CSD2250/MAT250 Lecture 3 (Wed 18/5)

Topics covered:
- Matrix operations and rules
- Inverse of a matrix
- Transpose of a matrix

Matrix operations and rules

Let A be a $m \times n$ matrix.

$$\begin{array}{ccc} m & \times & n \\ \text{rows} & & \text{columns} \end{array}$$

Recall: A can be added to another $m \times n$ matrix B by adding their entries "component-wise":

Eg. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$.

A can also be multiplied by a constant c by multiplying each entry by c:

Eg. $7\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 14 & 21 \\ 28 & 35 & 42 \end{bmatrix}$.

What about matrix multiplication?

Let A be an $m\times n$ matrix, and B be a $p\times q$ matrix.

*** Important test:**

- $A\cdot B$ is valid if A has the same number of columns as the number of rows of B, i.e. $\underline{n=p}$.
- Similarly, $B\cdot A$ is valid if B has the same number of columns as the number of rows of A, i.e. $\underline{q=m}$.

$$(m\times n)(\underline{n}\times q)=(m\times q)$$

if $p=n$

AB is an $m\times q$ matrix.

Likewise, BA is a $p\times n$ matrix (if $q=m$)

Observation: We can see that AB may have a different size from BA.

★ How to multiply two matrices A and B?

① First, check if the sizes of A and B match (see previous page).

② Every entry of AB, which can be described as $(AB)_{ij}$, can be computed in the following fashion:

row i, column j entry of the matrix AB

$$(AB)_{ij} = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B)$$

dot product

has the same no. of entries corresponding to the no. of columns of A

has the same no. of entries corresponding to the no. of rows of B.

must be the same for dot product to work!

Exercise

1. Let

$$A = \begin{bmatrix} 2 & 4 & 5 \\ -1 & 2 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & -3 & 6 \\ 6 & 1 & -2 \\ 2 & 4 & -5 \end{bmatrix}$$

(a) Express $(AB)_{23}$ in terms of the dot product of a row of A and a column of B.

(b) Calculate AB and BA. If either of these multiplications are not possible, explain why.

Laws for matrix multiplication

In this part, and in each case, we assume that the matrices A, B and C are of suitable sizes for either addition or multiplication. Also, let α be a constant.

Addition laws

1. $A+B = B+A$ (commutative law)
2. $\alpha(A+B) = \alpha A + \alpha B$ (distributive law)
3. $A+(B+C) = (A+B)+C$ (associative law)

Multiplication laws

1. $C(A+B) = CA + CB$ (left distributive law)
2. $(A+B)C = AC + BC$ (right distributive law)
3. $(AB)C = A(BC)$ (associative law)

*NOTE: AB is usually NOT equal to BA, i.e. the commutative law for multiplication is broken.

Inverse matrix

Let A be a $n \times n$ matrix (we refer to this matrix as a square matrix of size n). We search for a matrix A^{-1} of the same size as A so that A^{-1} multiplied by A (either way AA^{-1} or $A^{-1}A$) equals I_n.

If such a matrix exists (it doesn't always exist), then solving $A\underset{\sim}{x} = \underset{\sim}{b}$ becomes trivial:

$\Rightarrow$ Multiply A^{-1} to both sides of $A\underset{\sim}{x} = \underset{\sim}{b}$ gives

$$A^{-1}A\underset{\sim}{x} = A^{-1}\underset{\sim}{b} \quad \Rightarrow \quad \underset{\sim}{x} = A^{-1}\underset{\sim}{b},$$

i.e. we have essentially solved the system $A\underset{\sim}{x} = \underset{\sim}{b}$.

We introduce some important notes on the inverse A^{-1}, before delving into the method of finding A^{-1} (if such a matrix exists).

**A square matrix A is invertible if there exists a matrix A^{-1} such that

$$A^{-1}A = I_n \quad \text{and} \quad AA^{-1} = I_n$$

Notes on A^{-1}

① A^{-1} exists if and only if elimination produces n pivots (note that pivots must be nonzero).

equal to the size of A.

② If A^{-1} exists, then it must be unique.

*③ The system $A\underset{\sim}{x} = \underset{\sim}{0}$ has at least one solution: $\underset{\sim}{x} = \underset{\sim}{0}$ is a solution. If $\underset{\sim}{x} = \underset{\sim}{0}$ is not the only solution to $A\underset{\sim}{x} = \underset{\sim}{0}$, then A is not invertible.

zero vector

Why? Suppose $\underset{\sim}{x}' \neq \underset{\sim}{0}$ is a solution to $A\underset{\sim}{x} = \underset{\sim}{0}$, then $A\underset{\sim}{x}' = \underset{\sim}{0}$. If A^{-1} exists then multiply to this equation:

any matrix multiplied by $\underset{\sim}{0}$ must be $\underset{\sim}{0}$

$$A^{-1}A\underset{\sim}{x}' = A^{-1}\underset{\sim}{0} = \underset{\sim}{0} \Rightarrow \underset{\sim}{x}' = \underset{\sim}{0}.$$

But this contradicts $\underset{\sim}{x}' \neq \underset{\sim}{0}$, so A^{-1} cannot exist.

We will see this later in the nullspace of A.

*④ The inverse of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ provided $ad-bc \neq 0$.

Here, $ad-bc$ is the determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Important: Since A^{-1} is rather tedious to find, it is often enough to show that A^{-1} exists through ①, ③ and ④. In ④, as we will see later, A^{-1} exists if and only if the determinant of A is nonzero.

Product of invertible matrices

Let A and B be invertible matrices of size n. Then AB is also invertible.

The inverse of AB is

$$(AB)^{-1} = B^{-1}A^{-1}.$$

The shoe-and-sock principle.

We recall that matrices perform actions on other matrices/vectors. As a way to remember this, we can think of

A = "putting on shoes" and

B = "putting on socks"

then logically

A^{-1} = "taking off shoes" and

B^{-1} = "taking off socks".

AB here means that we put on socks first, followed by our shoes. (Matrices are "read" from right to left), so naturally to reverse this, we perform $B^{-1}A^{-1}$, taking off our shoes, then our socks.

Exercises

What is the inverse of

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$?

(b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$?

(c) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$?

(Slightly harder)

Hint: What actions do these matrices perform?

Calculating A^{-1} using Gauss-Jordan

Gauss-Jordan algorithm is technically the elimination + back-substitution algorithm we have learnt last week. We now show how it can be used to compute A^{-1}, using an example we are familiar with:

$$\text{Let } A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$$

Place A and I_3 into a matrix, side by side:

$$\begin{array}{cc} A & I_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

Perform Gauss-Jordan on this matrix until I_3 appears on the left side of this matrix. Then the matrix of the right side becomes A^{-1}.

Demonstration to find A^{-1} using Gauss-Jordan

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \text{ Eqn } 2 - 2 \times \text{Eqn } 1$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \text{ Eqn } 3 + \text{Eqn } 1$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \text{ Eqn } 3 - \text{Eqn } 2$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right]$$

$$\Big\downarrow \ \frac{1}{4} \times \text{Eqn } 3$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \end{array}\right]$$

$$\downarrow \text{Eqn 2} - \text{Eqn 1}$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \end{array}\right]$$

$$\downarrow \tfrac{1}{2} \times \text{Eqn 1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \end{array}\right]$$

$$\downarrow \text{Eqn 1} + \text{Eqn 3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{5}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \end{array}\right]$$

$$\downarrow \text{Eqn 1} - 2 \times \text{Eqn 2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{27}{4} & -\frac{11}{4} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \end{array}\right]$$

I_3, so algorithm stops

this is A^{-1} !

Exercises

Use the Gauss-Jordan algorithm to find the inverses of the following matrices:

(a) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Transpose of a matrix

We cover another useful matrix, the transpose of another matrix A, which is denoted by A^T. ← stands for transpose

Simply put, the rows and columns of A are the columns and rows of A^T respectively.

E.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}$

Important rules for transpose

1. $(A+B)^T = A^T + B^T$
2. $(A \cdot B)^T = B^T A^T$ (swapped, like inverse)
3. $(A^{-1})^T = (A^T)^{-1}$

↑ The transpose of the inverse of A is exactly the inverse of the transpose of A.

Exercises

① Let

$$\underset{\sim}{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{y} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

Compute

(a) $\underset{\sim}{x}^T \underset{\sim}{y}$, and

(b) $\underset{\sim}{x}\, \underset{\sim}{y}^T$.

For (a), what does this remind you of?

② Let A be an $m \times n$ matrix of your choosing. Compute

(a) $A^T A$, and

(b) $A A^T$.

What are the sizes of $A^T A$ and $A A^T$? What other characteristics of $A^T A$ and $A A^T$ do you observe?