EXPLICIT FORM OF L^2 IN SPHERIC COORDINATES

$$\boxed{z = r\cos\vartheta} \qquad \boxed{y = r\sin\vartheta\sin\varphi} \qquad \boxed{x = r\sin\vartheta\cos\varphi}$$

$$\boxed{r = \sqrt{x^2+y^2+z^2}} \qquad \cos\vartheta = \frac{z}{r} \qquad \tan\varphi = \frac{y}{x}$$

$$\Rightarrow \boxed{\vartheta = \operatorname{ACOS}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right]} \qquad \boxed{\varphi = \operatorname{ATAN}\left[\frac{y}{x}\right]}$$

$$d\left(\operatorname{ACOS}(w)\right) = \frac{-1}{\sqrt{1-w^2}} \qquad d\left(\operatorname{ATAN}(w)\right) = \frac{1}{1+w^2}$$

$$\frac{\partial\vartheta}{\partial x} = \frac{-1}{\sqrt{1-\frac{z^2}{r^2}}}\cdot z\cdot\frac{\partial}{\partial x}\left(\frac{1}{r}\right) =$$

$$= \frac{-1}{\sqrt{1-\frac{z^2}{r^2}}}\cdot z\cdot\left(-\frac{1}{r^2}\right)\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)$$

$$= \frac{1}{\sqrt{1-\frac{z^2}{r^2}}}\cdot z\cdot\left(+\frac{1}{r^2}\right)\cdot\frac{1}{\cancel{2}}\cdot\frac{1}{r}\cdot\cancel{2}x = +\frac{zx}{\sqrt{1-\cos^2\vartheta}\, r^3}$$

$$= \frac{\cancel{r^2}\cos\vartheta\sin\vartheta\cos\varphi}{\sqrt{1-\cos^2\vartheta}\,\cancel{r^3}} = +\frac{\cos\vartheta\,\cancel{\sin\vartheta}\cos\varphi}{\cancel{\sin\vartheta}\, r} = +\frac{\cos\vartheta\cos\varphi}{r}$$

$$\boxed{\frac{\partial\vartheta}{\partial x} = +\frac{\cos\vartheta\cos\varphi}{r}}$$

$$\frac{\partial\vartheta}{\partial y} = +\frac{zy}{\sin\vartheta\, r^3} = +\frac{\cancel{r^2}\cos\vartheta\,\cancel{\sin\vartheta}\sin\varphi}{\cancel{\sin\vartheta}\,\cancel{r^3}} \Rightarrow \boxed{\frac{\partial\vartheta}{\partial y} = +\frac{\cos\vartheta\sin\varphi}{r}}$$

$$\frac{\partial\vartheta}{\partial z} = \frac{-1}{\sqrt{1-\frac{z^2}{r^2}}}\left[\frac{1}{r} - \frac{z^2}{r^3}\right] = \frac{-1}{\sin\vartheta}\,\frac{r^2-z^2}{r^3} = \frac{-1}{r\sin\vartheta}\left(1-\frac{z^2}{r^2}\right)$$

$$= \frac{-1}{r\sin\vartheta}\sin^2\vartheta$$

$$\boxed{\frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r}}$$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r} = \frac{r\sin\theta\cos\varphi}{r} \Rightarrow \boxed{\frac{\partial r}{\partial x} = \sin\theta\cos\varphi}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\varphi \qquad \boxed{\frac{\partial r}{\partial y} = \sin\theta\sin\varphi}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta \qquad \boxed{\frac{\partial r}{\partial z} = \cos\theta}$$

$$\boxed{\frac{\partial \varphi}{\partial z} = 0}$$

$$\frac{\partial \varphi}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = -\frac{\cancel{x^2}}{x^2+y^2} \cdot \frac{y}{\cancel{x^2}}$$

$$= -\frac{\cancel{r}\,\cancel{\sin\theta}\sin\varphi}{r^{\cancel{2}}\sin^{\cancel{2}}\theta} = -\frac{\sin\varphi}{r\sin\theta}$$

$$\Rightarrow \boxed{\frac{\partial \varphi}{\partial x} = -\frac{\sin\varphi}{r\sin\theta}}$$

$$\frac{\partial \varphi}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{\cos\varphi}{r\sin\theta}$$

$$\boxed{\frac{\partial \varphi}{\partial y} = \frac{\cos\varphi}{r\sin\theta}}$$

$$L_z = -i\hbar\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right] = -i\hbar\left[r\sin\theta\cos\varphi\left(\frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi} + \right.\right.$$

$$\left. + \frac{\partial r}{\partial y}\frac{\partial}{\partial r}\right) - r\sin\theta\sin\varphi\left(\frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi} + \frac{\partial r}{\partial x}\frac{\partial}{\partial r}\right) =$$

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$$= -i\hbar \Big[\cancel{r \sin\theta \cos\varphi} \frac{\cos\theta \sin\varphi}{\cancel{r}} \partial_\theta + \cancel{r \sin\theta} \cos\varphi \frac{\cos\varphi}{\cancel{r \sin\theta}} \partial_\varphi +$$

$$+ \cancel{r \sin\theta \cos\varphi \sin\theta \sin\varphi \partial_r} - \cancel{r \sin\theta \sin\varphi \frac{\cos\theta \cos\varphi}{r}} \partial_\theta +$$

$$+ \cancel{r \sin\theta} \sin\varphi \frac{\sin\varphi}{\cancel{r \sin\theta}} \partial_\varphi - \cancel{r \sin\theta \sin\varphi \sin\theta \cos\varphi \partial_r} \Big] =$$

$$= -i\hbar \left(\cos^2\varphi + \sin^2\varphi \right) \partial_\varphi = -i\hbar \partial_\varphi$$

$$\Rightarrow \boxed{L_z = -i\hbar \partial_\varphi}$$

$$L_x = -i\hbar \left[y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right] = -i\hbar \Big[r \sin\theta \sin\varphi \left(\frac{\partial\theta}{\partial z} \partial_\theta + \frac{\partial r}{\partial z} \partial_r + \frac{\partial\varphi}{\partial z} \partial_\varphi \right)$$

$$- r\cos\theta \left(\frac{\partial\theta}{\partial y} \partial_\theta + \frac{\partial r}{\partial y} \partial_r + \frac{\partial\varphi}{\partial y} \partial_\varphi \right) \Big] = -i\hbar \Big[- \frac{\cancel{r} \sin^2\theta \sin\varphi}{\cancel{r}} \partial_\theta + \cancel{r \sin\theta \sin\varphi \cos\theta \partial_r}$$

$$- \frac{\cancel{r} \cos^2\theta \sin\varphi}{\cancel{r}} \partial_\theta - \cancel{r \cos\theta \sin\theta \sin\varphi \partial_r} - \frac{\cancel{r} \cos\theta \cos\varphi}{\cancel{r} \sin\theta} \partial_\varphi \Big]$$

$$= -i\hbar \left[-\sin\varphi \partial_\theta - \cot\theta \cos\varphi \partial_\varphi \right]$$

$$\Rightarrow \boxed{L_x = i\hbar \left[\sin\varphi \partial_\theta + \cot\theta \cos\varphi \partial_\varphi \right]}$$

$$L_y = -i\hbar \left[z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right] = -i\hbar \Big[r\cos\theta \left(\frac{\partial\theta}{\partial x} \partial_\theta + \frac{\partial\varphi}{\partial x} \partial_\varphi + \frac{\partial r}{\partial x} \partial_r \right)$$

$$- r \sin\theta \cos\varphi \left(\frac{\partial\theta}{\partial z} \partial_\theta + \frac{\partial r}{\partial z} \partial_r + \frac{\partial\varphi}{\partial z} \partial_\varphi \right) =$$

$$= -i\hbar\left[\frac{\cancel{r}\cos^2\theta\cos\varphi}{\cancel{r}}\partial_\theta + \frac{\cancel{r}\cos\theta\sin\varphi}{\cancel{r}\sin\theta}\partial_\varphi + \cancel{r\cos\theta\sin\theta\cos\varphi\,\partial_r}\right.$$

$$\left. + \frac{\cancel{r}\sin\theta\cos\varphi\sin\theta}{\cancel{r}} - \cancel{r\sin\theta\cos\varphi\cos\theta\,\partial_r}\right] =$$

$$= i\hbar\left[\cot\theta\sin\varphi\,\partial_\varphi - \cos\varphi\,\partial_\theta\right]$$

$$\Rightarrow \boxed{L_y = i\hbar\left[\cot\theta\sin\varphi\,\partial_\varphi - \cos\varphi\,\partial_\theta\right]}$$

$\Rightarrow$

$$L_x^2 = -\hbar^2\left[\sin^2\varphi\,\partial_\theta^2 + \cot^2\theta\cos^2\varphi\,\partial_\varphi^2 + \sin\varphi\cos\varphi\,\partial_\theta(\cot\theta\,\partial_\varphi)\right.$$
$$\left. + \cot\theta\cos\varphi\,\partial_\varphi(\sin\varphi\,\partial_\theta)\right]$$

$$L_y^2 = -\hbar^2\left[\cot^2\theta\sin^2\varphi\,\partial_\varphi^2 + \cos^2\varphi\,\partial_\theta^2 - \cot\theta\sin\varphi\,\partial_\varphi(\cos\varphi\,\partial_\theta)\right.$$
$$\left. - \cos\varphi\sin\varphi\,\partial_\theta(\cot\theta\,\partial_\varphi)\right]$$

$$\Rightarrow L_x^2 + L_y^2 = -\hbar^2\left[\partial_\theta^2 + \cot^2\theta\,\partial_\varphi^2 + \cot\theta\cos\varphi\,\partial_\varphi(\sin\varphi\,\partial_\theta)\right.$$

$$\left. - \cot\theta\sin\varphi\,\partial_\varphi(\cos\varphi\,\partial_\theta)\right] = -\hbar^2\left[\partial_\theta^2 + \cot^2\theta\,\partial_\varphi^2\right.$$

$$+ \cot\theta\cos^2\varphi\,\partial_\theta + \cot\theta\sin^2\varphi\,\partial_\theta + \cancel{\cot\theta\cos\varphi\sin\varphi\,\partial_\varphi\partial_\theta}$$

$$\left. - \cancel{\cot\theta\sin\varphi\cos\varphi\,\partial_\varphi\partial_\theta}\right] = -\hbar^2\left[\partial_\theta^2 + \cot^2\theta\,\partial_\varphi^2 + \cot\theta\,\partial_\theta\right]$$

$$\Rightarrow L^2 = -\hbar^2\left[\partial_\theta^2 + \cot^2\theta\,\partial_\varphi^2 + \cot\theta\,\partial_\theta\right] - \hbar^2\partial_\varphi^2$$

$$\cot^2\theta + 1 = \frac{\cos^2\theta}{\sin^2\theta} + 1 = \frac{1}{\sin^2\theta}$$

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$$L^2 = -\hbar^2 \left[+ \partial_\theta^2 + \cot\theta \, \partial_\theta + \frac{1}{\sin^2\theta} \partial_\varphi^2 \right]$$

$$\partial_\theta \left(\sin\theta \, \partial_\theta \right) = \sin\theta \, \partial_\theta^2 + \cos\theta \, \partial_\theta$$

$$\Rightarrow \frac{1}{\sin\theta} \partial_\theta \left(\sin\theta \, \partial_\theta \right) = \partial_\theta^2 + \cot\theta \, \partial_\theta$$

$$\Rightarrow \boxed{L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \partial_\theta \left(\sin\theta \, \partial_\theta \right) + \frac{1}{\sin^2\theta} \partial_\varphi^2 \right]}$$