Problem 1

Part 1

① $y = -0.5x + 2 \rightarrow 0.5x + y - 2z = 0$

① $3x + 6y = 5 \Rightarrow 3x + 6y - 5z = 0$

$L_1 = (0.5, 1, -2)$

$L_2 = (3, 6, -5)$

$$L_1 \times L_2 = \begin{array}{c} \begin{matrix} i & j & k \end{matrix} \\ \begin{bmatrix} 0.5 & 1 & -2 \\ 3 & 6 & -5 \end{bmatrix} \end{array}$$

$$\Delta = (1(-5) - (-2)(6))i + [((-2)(3) - (0.5)(-5))j + ((0.5)(6) - (1)(3))$$

$$(-5 + 12)i + [-6 + 2.5]j + [3 - 3]k$$

$$7i - 3.5j + 0k$$

$$(7, 3.5, 0)$$

* no intersection since $z = 0$

Part 2

ideal point $= [x, y, 0]$ line at infinity $w = 0$

$[7, 3.5, 0]$ since z or $w = 0$ it lies on the line of infinity

Part 3

in homogeneous form

$(0.5x + y - 2)(3x + 6y - 5) = 0$

$(0.5x)(3x) + (0.5x)(6y) + (0.5x)(-5) + (y)(3x) + (y)(6y) + (y)(-5) + (-2)(3x) + (-2)(6y) + (-2)(-5) = 0$

$1.5x^2 + 3xy - 2.5x + 3xy + 6y^2 - 5y - 6x - 12y + 10 = 0$

$1.5x^2 + 6xy - 8.5x + 6y^2 - 17y + 10 = 0$

homogeneous form

$$1.5(x/w)^2 + 6(x/w)(y/w) + 6(y/w)^2 - 8.5(x/w) - 17(y/w) + 10 = 0$$

multiply by w^2

$$1.5x^2 + 6xy + 6y^2 - 8.5xw - 17yw + 10w^2 = 0$$

Matrix form

$$C = \begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix} = \begin{bmatrix} 1.5 & 3 & -4.25 \\ 3 & 6 & -8.5 \\ -4.25 & -8.5 & 10 \end{bmatrix}$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} 1.5 & 3 & -4.25 \\ 3 & 6 & -8.5 \\ -4.25 & -8.5 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

Problem 2

Ideal point : $(x, y, 0)$

2D transformation 3x3 matrix : $\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$

Apply transformation : $[x \ y \ 0]$

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} h_{11} \cdot x + h_{12} \cdot y + h_{13} \cdot 0 \\ h_{21} \cdot x + h_{22} \cdot y + h_{23} \cdot 0 \\ h_{31} \cdot x + h_{32} y + h_{33} \cdot 0 \end{bmatrix} = \begin{bmatrix} h_{11} \cdot x + h_{12} \cdot y \\ h_{21} \cdot x + h_{22} \cdot y \\ h_{31} \cdot x + h_{32} y \end{bmatrix}$$

line in homogeneous coordinates ; $\ell^T \cdot x = 0$

$\ell'^T \cdot x' = 0$

$\ell'^T \cdot H_x = 0$

To ensure that $\ell^T \cdot x = 0$

$\ell' = H^{-T} \cdot \ell$

verify : $C' = H^{-T} \cdot C \cdot H^{-1}$

Conic in homogeneous coordinates : $x^T \cdot C_x = 0$

$x'^T \cdot C' \cdot x' = 0$

$(H_x)^T \cdot C' \, H_x = 0$

$C' = H^{-T} \cdot C \cdot H^{-1}$

Problem 3

Projective

general form = $\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

degree of freedom = in 2D it has 8 degree of freedom since the matrix has 9 parameters but is defined up to a scalar multiple

invariant properties = preserves collinearity and cross ratio of points but do not preserve angles, lengths or parallelism

Affine Transformations

general form = $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

degree of freedom = in 2D has 6 degrees of freedom, 2 for translation, 1 for rotation, 1 for scalling and 2 for Shearing

invariant Properties = Preserves points, straight lines and planes and also preserves parallellism and ratios of segments on parrallel lines

Similarity Transformation

general form = $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

degree of Freedom = in 2D has 4 degrees of freedom 1 for uniform scalling, 1 for rotation an 2 for translation

invariant properties = Preserve Shape of geometrical figures. They maintain the proportionality of lengths and angles. They enlarge or reduce the figures without changing the shape.

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} ax + by \\ dx + ey \\ 0 \end{bmatrix}$$

w' is still $= 0$

Line at infinity = $l_1 x + l_2 y = 0$ $\therefore l_{3u} = 0$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \qquad A^{-T} = \begin{bmatrix} a & d \\ b & e \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1}$$

$$A^{-T} l = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} l'_1 \\ l'_2 \end{bmatrix}$$

Problem 4

for circular points: $I = [1, i, 0]^T$ $\quad J = [1, -i, 0]^T$

for a circle: $x^2 + y^2 + Ax + By + C = 0$

Intersection with I

$1^2 + i^2 + A + Bi + C = 0$

$1 - 1 + A + Bi + C = 0$

Real $= A + C = 0$

imaginary $= B = 0$

Intersection with B

$1^2 + (-i)^2 + A - Bi + C = 0$

$1 - 1 + A - Bi + C = 0$

real : $A + C = 0$

imaginary : $B = 0$

* Every circle intersects the infinite line at the circular points

Under point transformation: $x' = H_s \cdot X$

$$C_\infty^{*'} = H_s C_\infty^* H_s^T = C_\infty^*$$

$$\Rightarrow I^T l_\infty = J^T l_\infty = 0$$

$$C_\infty^* l_\infty = (IJ^T + JI^T) l_\infty = I(J^T l_\infty) + J(I^T l_\infty) = 0$$

For 2 orthogonal lines:

$l = (l_1, l_2, l_3)^T$ $\quad m = (m_1, m_2, m_3)^T$

Projective Space: $\cos\theta = \dfrac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$

lines l and m are orthogonal if $l^T C_\infty^* m = 0$