CSD2250 Linear Algebra Week 3 Homework

27th May 2022

You are given until 3rd of June 2022, 2359 HRS to submit this homework.

Question 1 (Subspaces)

For (a) and (b), show that these sets are subspaces of $\mathbb{R}^3$.

(a) $\{(a, b, c) : 4a = 3b\}$

(b) $\{(x, y - 1, z) : x + y - 2z = 1\}$.

For (c) and (d), show that these sets are subspaces of $M_2(\mathbb{R})$.

(c) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : b = 0 \right\}$

(d) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b - c - d = 0 \right\}$.

Suggested Solution:

(a) Let $S_1 = \{(a, b, c) : 4a = 3b\}$.

(1) Firstly, $(0, 0, 0) \in S_1$ because $4 \cdot 0 = 3 \cdot 0$.

(2) Let $(a', b', c'), (a'', b'', c'') \in S_1$. Then $4a' = 3b'$ and $4a'' = 3b''$. We show that

$$(a', b', c') + (a'', b'', c'') = (a' + a'', b' + b'', c' + c'') \in S_1$$

by showing that $4(a' + a'') = 3(b' + b'')$. Now,

$$\begin{aligned} 4(a' + a'') &= 4a' + 4a'' \quad \text{(distributive law for numbers)} \\ &= 3b' + 3b'' \quad \text{(since } 4a' = 3b' \text{ and } 4a'' = 3b''\text{)} \\ &= 3(b' + b'') \quad \text{(distributive law for numbers).} \end{aligned}$$

(3) Let $(a', b', c') \in S_1$ and α be a scalar. Then $4a' = 3b'$. We show that

$$\alpha(a', b', c') = (\alpha a', \alpha b', \alpha c') \in S_1$$

by showing that $4\alpha a' = 3\alpha b'$. Then

$$\begin{aligned} 4\alpha a' &= \alpha 4a' \quad \text{(commutative law for numbers)} \\ &= \alpha 3b' \quad \text{(since } 4a' = 3b'\text{)} \\ &= 3\alpha b' \quad \text{(commutative law for numbers).} \end{aligned}$$

Since all three properties of a subspace is satisfied, S_1 is a subspace of $\mathbb{R}^3$.

(b) Let $S_2 = \{(x, y - 1, z) : x + y - 2z = 1\}$.

(1) Firstly, $(0, 0, 0) \in S_2$ because $(0, 0, 0) = (0, 1 - 1, 0)$ so $0 + 1 - 2 \cdot 0 = 1$.

(2) Let $(x', y' - 1, z'), (x'', y'' - 1, z'') \in S_2$. Then $x' + y' - 2z' = 1$ and $x'' + y'' - 2z'' = 1$. We show that

$$(x', y' - 1, z') + (x'', y'' - 1, z'') = (x' + x'', (y' + y'' - 1) - 1, z' + z'') \in S_2$$

by showing that $(x' + x'') + (y' + y'' - 1) - 2(z' + z'') = 1$. Now,

$$\begin{aligned} (x' + x'') + (y' + y'' - 1) - 2(z' + z'') &= (x' + y' - 2z') + (x'' + y'' - 2z'') - 1 \\ &= 1 + 1 - 1 = 1. \end{aligned}$$

The last second equality is due to the fact that $x' + y' - 2z' = 1$ and $x'' + y'' - 2z'' = 1$.

(3) Let $(x', y' - 1, z') \in S_2$ and c be a scalar. Then $x' + y' - 2z' = 1$. We show that

$$c(x', y' - 1, z') = (cx', cy' - c, cz') = (cx', (cy' - c + 1) - 1, cz') \in S_2.$$

by showing that $cx' + (cy' - c + 1) - 2cz' = 1$.

$$\begin{aligned} cx' + (cy' - c + 1) - 2cz' &= c(x' + y' - 2z') - c + 1 \\ &= c - c + 1 \quad (\text{since } x' + y' - 2z' = 1) \\ &= 1. \end{aligned}$$

Since all three properties of a subspace is satisfied, S_2 is a subspace of $\mathbb{R}^3$.

(c) Let $S_3 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : b = 0 \right\}$.

(1) Clearly, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S_3$ since the $(1,2)$-entry of the zero matrix is 0.

(2) Let $\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} \in S_3$. Then $b' = b'' = 0$. We show that

$$\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} = \begin{bmatrix} a' + a'' & b' + b'' \\ c' + c'' & d' + d'' \end{bmatrix} \in S_3.$$

This is true because $b' + b'' = 0 + 0 = 0$.

(3) Let $\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in S_3$ and α be a scalar. Then $b' = 0$. We show that

$$\alpha \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} \alpha a' & \alpha b' \\ \alpha c' & \alpha d' \end{bmatrix} \in S_3.$$

This is true because $\alpha b' = \alpha \cdot 0 = 0$.

Since all three properties of a subspace is satisfied, S_3 is a subspace of $\mathbb{R}^3$.

(d) Let $S_4 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b - c - d = 0 \right\}$.

(1) Clearly, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S_3$ since $0 + 0 - 0 - 0 = 0$.

(2) Let $\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} \in S_4$. Then $a'+b'-c'-d' = 0$ and $a''+b''-c''-d'' = 0$. We show that

$$\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} = \begin{bmatrix} a'+a'' & b'+b'' \\ c'+c'' & d'+d'' \end{bmatrix} \in S_4.$$

This is true because $(a'+a'')+(b'+b'')-(c'+c'')-(d'+d'') = (a'+b'-c'-d')+(a''+b''-c''-d'') = 0+0 = 0$.

(3) Let $\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in S_4$ and α be a scalar. Then $a'+b'-c'-d' = 0$. We show that

$$\alpha \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} \alpha a' & \alpha b' \\ \alpha c' & \alpha d' \end{bmatrix} \in S_4.$$

This is true because $\alpha a' + \alpha b' - \alpha c' - \alpha d' = \alpha(a'+b'-c'-d') = \alpha \cdot 0 = 0$.

Since all three properties of a subspace is satisfied, S_4 is a subspace of $\mathbb{R}^3$.

Grading policy (for graders):

(1) 4 marks for (b), 2 marks for the other parts.

(2) How you grade each part is entirely up to your discretion.

Question 2

In this question, we demonstrate a technique to show that a subset S of a vector space V is **not** a subspace of V.

(a) Write down examples of 2 by 2 matrices A_1 and A_2 such that $\det(A_1) = \det(A_2) = 1$ **but** $\det(A_1 + A_2) \neq 1$.

(b) Use (a) to show that

$$S = \{A \in M_2(\mathbb{R}) : \det(A) = 1\}$$

is not a subspace of $M_2(\mathbb{R})$.

Suggested Solution:

(a) Take $A_1 = A_2 = I_2$. Then $\det(A_1) = \det(A_2) = 1$. But

$$\det(A_1 + A_2) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \neq 1.$$

(b) Part (a) violates the 2nd property of a subspace; $A_1, A_2 \in S$ but $A_1 + A_2 \notin S$.

Grading policy (for graders):

(1) 5 marks for each part.

(2) If students did not use (a) to solve part (b), then deduct 5 marks.

(3) There are many pairs of A_1, A_2 that satisfy this requirement. Just make sure you check that the pair that the students provide are valid.

(4) How you grade each part is entirely up to you.

Question 3 (Column spaces)

Let A be the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 5 \end{bmatrix}.$$

(a) What is $C(A)$ a subspace of?

(b) Describe $C(A)$. Is it a line or a plane?

Suggested Solution:

(a) $C(A)$ is a subspace of $\mathbb{R}^3$, since A has 3 rows/each column of A has 3 entries.

(b) The description of $C(A)$ is that $C(A)$ is the set of all linear combinations of the columns of A:

$$C(A) = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}.$$

To figure out if $C(A)$ is a line or plane, we examine the elements of $C(A)$ closer. The elements of $C(A)$ are

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$$

where x_1, x_2 are arbitrary real numbers. Then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 5x_2 \end{bmatrix}.$$

Then $z = 5y$ and x is arbitrary, thus $C(A)$ is a plane (which contains the origin $(0,0,0)$). If x were fixed here, then $C(A)$ would be a line. A figure of the plane can be found in the next page.

Grading policy (for graders):

(1) 5 marks for each part, for part (b), the description of $C(A)$ is worth 3 marks, and whether is it a line or plane, is worth 2 marks.

(2) For part (a), students must be able to recognise that $C(A)$ is a subspace the linear combinations of the vectors whose **coordinates correspond to the rows of the matrix** A. I expect a common mistake that $C(A)$ is a subspace of $\mathbb{R}^2$ (column entries instead of row entries).

(3) The description required from part (b) is the set

$$C(A) = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}.$$

(4) Within each part of the question, how you grade is entirely up to you.

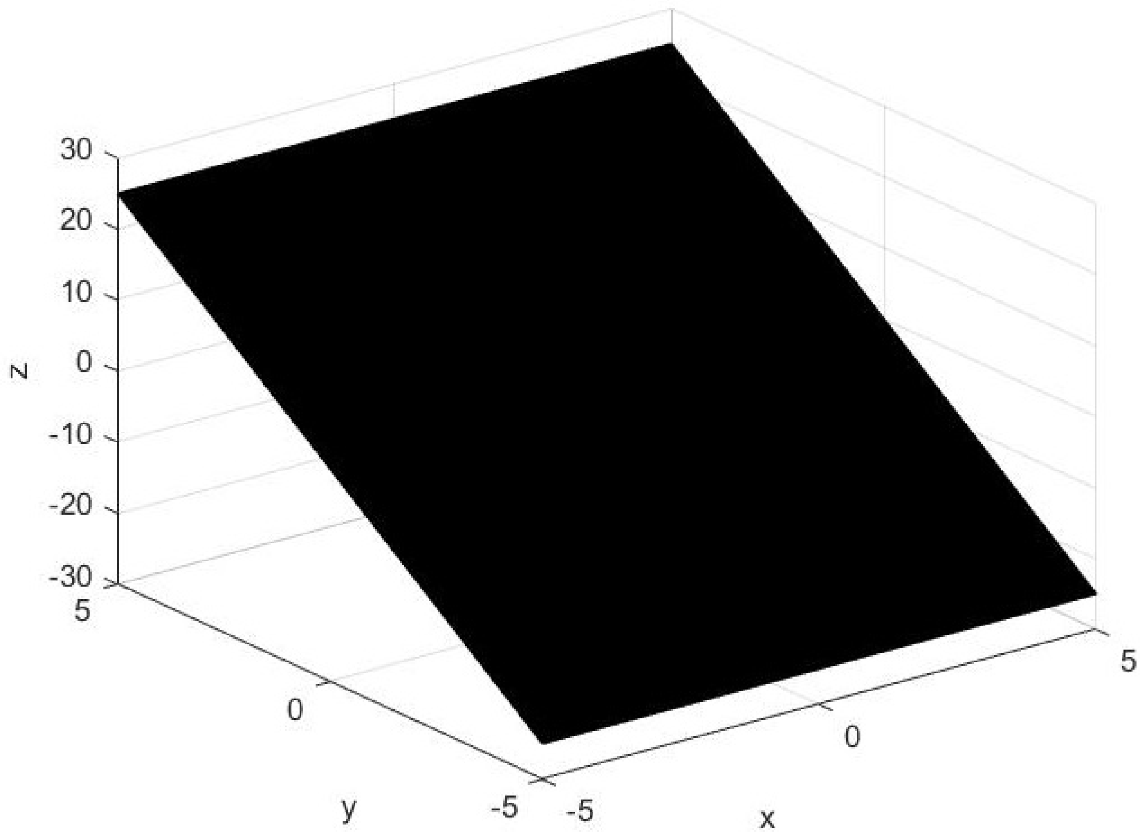

Figure 1: The plane $z = y$, which is the visualization of the column space of the matrix is Question 3.

Question 4 (Nullspaces)

Let A is the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix}.$$

(a) Find the special solutions of A using the algorithm found in the lecture notes. How many of such special solutions are there?

(b) How many pivots/pivot columns of this matrix are there?

(c) Using the special solutions you found in (b), describe $N(A)$.

Suggested Solution:

(a) We first use elimination and back-substitution to bring the matrix to reduced row echelon form. The reduced row echelon form of A is (please include your workings)

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Columns 1 and 3 are pivot columns, while columns 2 and 4 are the free columns. The corresponding free variables are x_2 and x_4. We have two cases here:

(1) $x_2 = 1$ and $x_4 = 0$, and

(2) $x_2 = 0$ and $x_4 = 1$.

We note that the system of equations for the reduced row echelon form is

$$\begin{aligned} x_1 + 2x_2 + 2x_4 &= 0 \\ x_3 + x_4 &= 0. \end{aligned}$$

In the first case, we get $x_1 = -2$ and $x_3 = 0$, and hence the first special solution is

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

In the second case, we get $x_1 = -2$ and $x_3 = -1$, and hence the second special solution is

$$\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

There are 2 special solutions, which correspond to the 2 free columns in A.

(b) There are two pivot columns in this matrix, columns 1 and 3.

(c) The nullspace of A can be described as

$$N(A) = \left\{ a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Grading policy (for graders):

(1) 5 marks for part (a), 2 marks for part (b), and 3 marks for part (c).

(2) For part (a), students should have workings for their rref matrix.

(3) How you grade each part is at your own discretion, but do make sure you pay attention to part (a) as the special solutions students obtain may differ. **Any multiple of the special solutions are fine**, provided their working results in those special solutions.

Question 5

Repeat the same exercises (a) to (c) in Question 4, but using the following matrix

$$A = \begin{bmatrix} 4 & 6 & 8 & 10 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 3 & 3 \end{bmatrix}.$$

For both matrices in Question 4 and 5, what do you observe about the number of pivots/pivot columns + number of special solutions?

Suggested Solution:

(a) We first use elimination and back-substitution to bring the matrix to reduced row echelon form. The reduced row echelon form of A is (please include your workings)

$$\begin{bmatrix} 1 & 0 & 0 & -16 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Columns 1,2 and 3 are pivot columns, while column 4 is a free column. The corresponding free variable is x_4. We have only one case here:

(1) $x_4 = 1$.

We note that the system of equations for the reduced row echelon form is

$$\begin{aligned} x_1 - 16x_4 &= 0 \\ x_2 + 7x_4 &= 0 \\ x_3 + 4x_4 &= 0. \end{aligned}$$

The only special solution is

$$\begin{bmatrix} 16 \\ -7 \\ -4 \\ 1 \end{bmatrix}.$$

(b) There are three pivot columns in this matrix, columns 1, 2 and 3.

(c) The nullspace of A can be described as

$$N(A) = \left\{ a \begin{bmatrix} 16 \\ -7 \\ -4 \\ 1 \end{bmatrix} : a \in \mathbb{R} \right\}.$$

In question 4, the number of pivot columns + number of special solutions = 2+2 = 4, while in question 5, the number of pivot columns + number of special solutions = 3 + 1 = 4. Both of these add up to the size of the vector space $\mathbb{R}^4$, which is the vector space with "dimension" = 4.

Grading policy (for graders):

(1) Same grade policy as Question 4.

(2) The question at the end of this question is optional, students may or may not know. If they happen to know, at your own discretion, you may choose to be more lenient with your grading of Questions 4 and 5.