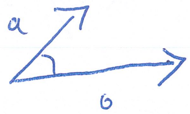
DIRAC FORMALISM

$$\vec{x}\cdot\vec{y} = \sum_i x_i y_i$$



$|a||b|\cos\theta$

$\vec{x}\in\mathbb{R}^n$ $\quad \vec{x}\cdot\vec{x} = |x|^2 \geq 0$, >0 IF $\vec{x}\neq 0$

COMPLEX CASE

$$\vec{x}\in\mathbb{C}^n \Rightarrow \vec{x}\cdot\vec{y} = \sum_i \bar{x}_i y_i$$

SINCE $\vec{x}\cdot\vec{x} = \sum_i |x_i|^2 \geq 0$

FORMAL DEFINITION OF SCALAR (INNER) PRODUCT

$$\begin{cases} \langle x,y\rangle \in \mathbb{C} & \langle x,y\rangle = \overline{\langle y,x\rangle} \\ \langle x,x\rangle \geq 0 & \langle x,x\rangle = 0 \text{ IFF } x=0 \\ \langle x,\alpha y\rangle = \alpha\langle x,y\rangle & \\ \langle x,y+z\rangle = \langle x,y\rangle + \langle x,z\rangle & \end{cases}$$

NOTE

$$\langle \alpha x,y\rangle = \overline{\langle y,\alpha x\rangle} = \overline{\alpha\langle y,x\rangle} = \bar{\alpha}\,\overline{\langle y,x\rangle} = \bar{\alpha}\langle x,y\rangle$$

IF A IS A LINEAR OPERATOR WE DEFINE $A^\dagger$ SUCH THAT

$$\langle x,Ay\rangle = \langle A^\dagger x,y\rangle \qquad \forall\, x,y$$

IN THE CASE OR

$$\langle x, y\rangle = x \cdot y = \sum_i \bar{x}_i y_i$$

$$\langle x, Ay\rangle = \overline{x_i} A_{ij} y_j$$

$$\langle A^+ x, y\rangle = \overline{A^+_{ji} x_i}\, y_j \Rightarrow A^+_{ij} = \overline{A_{ji}}$$

SELF ADJ $\quad A^+ = A$

IF $\quad A v_m = \lambda_m v_m \quad$ AND $\quad A^+ = A \Rightarrow \lambda_m \in \mathbb{R}$

$$\langle v_m, A v_m\rangle = \langle v_m, \lambda_m v_m\rangle = \lambda_m \langle v_m, v_m\rangle$$

$$\|$$

$$\langle A^+ v_m, v_m\rangle = \langle A v_m, v_m\rangle = \langle \lambda_m v_m, v_m\rangle = \bar{\lambda}_m \langle v_m, v_m\rangle$$

$$\Rightarrow \lambda_m = \bar{\lambda}_m$$

IF $\quad A v_m = \lambda_m v_m \qquad A v_n = \lambda_n v_n$

AND $\quad A^+ = A \Rightarrow \langle v_m, v_n\rangle = 0$

$$\langle v_m, A v_n\rangle = \lambda_n \langle v_m, v_n\rangle$$

$$\|$$

$$\langle A v_m, v_n\rangle = \lambda_m \langle v_m, v_n\rangle \Rightarrow \#$$

LET US ASSUME (IT IS POSSIBLE TO SHOW) $A^{+} = A$

$$A v_n = \lambda_n v_n \qquad \text{(MAY BE } \lambda_n = \lambda_m \text{)}$$

$$\langle v_n, v_m \rangle = \delta_{n,m}$$

v_n BASIS

$$\forall a \quad , \quad a = \sum_n c_n v_n$$

$$\Rightarrow \quad \langle v_m, a \rangle = \langle v_m, \sum_n c_n v_n \rangle = \sum_n c_n \delta_{n,m} = c_m$$

$$\Rightarrow \quad a = \sum_n \langle v_n, a \rangle v_n$$

$\langle \cdot , \cdot \rangle$ BUT ALSO $\langle v, \cdot \rangle \to \mathbb{R}$

$\Rightarrow$ $\langle v, \cdot \rangle$ IS A LINEAR OPERATOR ON VECTORS TO $\mathbb{R}$

DIRAC CALLS

$$v \equiv |v\rangle \qquad \langle v, \;\rangle \equiv \langle v|$$

$$\Rightarrow \quad \langle v, u \rangle = \langle v | u \rangle$$

WE HAVE

$$|a\rangle = \sum_n \langle v_n | a \rangle |v_n\rangle = \sum_n |v_n\rangle \langle v_n | a \rangle$$

$$\Rightarrow \quad \sum_n |v_n\rangle \langle v_n| = \mathbb{1}$$

$$\langle x, Ay \rangle$$

MAY BE WRITTEN AS

$$\langle x | A | y \rangle$$

THE NOTATION FOR

$$\langle Ax, y \rangle$$ IS

$$\langle Ax | y \rangle \quad \text{OR} \quad \langle x | A^\dagger | y \rangle$$

IF WE DEFINE

$$\langle n | A | m \rangle = A_{nm}$$ WHERE $|n\rangle$ IS A SHORT FOR $|\psi_n\rangle$

THEN

$$\langle n | AB | m \rangle \equiv (AB)_{nm}$$

$$= \sum_k \langle n | A | k \rangle \langle k | B | m \rangle = \sum_k A_{nk} B_{km}$$

VERY USEFUL TO SHOW THE RELATION BETWEEN MATRICES AND LIN OP.

AGAIN

$$A^\dagger_{nm} \equiv \langle n | A^\dagger | m \rangle = \langle n | A^\dagger | m \rangle = \overline{\langle m | A | n \rangle} = \overline{A_{mn}}$$

CHANGE OF BASIS $|n'\rangle$ AND $|n\rangle$

$$A'_{nm} = \langle n' | A | m' \rangle = \sum_{n,m} \langle n' | n \rangle \langle n | A | m \rangle \langle m | m' \rangle$$

$$= U_{n'n} A_{nm} U_{mm'}$$

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$$U_{m'm} = \langle m' | m \rangle = \overline{\langle m | m' \rangle} = \overline{U_{m'm}}$$

$$\Rightarrow \quad A'_{nm} = U_{n'n} A_{nm} U^{\dagger}_{m'm}$$

U IS A UNITARY OPERATOR SINCE

$$(U^{\dagger}U)_{mn} = \sum_{m'} \langle m | m' \rangle \langle m' | n \rangle = \delta_{mn}$$

ALL THIS CAN BE GENERALIZED TO SPACES OF FUNCTIONS.

EX

PERIODIC FUNCTIONS ON $[0, 2\pi]$ THAT MAY BE DIFF. IN THEIR
(CANNOT BE NNA)

VECTOR SPACE $\alpha \in \mathbb{C}(\mathbb{R})$

$g(x) = \alpha f(x)$ IS IN THE SPACE

$g(x) = \alpha f(x) + \beta h(x)$ ...

LIN. OP.

$$A(\alpha f(x) + \beta g(x)) = \alpha A f(x) + \beta A g(x)$$

EX ∂_x

HOW DO WE DEFINE $\langle \, , \, \rangle$?

$$\langle \varphi, \phi \rangle = \int_0^{2\pi} dx \; \overline{\varphi(x)} \; \phi(x)$$

$$\langle \varphi, \varphi \rangle = \int_0^{2\pi} dx \; |\varphi(x)|^2 \geq 0$$ (Re Recur?)

$= 0$ iff $\varphi(x) = 0$

$$\langle \phi, \varphi \rangle = \overline{\langle \varphi, \phi \rangle}$$

Also linearity . .

Let is comm

$$A = \partial^2 x \qquad \langle \varphi, A \phi \rangle$$

$$\int_0^{2\pi} \overline{\varphi(x)} \; \partial^2 x \; \phi(x) = \int_0^{2\pi} \partial x \left[\overline{\varphi(x)} \; \partial x \; \phi(x) \right]$$

$$- \int_0^{2\pi} \partial x \; \overline{\varphi(x)} \; \partial x \; \phi(x)$$

$$= \cancel{\overline{\varphi(2\pi)} \; \partial x \; \phi(2\pi)} - \cancel{\overline{\varphi(0)} \; \partial x \; \phi(0)} - \int_0^{2\pi} \partial x \; \overline{\varphi(x)} \; \partial x \; \phi(x)$$

$$= - \cancel{\int_0^{2\pi} \partial x \left[\left(\partial x \; \overline{\varphi(x)} \right) \phi(x) \right]} + \int_0^{2\pi} \partial^2 x \; \overline{\varphi(x)} \; \phi(x) =$$

$$= \int_0^{2\pi} \overline{\partial^2 x \; \varphi(x)} \quad \phi(x)$$

$$\Rightarrow \quad A^\dagger = A$$

WE MAY EXPECT

$$A\,\varphi_n(x) = n\,\varphi_n(x)$$

TO BE SUCH THAT

$n \in \mathbb{R}$, $\langle \varphi_n(x), \varphi_m(x)\rangle = \delta_{nm}$

$$f(x) \overset{\text{in the space}}{=} \sum_n c_n\,\varphi_n(x)$$

BUT WE KNOW SUCH FUNCTIONS, THEY ARE THE

$$\varphi_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx} = |n\rangle \qquad \left(c_n, \delta_n \in \mathbb{R} \right)$$

$\Rightarrow$

$$f(x) = |f\rangle$$

$\Rightarrow$ $$|f\rangle = \sum_n |n\rangle\langle n|f\rangle$$

$$\langle n|f\rangle = \int_0^{2\pi} \varphi_n^*(x)\,f(x)$$

etc...