CSD2250/MAT250 Lecture 6 (Fri 27/5)

Topics covered:
- Nullspace $N(A)$
- Rank of a matrix A

Notational shift

Last lecture, we covered subspaces. We described subspaces in the following way:

e.g. For $V = \mathbb{R}^2$, let S be the subset where (x_1, x_2) satisfies $x_2 = 2x_1$.

** We transition it to this set form: curly bracket means set

$$S = \{ (x_1, x_2) \in \mathbb{R}^2 : x_2 = 2x_1 \}.$$

vectors in the vector space V satisfies/such that   condition

which is more succinct.

Exercise For each of the previous examples/exercises in the last lecture, write them in the set form.

For example, the column space of an $m \times n$ matrix A can be written in the set form

$$C(A) = \{ x_1 \underset{\sim}{a}_1 + x_2 \underset{\sim}{a}_2 + \dots + x_n \underset{\sim}{a}_n : \underset{\sim}{a}_1, \underset{\sim}{a}_2, \dots, \underset{\sim}{a}_n \text{ are columns of } A \text{ and } x_1, x_2, \dots, x_n \in \mathbb{R} \}$$

- Elements of $C(A)$ are the linear combinations of columns of A.

Nullspace of A

Let A be an $m \times n$ matrix. Recall that $C(A)$ is a subspace of $\mathbb{R}^m$. Here, we consider a subspace of $\mathbb{R}^n$, the nullspace of A.

☆☆ The nullspace of A, denoted by $N(A)$, is a subspace of $\mathbb{R}^n$ such that

$$N(A) = \{ \underset{\sim}{x} \in \mathbb{R}^n : A\underset{\sim}{x} = \underset{\sim}{0} \}.$$

In other words, $N(A)$ consists of all solutions $\underset{\sim}{x}$ of the equation $A\underset{\sim}{x} = \underset{\sim}{0}$.

Exercise: Show that $N(A)$ is a subspace of $\mathbb{R}^n$.

How to find $N(A)$ for an $m \times n$ matrix A?

<u>Eg.</u> ① $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Notice $2 \times$ column 1 $=$ column 2. (or rows too)

We apply elimination to

$$A\underset{\sim}{x} = \underset{\sim}{0} \iff \begin{aligned} x_1 + 2x_2 &= 0 \\ 3x_1 + 6x_2 &= 0 \end{aligned}$$

Eqn 2 $- 3 \times$ Eqn 1

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{aligned} x_1 + 2x_2 &= 0 \\ 0x_2 &= 0 \end{aligned}$$

*free variable

pivot column, corresponding to the variable x_1

free column, corresponding to the variable x_2

Note that there is only one equation left, and the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ is already in reduced row echelon form. This system has infinitely many solutions. So how do we find every solution of this system? Simple. We first notice that $x_1 + 2x_2 = 0$ is an equation of a line. <u>Find a point on the line, then all other solutions are multiples of that point!</u>

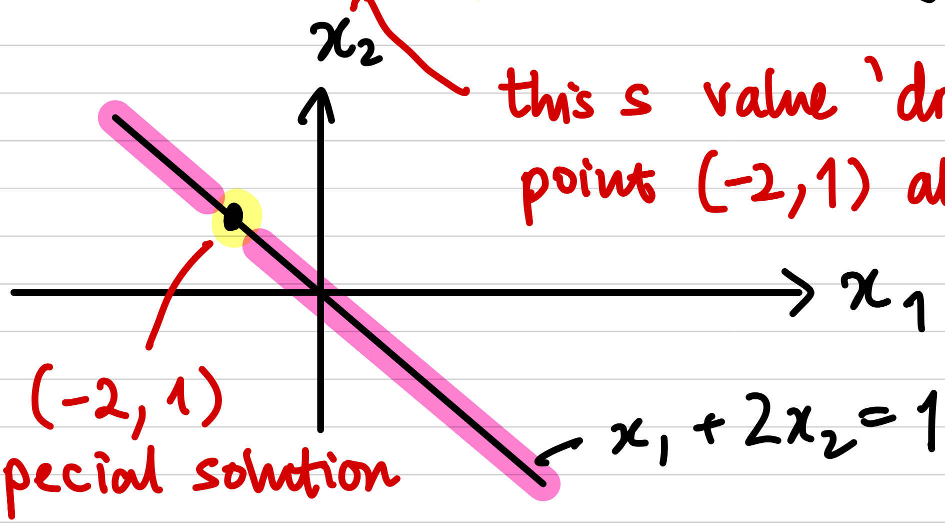
① Find a point on the line:

Set free variable $x_2 = 1 \Rightarrow x_1 = -2$.

② Every solution is a multiple of that point:

$$\therefore N(A) = \left\{ s \begin{bmatrix} -2 \\ 1 \end{bmatrix} : s \in \mathbb{R} \right\}.$$

this s value 'drags' the point (-2,1) along every point on the line



Eg. ② A more complicated example

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$$

$\downarrow$ Eqn 2 − 3 × Eqn 1 and then $\frac{1}{2}$ × Eqn 2

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

pivots circled in red

pivot columns x_1, x_2 free columns x_3, x_4

Bring it to rref form

$$\xrightarrow{\text{Eqn 1} - 2\times \text{Eqn 2}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$ rref form liao

Algorithm to find N(A)

1. Bring the matrix A to a reduced row echelon form.

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

2. Identify the free columns and the corresponding free variables from the rref.

Here, the 3rd and 4th columns are the free columns. So x_3 and x_4 are free variables.

3. Set one free variable as 1 and the rest to be 0.

$$\left.\begin{array}{l} \text{Case 1}: \ x_3 = 1 \text{ and } x_4 = 0 \\ \text{Case 2}: \ x_3 = 0 \text{ and } x_4 = 1 \end{array}\right\}$$

No. of cases = no. of free variables.

4. Find a special solution for each of the cases identified in the previous step.

For case 1: $x_3 = 1$, $x_4 = 0$,

we have

$$\begin{aligned} x_1 + \quad + 2x_3 \quad &= 0 \\ x_2 \quad + 2x_4 &= 0 \end{aligned}$$

$\Rightarrow \quad x_1 = -2$ and $x_2 = 0$

First special solution = $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

For case 2: $x_3 = 0$, $x_4 = 1$,

we have

$$x_1 = 0, \quad x_2 = -2$$

Second and last special solution = $\begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$.

5. <u>N(A) consists of linear combinations of all the special solutions.</u>

$$\Rightarrow N(A) = \left\{ x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\}.$$

Exercises

① Find the nullspaces of the following matrices:

(a) $\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$.

(b) $\begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix}$.

② What is the null space of an invertible matrix?

③ For a 4×6 matrix, what is the <u>maximum</u> number of pivot columns?

Rank of an $m \times n$ matrix A

We have seen some systems of equations for which some equations actually do not matter:

$$\begin{aligned} x - 2y &= 1 \\ 4x - 8y &= 4 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x - 2y &= 1 \\ 0y &= 0 \end{aligned}$$

The second equation does not actually matter; the first and second equation are essentially <u>the same</u>! So, what is the <u>size</u> of this system in actuality?

☆ The <u>rank</u> of an $m \times n$ matrix A is the number of pivot columns/variables of A. We denote this number by <u>r</u>.

As you may have already noticed from our previous examples,

☆ no. of pivot columns + no. of free columns

= no. of columns = n

* How to find the rank of an $m \times n$ matrix A?

① Use elimination to bring A to <u>row echelon form</u> (rref also can, but overkill).

② The first '1' entry of every row of A are the pivots. The pivot columns are the columns containing these '1' entries.

<u>Exercises</u>

① Let $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{bmatrix}$

(a) Find the rank of A.

(b) Find the rank of A^T.

② Let $A = \begin{bmatrix} 1 & 0 & 5 \\ 1 & 0 & 2 \end{bmatrix}$.

(a) Find the rank of A.

(b) Find the rank of $A \cdot A^T$ and $A^T A$.

What do you observe?

③ Choose q (if possible) so that the rank of

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}$$

(a) is 1.

(b) is 2.

(c) is 3.