ABOUT ORTHOGONALITY (STURM-LIOUVILLE)

$$\int_0^{2\pi}\left(\frac{d^2}{dx^2} S_n(x)\right) S_m(x)\,dx = n^2 \int_0^{2\pi} S_n(x)\, S_m(x)\,dx$$

$$= \int_0^{2\pi} \frac{d}{dx}\underbrace{\left(\left(\frac{d}{dx} S_n(x)\right) S_m(x)\right)}_{}^{=0} dx - \int_0^{2\pi}\left(\frac{d}{dx} S_n(x)\right)\left(\frac{d}{dx} S_m(x)\right)$$

$$= 0 + \int_0^{2\pi} S_n(x) \frac{d^2}{dx^2} S_m(x) - \int_0^{2\pi} \frac{d}{dx}\left(S_n(x) \frac{d}{dx}\left(S_m(x)\right)\right)$$

$= 0$

$$= m^2 \int_0^{2\pi} S_n(x)\, S_m(x)\,dx$$

$$\Rightarrow \left(n^2 - m^2\right)\int_0^{2\pi} S_n(x)\, S_m(x)\,dx = 0$$

$\Rightarrow$ IF $n \neq m$

$$\int_0^{2\pi} S_n(x)\, S_m(x)\,dx = 0$$

$\Rightarrow$ IF WE CHOOSE $\int_0^{2\pi} S_n^2(x) = 1 \Rightarrow \int_0^{2\pi} S_n(x)\, S_m(x)\,dx = \delta_{nm}$

NOTE: WE ARE JUST DOING

$$\langle n|L|m\rangle = \langle n|L|m\rangle = m\langle n|m\rangle$$

$$\parallel$$

$n\langle n|m\rangle \Rightarrow$ #

$\boxed{\text{SINCE } L^\dagger = L}$

THE GENERAL S-L PROBLEM

$$\alpha(x)\frac{d^2}{dx^2}\psi(x)+\beta(x)\frac{d}{dx}\psi(x)+\gamma(x)\psi(x)=\lambda\psi(x)$$

On

$$\hat{L}\psi_\lambda(x)=\lambda\psi_\lambda(x)$$

$$\hat{L}\equiv\left(\alpha(x)\frac{d^2}{dx^2}+\beta(x)\frac{d}{dx}+\gamma(x)\right)$$

α, β, γ REAL FUNCTIONS

TO ANALYSE THE PROBLEM, WE DEFINE THE FOLLOWING SCALAR PRODUCT

$$\langle\phi,\psi\rangle_w=\int dx\,\overline{\phi(x)}\,\psi(x)\,w(x)\,dx$$

$w(x)$ IS A REAL FUNCTION, $w(x)\geq 0$

$$\Rightarrow\quad\langle\psi,\phi\rangle=\overline{\langle\phi,\psi\rangle}$$

$$\langle\psi,\alpha\phi_1+\beta\phi_2\rangle=\alpha\langle\psi,\phi_1\rangle+\beta\langle\psi,\phi_2\rangle$$

$$\langle\phi,\phi\rangle\geq 0$$

$\langle\phi,\phi\rangle=0$ MEANS $\int dx\,|\phi(x)|^2\,w(x)\,dx$

$\Rightarrow\quad\phi(x)\neq 0$ ONLY IN POINTS WITH $w(x)=0$

$\Rightarrow$ WE MAY SAY

$\phi(x) \perp 0$ WITH RESPECT TO THE MEASURE DEFINED BY

$w(x)$

PROVIDED

$\langle \psi, \phi \rangle_w$ AND $\hat{L}$

WE DEFINE L^+ (WE COULD ALSO WRITE L^+_w TO SPECIFY THAT IT IS ADJ- WITH RESPECT TO $\langle \, , \rangle_w$) AS THE LIN. OP. S.T. $\forall \psi, \phi$

$$\langle L^+\psi, \phi \rangle = \langle \psi, L\phi \rangle$$

OR

$$\int \left(\overline{L^+ \psi(x)} \right) \phi(x) \, w(x) \, dx = \int \overline{\psi(x)} \left(L \phi(x) \right) w(x) \, dx$$

WE NOW DERIVE AN EXPRESSION OF $w(x)$ SUCH THAT

$$L^+ = L$$

NOTE ALL INTEGRALS $\int_a^b$ ARE OVER THE DOMAIN OF OUR VECTOR SPACE (WHERE WE SOLVE D.E.!)

LET US WRITE

$$\int \overline{\psi(x)}\left(L\phi(x)\right)dx = \int \overline{\psi(x)}\,\alpha(x)\left(\frac{d^2}{dx^2}\phi(x)\right)w(x)\,dx$$

$$+\int \overline{\psi}(x)\,\beta(x)\left(\frac{d}{dx}\phi(x)\right)w(x)\,dx + \int \overline{\psi(x)}\,\gamma(x)\,\phi(x)\,w(x)\,dx$$

$$= \int \frac{d}{dx}\left[\overline{\psi(x)}\,\alpha(x)\left(\frac{d}{dx}\phi(x)\right)w(x)\right]dx - \int \frac{d}{dx}\left(\overline{\psi}(x)\,\alpha(x)\,w(x)\right)\frac{d}{dx}\left(\phi(x)\right)dx$$

$$+\int \frac{d}{dx}\left[\overline{\psi}(x)\,\beta(x)\,\phi(x)\,w(x)\right]dx - \int \frac{d}{dx}\left(\overline{\psi}(x)\,\beta(x)\,w(x)\right)\phi(x)$$

$$+\int \overline{\psi(x)}\,\gamma(x)\,\phi(x)\,w(x) =$$

$$= \left[\overline{\psi(x)}\,\alpha(x)\frac{d}{dx}\left(\phi(x)\right)w(x)\right]_a^b + \left[\overline{\psi(x)}\,\beta(x)\,\phi(x)\,w(x)\right]_a^b$$

$$-\int\left(\frac{d}{dx}\overline{\psi(x)}\right)\left(\frac{d}{dx}\phi(x)\right)\alpha(x)\,w(x)\,dx - \int \overline{\psi}(x)\frac{d}{dx}\left(\alpha(x)\,w(x)\right)\frac{d}{dx}\left(\phi(x)\right)dx$$

$$-\int\left(\frac{d}{dx}\overline{\psi(x)}\right)\beta(x)\,w(x)\,\phi(x) - \int \overline{\psi(x)}\frac{d}{dx}\left(\beta(x)\,w(x)\right)\phi(x)$$

$$+\int \overline{\psi(x)}\,\gamma(x)\,w(x)\,\phi(x)$$

IF 1) ϕ, ψ ARE 0 IN a, b

OR 2) ϕ, ψ ARE PERIODIC

OR 3) $w(x)$ IS 0 IN a, b

$\Rightarrow \left[\quad\right]_a^b = 0$ WE ASSUME THIS

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$\Rightarrow$

$$\langle \psi, L\phi \rangle = -\int \left(\frac{d}{dx}\overline{\psi(x)}\right)\left(\frac{d}{dx}\phi(x)\right)\alpha(x)\,w(x)\,dx$$

$$-\int \overline{\psi(x)}\frac{d}{dx}\left(\alpha(x)\,w(x)\right)\left(\frac{d}{dx}\phi(x)\right)dx - \int\left(\frac{d}{dx}\overline{\psi(x)}\right)\beta(x)\,w(x)\,\phi(x)\,dx$$

$$-\int \overline{\psi(x)}\frac{d}{dx}\left(\beta(x)\,w(x)\right)\phi(x)\,dx + \int \overline{\psi(x)}\,\gamma(x)\,w(x)\,\phi(x)\,dx$$

IN A SIMILAR WAY

$$\langle L\psi, \phi \rangle =$$

$$= \int \alpha(x)\frac{d^2}{dx^2}\left(\overline{\psi(x)}\right)\phi(x)\,w(x)\,dx + \int \beta(x)\frac{d}{dx}\left(\overline{\psi(x)}\right)\phi(x)\,w(x)\,dx$$

$$+\int \overline{\psi(x)}\,\gamma(x)\,w(x)\,\phi(x)\,dx$$

THE ONLY THING CHANGING IS WHERE THE DERIVATIVE OPERATES

$\Rightarrow$ (BY REPEATING POSITION OF ARG)

$$\langle L\psi, \phi \rangle = -\int\left(\frac{d}{dx}\overline{\psi(x)}\right)\left(\frac{d}{dx}\phi(x)\right)\alpha(x)\,w(x)\,dx$$

$$-\int\left(\frac{d}{dx}\overline{\psi(x)}\right)\frac{d}{dx}\left(\alpha(x)\,w(x)\right)\phi(x)\,dx - \int \overline{\psi(x)}\,\beta(x)\,w(x)\left(\frac{d}{dx}\phi(x)\right)dx$$

$$-\int \overline{\psi(x)}\frac{d}{dx}\left(\beta(x)\,w(x)\right)\phi(x) + \int \overline{\psi(x)}\,\gamma(x)\,w(x)\,\phi(x)$$

$$\Rightarrow \quad \langle L\psi, \phi \rangle - \langle \psi, L\phi \rangle =$$

$$= \int \overline{\psi(x)} \left(\frac{d}{dx}\phi(x)\right) \left[\frac{d}{dx}\left(\alpha(x)\, w(x)\right) - \beta(x)\, w(x) \right] dx$$

$$+ \int \left(\frac{d}{dx}\overline{\psi(x)}\right) \phi(x) \left[\beta(x)\, w(x) - \frac{d}{dx}\left(\alpha(x)\, w(x)\right)\right]$$

THIS TERM EQUALS 0 (AND THUS $L = L^{+}$) IF

$$\frac{d}{dx}\left[\alpha(x)\, w(x)\right] = \beta(x)\, w(x)$$

$$\Rightarrow \left(\frac{d}{dx}\alpha(x)\right) w(x) + \alpha(x)\left(\frac{d}{dx} w(x)\right) = \beta(x)\, w(x)$$

$$\Rightarrow \frac{d}{dx} w(x) = \frac{w(x)\left[-\left(\frac{d}{dx}\alpha(x)\right) + \beta(x)\right]}{\alpha(x)} = g(x)\, w(x)$$

WHERE WE DEFINE

$$g(x) \equiv \frac{\beta(x) - \alpha'(x)}{\alpha(x)}$$

THIS IS SOLVED BY

$$w(x) = C_1 \exp\left[\int g(x')\, dx'\right]$$

CHECK $\quad w'(x) = g(x)\, C_1 \exp\left[\int g(x')\, dx'\right] =$

$$= g(x)\, w(x)$$

WE HAVE THUS FOUND

$w(x)$ S.T. $L^{+}_{w} = L$

WE HAVE ALSO A WAY TO DERIVE THE FORM OF

$w(x)$ EASILY, WITH NO NEED OF INTEGRATION

SINCE

$$\frac{d}{dx}\left(\alpha(x)\, w(x) \right) = \beta(x)\, w(x)$$

$\Rightarrow$

ACTS ON ALL

$$\frac{1}{w(x)}\left(\frac{d}{dx}\left[w(x)\,\alpha(x)\frac{d}{dx}\right]\right)\psi(x) + \gamma(x)\psi(x) = \lambda\,\psi(x)$$

IS EQUAL TO

$$\frac{1}{w(x)}\left(\frac{d^2}{dx^2}\psi(x)\right)\left(w(x)\,\alpha(x)\right) + \frac{1}{w(x)}\,\beta(x)\,w(x)\frac{d}{dx}\psi(x) + \gamma(x)\,\psi(x)$$

$$= \lambda\,\psi(x)$$

$$= \alpha(x)\frac{d^2}{dx^2}\psi(x) + \beta(x)\frac{d}{dx}\psi(x) + \gamma(x)\,\psi(x) = \lambda\,\psi(x)$$

THIS IS AN ALTERNATIVE DEFINITION OF S-L PROB

$$\frac{1}{w(x)}\left(\frac{d}{dx}\left[\eta(x)\frac{d}{dx}\right]\right)\psi(x) + \gamma(x)\,\psi(x) = \lambda\,\psi(x)$$

WITH $\eta(x) \equiv \alpha(x)\, w(x)$

NOW IF

$$\hat{L}\,\psi_\lambda(x) = \lambda\,\psi_\lambda(x)$$

$$\hat{L}\,\psi_{\lambda'}(x) = \lambda'\,\psi_{\lambda'}(x)$$

$$\Rightarrow \quad \langle \psi_\lambda, \hat{L}\,\psi_{\lambda'} \rangle_w = \lambda' \langle \psi_\lambda, \psi_{\lambda'} \rangle_w$$

$$\parallel$$

$$\langle L\,\psi_\lambda, \psi_{\lambda'} \rangle = \bar{\lambda}^{*} \langle \psi_{\lambda'}, \psi_\lambda \rangle$$

BUT

$$\langle \psi_\lambda, L\,\psi_\lambda \rangle_w = \lambda \langle \psi_\lambda, \psi_\lambda \rangle_w$$

$$\parallel$$

$$\langle L\,\psi_\lambda, \psi_\lambda \rangle = \bar{\lambda} \langle \psi_\lambda, \psi_\lambda \rangle \quad \Rightarrow \quad \bar{\lambda} = \lambda$$

$$\Rightarrow \quad (\lambda' - \lambda) \langle \psi_{\lambda'}, \psi_\lambda \rangle = 0$$

$$\Rightarrow \quad \langle \psi_\lambda, \psi_{\lambda'} \rangle = 0 \quad \text{IF} \quad \lambda \neq \lambda'$$

PREVIOUSLY WE STUDIED

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} \right) \Theta(\theta) = \lambda\, \Theta(\theta)$$

THIS IS A S.L. WITH $\alpha(x) = 1$ $\quad \gamma(x) = \ell$

$w(x) = \sin x$

$$\Rightarrow \int \Theta_\lambda(\theta)\, \Theta_{\lambda'}(\theta) \sin\theta \, d\theta = \delta_{\lambda\lambda'}$$

BUT Θ_λ ARE THE $P_\ell(\cos\theta)$, ORTHOGONAL WHEN INTEGRATED IN $d^3x = r^2 d\theta \sin\theta \, d\varphi \, dr$