simeq \frac{1}{n^2} f(\mu) \Rightarrow f(\mu) \simeq e^{\pm \frac{\mu}{n}}$$

SINCE $\mu > 0$, $n > 0$ $\Rightarrow$ $f(\mu) \simeq e^{-\frac{\mu}{n}}$ (OTHERWISE DIVERGES)

WE ALSO CHECK THE BEHAVIOUR FOR

$u \to 0 \quad (r \to 0)$

$$\partial_u^2 \xi(u) \simeq \frac{\ell(\ell+1)}{u^2} \xi(u)$$

WE TRY ASSUMPTION AS

$$\xi(u) = u^k$$

$$\Rightarrow \quad \partial_u^2 \xi(u) = k(k-1) u^{k-2}$$

$$\Rightarrow \quad k(k-1) u^{k-2} = \ell(\ell+1) u^{k-2}$$

$$\Rightarrow \quad k = \ell + 1$$

WE ASSUME THE CORRECT FORM TO BE

$$\boxed{\xi(u) = u^{\ell+1} e^{-\frac{u}{n}} f(u)}$$

FROM THIS ASSUMPTION, WE HAVE

$$\partial_u(\xi(u)) = (\ell+1) u^{\ell} e^{-\frac{u}{n}} f(u) + u^{\ell+1}\left(-\frac{1}{n}\right) e^{-\frac{u}{n}} f(u)$$

$$+ u^{\ell+1} e^{-\frac{u}{n}} \partial_u(f(u)) =$$

$$= u^{\ell+1} e^{-\frac{u}{n}} f(u) \left[\frac{\ell+1}{u} - \frac{1}{n} + \frac{\partial_u(f(u))}{f(u)}\right] =$$

$$= \xi(u) \left[\frac{\ell+1}{u} - \frac{1}{n} + \frac{\partial_u(f(u))}{f(u)}\right]$$

AND THUS

$$\partial^2_\mu\left(\xi(\mu)\right) = \partial_\mu\left(\xi(\mu)\right)\left[\frac{l+1}{\mu} - \frac{1}{n} + \frac{\partial_\mu(f(\mu))}{f(\mu)}\right] +$$

$$+ \xi(\mu)\left[-\frac{l+1}{\mu^2} + \frac{\partial^2_\mu(f(\mu))}{f(\mu)} - \frac{\left(\partial_\mu(f(\mu))\right)^2}{f^2(\mu)}\right] =$$

$$= \xi(\mu)\left[\left(\frac{l+1}{\mu} - \frac{1}{n} + \frac{\partial_\mu(f(\mu))}{f(\mu)}\right)^2 - \frac{l+1}{\mu^2} + \frac{\partial^2_\mu(f(\mu))}{f(\mu)} - \frac{\left(\partial_\mu(f(\mu))\right)^2}{f^2(\mu)}\right] =$$

$$= \xi(\mu)\left[\frac{(l+1)^2}{\mu^2} + \frac{1}{n^2} + \frac{\left(\partial_\mu(f(\mu))\right)^2}{\left(f(\mu)\right)^2} - \frac{2(l+1)}{n\mu} + \frac{2(l+1)\partial_\mu(f(\mu))}{\mu f(\mu)}\right.$$

$$\left. - \frac{2\partial_\mu f(\mu)}{n f(\mu)} - \frac{l+1}{\mu^2} + \frac{\partial^2_\mu(f(\mu))}{f(\mu)} - \frac{\left(\partial_\mu(f(\mu))\right)^2}{f^2(\mu)}\right]$$

USING $\quad (l+1)^2 = (l+1)(l+1) = (l+1) + l(l+1) \quad$ WE GET

2)

$$\partial^2_\mu\left(\xi(\mu)\right) = \xi(\mu)\left[\frac{l+1}{\mu^2} + \frac{l(l+1)}{\mu^2} - \frac{l+1}{\mu^2} - \frac{2(l+1)}{n\mu} +\right.$$

$$\left. + \frac{2(l+1)\partial_\mu(f(\mu))}{\mu f(\mu)} - \frac{2\partial_\mu(f(\mu))}{n f(\mu)} + \frac{\partial^2_\mu(f(\mu))}{f(\mu)}\right]$$

SO THAT

$$\partial^2_\mu\left(\xi(\mu)\right) + \left[-\frac{l(l+1)}{\mu^2} + \frac{2}{\mu} - \frac{1}{n^2}\right]\xi(\mu) = 0$$

BECOMES

$$\xi(\mu)\left[\cancel{\frac{\ell(\ell+1)}{\mu^2}} - \frac{2(\ell+1)}{n\mu} + \frac{2(\ell+1)\,\partial_\mu(f(\mu))}{\mu f(\mu)} - \frac{2\,\partial_\mu(f(\mu))}{n f(\mu)} + \frac{\partial^2_\mu(f(\mu))}{f(\mu)} - \cancel{\frac{\ell(\ell+1)}{\mu^2}} + \frac{2}{\mu} - \frac{1}{n^2}\right] = 0$$

THIS MAY BE REWRITTEN AS $[\quad] = 0$ OR

PUTTING THE ∂^2_μ AND ∂_μ TERMS TOGETHER

$\Rightarrow$

$$\frac{1}{f(\mu)}\partial^2_\mu(f(\mu)) + \frac{2(\ell+1)n - 2\mu}{n\mu f(\mu)}\partial_\mu(f(\mu)) + \frac{2n - 2(\ell+1)}{n\mu} = 0$$

$\Rightarrow$

$$\partial^2_\mu(f(\mu)) + \frac{2[(\ell+1)n - \mu]}{n\mu}\partial_\mu(f(\mu)) + 2\frac{n-\ell-1}{n\mu}f(\mu) = 0$$

TO SIMPLIFY THIS EQUATION WE CHANGE AGAIN VARIABLE TO

$$\boxed{k = \frac{2\mu}{n}} \Rightarrow \boxed{\mu = \frac{nk}{2}} \Rightarrow \partial_\mu = \partial_k\frac{\partial k}{\partial\mu} = \frac{2}{n}\partial_k$$

$\Rightarrow$

$$\frac{4}{n^2}\partial^2_k(f(\mu)) + \frac{2}{n}\cdot 2\frac{(\ell+1)n - \frac{nk}{2}}{\frac{n^2 k}{2}}\partial_k(f(\mu)) + 2\frac{n-\ell-1}{n^2\frac{k}{2}}f(\mu) = 0$$

$\Rightarrow$

THAT'S WHY WE MOVED TO k

$$\Rightarrow \frac{4}{n^2}\left[\partial^2_k\left(f(k)\right) + \frac{2(l+1)-k}{k}\partial_k\left(f(k)\right) + \frac{n-l-1}{k} f(k)\right] = 0$$

WE DELETE $\frac{4}{n^2}$ AND MULTIPLY BY k

$$\Rightarrow k\,\partial^2_k\left(f(k)\right) + \left(2(l+1)-k\right)\partial_k f(k) + \left(n-l-1\right) f(k) = 0$$

WE DEFINE

$$\alpha \equiv 2(l+1) \qquad \beta \equiv n-l-1$$

$$\Rightarrow k\,\partial^2_k\left(f(k)\right) + (\alpha-k)\,\partial_k\left(f(k)\right) + \beta f(k) = 0$$

LET US TRY TO SOLVE IT USING ANSATZ

$$f(k) = \sum_{k=0}^{\infty} c_j k^j$$

$$\Rightarrow k\,\partial^2_k\left(f(k)\right) = \sum_{j=0}^{\infty} c_j\left[j(j-1)\right] k^{j-1}$$

$$= \sum_{j=1}^{\infty} c_j\left[j(j-1)\right] k^{j-1}$$

$\rightarrow$ SINCE THE $J=0$ TERM IS 0

$$(\alpha - u)\,\partial_u (f(u)) = \sum_{j=0}^{\infty} \alpha(j)\, c_j\, u^{j-1} - \sum_{j=0}^{\infty} j\, c_j\, u^j$$

$$= \sum_{j=1}^{\infty} \alpha\, j\, c_j\, u^{j-1} - \sum_{j=0}^{\infty} j\, c_j\, u^j$$

AGAIN NOTE: I DON'T MATTER AT ALL, ONLY IF WRITTEN

$$\beta f(u) = \sum_{j=0}^{\infty} \beta\, c_j\, u^j$$

$\Rightarrow$

$$\sum_{j=1}^{\infty} \left(c_j\, [j(j-1)]\, u^{j-1} + \alpha(j)\, u^{j-1} \right) + \sum_{j=0}^{\infty} (\beta - j)\, c_j\, u^j$$

$$j' = j - 1 \quad , \quad j = j' + 1$$

$\Rightarrow$

$$\sum_{j'=0}^{\infty} c_{j'+1} \left[(j'+1)\,[j'+\alpha] \right] u^{j'} + \sum_{j=0}^{\infty} (\beta - j)\, c_j\, u^j$$

$$\Rightarrow \sum_{j=0}^{\infty} u^j \left[c_{j+1} \left[(j+1)(j+\alpha) \right] - c_j (j - \beta) \right] = 0$$

THIS IS TRUE IF

$$\boxed{c_{j+1} = c_j \, \frac{(j-\beta)}{(j+1)(j+\alpha)}}$$

WE NOTE THAT

$$j \to \infty \qquad c_{j+1} \simeq c_j \frac{1}{j}$$

THE SERIES BEHAVES AS THE EXPONENTIAL ASYMPTOTICALLY.

BUT THE EXP. DIVERGES IF $r \to \infty$ $(u \to \infty)$

WE DO NOT WANT IT

WE NEED SOME $c_{\tilde{j}} = 0$

$$\Rightarrow \quad \tilde{j} = \beta \qquad (\tilde{j} \text{ IS AN INTEGER})$$

BUT

$$\beta = n - \ell - 1$$

$$\Rightarrow \quad n = \ell + 1 + \tilde{j}$$

$$\Rightarrow \quad \boxed{n \in \mathbb{N}, \quad n > \ell}$$

$$\boxed{E = -\frac{E_0}{n^2} \qquad \text{MINIMUM} \quad E = -E_0}$$

PHYSICAL STATES

$$|n, \ell, m\rangle$$

$n = 1, 2, 3, 4 \ldots$

$\ell = 0, 1, \ldots, n-1$

$m = -\ell, \ldots, \ell$

BROKEN DEG.

$$f(u) = 1 + \frac{\beta}{\alpha} u + \frac{\beta(\beta+1)}{\alpha(\alpha+1)2!} + \frac{\beta(\beta+1)(\beta+2)}{\alpha(\alpha+1)(\alpha+2)3!} u^3 + \cdots$$

$$r = \frac{2u}{n} = \frac{2r}{n r_0}$$

$$R(r) = \frac{1}{r}\left(\frac{r}{r_0}\right)^{l+1} e^{-\frac{r}{n r_0}} f\left(\frac{2r}{n r_0}\right) = R_{nl}(r)$$

$n = 1 \quad R \propto e^{-\frac{r}{r_0}} \qquad l = 0 \quad m = 0$

$$\psi = e^{-\frac{r}{r_0}}$$