CSD2250 Linear Algebra Week 1 Homework Suggested Solutions

20th May 2022

Question 1 (Column form)

Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}.$$

Compute $A \cdot \boldsymbol{b}$ by first expressing it in terms of the column form.

Suggested Solution:

$$\begin{aligned} A \cdot \boldsymbol{b} &= 1 \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2+1 \\ 1+2+1 \\ 1+2+2 \\ 1+4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 5 \end{bmatrix}. \end{aligned}$$

Question 2 (Method of Elimination)

Solve the following linear system by the method of elimination and back-substitution.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

Suggested Solution:
The coefficient of x_1 in the first equation is 1, hence the first pivot is 1. The variable to eliminate first will be $-x_1$ in the second equation. Hence, the multiplier $l_{2,1} = -1/1 = -1$. Thus we subtract -1 times of equation 1 from equation 2, i.e. we add the first and second equation. The resulting system becomes

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_2 + 5x_3 &= 9 \\ 3x_1 - 7x_2 + 4x_3 &= 10. \end{aligned}$$

Next, the variable to eliminate next is $3x_1$ in the third equation. The multiplier $l_{3,1} = 3/1 = 3$. Thus we subtract 3 times of equation 1 from equation 3, resulting in the system

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_2 + 5x_3 &= 9 \\ -10x_2 - 2x_3 &= -14. \end{aligned}$$

The coefficient of x_2 in the second equation is 1, thus the second pivot is -1. The variable to eliminate next is $-10x_2$ in the third equation. Hence, the multiplier $l_{3,2} = -10/-1 = 10$. Thus we subtract 10 times of equation 2 from equation 3, resulting in the system

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_2 + 5x_3 &= 9 \\ -52x_3 &= -104. \end{aligned}$$

Solving by back-substitution gives

$$\begin{aligned}
x_3 &= \frac{-104}{-52} = 2, \\
x_2 &= 5x_3 - 9 = 10 - 9 = 1, \\
x_1 &= 8 - x_2 - 2x_3 = 8 - 1 - 2 \cdot 2 = 3.
\end{aligned}$$

Note: The last pivot here is -52, thus there are 3 pivots in a system of 3 equations, hence the solution we obtained here by back-substitution is unique.

Question 3 (Elimination matrices)

(a) Write down the elimination matrix E_{21} which subtracts 5 times of row 1 from row 2 of a 3 by 3 matrix A.

(b) Write down the elimination matrix E'_{21} which *adds* 5 times of row 1 to row 2 of a 3 by 3 matrix A.

(c) Show that $E_{21} \cdot E'_{21} = I_3$ **and** $E'_{21} \cdot E_{21} = I_3$, where I_3 is the 3 by 3 identity matrix.

Suggested Solution:

(a) The elimination matrix E_{21} is $\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) If the elimination matrix E'_{21} adds 5 times of row 1 to row 2, that also means to say that E'_{21} subtracts -5 times of row 1 from row 2. Thus, the elimination matrix E'_{21} is

$$\begin{bmatrix} 1 & 0 & 0 \\ -(-5) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c)

$$E_{21} \cdot E'_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $\boldsymbol{c}_1', \boldsymbol{c}_2', \boldsymbol{c}_3'$ be the columns of E_{21}', from left to right. Then

$$E_{21} \cdot \boldsymbol{c}_1' = 1 \cdot \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix} + 5 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$E_{21} \cdot \boldsymbol{c}_2' = 0 \cdot \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$E_{21} \cdot \boldsymbol{c}_3' = 0 \cdot \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Notice that these are the columns of the 3 by 3 identity matrix, thus $E_{21} \cdot E_{21}' = I_3$. $E_{21}' \cdot E_{21} = I_3$ can also be shown in a similar fashion (do it as an exercise!).

Question 4 (Permutation matrices)

(a) Write down the permutation matrix P_{31} which swaps row 1 with row 3 of a 3 by 3 matrix A.

(b) Show that $P_{31} \cdot P_{31} = I_3$, where I_3 is the 3 by 3 identity matrix.

Suggested Solution:

(a) The permutation matrix P_{31} is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

(b) Let $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ be the columns of P_{31}, from left to right. Then

$$P_{31} \cdot \boldsymbol{p}_1 = 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$P_{31} \cdot \boldsymbol{p}_2 = 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$P_{31} \cdot \boldsymbol{p}_3 = 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Again, notice that these are the columns of the 3 by 3 identity matrix, therefore $P_{31} \cdot P_{31} = I_3$.

Question 5 (Augmented matrices)

(a) Write down the augmented matrix form $[A \;\; \boldsymbol{b}]$ for the system in Question 2.

(b) Let

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Compute $E_{31} \cdot [A \;\; \boldsymbol{b}]$.

Suggested Solution:

(a) The augmented matrix form for the system in Question 2 is

$$[A \;\; \boldsymbol{b}] = \begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}.$$

(b) We first compute $E_{31} \cdot A$ and $E_{31} \cdot \boldsymbol{b}$ separately, then combine them into an augmented matrix form. Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ be the columns of A, from left to right.

Then

$$E_{31} \cdot \boldsymbol{a}_1 = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$E_{31} \cdot \boldsymbol{a}_2 = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-7) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -10 \end{bmatrix}$$

$$E_{31} \cdot \boldsymbol{a}_3 = 2 \cdot \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

$$E_{31} \cdot \boldsymbol{b} = 8 \cdot \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 10 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ -14 \end{bmatrix}.$$

Hence,

$$E_{31} \cdot A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 0 & -10 & -1 \end{bmatrix} \quad \text{and} \quad E_{31} \cdot \boldsymbol{b} = \begin{bmatrix} 8 \\ 1 \\ -14 \end{bmatrix}.$$

Therefore,

$$E_{31} \cdot [A \;\; \boldsymbol{b}] = [E_{31} \cdot A \;\; E_{31} \cdot \boldsymbol{b}] = \begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 0 & -10 & -2 & -14 \end{bmatrix}.$$

Observe that in the last row of $E_{31} \cdot [A \;\; \boldsymbol{b}]$ is the same as the last row of the matrix form of the system in Question 2 after subtracting 3 times of equation 1 from equation 3.