CSD2250/MAT250 Lecture 9 (Wed 8/6)

Topics covered:
- Orthogonality
- Orthonormality

Some notations

Let $\underset{\sim}{v} \in \mathbb{R}^k$. Then we can write $\underset{\sim}{v}$ as a $k \times 1$ matrix $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}$.

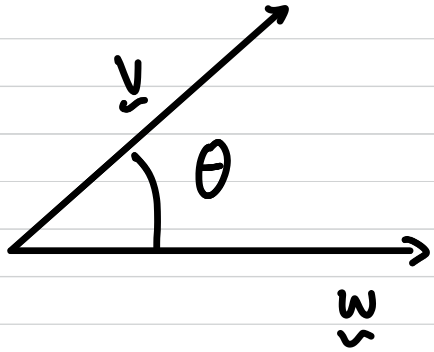
We define $\|\underset{\sim}{v}\|^2 = \underset{\sim}{v}^T \underset{\sim}{v} \Rightarrow \|\underset{\sim}{v}\| = \sqrt{\underset{\sim}{v}^T \underset{\sim}{v}}$.

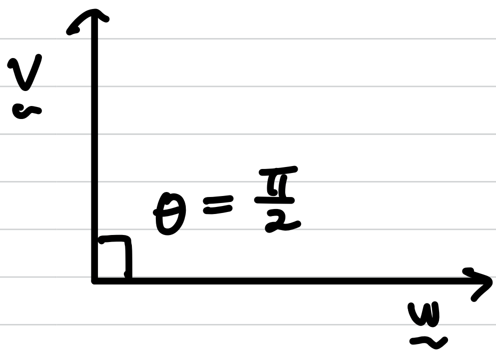
Here, $\|\underset{\sim}{v}\|$ symbolizes the length of the vector $\underset{\sim}{v}$, and $\underset{\sim}{v}^T \underset{\sim}{v}$ is actually another way of writing the dot product $\underset{\sim}{v} \cdot \underset{\sim}{v}$.

☆ Orthogonal Vectors

Let $\underset{\sim}{v}, \underset{\sim}{w} \in \mathbb{R}^k$, then $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are said to be orthogonal if $\underbrace{\underset{\sim}{v}^T}_{1 \times k} \underbrace{\underset{\sim}{w}}_{k \times 1} = \underbrace{0}_{\text{number}}$.

Note that $\underset{\sim}{v}^T \underset{\sim}{w} = \underset{\sim}{v} \cdot \underset{\sim}{w}$ (dot product).





Then $\underset{\sim}{v}^T \underset{\sim}{w} = \|\underset{\sim}{v}\| \cdot \|\underset{\sim}{w}\| \cdot \cos\theta$ where θ is the acute angle between $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

When $\theta = \frac{\pi}{2}$ (i.e. $90°$), then $\cos\theta = 0$

$\Rightarrow \underset{\sim}{v}^T \underset{\sim}{w} = 0$.

<u>Disclaimer</u>: we will not use this formula in this course.

<u>Eg</u>. $\underset{\sim}{v} = \begin{bmatrix} -1 \\ 2 \\ 4 \\ 3 \end{bmatrix}$ and $\underset{\sim}{w} = \begin{bmatrix} -1.2 \\ -2.1 \\ 0 \\ 1 \end{bmatrix}$ are orthogonal.

$$\underset{\sim}{v}^T \underset{\sim}{w} = \begin{bmatrix} -1 & 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} -1.2 \\ -2.1 \\ 0 \\ 1 \end{bmatrix}$$

$$= 1.2 - 4.2 + 3 = -3 + 3 = 0$$

$\therefore \underset{\sim}{v}$ and $\underset{\sim}{w}$ are orthogonal.

Orthogonal subspaces

Let V and W be subspaces of $\mathbb{R}^k$. We say that V and W are orthogonal if (or subsets also can)

$\underset{\sim}{v}^T \underset{\sim}{w} = 0$ for every $\underset{\sim}{v} \in V$ and for every $\underset{\sim}{w} \in W$.

Eg. $V = \{ z(0,0,1) : z \in \mathbb{R} \}$

$W = \{ x(1,0,0) + y(0,1,0) : x, y \in \mathbb{R} \}$

Both V and W are subspaces of $\mathbb{R}^3$.

Note that bases for V and W are $\{(0,0,1)\}$ and $\{(1,0,0), (0,1,0)\}$ respectively.

We can show V and W are orthogonal in two ways:

(1) Show directly $\underset{\sim}{v}^T \underset{\sim}{w} = 0$ for all $\underset{\sim}{v} \in V$ and for all $\underset{\sim}{w} \in W$.

Let $\underset{\sim}{v} \in V$ and $\underset{\sim}{w} \in W$. Then

$\underset{\sim}{v} = (0,0,z)$ for some $z \in \mathbb{R}$ and

$\underset{\sim}{w} = (x,y,0)$ for some $x, y \in \mathbb{R}$.

$$\underset{\sim}{v}^T \underset{\sim}{w} = \begin{bmatrix} 0 & 0 & z \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = 0.$$

Hence V and W are orthogonal.

② <u>Show that the bases for V and W are orthogonal to each other.</u>

Basis for V: $\{(0,0,1)\}$

Basis for W: $\{(1,0,0), (0,1,0)\}$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$\Rightarrow$ The bases for V and W are orthogonal to each other. $\Rightarrow$ V and W are orthogonal.

<u>Qn</u>: Why does basis for V and basis for W orthogonal imply V and W are orthogonal?

<u>Ans</u>: Because every element of V and W can be written as a linear combinations of the vectors in the basis for V and basis for W respectively.

E.g. Let A be an $m \times n$ matrix. Recall that the row space of A is the column space of A^T:

$$A = \begin{bmatrix} — & \underset{\sim}{a}_1' & — \\ & \vdots & \\ — & \underset{\sim}{a}_m' & — \end{bmatrix}$$

$\underset{\sim}{a}_1', \dots, \underset{\sim}{a}_m'$ are the rows of A

$$\Rightarrow A^T = \begin{bmatrix} | & & | \\ (\underset{\sim}{a}_1')^T & \cdots & (\underset{\sim}{a}_m')^T \\ | & & | \end{bmatrix}$$

$\Rightarrow$ Row space of A

$$= C(A^T) = \{ \underbrace{x_1 (\underset{\sim}{a}_1')^T + \dots + x_m (\underset{\sim}{a}_m')^T}_{\text{column form for } A^T \underset{\sim}{x},\ \underset{\sim}{x} \in \mathbb{R}^m} : x_1, \dots, x_m \in \mathbb{R} \}$$

Subspace of $\mathbb{R}^n$

$$= \{ A^T \underset{\sim}{x} : \underset{\sim}{x} \in \mathbb{R}^m \}.$$

while the null space of A is

$$N(A) = \{ \underset{\sim}{y} \in \mathbb{R}^n : A\underset{\sim}{y} = \underset{\sim}{0} \}$$

$C(A^T)$ and $N(A)$ are orthogonal subspaces!

Let $\underset{\sim}{y} \in N(A)$ and $A^T \underset{\sim}{x} \in C(A^T)$.

Then $\underset{\sim}{y}^T \cdot A^T \underset{\sim}{x} = (A\underset{\sim}{y})^T \underset{\sim}{x} \underset{\underset{\sim}{y} \in N(A)}{=} \underset{\sim}{0}^T \underset{\sim}{x} = 0.$

* Orthonormal vectors

Let $\underset{\sim}{v}, \underset{\sim}{w} \in \mathbb{R}^k$. Then $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are said to be orthonormal if

① $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are orthogonal, and

② $\|\underset{\sim}{v}\| = \|\underset{\sim}{w}\| = 1$ (their lengths are 1).

Recall:

The length of a vector $\underset{\sim}{v} = (v_1, \ldots, v_k) \in \mathbb{R}^k$ is the square root of the sum of squares of each component of $\underset{\sim}{v}$:

$$\|\underset{\sim}{v}\| = \sqrt{v_1^2 + \ldots + v_k^2} = \sqrt{\sum_{i=1}^{k} v_i^2}.$$

Eg. Let $\underset{\sim}{v} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and $\underset{\sim}{w} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

Then $\underset{\sim}{v}^T \underset{\sim}{w} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$$= \frac{1}{2} - \frac{1}{2} = 0.$$

and $\|\underset{\sim}{v}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = 1,$

$$\|\underset{\sim}{w}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1.$$

Therefore $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are orthonormal.

Exercises

1. (a) Show that the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ are orthogonal.

   (b) Using the vectors in (a), find a pair of orthonormal vectors.

2. The left nullspace of an $m \times n$ matrix is the set

$$N(A^T) = \{ \underset{\sim}{x} \in \mathbb{R}^m : A^T \underset{\sim}{x} = \underset{\sim}{0} \}.$$

Show that $N(A^T)$ and $C(A)$ are orthogonal subspaces of $\mathbb{R}^m$.