THE HYDROGEN ATOM

1 ANGULAR MOMENTUM, ALGEBRAIC TREATMENT

IN CLASSICAL MECHANICS (BEFORE EINSTEIN AND SCHRÖDINGER) THE ENERGY OF AN ELECTRON IN THE PROTON'S $\frac{e}{r}$ ELECTRIC FIELD IS

$$\vec{p} = m\vec{v}$$

$$\frac{1}{2} m v^2 - \frac{e^2}{r}$$

$$\Rightarrow \quad \frac{p^2}{2m} - \frac{e^2}{r} = E$$

SCHRÖDINGER SAYS:

$$E = i\hbar \frac{\partial}{\partial t} \qquad p^2 = -\hbar^2 \nabla^2$$

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m} \nabla^2 \Psi - \frac{e^2}{r} \Psi$$

SINCE THE POTENTIAL DEPENDS ON r, WE USE r, θ, φ

$$i\hbar \frac{\partial}{\partial t} \Psi(r,\theta,\varphi,t) = -\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \partial_r \left(r^2 \partial_r \right) + \frac{1}{r^2 \sin\theta} \partial_\theta \left(\sin\theta \, \partial_\theta \right) + \frac{1}{r^2 \sin^2\theta} \partial_\varphi^2 \right] \Psi(r,\theta,\varphi,t) - \frac{e^2}{r} \Psi(r,\theta,\varphi,t)$$

WE USE SEPARATION OF VARIABLES FIRST

$$\Psi = T(t)\,\psi(r,\theta,\varphi)$$

$$\Rightarrow \quad i\hbar\,\partial_t T(t) = E\,T(t)$$

IN SCHR. EQ., SEP. VAR. IN t IS EXTREMELY IMPORTANT, BECAUSE IT IDENTIFIES STATES OF FIXED ENERGY (WHAT S. WANTS TO USE!)

$$T(t) = e^{-i\frac{E}{\hbar}t}$$

$$\Rightarrow \quad \left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{r}\right)\psi = E\psi$$

TO SOLVE THIS WE USE AGAIN SEP. OF VARIABLES. WE KNOW ~~THAT~~ THE

$$\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\right) + \frac{1}{\sin^2\theta}\partial_\varphi^2$$ PART HAS EIGEN FUNCTIONS

Y_{lm} WITH EIGENVALUES $l(l+1)$

AND WE JUST HAVE TO ~~SOLVE~~ SOLVE THE RADIAL PART (SINCE THE POTENTIAL IS RADIAL!)

BUT THIS TIME WE SHOW HOW TO GET THE SPECTRUM (POSSIBLE E.V.) OF THE RADIAL EQ. USING THE THEORY OF QUANTUM ANGULAR MOMENTUM

$$\vec{L} = \vec{X} \wedge \vec{P} \Rightarrow$$

$$L_i = \sum_{j,k} \varepsilon_{ijk}\, X_j P_k$$

$\varepsilon_{123} = 1 \qquad \varepsilon_{122} = 0 \qquad \varepsilon_{132} = -1 \ldots$

$$L_z \equiv L_3 = \varepsilon_{312}\, x P_y + \varepsilon_{321}\, y P_x = x P_y - y P_x$$ ③

RM.

THERE IS A USEFUL FUNCTION IN CLASSICAL PHYSICS, THE LAGRANGIAN, THAT ALLOWS US TO WRITE THE ENERGY IN DIFFERENT COORDINATE SYSTEMS

CARTESIAN

POTENTIAL ENERGY

$$L = \frac{1}{2} m v^2 - U(\vec{x})$$

$$v_x^2 + v_y^2 + v_z^2 = \left(\frac{d}{dt}x\right)^2 + \left(\frac{d}{dt}y\right)^2 + \left(\frac{d}{dt}z\right)^2$$

THEN WE DEFINE

$$P_x = \frac{\partial L}{\partial v_x} = \frac{1}{2} m\, 2 v_x = m v_x \quad , \quad P_y \quad , \quad P_z$$

THE EQUATIONS OF NEWTON ARE REPLACED BY

$$\frac{d}{dt}(P_i) = \frac{\partial L}{\partial x_i} \Rightarrow \boxed{m \frac{d^2}{dt^2} x_i = -\frac{\partial U}{\partial x_i}}$$

AND THE ENERGY IS GIVEN BY

$$H = \sum_i P_i v_i - L = \sum_i P_i^2/m - \sum_i \frac{P_i^2}{2m} + U(x)$$

we use $v_i = P_i/m$

WE OBTAIN THE SAME RESULTS AS NEWTON, BUT THIS MAY BE USED FOR ALL COORDINATES

$\Rightarrow$ IR $U(\vec{x}) = U(r, \vartheta, \varphi) = U(r)$

WE WRITE

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U(r)$$

$$\Downarrow$$

$$\left(\frac{ds}{dt}\right)^2 = \frac{dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2}{dt^2}$$

$$\Rightarrow L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin^2\vartheta\,\dot{\varphi}^2\right) - U(r)$$

NOTE: ALSO FORMALLY AS

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial \vartheta}\dot{\vartheta} + \dots$$

SAME ~~VAR~~ THAT

$\Rightarrow$

$$P_r = \frac{\partial L}{\partial(\dot{r})} = m\dot{r} \qquad P_\vartheta = \frac{\partial L}{\partial(\dot{\vartheta})} = m r \dot{\vartheta}$$

$$P_\varphi = \frac{\partial L}{\partial(\dot{\varphi})} = m r^2 \sin^2\vartheta\,\dot{\varphi}$$

$\Rightarrow$ FROM $U = U(r)$ $\Rightarrow$ $\frac{\partial L}{\partial \vartheta} = \frac{\partial L}{\partial \varphi} = 0$ $\Rightarrow$ $P_\vartheta = \text{COST}$, $P_\varphi = \text{COST}$

LET US CALL x, y THE PLANE ~~GIVEN~~ GIVEN BY $\vec{r}, \vec{v}$ AT $t=0$

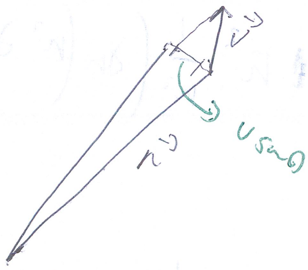
$\Rightarrow$ $\sin\vartheta = 1$, $\dot{\vartheta} = 0$

$$P_\vartheta = 0 \qquad P_\varphi = m r^2 \dot{\varphi} = \text{COST}$$

$L_x = \epsilon_{123}\, \cancel{y} P_z + \epsilon_{132}\, z P_y = 0$ since $z, P_z = 0 \;\; \forall t$

$\Rightarrow L_y = 0$

$\Rightarrow \boxed{L = L_z}$



$L_z = \vec{r} \wedge \vec{v} = m|r||v| \sin\theta$

$\Rightarrow$

$v \sin\theta \simeq r\dot{\theta}$ $\left(v \sin\theta = \frac{dx}{dt} \text{ since } dx = r\, d\theta \right)$

$\Rightarrow L_z = m r^2 \dot{\varphi} = P_\varphi \Rightarrow \dot{\varphi} = \frac{L_z}{m r^2} = \frac{L}{m r^2} = \frac{P_\varphi}{m r^2}$

$\Rightarrow L = \frac{P_r^2}{2m} + \frac{1}{2m} r^2 \dot{\varphi}^2 - U(r)$

$= \frac{1}{2m} P_r^2 + \frac{m}{2} r^2 \frac{P_\varphi^2}{m^2 r^4} - U(r)$

$= \frac{P_r^2}{2m} + \frac{P_\varphi^2}{2m r^2} - U(r)$

$H = P_r \dot{r} + P_\varphi \dot{\varphi} - L = \frac{P_r^2}{m} + \frac{P_\varphi^2}{m r^2} - \frac{P_r^2}{2m} - \frac{P_\varphi^2}{2m r^2} + U(r)$

$= \frac{P_r^2}{2m} + \frac{P_\varphi^2}{2m r^2} + U(r) = \frac{P_r^2}{2m} + \frac{L^2}{2m r^2} + U(r)$

COMPARING TO THE EXPRESSION FOR ∇^2 IN ~~PLANE~~ SPH COORDINATES,

WE MAY ASSUME IN Q.M.

$$P_r^2 = -\hbar^2 \frac{1}{r^2}\left(\partial_r\left(r^2 \partial_r\right)\right)$$

AND

$$L^2 = -\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\right) - \frac{1}{\sin^2\theta}\partial_\varphi^2$$

THE EIGENFUNCTIONS OF L^2 ARE THE

$$Y_{lm} \quad \text{WITH E.V.} \quad l(l+1)$$

AT THE SAME TIME

$$\hat{X} = \text{MULTIPLICATION BY } x$$

$$\hat{P}_x = -i\hbar\,\partial_x$$

$$\Rightarrow \quad \hat{L}_i = \varepsilon_{ijk}\hat{X}_j\hat{P}_k = -i\hbar\,\varepsilon_{ijk}\,x_j\,\partial_k$$

LET US STUDY THE CONSEQUENCES

COMMUTATOR

$$[A,B] = AB - BA$$

NOT NECESSARILY 0 FOR OPERATORS!

EX.

$[x, p_x] = ?$

$$[x, p_x]\psi(x) = -i\hbar\left(x\partial_x\psi(x) - \partial_x(x\psi(x))\right) =$$

$$= -i\hbar\,\cancel{x\partial_x\psi(x)} + i\hbar\,\cancel{x\partial_x\psi(x)} + i\hbar\psi(x)$$

$$\Rightarrow \quad [x, p_x] = i\hbar\mathbb{1}$$

ALSO $\quad [y, p_y] = [z, p_z] = i\hbar\mathbb{1}$

BUT y PASSES THROUGH ∂_x ETC.

$$\Rightarrow \boxed{[x_j, p_k] = i\hbar\delta_{jk}}$$

ALSO $\quad [x_i, x_j] = 0$

$[p_i, p_j] = 0$

GIVEN

$$L_x = yp_z - zp_y$$
$$L_y = zp_x - xp_z$$

$$[L_x, L_y] = [yp_z - zp_y,\ zp_x - xp_z]$$

NOW

$$[A+B, C] = (A+B)C - C(A+B) = AC - CA + BC - CB$$
$$= [A, C] + [B, C]$$

$$\Rightarrow \quad [L_x, L_y] = [yP_z, zP_x] - [yP_z, xP_z] +$$

WE USED THE FACT THAT THE [illegible] $[x_i, P_j]$ [illegible] PASS THROUGH

$$[zP_y, xP_z] - [zP_y, zP_x] = yP_x[P_z, z] + xP_y[z, P_z] =$$

$$= -i\hbar\, y P_x + i\hbar\, x P_y = i\hbar L_z$$

ONE MAY STUDY EXPLICITLY ALL THE

$$[L_i, L_j]$$

BUT LET US INTRODUCE THE FOLLOWING RELATIONS

$$[AB, C] = ABC - CAB = ABC - ACB + ACB - CAB = A[B,C] + [A,C]B$$

AND

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

(IS ONLY i'S THAT DO NOT ENTER IN j, k, l, m WILL SURVIVE THEN, EITHER $j = l$ AND $k = m$, OR THE OTHER. IF THE OTHER, THERE IS A CHANGE IN SIGN

$\Rightarrow$

$$[L_j, L_k] = [\varepsilon_{jlm} X_l P_m, \varepsilon_{knq} X_n P_q] =$$

$$= \varepsilon_{jlm}\varepsilon_{knq}[X_l P_m, X_n P_q] = \varepsilon_{jlm}\varepsilon_{knq}\big(X_l[P_m, X_n P_q] +$$

$$+ [X_l, X_n P_q]P_m = i\hbar\varepsilon_{jlm}\varepsilon_{knq}\big(-X_l\delta_{mn}P_q + X_n\delta_{lq}P_m\big)$$

$$= i\hbar\big(-\varepsilon_{mjl}\varepsilon_{mqk} X_l P_q + \varepsilon_{lmj}\varepsilon_{lkn} X_n P_m\big)$$

$$= i\hbar\Big[\delta_{mk}\delta_{jn} X_n P_m - \delta_{mn}\delta_{jk} X_n P_m \Big\} = \text{[illegible]}$$

$$- \delta_{jq}\delta_{lk} X_l P_q + \delta_{jk}\delta_{lq} X_l P_q\Big]$$

RENAME

$$\delta_{jq}\, \delta_{ln}\, x_l\, p_q \qquad l \to m \quad q \to m$$

$$\Downarrow$$

$$\delta_{mk}\, \delta_{jm}\, x_m\, p_m$$

2)

$$= i\hbar \left[\left(\delta_{jn}\,\delta_{km} - \delta_{jm}\,\delta_{km} \right) x_m\, p_m \right. =$$

$$= i\hbar\, \varepsilon_{nju}\, \varepsilon_{unm}\, x_m\, p_m = i\hbar\, \varepsilon_{nju}\, L_n$$

$$\Rightarrow \boxed{[L_j, L_u] = i\hbar\, \varepsilon_{nju}\, L_n}$$

NOW IF WE CALL

$$L_+ \equiv L_x + i L_y$$

$$L_- \equiv L_x - i L_y$$

WE HAVE

$$[L_z, L_+] = [L_z, L_x] + i\,[L_z, L_y] =$$

$$= i\hbar\left(\varepsilon_{312}\, L_y + i\, \varepsilon_{321}\, L_x \right) = i\hbar\, L_y + \hbar\, L_x$$

$$= \hbar\, L_+$$

AND

$$[L_z, L_-] = [L_z, L_x] - i\,[L_z, L_y] = \hbar\left(i L_y - L_x \right)$$

$$= -\hbar\, L_-$$

LET US ASSUME

$|m\rangle$ s.t.

$\langle n|m\rangle = \delta_{nm}$

$L_z|m\rangle = \hbar m|m\rangle$

Note:
WE SHALL CARE THAT

$L_z = -i\hbar\frac{\partial}{\partial\varphi} \Rightarrow$

$|m\rangle = e^{im\varphi}$

$L_z|m\rangle = \hbar m e^{im\varphi}$, $m \in \mathbb{N}$

$\Rightarrow$

$$L_z(L_+|m\rangle) = [L_zL_+ - L_+L_z + L_+L_z]|m\rangle$$

$$= ([L_z, L_+] + L_+L_z)|m\rangle = (\hbar L_+ + \hbar m L_+)|m\rangle$$

$$= \hbar(m+1)(L_+|m\rangle)$$

$$\Rightarrow \boxed{L_+|m\rangle \propto |m+1\rangle}$$

SIMILARLY

$$L_z(L_-|m\rangle) = ([L_z, L_-] + L_-L_z)|m\rangle = (\hbar L_- + \hbar m L_-)|m\rangle$$

$$= \hbar(m-1)(L_-|m\rangle)$$

$$\Rightarrow \boxed{L_-|m\rangle \propto |m-1\rangle}$$

WE STARTED THIS DISCUSSION TO STUDY

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

WE MAY SHOW

$$[L_i, L^2] = 0$$

WE ARE IN PARTICULAR INTERESTED IN

$$[L_z, L_x^2 + L_y^2 + L_z^2] = [L_z, L_x^2] + [L_z, L_y^2] =$$

$$= L_x [L_z, L_x] + [L_z, L_x] L_x + L_y [L_z, L_y] + [L_z, L_y] L_y =$$

$$= i\hbar \left[\cancel{L_x L_y} + \cancel{L_y L_x} - \cancel{L_y L_x} - \cancel{L_x L_y} \right] = 0$$

THE GENERAL PROOF IS THE FOLLOWING

$$[L_i, \sum_j L_j^2] = \sum_j \left([L_i, L_j] L_j + L_j [L_i, L_j] \right) =$$

$$= i\hbar \left[\sum_j \varepsilon_{ijn} L_n L_j + \varepsilon_{ijn} L_j L_n \right] = 0 \quad \text{BY ASYMMETRY}$$

$$\parallel$$

$$(\varepsilon_{ijn} + \varepsilon_{inj}) L_j L_n = 0$$

SELF-ADJ ✓

WHEN TWO LIN. OPERATORS COMMUTE, WE CAN SIMULTANEOUSLY DIAGONALISE THEM,

$$[A, B] = 0 \Rightarrow |n, m\rangle \quad \text{s.t.}$$

$$\langle n, m | l, s \rangle = \delta_{n,l} \delta_{m,s}$$

$$\sum_{n,m} |n, m\rangle \langle n, m| = \mathbb{1}$$

$$A |n, m\rangle = n |n, m\rangle$$

$$B |n, m\rangle = m |n, m\rangle$$

INDEED IF

$$B|b\rangle = b|b\rangle$$

$$AB|b\rangle = BA|b\rangle \quad \Rightarrow \quad B(A|b\rangle) = b(A|b\rangle)$$
$$\shortparallel$$
$$bA|b\rangle$$

IN GENERAL THIS DOES NOT MEAN THAT

$$A|b\rangle = \alpha|b\rangle$$

UNLESS WE TRIVIALLY HAVE $A = f(B)$

THERE WOULD BE A SUBSPACE OF VECTORS WITH

$$B|b\rangle = b|b\rangle$$

WE ~~MAY~~ CONSIDER A BASIS OF SUCH SUBSPACE

$$|b,i\rangle \quad , \quad B|b,i\rangle = b|b,i\rangle$$

$$\langle b,i|b,j\rangle = \delta_{ij}$$

CLEARLY

$$A|b,i\rangle = \sum_j c_j|b,j\rangle$$

LET US GIVE A SMARTER NAME TO THE c_j

$$\Rightarrow \quad A|b,i\rangle = A^{(b)}_{ij}|b,j\rangle$$

THIS IS A REASONABLE NAME SINCE

$$\langle b,i|A|b,j\rangle = A^{(b)}_{ij}$$ ACCORDING TO OUR DEFINITION OF MATRIX ELEMENTS

BUT ALSO IF WE JUST DEFINE

$$A|6,i\rangle = \tilde{A}^{(6)}_{ij}|6,j\rangle$$

$$\langle 6,u|A|6,i\rangle = \delta_{uj}\tilde{A}^{(6)}_{ij} = \tilde{A}^{(6)}_{iu}$$

IN GENERAL, IF $|\alpha\rangle$ IS IN THE 6 SUBSPACE

$$|\alpha\rangle = \sum_i \alpha_i |6,i\rangle$$

AND

$$|\alpha\rangle = \sum_i |6,i\rangle\langle 6,i|\alpha\rangle$$

$$\Rightarrow \quad \alpha_i = \langle 6,i|\alpha\rangle$$

$\Rightarrow$ GIVEN $|\alpha\rangle, |\beta\rangle$ IN THE SUBSPACE

$$\langle\alpha|A|\beta\rangle = \sum_{i,j} \langle\alpha|6,i\rangle\langle 6,i|A|6,j\rangle\langle 6,j|\beta\rangle$$

$$= \sum_{i,j} \bar{\alpha}_i A^{(6)}_{ij}\beta_j$$

BUT $\langle\alpha|A|\beta\rangle = \langle\alpha|A||\beta\rangle = \overline{\langle\beta|A|\alpha\rangle} = \overline{\bar{\beta}_j A_{ji}\alpha_i}$.

$$= \alpha_i \bar{A}_{ji}\bar{\beta}_j = \bar{\alpha}_i A^{+}_{ij}\beta_j$$

$$\Rightarrow \boxed{A^{+}_{ij} = A_{ij}}$$

WE DIAGONALISE THE SUBSPACE AS

$$|6,a\rangle \quad \Rightarrow \quad \begin{aligned} B|6,a\rangle &= 6|6,a\rangle \\ A|6,a\rangle &= a|6,a\rangle \end{aligned}$$

A IS SPLITTING THE DEGENERATION OF B

IF $\exists$ C , $[A,C]=0=[A,B]=[B,C]$

THEN WE MAY SPLIT FURTHER

$|a,b,c\rangle$

SUCH SETS OF COMMUTING (COMPLETE) $\boxed{\text{SELF-ADJ LIN. OP.}}$ (OBSERVABLES) ARE EXTREMELY USEFUL

IN Q.M.

SINCE $[L^2, L_z]=0$

WE TRY TO WRITE

~~[illegible]~~ $|a,m\rangle$ S.T.

$L^2|a,m\rangle = a|a,m\rangle$

$L_z|m\rangle = \hbar|m\rangle$

SPOILER ! $a=\ell(\ell+1)$, $|a,m\rangle = Y_{\ell m}$

WE NOTE THAT

$$L_x^2+L_y^2 = \frac{1}{2}\left(L_+L_- + L_-L_+\right) =$$

$$= \frac{1}{2}\left(2L_x^2+2L_y^2 + iL_yL_x - iL_xL_y - iL_yL_x + iL_xL_y\right)$$

$$\Rightarrow \quad L^2 = \frac{1}{2}\left(L_+L_- + L_-L_+\right) + L_z^2$$

WE ALSO NOTE

$$(L_+)^\dagger = (L_x + i L_y)^\dagger = L_x - i L_y = L_-$$

VERY IMPORTANT

$$\langle x, ABy \rangle = \langle A^\dagger x, By \rangle = \langle B^\dagger A^\dagger x, y \rangle$$

$$\Rightarrow \quad (AB)^\dagger = B^\dagger A^\dagger$$

IF $\quad A = A^\dagger, \quad B = B^\dagger$ AND $\quad [A, B] = 0$

$$\Rightarrow (AB)^\dagger = AB$$

$$L_i = \epsilon_{ijk} x_j p_k$$

DIFFERENT INDEXES, COMMUTE

$x^\dagger = x$

$$\langle x\psi, \varphi \rangle = \int x \overline{\psi(x)} \varphi(x)\, dx = \langle \psi, x\varphi \rangle$$

$p^\dagger = p$

$$\langle \psi, p\varphi \rangle = \hbar \int \overline{\psi(x)}\, i \partial_x \varphi(x) = \hbar \int \overline{-i\psi(x)}\, \partial_x \varphi(x) =$$

$$= -\hbar \int \partial_x (\text{—}) + \hbar \int \overline{i \partial_x \psi(x)}\, \varphi(x) = \langle p_x \psi, \varphi \rangle$$

ASSUME PERIODIC, OR

0 ...

$\Rightarrow \quad L_x, L_y, L_z$ SELF ADJ

L_x^2, L_y^2, L_z^2 SELF ADJ

L^2 SELF ADJ

$$[L_+, L_-] = [L_x + iL_y, L_x - iL_y] =$$

$$= i[L_y, L_x] - i[L_x, L_y] = -2i(i)\varepsilon_{123} L_z = 2L_z$$

$\Rightarrow$

$$L^2 = \frac{1}{2}\left(L_+L_- + (L_+L_- - L_+L_-) + L_-L_+\right) + L_z^2 =$$

$$= L_+L_- - L_z + L_z^2$$

AND

$$L^2 = \frac{1}{2}\left(L_+L_- + L_-L_+ + (L_-L_+ - L_-L_+)\right) + L_z^2$$

$$= L_-L_+ + L_z + L_z^2$$

WE ALSO NOTE THAT

$$\langle a, m | L^2 | a, m \rangle = a$$

$$\langle a, m | L || L | a, m \rangle = || L | a, m \rangle ||^2 \geq 0$$

$$\Rightarrow \quad a \geq 0$$

SIMILARLY

$$\langle a, m | L_z^2 | a, m \rangle = m^2 \hbar^2$$

$$\langle a, m | L_x^2 | a, m \rangle = \langle a, m | L_x || L_x | a, m \rangle = || L_x | a, m \rangle ||^2 \geq 0$$

$$\langle a, m | L_y^2 | a, m \rangle \geq 0$$

$\Rightarrow$

$$0 \leq a = \langle a, m | L_x^2 + L_y^2 + L_z^2 | a, m \rangle =$$

$$= || L_x | a, m \rangle ||^2 + || L_y | a, m \rangle ||^2 + m^2 \hbar^2$$

$$\Rightarrow \quad a \geq m^2 \qquad \text{OR} \qquad |m| < \sqrt{a}$$

LET US CONSIDER $\hbar m > -\sqrt{a}$, $n \in \mathbb{N}$, $n > \frac{2\sqrt{a}}{\hbar}$

$$\Rightarrow \quad L_+^n | a, m \rangle = | a, m + n \rangle$$

$$\hbar^2 (m + n)^2 > a \quad !$$

HOW IS THAT POSSIBLE

WE KNOW

$$L_z(L_+|m\rangle) = \hbar(m+1)\, L_+|m\rangle$$

SO WE ASSUMED $\quad L_+|m\rangle \propto |m+1\rangle$

THERE IS ANOTHER POSSIBILITY,

THAT $\forall\ \bar{m}$ SUCH THAT

$$L_+|\bar{m}\rangle = 0$$

THE SAME ANALYSIS MAY BE DONE STARTING FROM $m < \sqrt{a}$ AND APPLYING $n > \frac{2\sqrt{a}}{\hbar}$ TIMES

L_-

$\Rightarrow \quad \exists \quad |a, m_{MAX}\rangle \quad |a, m_{MIN}\rangle$

S.t. $\quad L_+|a, m_{MAX}\rangle = 0 \quad L_-|a, m_{MIN}\rangle = 0$

$$a = \langle a, m_{MAX}|L^2|a, m_{MAX}\rangle = \langle a, m_{MAX}|L_-L_+|a, m_{MAX}\rangle$$

$$+ \langle a, m_{MAX}|L_z + L_z^2|a, m_{MAX}\rangle = 0 + m_{MAX} + m_{MAX}^2$$

$$a = m_{MAX}(m_{MAX}+1)$$

$$a = \langle a, m_{MIN}|L^2|a, m_{MIN}\rangle = \langle a, m_{MIN}|L_+L_-|a, m_{MIN}\rangle$$

$$+ \langle a, m_{MIN}|L_z^2 - L_z|a, m_{MIN}\rangle = 0 + m_{MIN}^2 - m_{MIN} = m_{MIN}(m_{MIN}-1)$$

LET US SOLVE

$$x^2 - x = y^2 + y \Rightarrow x^2 - y^2 = x + y$$

$$\Rightarrow (x+y)(x-y) = x + y$$

EITHER $x - y = 1 \Rightarrow x = y + 1$ BUT THEN MIN > MAX

OR $x = -y$

$\Rightarrow m_{MAX} = \ell$, $m_{MIN} = -\ell$

$$m = -\ell, -\ell+1, \ldots, 0, \ldots \ell-1, \ell$$

$$a = \ell(\ell+1)$$ WE CALL THE STATES $|\ell, m\rangle$

WE HANDLE $\ell = 0$, $m = 0$ $\Rightarrow$ THE SOLUTION IS CONSTANT IN φ

$\Rightarrow |\ell, 0\rangle \propto P_\ell(\cos\theta)$

WE MAY USE THE OPERATORS L_+, L_- TO BUILD THE $|\ell, m\rangle$ FROM $|\ell, 0\rangle$

BUT (EXERCISE) $L_+|\ell, 0\rangle \propto |\ell, 1\rangle$

IF WE WANT $\langle \ell, m | \ell', m' \rangle = \delta_{\ell\ell'} \delta_{mm'}$

PROPORTIONALITY IS NOT ENOUGH

HOW CAN WE GET THE CONSTANT!

$$l(l+1) = \langle l,m| L^2 |l,m\rangle = \langle l,m| L_- L_+ | l,m\rangle + m + m^2$$

$$\Rightarrow \quad l(l+1) - m(m+1) = \sum_{m'=-l}^{l} \langle l,m| L_- |l,m'\rangle \langle l,m'| L_+ |l,m\rangle$$

$$= \langle l,m| L_- | \underbrace{l,m+1\rangle \langle l,m+1}_{\text{the other terms disappear}} | L_+ |l,m\rangle =$$

$$= \langle l,m| L_+^\dagger |l,m+1\rangle \langle l,m+1| L_+ | l,m\rangle =$$

$$\| \langle l,m+1| L_+ | l,m\rangle \|^2 \quad \text{if} \quad L_+ |l,m\rangle = \alpha^+_m |l,m+1\rangle$$

$$\| \langle l,m+1| L_+ | l,m\rangle \|^2 = \alpha^{+2}_m$$

$$\Rightarrow \quad \alpha^+_m = \sqrt{l(l+1) - m(m+1)} \qquad (\text{phase}, e^{i\phi})$$

$\rightarrow 0$ if $l = m$

Similarly

$$l(l+1) = \langle l,m| L_+ L_- | l,m\rangle - m + m^2 \Rightarrow$$

$$l(l+1) - m(m-1) = \sum_{m'} \langle l,m| L_+ |l,m'\rangle \langle l,m'| L_- | l,m\rangle =$$

$$= \langle l,m| L_+ |l,m-1\rangle \langle l,m-1| L_- | l,m\rangle =$$

$$\langle l,m| L_-^\dagger |l,m-1\rangle \langle l,m-1| L_- | l,m\rangle = \| \langle l,m-1| L_- |l,m\rangle \|^2$$

$$\alpha^-_m = \sqrt{l(l+1) - m(m-1)}$$

$m = -l \qquad -(-l)(-l-1) = -(l+1)l \quad \Rightarrow 0$