CSD2250/MAT250 Lecture 8 (Fri 3/6)

Topics covered:
- Basis of a vector space
- Dimension of a vector space

In our last Wed lecture, we covered what it means for a set of vectors to be linearly independent, and also what it means for a set of vectors to span a vector space.

Today, we unify these two concepts.

★ A **basis** for a vector space is a set of vectors (in the vector space) that
1. are linearly independent
2. span the vector space.

Example

(a) Let $B_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$.

① These vectors are linearly independent

as $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ implies $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

② These vectors span $\mathbb{R}^3$.

* We can do this by showing that every (a,b,c) in $\mathbb{R}^3$ can be written as a linear combination of $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

We can see that

$$\left\{ \begin{aligned} &\underline{a}(1,0,0) + \underline{b}(0,1,0) + \underline{c}(0,0,1) \\ &= (a,b,c) \end{aligned} \right.$$

These coefficients are unique to every vector in $\mathbb{R}^3$ too!

and this holds for ANY $(a,b,c) \in \mathbb{R}^3$, so these vectors span $\mathbb{R}^3$.

Hence by ① and ②, B_1 is a basis for $\mathbb{R}^3$.

Observations

① We see that the vectors in a basis are like building blocks of the vector space; every vector in $\mathbb{R}^3$ can be 'built' from these 3 vectors.

② This equation can also be interpreted in the matrix form:

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

where the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Recall how to solve this?

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

So now, notice that this argument can be 'reused' for an <u>invertible 3x3 matrix A</u>:

$$A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Now this implies that the columns of an invertible 3x3 matrix A spans $\mathbb{R}^3$, and also are linearly independent (cuz $r = 3 = n$)! So <u>the columns of any invertible A form a basis of $\mathbb{R}^3$!</u>

We can generalize this argument for an invertible $n \times n$ matrix to $\mathbb{R}^n$:

$\boxed{\text{★ The columns of } \underline{\text{any}} \text{ invertible square matrix of size } n \text{ form a basis for } \mathbb{R}^n.}$

This implies that there are infinitely many bases (plural for basis) for $\mathbb{R}^n$!

But notice that each of these bases have the <u>same number of vectors $= n$</u>.

Can another basis of $\mathbb{R}^n$ have more than n vectors? Or less than n vectors? <u>No</u>.

For the first question, we have already gone through: let's suppose we have a set of k vectors in $\mathbb{R}^n$, where $k > n$. Then if we place them into the columns of a matrix, then

$$\begin{array}{c} n = \\ \text{rows} \end{array} \left\{ \underbrace{\begin{bmatrix} \underset{\sim}{a_1} & \underset{\sim}{a_2} & \cdots & \underset{\sim}{a_k} \end{bmatrix}}_{k \text{ columns}} \right.$$

this matrix has more columns than rows, therefore these vectors cannot be linearly independent and hence this set of k vectors cannot be a basis for $\mathbb{R}^n$.

For the second question, a basis for $\mathbb{R}^n$ cannot have less than n vectors. A set containing less than n vectors cannot span $\mathbb{R}^n$.

Example This is the example from this Wed lecture:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Since the columns are linearly dependent, we toss $(2,4)$ out. But then the span of $\{(1,2)\}$ is only a line, not the whole $\mathbb{R}^2$!

This "no more, no less" approach for bases is actually true for all vector spaces, not just $\mathbb{R}^n$.

$\boxed{\text{* If } \underset{\sim}{a_1}, \dots, \underset{\sim}{a_n} \text{ and } \underset{\sim}{b_1}, \dots, \underset{\sim}{b_m} \text{ are bases for the same vector space } V \text{, then } m=n \text{, i.e. the size of every basis for } V \text{ is the same.}}$

$\star$ The dimension of a vector space V is the number of vectors in any basis for V.

$\star\star\star$ Unifying example

Let
$$A = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 6 & 5 & 2 & 1 \end{bmatrix}.$$

We want to find

(a) A basis for $C(A)$.

(b) A basis for $C(A^T)$.

(c) A basis for $N(A)$.

(a) and (b) can be found by first converting A to rref:

$$\xrightarrow{R_2 - 2R_1} \begin{bmatrix} 3 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} \xrightarrow[-\frac{1}{3}R_2]{\frac{1}{3}R_1} \begin{bmatrix} 1 & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

$$\xrightarrow{R_1 - \frac{4}{3}R_2}$$

$$\begin{array}{c} \text{pivot} \to \\ \text{rows} \to \end{array} \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{4}{9} \\ 0 & 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

$$\begin{array}{cccc} \text{pivot} & & \text{free} & \end{array}$$

(a) A basis for $C(A)$ is then the columns of A corresponding to the pivot columns in the rref:

Basis for $C(A) = \{(3,6), (4,5)\}$

(which happens to be a basis for $\mathbb{R}^2$ too!)

(b) A basis for $C(A^T)$ is then the rows of A corresponding to the pivot rows in the rref:

Basis for $C(A^T) = \{(3,4,1,0), (6,5,2,1)\}$

(c) The special solutions form a basis of $N(A)$.

The free variables are x_3 and x_4.

rref system:

$$x_1 + \frac{1}{3}x_3 + \frac{4}{9}x_4 = 0$$

$$x_2 - \frac{1}{3}x_4 = 0$$

Case 1 $x_3 = 1$, $x_4 = 0$

$\Rightarrow x_1 = -\frac{1}{3}$, $x_2 = 0$

First special solution $= \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

Case 2 $x_3 = 0$, $x_4 = 1$

$\Rightarrow x_1 = -\frac{4}{9}$, $x_2 = \frac{1}{3}$

Second special solution $= \begin{bmatrix} -\frac{4}{9} \\ \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$.

Therefore a basis for N(A) is

$$\left\{ \left(-\frac{1}{3}, 0, 1, 0\right), \left(-\frac{4}{9}, \frac{1}{3}, 0, 1\right) \right\}.$$

* Note that if we combine vectors from the basis for the row space of A along with vectors from the basis for the null space of A, we get a basis for $\mathbb{R}^4$!

Exercises

① Show that

$$\{(3,4,1,0),(6,5,2,1),\left(-\tfrac{1}{3},0,1,0\right),\left(-\tfrac{4}{9},\tfrac{1}{3},0,1\right)\}$$

forms a basis for $\mathbb{R}^4$.

② We know that

$\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for $\mathbb{R}^3$.

It is called the standard basis for $\mathbb{R}^3$.

Find a basis for $\mathbb{R}^3$ that is not the standard basis for $\mathbb{R}^3$.

③ Using the matrix

$$A=\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 3 & 5 & 6 & 1 \\ 3 & 4 & 7 & 8 & 1 \end{bmatrix}$$

find a basis for $\mathbb{R}^5$.