LAPLACE EQUATION

HEAT, WAVE EQS. IN THE STATIONARY CASE

$\partial_t = 0$

$\Rightarrow$

$$\boxed{\nabla^2 \phi = 0}$$

IMPORTANT IN ELECTROSTATICS

(or $4\pi\rho$, $4\pi\varepsilon_0\rho$
DEFINITION OF ρ!)

$$\boxed{\nabla \cdot E = \rho}$$

ALWAYS TRUE!

$$\boxed{E = -\vec{\nabla}\phi}$$

TRUE ONLY IF FIELDS DO NOT CHANGE IN TIME!

$$-\nabla \cdot \vec{\nabla}\phi = \rho \Rightarrow \nabla^2\phi = -\rho$$

IN VACUUM $\rho = 0$

$$\Rightarrow \nabla^2 \phi = 0$$

YOU MAY REMEMBER

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\vec{e}_r$$

$\Rightarrow$

$$\int_S \vec{E}\cdot\vec{n} = \cancel{4\pi r^2}\frac{q}{\cancel{4\pi}\varepsilon_0 \cancel{r^2}} = \frac{q}{\varepsilon_0} \quad \text{IF} \quad \vec{E} = -\nabla\phi$$

$\Rightarrow$

$$\int_S -\vec{\nabla}\cdot\phi = \int_V \nabla\cdot(-\vec{\nabla}\phi) = -\int_V \nabla^2\phi$$

$$\shortparallel$$

$$\int_S \vec{E}\cdot\vec{n} = \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_V \rho \implies \nabla^2\phi = -\frac{\rho}{\varepsilon_0}$$

IF WE DEFINE

$$q' = \frac{q}{4\pi\varepsilon_0} \implies \vec{E} = \frac{q'}{r^2}\vec{e}_r$$

$$\rho' = \frac{\rho}{4\pi\varepsilon_0} \implies \rho = 4\pi\varepsilon_0\rho'$$

$$\implies \nabla^2\phi = -\frac{4\pi\cancel{\varepsilon_0}\rho'}{\cancel{\varepsilon_0}} = -4\pi\rho'$$

IF $\quad q'' = q/\varepsilon_0 \implies \boxed{\nabla^2\phi = -\rho}$

$\Downarrow$

POISSON EQ.

MEANS $\boxed{(\bar{\varphi} - \varphi) \propto (-\rho)}$

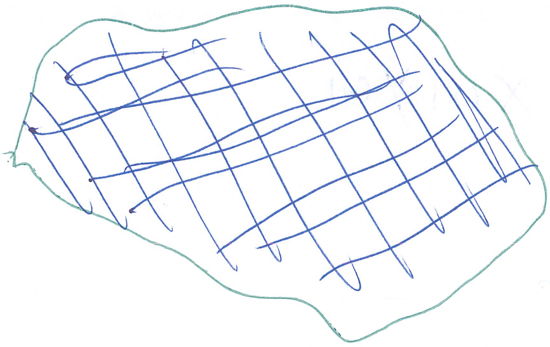
IN THE VACUUM (LAPLACE)

$\implies \boxed{\bar{\varphi} = \varphi}$

φ CANNOT HAVE A MAXIMUM OR MINIMUM!

$\Rightarrow$

IF WE GIVE ϕ_{BOUNDARY}, $\nabla^2\phi = 0$



$\Rightarrow$ ϕ HAS A UNIQUE SOLUTION

$\phi_1 = \phi_{\text{BOUNDARY}}$ $\qquad \phi_2 = \phi_{\text{BOUNDARY}}$, $\nabla^2\phi_1 = \nabla^2\phi_2$ INSIDE

$\tilde{\phi} \equiv \phi_1 - \phi_2$ $\qquad \tilde{\phi} = 0$ ON BOUNDARY

$\nabla^2\tilde{\phi} = \nabla^2\phi_1 - \nabla^2\phi_2 = 0$ INSIDE

$\Rightarrow$ $\tilde{\phi}$ HAS NO MAX OR MIN INSIDE, IT'S 0 ON BOUNDARY $\Rightarrow$

$\tilde{\phi} = 0 \Rightarrow \boxed{\phi_1 = \phi_2}$

LET US STUDY LAPLACE ON

WE GIVE

$\phi(0,y) = f(y)$

$\phi(L_x,y) = g(y)$

$\phi(x,0) = h(x)$

$\phi(x,L_y) = \ell(x)$

THIS PROBLEM CANNOT BE SOLVED IN A STRAIGHTFORWARD WAY WITH SEP. VAR.

LET US SOLVE A SIMPLER ONE

$$y = h_y = \ell = 0 \qquad \phi(0,y) = f(y)$$

$$\nabla^2 \phi = 0 \qquad \phi = X(x)\, Y(y)$$

$$\Rightarrow \quad \frac{\partial_x^2 X}{X} = k = -\frac{\partial_y^2 Y}{Y}$$

WE HAVE

$$\phi(x,0) = \phi(x, L_y) = 0 \qquad \forall x$$

PROCEEDING IN THE USUAL WAY

$$\partial_y^2 Y(y) = -\lambda^2 Y(y)$$

$$\lambda = \frac{m_y \pi}{L_y} \qquad k = \frac{m_y^2 \pi^2}{L_y^2} > 0 \equiv \gamma^2$$

$$Y(y) = A_{m_y} S_{m_y}(y)$$

$$\partial_x^2 X = \gamma^2 X$$

$$X = A \sinh(\gamma x) + B \cosh(\gamma x)$$

$$x' = (L_x - x)$$

$$\cancel{\Rightarrow X(x') = A \sinh(\gamma [L_x - x']) + B \cosh(\gamma [L_x - x'])}$$

$$\Rightarrow \partial_{x'}^2 X(x') = \gamma^2 X(x')$$

$$\cancel{X(x=0)=0}$$

$$X(x') = A\sinh(\gamma x') + B\cosh(\gamma x')$$

$$X(x=L_x)=0 \Rightarrow X(x'=0)=0$$

$$\Rightarrow \boxed{B=0}$$

$$X(x') = A\sinh(\gamma x') \qquad x' = L_x - x$$

$$\Rightarrow X(x) = A\sinh(\gamma[L_x - x])$$

$$\Rightarrow \phi(x,y) = \sum_n A_n \sinh(\gamma_n[L_x - x])\, S_n(y)$$

$$\gamma_n \equiv \frac{n\pi}{L_y}$$

$$\Rightarrow \phi(0,y) = \sum_n A_n \sinh(\gamma_n L_x)\, S_n(y) = f(y)$$

$$\Rightarrow A_n = \int f(y)\, S_n(y) \frac{1}{\sinh(\gamma_n L_x)}$$

LET US CALL THIS ϕ AS ϕ_1

IN THE SAME WAY ~~WE~~ MAY SOLVE

ϕ_2 FOR $f = h = l = 0$, $\phi(L_x, y) = g(y)$

ϕ_3 FOR $f = g = l = 0$, $\phi(x, 0) = h(x)$

ϕ_4 FOR $f = g = h = 0$, $\phi(x, L_y) = l(x)$

$$\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$$

HAS $\nabla^2 \phi = 0$, AND SATISFIES BOUNDARY!

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LAPLACE EQUATION IN POLAR COORDINATES

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\phi\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\phi = 0$$

WE TRY AGAIN WITH

SEP. VAR.

$$\phi = \Theta(\theta) R(r)$$

$\Rightarrow$

$$\Theta \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} R\right) + \frac{R}{r^2}\frac{\partial^2}{\partial \theta^2}\Theta = 0$$

& DIVIDE BY $R\Theta$

$$\frac{1}{R}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}R\right) = -\frac{1}{\Theta}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\Theta$$

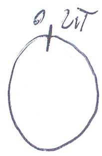
I MULTIPLY BY r^2

$\Rightarrow$

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}R\right) = -\frac{1}{\Theta}\frac{\partial^2}{\partial \theta^2}\Theta = k$$

$\Rightarrow$ $$\frac{\partial^2}{\partial \theta^2}\Theta = -k\Theta$$

THIS HAS TO BE SOLVED



$\Rightarrow$ $k = m^2$

$$\Theta(\theta) = \sum_{m=0}^{\infty} A_m C_m(\theta) + \sum_{m=1}^{\infty} B_m S_m(\theta)$$

$\Rightarrow$ FOR THE RADIAL PART WE HAVE

$$r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}R\right) = m^2 R$$

LET US TRY THE SOLUTION

$$R(r) = r^{\alpha}$$

$$\partial_r r^\alpha = \alpha r^{\alpha-1}$$

$$r \partial_r r^\alpha = \alpha r^\alpha$$

$$\partial_r (r \partial_r (r^\alpha)) = \alpha^2 r^{\alpha-1}$$

$$\Rightarrow \quad r(\partial_r (r \partial_r (r^\alpha))) = \alpha^2 r^{\alpha}$$

$$\Rightarrow \quad \boxed{\alpha^2 r^\alpha = m^2 r^\alpha} \quad \Rightarrow \quad \alpha = \pm m$$

WE GET 2 SOL FOR $m \neq 0$ / 1 SOL FOR $m = 0$

BUT

$$r \partial_r (r \partial_r (\;)) = 0$$

IS SATISFIED NOT ONLY BY $R \propto r^0 = \text{cost}$

BUT ALSO $\quad r \partial_r (R) = \text{cost} \Rightarrow \partial_r R = \frac{\text{cost}}{r}$

$$\Rightarrow \quad R(r) = A \ln r$$

$\Rightarrow$

$$\phi(r,\theta) = A_0 + B_0 \ln r + \sum_{m=1}^{\infty} A_m C_m(\theta) r^m + \sum_{m=1}^{\infty} \bar{A}_m C_m(\theta) r^{-m}$$

$$+ \sum_{m=1}^{\infty} B_m S_m(\theta) r^m + \sum_{m=1}^{\infty} \bar{B}_m S_m(\theta) r^{-m}$$

HOW DO WE GET THE A_n, B_n ${}^{\text{FROM}}$ B.C. ?

WE CHOOSE r, θ SO WE PROPERLY KNOW

$$\phi(R,\theta) = h(\theta)$$

THE OTHER B.C. IS GIVEN BY $r \to 0$, $r \to \infty$

WE SPLIT THE PROBLEM IN 2

$r \leq R$

IF r^{-n}, $r^{-n} \to \infty$

ALSO $\ln r \to -\infty$

$\Rightarrow$

$$\phi(r,\theta) = A_0 + \sum_{n\geq 1}^{\infty} A_n C_n(\theta) r^n + \sum_{n\geq 1}^{\infty} B_n S_n(\theta) r^n$$

$$\Downarrow$$

$$\phi(R,\theta) = h(\theta)$$

WE FIND.

$$A_n = \int h(\theta)\, C_n(\theta) / R^n$$

$$B_n = \int h(\theta)\, S_n(\theta) / R^n$$

$r \geq R$

$\ln r$, r^n NOT GOOD

$\Rightarrow$

$$\phi(r,\theta) = A_0 + \sum_{n\geq 1}^{\infty} \bar{A}_n C_n(\theta) r^{-n} + \sum_{n\geq 1}^{\infty} B_n S_n(\theta) r^{-n}$$