ANOTHER APPROACH TO THE HEAT-DIFFUSION EQ.

CONSIDER A FIXED VOLUME V. THE AMOUNT OF ... IN V

IS $Q = \int_V dx^3 \rho(\vec{x}, t) = Q_V(t)$

$$\Rightarrow \frac{dQ_V}{dt} = \int dx^3 \frac{\partial \rho}{\partial t}$$

BUT THE CHANGE IN Q_V IS ALSO GIVEN BY THE FLUX OF .... THROUGH

THE SURFACE $\vec{\Phi} = \rho \vec{v}$

SINCE $\vec{n}$ IS OUT-GOING

$$\frac{dQ_V}{dt} = - \int_S \vec{\Phi} \cdot \vec{n}\, dS$$

S → BOUNDARY OF V

BUT $\int_S \vec{\Phi} \cdot \vec{n}\, dS = \int_V \operatorname{div} \vec{\Phi}$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \operatorname{div} \vec{\Phi} = 0$$

ASSUMING $\vec{\Phi} = -k \vec{\nabla} \rho \Rightarrow$ $\boxed{\partial_t \rho = k \nabla^2 \rho}$

— 0 —

SOLUTION OF P.D.E. : SEPARATION OF VARIABLES

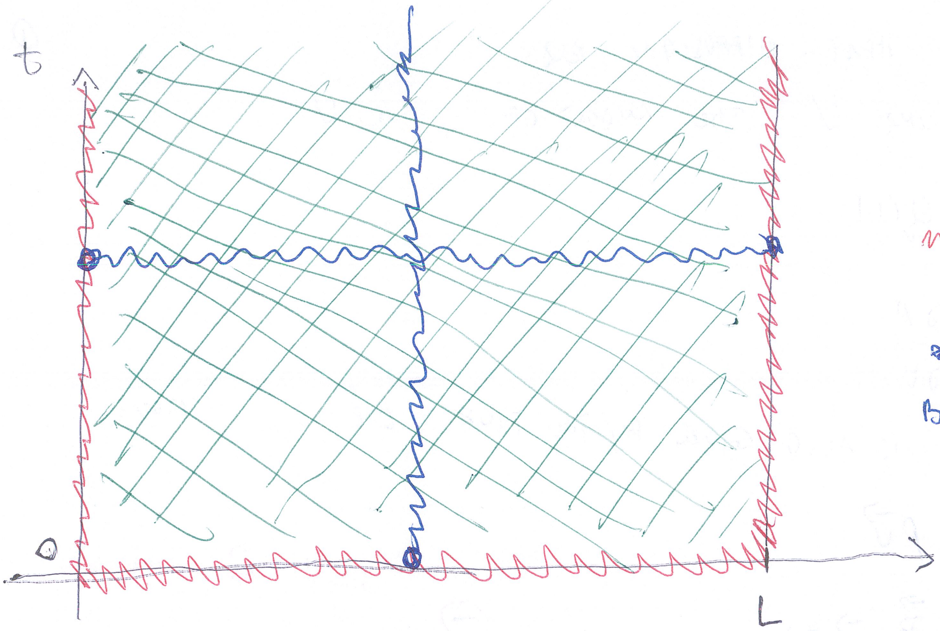
WE CONSIDER AGAIN THE HEAT-DIFFUSION EQ.

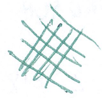
IT IS EXPRESSED AS A RELATION BETWEEN P.D., ρ GIVING A SOLUTION OF THE (MODELED) PHYSICAL PROBLEM WILL SATISFY SUCH A RELATION IN A GIVEN AREA OF SPACE AND TIME (NOT SPACE-TIME ...)

WE MAY FOR EX. CONSIDER THE TIME EVOLUTION OF THE TEMPERATURE ON A METAL BAR. THE BAR HAS A FINITE LENGHT L. WE GIVE THE INITIAL TEMPERATURE FOR EACH $x \in [0, L]$ (THIS IS A 1D PROBLEM, i.e. WE CONSIDER THE BAR AS INFTY THIN), AND PROVIDE SOME INFORMATION ABOUT THE BEHAVIOUR OF THE TEMPERATURE FOR ALL $t \geq 0$ IN $x = 0$ AND $x = L$.

THIS WILL BE NEEDED AND ENOUGH TO SOLVE THE PROBLEM. WHY?

WITH ODE WE NEEDED B.C. ....



In  we find the solution, with ~~~~ given.

~~~~ we see behavior at fixed t or x

N.B. All the following discussion is extremely not-rig. We just give plausible arguments, not proofs. We also assume functions to be well behaved (smooth) etc.

Let us first consider the behaviour at fixed $x = \bar{x}$

$$\partial_t \rho(t, \bar{x}) = \nabla^2 \rho(t, \bar{x})$$

If we ignore the x dependence, this looks like a 1st order ODE in t. It needs an initial condition $\rho(0, \bar{x})$

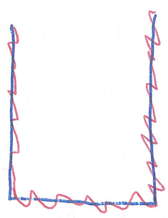
But this is true $\forall \bar{x}$ so we need a function

$$\rho(0, x), \quad x \in [0, L]$$

What about the behaviour at fixed $\bar{t}$?

This looks like a 2nd order ODE, for which we ask 2 I.C.

$$f''(t) = F(f'(t), f(t), t) \Rightarrow f'(0) = \dots \quad f(0) = \dots$$

But here we are studying  we would like to tell what happens here in L

For a 2nd order ODE we may also give the initial and final value of f (or f') $\Rightarrow f(0) = \dots \quad f(L) = \dots$

Think about a ballistic problem, defined by ↗ $x(0), v(0)$

but also 
~~~~

$\Rightarrow \rho(\bar{t},0)\,,\ \rho(\bar{t},L)$ OR ARE

$\rho(t,0) \qquad \rho(t,L] \quad \forall t$

THE ~~TIME~~ B.C. IS EASY TO UNDERSTAND: WE NEED TO SPECIFY WHAT ~~[illegible]~~ IS THE INITIAL CONDITION OF THE SYSTEM (NOT OF TIME). BUT IS CLEAR THAT ALSO THE PHYSICAL CONDITION AT THE BOUNDARY HAVE TO BE SPECIFIED: WHAT TEMPERATURE IN $0, L$? OR WHAT FLOW CONDITIONS? $(\partial_x \rho)$

LET US, FOR EXAMPLE, ASSUME THAT WE FIX THE DENSITY-TEMPERATURE AT 0

$$\rho(t,0) = \rho(t,L) = 0$$

LET US SOLVE THIS PROBLEM!

BUT FIRST, FOR REASONS THAT WILL BECOME CLEAR LATER, LET US SCALE

$[0,L] \Rightarrow [0,\pi]$

WE CALL x' THE VARIABLE THAT RUNS BETWEEN 0 AND L, FOR WHICH

$$\partial_{t'} \rho(t',x') = K \partial^2_{x'} \rho(t',x')$$

WE INTRODUCE X THAT RUNS BETWEEN $[0,\pi]$

$$\boxed{X = \frac{\pi}{L} x'} \Rightarrow x' = \frac{L}{\pi} X$$

FOR SIMPLICITY OF NOTATION, WE CALL $\alpha \equiv \frac{\pi}{L}$

WE HAVE $\frac{\partial}{\partial x'} = \frac{\partial}{\partial X}\frac{\partial X}{\partial x'} = \alpha \frac{\partial}{\partial X} \Rightarrow \frac{\partial^2}{\partial x'^2} = \alpha^2 \frac{\partial}{\partial X^2}$

$$\Rightarrow \partial_{t'} \rho(t',x) = K \alpha^2 \partial^2_x \rho(t',x)$$

WE NOW TRY TO SCALE ALSO THE TIME

$$t = \beta t' \,,\quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t}\frac{\partial t}{\partial t'} = \frac{\partial}{\partial t}\beta$$

$$\Rightarrow \beta\, \partial_t \rho(t,x) = K \alpha^2 \partial^2_x \rho(t,x)$$

$\Rightarrow$ IF $\beta = k\alpha^2 = k\frac{\pi^2}{L^2}$ $\Rightarrow$ $t = k\frac{\pi^2}{L^2}t'$

$t' = \frac{L^2}{k\pi^2}t$

$$\Rightarrow \boxed{\partial_t \rho = \partial_x^2 \rho}$$ $x \in [0,\pi]$

NOTE (DIM. AN.)

$\partial_{t'}\rho = k\,\partial_{x'^2}\rho$ $\Rightarrow [T]^{-1} = [k]\cdot[L]^{-2}$

$\Rightarrow [k] = \frac{[L]^2}{[T]}$

$\Rightarrow [t] = \frac{[L]^2}{[T]}\cdot\frac{1}{[L]^2}\cdot[T] = 0$

$[x] = [L]\cdot\frac{1}{[L]} = 0$

— o —

TO SOLVE

$\partial_{t'}\rho(t',x') = k\,\partial_{x'}^2\rho(t',x')$

$\rho(0,x') = f(x')$

$\rho(t',0) = 0 = \rho(t',L)$

WE GET

$\partial_t\rho = \partial_x^2\rho$

$\rho(0,x) = f\left(\frac{\pi}{L}x'\right)$

$\rho(t,0) = \rho(t,\pi) = 0$

THE SOLUTION

$$\rho(t,x) = \rho\left(k\frac{\pi^2}{L^2}t', \frac{\pi}{L}x'\right)$$

TO SOLVE IT, WE USE A METHOD CALLED

SEPARATION OF VARIABLES

WE ASSUME

$$\rho(t,x) = T(t)X(x)$$

$$\Rightarrow \quad \partial_t [T(t)X(x)] = \partial_x^2 [T(t)X(x)]$$

$$\Rightarrow \quad X(x)\,\partial_t T(t) = T(x)\,\partial_x^2 X(x)$$

WE DIVIDE BOTH SIDES BY $X(x)T(t)$

(WE DO NOT WORRY ABOUT ZEROS. WE APPLY THIS METHOD HOPING IT WILL PROVIDE A CORRECT SOLUTION)

$$\Rightarrow \quad \frac{\partial_t T(t)}{T(t)} = \frac{\partial_x^2 X(x)}{X(x)}$$

FUNCTION OF t = FUNCTION OF x $\quad \forall x, t$!!

$$\Rightarrow \quad \frac{\partial_t T(t)}{T(t)} = \overset{\text{CONST}}{C_1} = \frac{\partial_x^2 X(x)}{X(x)}$$

$$\Rightarrow \quad \partial_t T(t) = C_1 T(t) \qquad \partial_x^2 X(x) = C_1 X(x)$$

IF THIS IS TRUE $\forall x, t$

$$\partial_t X(x)T(t) = \partial_t \rho = \partial_x^2 \rho = \partial_x^2 X(x)T(t)$$

$$X(x)\,\partial_t T(t) = C_1 X(x) T(t) = T(t)\,\partial_x^2 X(x)$$

$\Rightarrow$ WE DIDN'T NEED TO WORRY ABOUT ZEROS

THE SOLUTION FOR t IS

$$\partial_t T(t) = C_1 T(t) \Rightarrow T(t) \propto e^{C_1 t}$$

BUT LET US FIRST

WE CONSIDER THE 3 CASES

I) $C = 0$

II) $C < 0$, $C = -\lambda^2$

III) $C > 0$, $C = \lambda^2$

I)

$$\partial^2_x X = 0 \Rightarrow \partial_x X = A \Rightarrow X(x) = Ax + B$$

BUT

$$p(t,x) = A' e^{0t} [Ax + B] = Ax + B$$

WE RE-DEFINE $A = A'A$

$$p(t,0) = 0 = B \qquad p(t,\pi) = 0 = A\pi$$

$$\Rightarrow p(t,x) = 0$$

THIS CAN SATISFY ONLY $f(x) = p(0,x) = 0$

THE TRIVIAL SOLUTION WE NOT CARE!

II)

$$\partial^2_x X = \lambda^2 X \Rightarrow X(x) = A e^{\lambda x} + B e^{-\lambda x}$$

$T(t) \propto e^{\lambda^2 t} \Rightarrow T(0) \propto 1 \ldots$ BUT $T(0)$ WILL ALWAYS BE ABSORBED... JUST RE-SCALE $X(x)$, NOT DIFFERENT FROM A, B

$$X(0) = A + B = 0 \Rightarrow B = -A \Rightarrow X(x) = A(e^{\lambda x} - e^{-\lambda x})$$

$$X(\pi) = A(e^{\lambda\pi} - e^{-\lambda\pi}) \neq 0 \,! \quad (\text{Since } (e^x)' > 0 \ \forall x)$$

NO SOL FOR $C > 0$

III)

$$\partial_x^2 X(x) = -\lambda^2 X(x)$$

$$\Rightarrow \quad X(x) = A \sin(\lambda x) + B \cos(\lambda x)$$

$$\Rightarrow \quad X(0) = B = 0 \qquad \Rightarrow \quad X(x) = A \sin(\lambda x)$$

$$X(\pi) = A \sin(\lambda \pi) = 0 \qquad \text{IF } \lambda \in \mathbb{Z}$$

BUT $\lambda = 0 \quad \Rightarrow \quad X(x) = 0$

$\lambda = -m \,, m \in \mathbb{N} \quad \Rightarrow \quad A \sin(\lambda \pi) = -A \sin(m\pi) = A' \sin(m\pi)$

WE MAY CONSIDER $m \in \mathbb{N}$

IF

$$X(x) = A \sin(m x)$$

$$\partial_x^2 X = -m^2 X$$

$$\Rightarrow \quad \partial_t T = -m^2 T$$

$$\Rightarrow \quad T(t) = e^{-m^2 t}$$

$$\Rightarrow \quad p(t,x) = A e^{-m^2 t} \sin(m x) \qquad \left[= A e^{-\frac{m^2 \kappa \pi^2}{L^2} t'} \sin\left(m \frac{\pi}{L} x'\right) \right]$$

$\hookrightarrow 0$ FOR $t \to \infty$

IS A SOLUTION OF OUR P.D.E

BUT, $p(0,x) = A \sin(mx)$ IT'S THE ONLY INITIAL SOLUTION THAT IT CAN SATISFY !

BUT OUR P.D.E IS LINEAR, SINCE

$$\partial_t [\alpha f(t,x) + \beta g(t,x)] = \alpha \partial_t f + \beta \partial_t g$$

$$\partial_x^2 [\alpha f(t,x) + \beta g(t,x)] = \alpha \partial_x^2 f + \beta \partial_x^2 g$$

LET US CALL

$$L = (\partial_t - \partial_x^2)$$

THIS 0 IS IMPORTANT

IF $Lf = 0$ AND $Lg = 0$

$Lf = 4$
$Lg = 4$
BUT $L(f+g) \neq 4$

HOMOGENEOUS

$$\Rightarrow L(\alpha f + \beta g) = 0$$

$$\Rightarrow p = \sum_{m=1}^{\infty} A_m e^{-m^2 t} \sin(mx)$$ IS A SOLUTION WITH

$$p(t,0) = 0 = p(t,\pi) \quad !$$

THIS 0 IS IMPORTANT.

$f(t,0) = 4$
$g(t,0) = 4$ $f+g\,(t,0) \neq 4$
AGAIN. !

$$p(t,x) = \sum_{m=1}^{\infty} A_m e^{-mt^2} \sin(mx)$$

$$p(0,x) = \sum_{m=1}^{\infty} A_m \sin(mx)$$

$$\parallel$$

$$f(x)$$

BUT THE FOURIER SERIES THEORY TELLS US THAT ANY REGULAR f WITH $f(0) = 0 = f(L)$ MAY BE WRITTEN LIKE THAT.

WE CONSIDER THIS AS GIVEN, BUT WE REMIND THE USE OF F.S.T. JUST FROM OUR P.D.E PERSPECTIVE.

WE WANT TO BE ABLE TO COMPUTE THE

A_n FROM $f(x)$ (SEE → S.L.)

WE FIRST NOTICE THAT

$$\int_0^{\pi} \sin(nx)\sin(mx) = 0$$

WHY? WE RECALL THE USEFUL RELATIONS

$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$$

IF $y=0$ $\sin(x) = \sin(x)$

IF $x=0$ $\sin(y) = \sin(y)$ ...

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

IF $y=0$ $\cos(x) = \cos(x)$ ...

$$\Rightarrow \cos(x-y) - \cos(x+y) = \cos(x)\cos(-y) - \sin(x)\sin(-y) - \cos(x)\cos(y) + \sin(x)\sin(y) = \cancel{\cos(x)\cos(y)} - \cancel{\cos(x)\cos(y)} + \sin(x)\sin(y) + \sin(x)\sin(y)$$

$$\Rightarrow \sin(x)\sin(y) = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

$$\Rightarrow \int_0^{\pi} \sin(nx)\sin(mx) = \frac{1}{2}\int_0^{\pi}\cos((n-m)x) + \frac{1}{2}\int_0^{\pi}\cos(n+m)x$$

IF $n \neq m$

$$\int_0^{\pi} \cos kx \;, k \neq 0 = \int_0^{\pi} dx \frac{d}{dx}\left[\frac{1}{k}\sin(kx)\right] = \frac{1}{k}\left[\sin(kx)\right]_0^{\pi} = 0$$

$$\Rightarrow \int_0^{\pi} \sin(nx)\sin(mx) = 0$$

IF $n = m$

$$\sin^2(mx) = \sin(mx)\sin(mx) = \frac{1}{2}\left[\cos(m-m) - \cos(2m)\right] = \frac{1}{2} - \frac{1}{2}\cos(2m)$$

$$\Rightarrow \int_0^{\pi} \sin^2(mx) = 0 + \frac{1}{2}\int_0^{\pi} dx = \frac{\pi}{2}$$

$\Rightarrow$

$$\int_0^{\pi} \sin(nx)\sin(mx) = \delta_{nm}\frac{\pi}{2}$$

$$\Rightarrow \int_0^{\pi} \left(\lambda \sin(nx)\right)\left(\lambda \sin(mx)\right) = \frac{\pi}{2}\lambda^2 \delta_{nm}$$

IF WE DEFINE

$$S_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$$

$$\Rightarrow \boxed{\int_0^{\pi} S_n(x)\, S_m(x) = \delta_{nm}}$$

OBVIOUSLY

$$P(t,x) = \sum_{n=1}^{\infty} A_n e^{-n^2 t} S_n(x)$$ IS STILL A SOLUTION

NOW WE ASSUME

$$f(x) = P(0,t) = \sum_{n=1}^{\infty} A_n S_n(x)$$

$$\int_0^{\pi} S_m(x) f(x) = \int_0^{\pi} S_m(x) \sum_{n=1}^{\infty} A_n S_n(x) = \sum_{n=1}^{\infty} A_n \int_0^{\pi} S_m(x) S_n(x)$$

$$= \sum_{n=1}^{\infty} A_n \delta_{nm} = A_m$$

$$\Rightarrow \quad A_m = \int_0^{\pi} S_m(x)\, f(x)\, dx$$

$$\Rightarrow \quad \rho(t,x) = \sum_{m=1}^{\infty} \left[\int_0^{\pi} S_m(x) \cancel{f(x)}\, \rho(0,x)\, dx \right] e^{-m^2 t}\, S_m(x)$$

Then modify to t', x'; if you want to work with t', x' to this

A_m, use

$$\delta_{mn} = \frac{2}{\pi} \int_0^{\pi} \sin(nx) \sin(mx)\, dx$$

$$x' = \frac{L}{\pi} x = \alpha^{-1} x \quad \Rightarrow \quad dx = \alpha\, dx'$$

$$\delta_{mn} = \frac{2}{\pi} \int_0^{L} \sin(n \alpha x') \sin(m \alpha x')\, \alpha\, dx'$$

$$\Rightarrow \quad \delta_{nm} = \frac{2}{\pi} \cdot \frac{\pi}{L} \int_0^{L} \sin\left(\frac{n\pi}{L} x'\right) \sin\left(\frac{m\pi}{L} x'\right)$$

$$\Rightarrow \quad S_m^L(x') = \sqrt{\frac{2}{L}} \sin\left(\frac{m\pi}{L} x'\right)$$

Let's go back to

Rather for all m, the last to resist is $\sin(x)$

$$\rho(t,x) = \sum_{m=1}^{\infty} A_m\, e^{-m^2 t} \sin(mx)$$

$$t \to \infty \quad \rho(t,x) \to 0 \quad \forall x$$

This $\rho = 0$ is the only solution of the stationary problem

$\rho(t,x) = \rho(x)$

$$\Rightarrow \quad \partial_t \rho = 0 = \partial^2_x \rho$$

$$\Rightarrow \quad \rho(x) = Ax + B \quad \Rightarrow \rho(x) = 0$$ GIVEN THE B.C.

WE MAY CONSIDER A DIFFERENT PHYSICAL PROBLEM, IN WHICH WE KEEP THE END $x = 0, L$ ISOLATED, i.e. NO FLUX

$$\vec{\Phi} \propto \vec{\nabla} \rho \quad \Rightarrow \Phi_x = \partial_x \rho$$

$$\Rightarrow \quad \boxed{\partial_x \rho(t,0) = \partial_x \rho(t,L) = 0}$$

LET US START FROM WHAT WE DID FOR THE $\rho(t,0) = \rho(t,L) = 0$ PROBLEM, i.e. THE ANALYSIS OF THE STATIONARY SOL.

$$\partial^2_x \rho = 0 \Rightarrow \quad \rho(x) = Ax + B$$

$$\Rightarrow \quad \rho'(x) = A = 0$$

$$\Rightarrow \quad \boxed{\rho(x) = B = \text{CONST}}$$

BUT WHICH B? HOW DOES IT RELATE TO $\rho(0,x)$?

(HERE I AM ASSUMING AGAIN THAT THE STATIONARY IS THE $\lim_{t\to\infty}$ OF $\rho(t,x)$)

LET US STUDY THE FULL PROBLEM!

THE SEPARATION OF VARIABLES IS THE SAME AS BEFORE

$$\frac{\partial_t T(t)}{T(t)} = C = \frac{\partial^2_x X(x)}{X(x)}$$

I) $C = \lambda^2 > 0$

$$\partial^2_x X = \lambda^2 X \Rightarrow X(x) = A e^{\lambda x} + B e^{-\lambda x}$$

$$\frac{d}{dx} X(x) = \lambda \left[A e^{\lambda x} - B e^{\lambda x} \right]$$

$$\frac{d}{dx} X(0) = \lambda (A - B) = 0 \Rightarrow B = +A$$

$$\Rightarrow X(x) = A \left(e^{\lambda x} + e^{-\lambda x} \right) \qquad \frac{d}{dx} X(x) = \lambda A \left(e^{\lambda x} - e^{-\lambda x} \right) \neq 0$$

for $x \neq 0$

NO SOL FOR $C > 0$

II) $C = 0$

$$\partial^2_x X = 0 \Rightarrow X(x) = Ax + B$$

$$X'(x) = A = 0$$

$$\Rightarrow X(x) = A$$

$$e^{0t} = 1 \qquad \rho(t,x) = A$$

IS A SOLUTION

III) $C = -\lambda^2$

$$\partial^2_x X(x) = -\lambda^2 X(x) \Rightarrow X(x) = A \sin(\lambda x) + B \cos(\lambda x)$$

$$\Rightarrow X'(x) = \lambda \left[A \cos(\lambda x) - B \sin(\lambda x) \right]$$

$X'(0) = \lambda A = 0 \quad \Rightarrow \quad X(x) = B\cos(\lambda x),$

$$\frac{d}{dx}X(x) = -\lambda B \sin(\lambda x)$$

$X(\pi) = 0 \quad \Rightarrow \quad \lambda = m \in \mathbb{N}$

WHY $\mathbb{N}$?, $\quad \cos(-m x) = \cos(m x),$

$\cos(0) = \text{const} \quad$ (THE $C = 0$ CASE)

THE DISCUSSION ABOUT L BEING LINEAR IS STILL VALID. ALSO $L\rho = 0$ IS ALWAYS TRUE, AND IF $f'(0) = 0$, $g'(0) = 0$

$\Rightarrow g' * f'(0) = 0$

$$\Rightarrow \boxed{A_0 + \sum_{m=1}^{\infty} B_m e^{-m^2 t} \cos(m x)} \quad \text{IS A SOLUTION}$$

WE WANT AGAIN TO COMPUTE B_m

$$\cos((m-n)x) + \cos((m+n)x) = \cos(m x)\cos(n x) + \sin(m x)\sin(n x) + \cos(m x)\cos(n x) - \sin(n x)\sin(m x)$$

$$= 2\cos(m x)\cos(n x) + 0$$

$$\Rightarrow \quad \cos(n x)\cos(m x) = \frac{1}{2}\left[\cos((m-n)x) + \cos((m+n)x)\right]$$

AGAIN

$$\int_0^{\pi} \cos(k x)\, dx = 0, \quad k \in \mathbb{N}$$

$\Rightarrow$ IF $n \neq m$

$$\int_0^{\pi} \cos(m x)\cos(n x) = 0$$

IF $n = m$

$$\cos^2(mx) = \frac{1}{2}\left[1 + \cos(2mx)\right]$$

$$\Rightarrow \int_0^{\pi} \cos^2(mx) = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\pi} \cos(nx)\cos(mx) = \frac{\pi}{2}\delta_{nm}$$

$$\Rightarrow \quad C_n \equiv \sqrt{\frac{2}{\pi}}\cos(nx) \qquad n \geq 1 \quad n \in \mathbb{N}$$

$$\int_0^{\pi} dx\, C_n(x)\, C_m(x) = \delta_{nm}$$

WHAT ABOUT THE FUNCTION $C_0(x)$? IT IS A CONSTANT, THE ONE THAT REFERS TO A_0

$$\int_0^{\pi} c\, dx = c\pi$$

$$\int_0^{\pi} c\cos(mx)\,dx = c\int_0^{\pi}\cos(mx)\,dx = 0$$

$$\Rightarrow \quad C_0 = \frac{1}{\sqrt{\pi}} \quad \text{GIVES}$$

$$\int_0^{\pi} C_n(x)\, C_m(x)\, dx = \delta_{n,m}$$

$$\Rightarrow \quad B_m = \int_0^{\pi} p(0,x)\, C_n(x)$$

FOR B_0 WE HAVE

$$B_0 = \int_0^{\pi} \frac{1}{\sqrt{\pi}}\, p(0,x) = \frac{1}{\sqrt{\pi}}\bar{p}\pi = \sqrt{\pi}\,\bar{p}$$

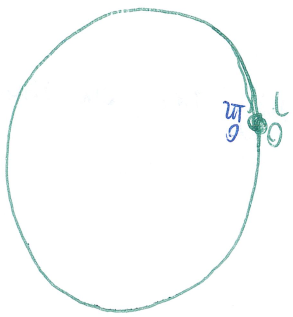
THE GENERAL SOLUTION (AGAIN FROM F.S.T) IS

$$\rho(t,x) = \sum_{n=0}^{\infty} \left[\int_0^{\pi} \rho(0,x)\, C_n(x) \right] e^{-n^2 t}\, C_n(x)$$

$t \to \infty$

$$\rho(t,x) \to A_0\, C_0(x) = \sqrt{\pi}\,\bar{\rho} \cdot \frac{1}{\sqrt{\pi}} = \bar{\rho}$$

LET US NOW BEND (BUT WITHOUT CHANGE ITS PHYSICAL PROPERTIES) THE BAR



WE USE A DIFFERENT SCALING HERE

$[0,L] \to [0,2\pi]$

$$x = x' \cdot \frac{2\pi}{L} \Rightarrow t = \frac{m^2 k\, 4\pi^2}{L^2}\, t'$$

$$\partial_t \rho = \partial_x^2 \rho$$

HERE IS CLEAR WHICH IS THE CORRECT B.C.

$$\rho(t, x+2\pi) = \rho(t,x)$$

WE OBTAIN AGAIN

$$\partial_t T = CT \qquad \partial_x^2 X = CX$$

$C = \lambda^2 > 0$

$$X(x) = A e^{\lambda x} + B e^{-\lambda x} \Rightarrow$$ NOT PERIODIC

WE NEED

$$C = -\lambda^2 \qquad \lambda^2 \geq 0$$

$C = 0$

$X(x) = Ax + B$ TO BE PERIODIC $A = 0$

FOR $C = -\lambda^2 < 0$ LET US USE AN EQUIVALENT BUT INSTRUCTIVE APPROACH

$e^{i\lambda x}$ IS A SOLUTION $\left(\partial^2_x e^{i\lambda x} = -\lambda^2 e^{i\lambda x}\right)$ /

PERIODIC ONLY IF $\lambda \in \mathbb{Z}$ ($\lambda = 0$ GIVES THE $C = 0$ CASE ABOVE)

LET US PUT TOGETHER m AND $-m$

$$X(x) = A' e^{imx} + B' e^{-imx}$$

WE WANT THE SOLUTION TO BE REAL

$$A' e^{imx} + B' e^{-imx} = X(x) = \bar{X}(x) = \bar{A}' e^{-imx} + \bar{B}' e^{imx}$$

$$\Rightarrow \quad A' = \bar{B}'$$

$$\Rightarrow \quad X(x) = (A'' + iB'') e^{imx} + (A'' - iB'') e^{-imx}$$

$$= 2A'' \frac{e^{imx} + e^{-imx}}{2} + \frac{2i}{i} B'' \frac{e^{imx} - e^{-imx}}{2} =$$

$$= B \cos(mx) + A \sin(mx)$$

IN THE CASE OF WORKING WITH COMPLEX FUNCTIONS, WE ~~SHOULD~~ DIRECTLY USE THE e^{imx}

$$= \int_0^{2\pi} \left(e^{-inx}\right) e^{imx} = \int_0^{2\pi} \overline{\left(e^{inx}\right)} e^{imx}$$

$$= \int_0^{2\pi} e^{i(m-n)x}$$

IF $m = n$

$$\int_0^{2\pi} 1 = 2\pi$$

IF $m - n$

$$\int_0^{2\pi} e^{ihx} = \frac{1}{ih}\left[e^{ihx}\right]_0^{2\pi} = 0$$

$$\Rightarrow \quad C_n(x) = \frac{1}{\sqrt{2\pi}} \Rightarrow \delta_{nm} = \int_0^{2\pi} C_n^*(x)\, C_m(x)$$

THE FACT THAT THE CONSTANT TERM COMES FROM C_0, WHILE THE COS AND SIN ARE LIN. COMB. OF C_n, C_{-n}, CREATES THE ASYMMETRY

USING ABOVE WE EASILY CHECK (BUT IT COULD BE DONE ALSO USING $\sin\cos = \sin(n+m) - \sin(n-m)$ FORMULA

$$\int_0^{2\pi} \sin(nx)\cos(mx) = \sum c \int e^{inx} e^{imx} = 0 \quad n \neq m$$

$$\int_0^{2\pi} \sin(nx)\cos(mx) = \overset{n+m\ \text{term}}{0} + \frac{1}{4i}\left[\int_0^{2\pi} e^{i(n-m)x} - e^{i(n-m)x}\right] = 0$$

WE ALSO HAVE

$$\int_0^{2\pi} dx = 2\pi \qquad \int_0^{2\pi} dx \sin(nx) = \left[-\frac{1}{n}\cos(nx)\right]_0^{2\pi}$$

$= 0$ (PERIODIC)

$$\int_0^{2\pi} \sin^2(mx) = 0 + \int_0^{2\pi} \frac{dx}{2} = \pi$$

$$\int_0^{2\pi} \cos^2(mx) = 0 + \int_0^{2\pi} \frac{dx}{2} = \pi$$

$\Rightarrow$

$$C_0 = \frac{1}{\sqrt{2\pi}}$$

$$C_m(x) = \frac{1}{\sqrt{\pi}} \cos(mx) \qquad m \in \mathbb{N}^+$$

$$S_m(x) = \frac{1}{\sqrt{\pi}} \sin(mx) \qquad m \in \mathbb{N}^+$$

$$p(t,x) = \sum_{m=0}^{\infty} B_m C_m(x) e^{-m^2 t} + \sum_{m=1}^{\infty} A_m S_m(x) e^{-m^2 t}$$

$$B_0 = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} p(0,x) = \sqrt{2\pi}\, \bar{p}$$

$$B_m = \int_0^{2\pi} p(0,x)\, C_m(x) \qquad A_m = \int_0^{2\pi} p(0,x)\, S_m(x)$$

$$p(t,x) \to \bar{p}$$

NOTE

C_m symmetric, S_m asymmetric

$$f(x) = \frac{1}{2}\left(f(x) + f(-x)\right) + \frac{1}{2}\left(f(x) - f(-x)\right) = \mathrm{SYM}(x) + \mathrm{ASYM}(x)$$