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State reconstruction of a delayed system affected by noise on the input measurement

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Consider the classic system:

$$\begin{cases} x[t+1] = Ax[t] + B_w w[t] + B_u \mathrm{u}[t-\tau] \\ y[t] = Cx[t] + D_w w[t] \end{cases} \tag{0.1}$$

But, in the condition where the control input is measured with a noise as $u_m[t] = u[t] + D_u n_u[t]$, where n_u is a unit-variance white process.

$$\begin{cases} x[t+1] = Ax[t] + B_w w[t] - B_u D_u n_u[t-\tau] + B_u u_m[t-\tau] \\ y[t] = Cx[t] + D_w w[t] \end{cases} \tag{0.2}$$

As first, the system state space realization with a new state vector x_τ and input noise signal w_τ and the corresponding matrices $A_\tau, B_{w\tau}, B_{u\tau}, C_\tau$ and $D_{w\tau}$ are computed.
Then the conditions that guarantee the existence of the solution of the DARE are checked.

After the DARE is solved an analysis on the poles of the Kalman filter is performed, in particular, focusing on the n-dimensional part in order to interpret the results.

As last matter a reduced order Kalman filter is obtained.

State vector 'x_τ' with new input signal 'w_τ'

Starting from the system (0.2):

$$\begin{cases} x[t+1] = Ax[t] + B_w w[t] - B_u D_u n_u[t-\tau] + B_u u_m[t-\tau] \\ y[t] = Cx[t] + D_w w[t] \end{cases} \tag{1.1}$$

Where $x[t] \in R^n$, $u[t] \in R^m$, $w[t] \in R^{m_w}$ and $y[t] \in R^p$

The new state vector of the system x_τ is a vector with the actual state of the system and all the real input from $\mathrm{u}[t-\mathrm{\tau}]$ to $\mathrm{u}[t-1]$, so $x_\tau \in R^{n+\tau m}$ and it represents the real state of the system, while the observed part is a vector $\mathrm{y}_\tau[t] \in R^{p+\tau m}$ with the observation $\mathrm{y}[t]$ and all the measured input from $u_m[t-\mathrm{\tau}]$ to $u_m[t-1]$ (where $u_m[t] = u[t] + D_u n_u[t]$). The new w_τ is a vector with the old process disturbance w_x and output noise w_y and $\tau+1$ realizations of the noise n_u on the measured input.

For the state evolution $x_\tau[t+1]$ all the equations have been written as function of $x_\tau[\mathrm{t}]$ and $w[t]$, the only exception is the last row for which are considered the noise $n_u[t]$ and the measured output $u_m[t]$.

The measured output is written as $u_m[t-k] = u[t-k] + D_u n_u[t-k]$, $k = 1 \dots \tau$ so all the $n_u[t-\tau] \dots n_u[t-1]$ are needed.

$$\begin{cases} \begin{bmatrix} x[t+1] \\ u[t-\tau+1] \\ \vdots \\ u[t-1] \\ u[t] \end{bmatrix} = \begin{bmatrix} A & B_u & 0 & \dots & 0 \\ 0 & 0 & I_m & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & I_m \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} x[t] \\ u[t-\tau] \\ \vdots \\ u[t-2] \\ u[t-1] \end{bmatrix} + \begin{bmatrix} B_w \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} w[t] + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -D_u \end{bmatrix} n_u[t] + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix} u_m[t] \\ \begin{bmatrix} y[t] \\ u_m[t-\tau] \\ \vdots \\ u_m[t-2] \\ u_m[t-1] \end{bmatrix} = \begin{bmatrix} C & 0 & \dots & 0 & 0 \\ 0 & I_m & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 0 & I_m \end{bmatrix} \begin{bmatrix} x[t] \\ u[t-\tau] \\ \vdots \\ u[t-2] \\ u[t-1] \end{bmatrix} + \begin{bmatrix} D_w \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} w[t] + \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ D_u & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 0 & D_u \end{bmatrix} \begin{bmatrix} n_u[t-\tau] \\ n_u[t-\tau+1] \\ \vdots \\ n_u[t-2] \\ n_u[t-1] \end{bmatrix} \end{cases} \tag{1.2}$$

Since n_u is a white noise there is no correlation between the different time realization of the signal, thus is like they were different noise and the condition $B_w D_w^T = \underline{0}$ is kept.

Since all state disturbances $w_x[t]$, output measurement noise $w_y[t]$ and input measurement noise $n_u[t]$ are white with unit-variance, they can be joined as the vector $w_\tau = \left[w_x\ w_y\ n_u[t-\tau] \dots n_u[t]\ \right]^T$.

Considering the new state vector $x_\tau[t]$, the new measured output $y_\tau[t]$ and the input signal vector $w_\tau[t]$, the whole enlarged system can be written as:

$$\left\{\begin{array}{l}
\begin{bmatrix} x[t+1] \\ u[t-\tau+1] \\ \vdots \\ u[t-1] \\ u[t] \end{bmatrix} = \overbrace{\begin{bmatrix} A & B_u & 0 & \dots & 0 \\ 0 & 0 & I_m & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & I_m \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}}^{A_\tau \in R^{(n+m\tau)x(n+m\tau)}} \begin{bmatrix} x[t] \\ u[t-\tau] \\ \vdots \\ u[t-2] \\ u[t-1] \end{bmatrix} + \overbrace{\begin{bmatrix} B_w & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -D_u \end{bmatrix}}^{B_{w\tau} \in R^{(n+m\tau)x(m_W+m_n(\tau+1))}} \begin{bmatrix} w[t] \\ n_u[t-\tau] \\ \vdots \\ n_u[t-1] \\ n_u[t] \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix}}^{B_{u\tau} \in R^{(n+m\tau)xm}} u_m[t] \\
\begin{bmatrix} y[t] \\ u_m[t-\tau] \\ \vdots \\ u_m[t-2] \\ u_m[t-1] \end{bmatrix} = \underbrace{\begin{bmatrix} C & 0 & \dots & 0 & 0 \\ 0 & I_m & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 0 & I_m \end{bmatrix}}_{C_\tau \in R^{(p+m\tau)x(n+m\tau)}} \begin{bmatrix} x[t] \\ u[t-\tau] \\ \vdots \\ u[t-2] \\ u[t-1] \end{bmatrix} + \underbrace{\begin{bmatrix} D_w & 0 & \dots & 0 & 0 \\ 0 & D_u & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & D_u & 0 \end{bmatrix}}_{D_{w\tau} \in R^{(p+m\tau)x(m_W+m_n(\tau+1))}} \begin{bmatrix} w[t] \\ n_u[t-\tau] \\ \vdots \\ n_u[t-1] \\ n_u[t] \end{bmatrix}
\end{array}\right. \tag{1.3}$$

Conditions on Du in order to guarantee the existence of a stabilizing solution of DARE.

Starting from the new formulation of the state space equation in form (1.1), it is necessary to verify that the conditions that guarantee the existence of a stabilizing solution of DARE exist.
It can be done by ensuring that:

a) The pair $(C_\tau,\ A_\tau)$ is detectable.
b) The matrix $\begin{bmatrix} A_\tau - e^{j\theta} I & B_{w,\tau} \\ C_\tau & D_{w,\tau} \end{bmatrix}$ has full row rank for all $\theta \in [-\pi, \pi]$, that means that the realization of $\left(A_\tau, B_{w,\tau}, C_\tau, D_{w,\tau}\right)$ has no invariant zeros in the unitary circle.

For the sake of simplicity, we also assume that the process noise and the measurement noise are uncorrelated, so we check that $B_{w,\tau} D_{w,\tau}^T = \underline{0}$.

a) The matrix $\begin{bmatrix} A_\tau - \lambda I \\ C_\tau \end{bmatrix}$ has full rank

$$rank \begin{bmatrix} A-\lambda I & B_u & 0 & \dots & 0 \\ 0 & -\lambda I & I_m & \ddots & 0 \\ 0 & 0 & -\lambda I & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & I_m \\ 0 & 0 & 0 & \dots & -\lambda I \\ C & 0 & 0 & \dots & 0 \\ 0 & I_m & 0 & \ddots & \vdots \\ 0 & 0 & I_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & I_m \end{bmatrix} = rank \left[\begin{array}{c|cccc} A-\lambda I & B_u & 0 & \dots & 0 \\ C & 0 & 0 & \dots & 0 \\ \hline 0 & -\lambda I & I_m & \ddots & 0 \\ 0 & 0 & -\lambda I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & I_m \\ \vdots & \ddots & \ddots & \ddots & -\lambda I \\ \hline \vdots & I_m & \ddots & \ddots & \vdots \\ \vdots & \ddots & I_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & I_m \end{array}\right] = rank \begin{bmatrix} A-\lambda I \\ C \end{bmatrix} + \tau m$$

From the assumption that the conditions were satisfied for (0.1), $rank \begin{bmatrix} A-\lambda I \\ C \end{bmatrix} = n$ and then the pair $(C_\tau,\ A_\tau)$ is detectable.

b) The matrix $\begin{bmatrix} A_\tau - e^{j\theta} I & B_{w,\tau} \\ C_\tau & D_{w,\tau} \end{bmatrix}$ has full row rank:

$$rank \begin{bmatrix} A - e^{i\theta} & B_u & 0 & \dots & 0 & B_w & 0 & \dots & 0 & 0 \\ 0 & -e^{i\theta} I_m & I_m & \ddots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -e^{i\theta} I_m & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & I_m & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & \dots & \dots & -e^{i\theta} I_m & 0 & 0 & \dots & 0 & -D_u \\ C & 0 & 0 & \dots & 0 & D_w & 0 & \dots & 0 & 0 \\ 0 & I_m & 0 & \ddots & \vdots & 0 & D_u & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_m & 0 & 0 & \dots & D_u & 0 \end{bmatrix} =$$

$$
= rank \begin{bmatrix} A-e^{i\theta} & B_w & B_u & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ C & D_w & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -e^{i\theta} & I_m & \ddots & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -e^{i\theta} & I_m & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & -e^{i\theta} & 0 & \dots & 0 & 0 & -D_u \\ 0 & 0 & I_m & 0 & 0 & 0 & D_u & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_m & 0 & 0 & \ddots & D_u & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & I_m & 0 & \dots & 0 & D_u & 0 \end{bmatrix} =
$$

$$
= rank \left[\begin{array}{cc|cccc|ccccc} A-e^{i\theta} & B_w & B_u & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ C & D_w & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \hline 0 & 0 & -e^{i\theta}I & I_m & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -e^{i\theta}I & I_m & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & -e^{i\theta}I & 0 & \ddots & \ddots & \ddots & -D_u \\ \hline \vdots & \vdots & \dots & \dots & \dots & 0 & D_u & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \ddots & D_u & 0 & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & D_u & 0 \end{array}\right]
$$

Here can be easily seen that under the assumption that for the system (0.1) the matrix $\begin{bmatrix} A-\mathrm{e}^{\mathrm{j}\theta}I & B_w \\ C & D_w \end{bmatrix}$ has full row rank, the row rank is given by:

$$
rank \begin{bmatrix} A-\mathrm{e}^{\mathrm{j}\theta}I & B_w \\ C & D_w \end{bmatrix} + \tau m + rank \begin{bmatrix} D_u & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & D_u & 0 \\ 0 & \dots & 0 & D_u \end{bmatrix}
$$

That is full only if

$$
D_u \neq \underline{0} \tag{2.1}
$$

It is immediate to see that $B_{w,\tau} D_{w,\tau}^T = \underline{0}$, indeed:

$$
\begin{bmatrix} B_w & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -D_u \end{bmatrix} \begin{bmatrix} {D_w}^T & 0 & \dots & 0 & 0 \\ 0 & {D_u}^T & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & {D_u}^T & 0 \\ 0 & 0 & \dots & 0 & {D_u}^T \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} = \begin{bmatrix} B_w {D_w}^T & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} =^* \underline{0}
$$

$=^*$ under the assumption that $B_w D_w^T = \underline{0}$

DARE solution and Kalman's filter poles analysis

The structure of the system does not change with the delay, but its dimension grows with it. Given a general square $Y \in R^{(n+m\tau)x(n+m\tau)}$, such that $Y = Y^T$, the DARE equation reads:

$$Y = A_\tau Y A_\tau^T + B_{w,\tau} B_{w,\tau}^T + (B_{w,\tau} D_{w,\tau}^T + A_\tau Y C_\tau^T)\left(D_{w,\tau} D_{w,\tau}^T + C_\tau Y C_\tau^T\right)(D_{w,\tau} B_{w,\tau}^T + C_\tau Y A_\tau^T)$$

Under the assumption that $B_{w,\tau} D_{w,\tau}^T = \underline{0}$ this reduce to

$$Y = A_\tau Y A_\tau^T + B_{w,\tau,} B_{w,\tau}^T + A_\tau Y C_\tau^T\left(D_{w,\tau} D_{w,\tau}^T + C_\tau Y C_\tau^T\right) C_\tau Y A_\tau^T$$

In the DARE there is the sum of three terms, two of these are pre-multiplied by A_τ and post multiplied by its transpose. It is important to analyse the behaviour of this operation as it plays a fundamental role in the solution of the extended system. The last row of matrix A_τ, then the last column of A_τ^T, is full of zero. Then the result of the multiplication will produce a matrix with a set of zero in the bottom row and the last column. $\tilde{Q} = B_{w,\tau,} B_{w,\tau}^T$ is the last term to be summed of the Riccati equation. It has a special structure with only one element in the upper-left n-square matrix, that is $B_w {B_w}^T$, and another in the m-square matrix in the bottom right, which is $D_u {D_u}^T$. Then the structure with the zeros in the last row and column is preserved, except for the last bottom-right block where only the element $D_u {D_u}^T$is present.

$$Y^* = \begin{bmatrix} Y^{n+m(\tau-1)xn+m(\tau-1)} & 0 \\ 0 & D_u {D_u}^T \end{bmatrix}$$

By the inclusion of this constraints into the general formulations of the matrix Y we obtain that also the elements outside the diagonal of the previous m rows and columns are equal to zero, because the elements on the row k depend on the elements of the row k+1. This operation can be iterated τ times and then we finally obtain the general structure for Y:

$$Y = \begin{bmatrix} \tilde{Y} & 0 & \dots & 0 \\ 0 & y_\tau & \ddots & \vdots \\ \vdots & \ddots & y_k & \vdots \\ 0 & \dots & \dots & y_1 \end{bmatrix} k = \tau - 1 \dots 2 \qquad (3.1)$$

Where $\tilde{Y} \in R^{nxn}$ and the of the τ terms $y_k \in R^{(m\tau)x(m\tau)}$.

Now with the knowledge on the structure of Y we analyse the solution, the first term of the sum in the right end:

$$A_\tau Y {A_\tau}^T = \begin{bmatrix} A\tilde{Y}A^T + B_u y_\tau B_u^T & \dots & \dots & \dots & 0 \\ 0 & y_{\tau-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & y_k & \ddots & \vdots \\ \vdots & \ddots & \ddots & y_1 & \vdots \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \qquad k = \tau - 2 \dots 2 \qquad (3.2)$$

Similarly, the third one.

Then, adding the $\tilde{Q}$ term the Riccati equation become:

$$\begin{bmatrix} \tilde{Y} & 0 & \dots & 0 \\ 0 & y_\tau & \ddots & \vdots \\ \vdots & \ddots & y_k & 0 \\ 0 & \dots & 0 & y_1 \end{bmatrix} = \begin{bmatrix} *_1 & 0 & \dots & 0 \\ 0 & y_{\tau-1} - y_{\tau-1}^2 (D_u D_u^T + y_{\tau-1})^{-1} & \ddots & \vdots \\ \vdots & \ddots & *_2 & 0 \\ 0 & \dots & 0 & D_u D_u^T \end{bmatrix}, \qquad k = \tau - 1 \dots 2 \tag{3.3}$$

$$*_1 = A\tilde{Y}A^T + B_w B_w^T + B_u \left(y_\tau - y_\tau^2 \left(D_u D_u^T + y_\tau\right)^{-1}\right) B_u^T - \left(B_w D_w^T + A\tilde{Y}C^T\right)\left(D_w D_w^T + C\tilde{Y}C^T\right)\left(D_w B_w^T + C\tilde{Y}A^T\right)$$

$$*_2 = y_{k-1} - y_{k-1}^2 \left(D_u D_u^T + y_{k-1}\right)^{-1}, \qquad k = \tau - 1 \dots 2$$

Clearly, the last τ elements of the Y solution are connected by these relationships:

$$\begin{cases} y_k = y_{k-1} - y_{k-1}^2 \left(D_u D_u^T + y_{k-1}\right)^{-1}, & k = \tau \dots 2 \\ y_1 = D_u D_u^T \end{cases} \tag{3.4a}$$

Thus,

$$\begin{cases} y_k = \dfrac{D_u D_u^T}{k}, & k = \tau \dots 2 \\ y_1 = D_u D_u^T \end{cases} \tag{3.4b}$$

Then the solution of Y is:

$$\begin{bmatrix} \tilde{Y} & 0 & \dots & 0 \\ 0 & y_\tau & \ddots & \vdots \\ \vdots & \ddots & y_k & 0 \\ 0 & \dots & 0 & y_1 \end{bmatrix} = \begin{bmatrix} * & 0 & \dots & 0 \\ 0 & \dfrac{D_u D_u^T}{\tau} & \ddots & \vdots \\ \vdots & \ddots & \dfrac{D_u D_u^T}{k} & 0 \\ 0 & \dots & 0 & D_u D_u^T \end{bmatrix}, k = \tau - 1 \dots 2 \tag{3.5}$$

$$* = A\tilde{Y}A^T + B_w B_w^T + \frac{B_u D_u D_u^T B_u^T}{\tau + 1} - (B_w D_w^T + A\tilde{Y}C^T)(D_w D_w^T + C\tilde{Y}C^T)(D_w B_w^T + C\tilde{Y}A^T)$$

This leaves unsolved only the n -dimensional block that has a special structure, it is another DARE for a system where:

$$\widetilde{Q^*} = B_w^* {B_w^*}^T = B_w B_w^T + \frac{B_u D_u D_u^T B_u^T}{\tau + 1}$$

Thus, the computation of the covariance matrix only requires the solution of a n dimensional DARE in order to compute the upper nxn submatrix values while the other τ blocks elements in the diagonal are readily computed from closed-form expressions (3.5).

A deeper look into the n-dimensional part of the solution.

The matrices of this formulation are clearly the ones of the system without delay, then we can try to solve this and understand if there is some correlation or is just a visual similarity.

Without delay the system would be:

$$\begin{cases} x[t+1] = Ax[t] + B_w w[t] - B_u D_u n_u[t] + B_u u_m[t] \\ y[t] = Cx[t] + D_w w[t] \end{cases}$$

$$\begin{array}{cc} B_w w_x \sim (0,\ \tilde{Q}_x) & \tilde{Q}_x = B_w B_w^T \\ D_w w_y \sim (0, \tilde{R}) & \tilde{R} = D_w D_w^T \\ B_u D_u n_u \sim (0, \widetilde{Q_n}) & \widetilde{Q_n} = B_u D_u D_u^T B_u^T \end{array}$$

Then, adding the noises and the disturbances affecting the state equation,

$$w_{xn} = B_w w_x + B_u D_u n_u$$

$$E(w_{xn}) = E(w_x) + E(n) = 0$$

$$cov(w_{xn}) = cov(w_x) + cov(n) = B_w B_w^T + B_u D_u D_u^T B_u^T$$

$$\begin{cases} x[t+1] = Ax[t] + w_{xn}[t] + B_u u_m[t] \\ y[t] = Cx[t] + w_y[t] \end{cases}$$

This system is solved by a DARE:

$$Y = AYA^T + B_w B_w^T + B_u D_u D_u^T B_u^T - (B_w D_w^T + AYC^T)(D_w D_w^T + CYC^T)^{-1}(D_w B_w^T + CYA^T)$$

Which is again like for the delayed system with exactly $\tau = 0$, i.e., no delay.

A possible interpretation of this is the following one. The measurement noise is both in the state and the output equations, exactly $\tau + 1$ time realizations of it appear at every time step in the system. Then, as delay increases, decreasing its influence in the covariance of the state disturbance, where it appears just once, it is needed to maintain the system “equilibrated” due to its increasing influence on the covariance of the output, where there are τ realizations.

The more the system is delayed the less the error on the single measurement is powerful on the state since many other disturbances have already acted in the previous time steps and this information are already stored inside the state of the system, as well as in the estimated state. The more memory the system has, the less influence the single disturbed element of the history has.

Back to delayed problem, even if it is the same.

The gain matrix L obtained from this solution is:

$$L = -\left(B_w\, D_w^T + A\tilde{Y}C^T\right)\left(D_w D_w^T + C\tilde{Y}C^T\right)^{-1}$$

$$L = \begin{bmatrix} \tilde{L} & -\dfrac{B_u}{\tau+1} & 0 & \dots & 0 \\ 0 & 0 & -\dfrac{1}{\tau} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\dfrac{1}{\tau-k} & 0 \\ \vdots & \ddots & \ddots & 0 & -\dfrac{1}{2} \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \qquad k = 1 \dots \tau - 3 \tag{3.6}$$

Like in the structure of Y, there is n-dimensional a component derived by the solution of the Riccati equation for an equivalent n-dimensional system and a fixed one to complete the last τm columns and rows.

Furthermore, $\hat{A} = A_\tau + \tilde{L}C_\tau$ also acquires a fixed structure that is:

$$\hat{A} = \begin{bmatrix} A+\tilde{L}C & \frac{\tau B_u}{\tau+1} & 0 & \dots & 0 \\ 0 & 0 & \frac{\tau-1}{\tau} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{\tau-k-1}{\tau-k} & 0 \\ \vdots & \ddots & \ddots & 0 & \frac{1}{2} \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \qquad k = 1 \dots \tau - 3 \tag{3.7}$$

The obtained Kalman filter will have τm poles at the origin and n in the solution of the corresponding n–dimensional equivalent system.

Kalman filter formulation

The Kalman filter formulation for the delayed system is:

$$\begin{bmatrix} \hat{x}[t+1] \\ \hat{u}[t-\tau+1] \\ \vdots \\ \hat{u}[t-1] \\ \hat{u}[t] \end{bmatrix} = \begin{bmatrix} A & B_u & 0 & \dots & 0 \\ 0 & 0 & I_m & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & I_m \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \hat{x}[t] \\ \hat{u}[t-\tau] \\ \vdots \\ \hat{u}[t-2] \\ \hat{u}[t-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix} u_m[t] - L \left(\begin{bmatrix} y[t] \\ u_m[t-\tau] \\ \vdots \\ u_m[t-2] \\ u_m[t-1] \end{bmatrix} - \begin{bmatrix} C & 0 & \dots & 0 & 0 \\ 0 & I_m & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & I_m & 0 \\ 0 & 0 & \dots & 0 & I_m \end{bmatrix} \begin{bmatrix} \hat{x}[t] \\ \hat{u}[t-\tau] \\ \vdots \\ \hat{u}[t-2] \\ \hat{u}[t-1] \end{bmatrix} \right) \quad (4.1a)$$

$$\begin{bmatrix} \hat{x}[t+1] \\ \hat{u}[t-\tau+1] \\ \vdots \\ \hat{u}[t-1] \\ \hat{u}[t] \end{bmatrix} = \begin{bmatrix} A+\tilde{L}C & \frac{\tau B_u}{\tau+1} & 0 & \dots & 0 \\ 0 & 0 & \frac{\tau-1}{\tau} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{\tau-k-1}{\tau-k} & 0 \\ \vdots & \ddots & \ddots & 0 & \frac{1}{2} \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}[t] \\ \hat{u}[t-\tau] \\ \vdots \\ \hat{u}[t-2] \\ \hat{u}[t-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix} u_m[t] - L \begin{bmatrix} y[t] \\ u_m[t-\tau] \\ \vdots \\ u_m[t-2] \\ u_m[t-1] \end{bmatrix} \quad (4.1b)$$

$$\begin{bmatrix} \hat{x}[t+1] \\ \hat{u}[t-\tau+1] \\ \vdots \\ \hat{u}[t-1] \\ \hat{u}[t] \end{bmatrix} = \begin{bmatrix} \mathrm{A}+\tilde{\mathrm{L}}\mathrm{C} & \frac{\tau \mathrm{B_u}}{\tau+1} & 0 & \dots & 0 \\ 0 & 0 & \frac{\tau-1}{\tau} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{\tau-\mathrm{k}-1}{\tau-\mathrm{k}} & 0 \\ \vdots & \ddots & \ddots & 0 & \frac{1}{2} \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}[t] \\ \hat{u}[t-\tau] \\ \vdots \\ \hat{u}[t-2] \\ \hat{u}[t-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix} u_m[t] + \begin{bmatrix} -\tilde{L}y[t] + \frac{B_u}{\tau+1} u_m[t-\tau] \\ \vdots \\ \frac{1}{3} u_m[t-2] \\ \frac{1}{2} u_m[t-1] \\ 0 \end{bmatrix} \quad (4.1c)$$

Given the presence of the τm poles at the origin in the obtained Kalman filter dynamics matrix, the estimation of the input value $\hat{u}$ are obtained in one step (deadbeat) and it results that u_m is the best approximation that it is possible to attain.

In fact, from (4.1c) the last m-row blocks reads:

$$\hat{u}[t] = u_m[t]$$

Similarly from (4.1c) computed at the previous time step t,

$$\hat{u}[t-1] = u_m[t-1]$$

so that from the second-last m-row block above results:

$$\hat{u}[t-1] = \frac{1}{2} u_m[t-1] + \frac{1}{2} u_m[t-1]$$

more in general the update rule is:

$$\hat{u}[t-k] = \frac{\tau-k-1}{\tau-k} u_m[t-k] + \frac{1}{\tau-k} u_m[t-k] = u_m[t-k]$$

The information of the noisy input measurement is not lost. In fact it can be noticed that with respect to the problem formulation that does not include the noise in the input commands, a correction term $\frac{B_u D_u D_u^T B_u^T}{\tau+1}$ is present in the noise covariance Q which reduces the bias in state estimation that otherwise would be caused by the presence of input noise.

These considerations leads to the following reduced order observer formulation.

Reduced order observer

$$\hat{x}[t+1] = \tilde{A}\hat{x}[t] + B_u u_m[t-\tau] - \tilde{L}y[t]$$

The solution of the reduced order formulation sets $\hat{u}[t-k] = u_m[t-k]$, this is the same to say that the best approximation that it is possible to attain about the history of the real input is the measured signal.

Further reasonings that led us to the same conclusion, are the following

A slightly different problem could have been formulated as a system with the noise affecting the input after the measurement on the input is taken:

$$\begin{cases} x[t+1] = Ax[t] + B_w w[t] + B_u D_u n_u[t-\tau] + B_u \mathrm{u}[t-\tau] \\ y[t] = Cx[t] + D_w w[t] \end{cases}$$

To solve this problem the history of u is used in the extended state and the noises and disturbances acting on the state equation are grouped into only one disturbance with a modified covariance as we did previuosly. Then since for this case the measurement is u and the state again uses u, the formulation of a reduced order Kalman filter more naturally.

$$\begin{cases} \begin{bmatrix} x[t+1] \\ u[t-\tau+1] \\ \vdots \\ u[t-1] \\ u[t] \end{bmatrix} = \begin{bmatrix} A & B_u & 0 & \dots & 0 \\ 0 & 0 & I_m & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & I_m \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} x[t] \\ u[t-\tau] \\ \vdots \\ u[t-2] \\ u[t-1] \end{bmatrix} + \overbrace{\begin{bmatrix} B_w \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} w[t] + \begin{bmatrix} B_u D_u \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} n_u[t]}^{B_w^* w_\tau[t]} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix} u[t] \\ \begin{bmatrix} y[t] \\ u[t-\tau] \\ \vdots \\ u[t-2] \\ u[t-1] \end{bmatrix} = \begin{bmatrix} C & 0 & \dots & 0 & 0 \\ 0 & I_m & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 0 & I_m \end{bmatrix} \begin{bmatrix} x[t] \\ u[t-\tau] \\ \vdots \\ u[t-2] \\ u[t-1] \end{bmatrix} + \begin{bmatrix} D_w \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} w[t] \end{cases}$$

Moreover, this alternative problem has the same formulation as (1.1), where $D_u = -D_u, u[t] = u_m[t]$. Thus, a possible formulation of a reduced order observer is not a surprise for the original problem either.
When $D_u \neq 0$ is possible to find a solution, then resort to a reduced order formulation.
When the conditions on $D_u$is checked, implicitly also the condition on the n-dimensional DARE is confirmed.
Another interpretation is the following one:

$$E(u_m) = E(u) + E(n_u) = E(u)$$

Then reconstructing u is the same as reconstructing u_m, this uncertainty affects the state with a different covariance matrix. With the same principle, it's not possible to perfectly reconstruct the y signal of a system after it's corrupted by a white noise. Instead, it is possible to use the information that are in $\tilde{y}$, together with the information of the variance of the noise affecting it, to reconstruct a state (4.2) observer that minimize the covariance of the estimation error, which is indeed the goal definition of a Kalman filter.