CSD2250/MAT250 Lecture 2 (Fri 13/5)

Topics covered:
- Echelon forms
- Matrix muliplication
- Elimination and Permutation matrices
- Augmented matrix

As seen in Wed (11/5) notes

Echelon forms

At the end of an elimination process, the matrix A in the matrix form of a system of equation will be upper triangular. We seek to bring it to a form where back substitution becomes slightly easier or trivial.

There are two such forms:

① Row echelon form

② Reduced row echelon form

* Row echelon form

A matrix is in <u>row echelon form</u> if these 3 conditions are satisfied

① All rows consisting of only zeros are at the bottom of the matrix.

from the left

② The first nonzero entry of a nonzero row is always to the right of the first nonzero entry of the row before it.

③ The first nonzero entry of every row must be 1 (can be achieved by dividing every entry of a row by its first nonzero entry).

Eg.

$$\begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

is not in row echelon form

③ fails, ①, ② OK

$$\begin{array}{cccc} \multicolumn{3}{c}{A} & b \\ \left[\begin{matrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}\right. & \left.\begin{matrix} 1 \\ 4 \\ 2 \end{matrix}\right] \end{array}$$

is in row echelon form.

We will see these matrices again later.

★ Reduced row echelon form (rref)

A matrix is in <u>reduced row echelon form</u> if

① It is in row echelon form.

② For all the columns that contain the first entry 1, all the other entries of those columns <u>must be 0</u>.

Eg.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ is in rref.

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ is in row echelon form but not in rref.

Matrix multiplication

We have seen how a vector $\underset{\sim}{x}$ can be (left) multiplied by a matrix A by using either column forms or row forms. What about a matrix A multiplied by another matrix B?

(Here, we note that certain conditions on A and B must be satisfied before we can multiply A w/ B, but let's keep it in view first)

E.g. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{array}{c} \begin{matrix} \underset{\sim}{b_1} & \underset{\sim}{b_2} & \underset{\sim}{b_3} \end{matrix} \\ \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \end{array}$$

Then the columns of AB ~~are~~ is

$$\begin{bmatrix} A\underset{\sim}{b_1} & A\underset{\sim}{b_2} & A\underset{\sim}{b_3} \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix}$$

We already know how to do this using column/row forms.

Some preliminary notes on matrix multiplication

① Associative law holds, i.e.

$$(AB)C = A(BC)$$

② Commutative law does not hold, i.e

AB is not always equal to BA.

Elimination matrices

i.e. These are matrices that perform the elimination process, specifically, they subtract a multiple of a row from other row of a matrix.

Consider

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{array}{c} \begin{array}{ccc} \underset{\sim}{a_1} & \underset{\sim}{a_2} & \underset{\sim}{a_3} \end{array} \\ \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \end{array}$$

$$E_{21}\underset{\sim}{a_1} = 2\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-2)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

first term of second row of A

$$= \begin{bmatrix} 2 \\ 4-4 \\ -2 \end{bmatrix}$$

$\frac{4}{2} = 2$ times of first term of first row of A

$$E_{21}\,\underset{\sim}{a}_2 = 4\begin{bmatrix}1\\-2\\0\end{bmatrix} + 9\begin{bmatrix}0\\1\\0\end{bmatrix} + (-3)\begin{bmatrix}0\\0\\1\end{bmatrix}$$

second term of second row of A

$$= \begin{bmatrix}4\\9-8\\-3\end{bmatrix}$$

2 times of second term of first row of A

$$E_{21}\,\underset{\sim}{a}_3 = -2\begin{bmatrix}1\\-2\\0\end{bmatrix} + (-3)\begin{bmatrix}0\\1\\0\end{bmatrix} + 7\begin{bmatrix}0\\0\\1\end{bmatrix}$$

third term of second row of A

$$= \begin{bmatrix}-2\\-3-(-2)\\7\end{bmatrix}$$

2 times the third term of first row of A

$$\Rightarrow E_{21}A = \begin{bmatrix}E_{21}\underset{\sim}{a}_1 & E_{21}\underset{\sim}{a}_2 & E_{21}\underset{\sim}{a}_3\end{bmatrix}$$

$$= \begin{bmatrix}2 & 4 & -2\\0 & 1 & 1\\-2 & -3 & 7\end{bmatrix}$$

* In all, the matrix E_{21} subtracted 2 times of the first row from the second row!

So essentially, the matrix E_{21} performed the first step of the elimination process!

To convince you further, we consider an arbitrary vector $\underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

$$E_{21} \cdot \underset{\sim}{b} = b_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix}.$$

★ In general, an elimination matrix E_{ij} is an identity matrix, along with an extra nonzero entry $-l$ in the i,j position.

This extra position has to be below the main diagonal: $-l$ here, zeros elsewhere.

$$E_{ij} = \begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

(left) multiplying E_{ij} to a matrix A, i.e. $E_{ij} \cdot A$ results in l times the jth row gets subtracted from row i.

Check: Find the next two elimination matrices to complete the elimination process for A.

Permutation matrices

There are also matrices which swap the rows of a matrix. These matrices are known as permutation matrices. They are much easier to visualize compared to elimination matrices, So we dive in straight to its definition:

* A permutation matrix P_{ij} is an identity matrix with its ith and jth row swapped.

(Left) multiplying P_{ij} to a matrix, i.e. $P_{ij} \cdot A$ results in **the ith and jth row of A being swapped.**

Check: The permutation matrix $P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

swaps the first and second row of

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}.$$

Augmented matrices

Recall that the matrix form of a system of equations is:

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

eg.

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\underset{\sim}{x}} = \underbrace{\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}}_{\underset{\sim}{b}}.$$

There is a simpler way to write this: the <u>augmented matrix</u>:

$$\left[\begin{array}{ccc c} \underbrace{\begin{matrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{matrix}}_{A} & \underbrace{\begin{matrix} 2 \\ 8 \\ 10 \end{matrix}}_{\underset{\sim}{b}} \end{array}\right],$$

or in general,

$$\left[A \quad \underset{\sim}{b} \right].$$

Matrix multiplication on augmented matrices

If we (left) multiply both sides of the matrix form $A\mathbf{x}=\mathbf{b}$ by a matrix E, we get

$$(EA)\mathbf{x} = E\mathbf{b}.$$

So the augmented matrix form of this equation is

$$[EA \quad E\mathbf{b}].$$

Therefore, if we left multiply an augmented matrix $[A \ \ b]$ by a matrix E, it is only logical to define it this way:

distributes

$$E \cdot [A \quad \mathbf{b}] = [EA \quad E\mathbf{b}].$$