CSD2250/MAT250 Lecture 1 (Wed 11/5)

Topics covered:
- System of linear equations
- Elimination method

System of linear equations

Powers of the unknowns = 1
no x^2, y^3, $\sin x$ for eg.

① System of 2 equations $\bar{w}$ 2 unknowns

Eg.
$$x - 2y = 1 \quad (1)$$
$$3x + 2y = 11 \quad (2)$$

→ How we would solve it in secondary school:

(1): $x = 1 + 2y$ sub into (2)

$\Rightarrow 3(1+2y) + 2y = 11 \Rightarrow 3 + 6y + 2y = 11$

$\Rightarrow 8y = 8 \Rightarrow y = 1$

$\Rightarrow x = 3.$

i.e. solution to this system is $x=3, y=1$.

*Problem $\bar{w}$ this method:

The method is not as straight forward if the system has at least 3 equations.

→ We need a standard method that works for any system of linear equations.

★ Important notations and terminologies

For the earlier system of linear equations

$$x - 2y = 1$$
$$3x + 2y = 11$$

the matrix equation/form is

$$\underbrace{\begin{bmatrix}1 & -2\\ 3 & 2\end{bmatrix}}_{\substack{\text{coefficient matrix}\\ A}} \underbrace{\begin{bmatrix}x\\ y\end{bmatrix}}_{\substack{\text{vector of}\\ \text{unknowns } \underset{\sim}{x}}} = \underbrace{\begin{bmatrix}1\\ 11\end{bmatrix}}_{\text{vector } \underset{\sim}{b}}$$

i.e. $A\underset{\sim}{x} = \underset{\sim}{b}$

$$\begin{bmatrix}1 & -2\\ 3 & 2\end{bmatrix}$$

Columns of A

$$\begin{bmatrix}1 & -2\\ 3 & 2\end{bmatrix}$$

rows of A

We already know that $x=3$, $y=1$, so

$$\begin{bmatrix}1 & -2\\ 3 & 2\end{bmatrix}\begin{bmatrix}3\\ 1\end{bmatrix} = \begin{bmatrix}1\\ 11\end{bmatrix}$$

There are two ways of representing the equation above.

1. Column form

Columns of A

$$A\underset{\sim}{x} = 3\begin{bmatrix}1\\3\end{bmatrix} + 1\begin{bmatrix}-2\\2\end{bmatrix}$$

$$= \begin{bmatrix}3\cdot 1\\3\cdot 3\end{bmatrix} + \begin{bmatrix}-2\\2\end{bmatrix}$$

$$= \begin{bmatrix}3\\9\end{bmatrix} + \begin{bmatrix}-2\\2\end{bmatrix}$$

$$= \begin{bmatrix}3+(-2)\\9+2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix} = \underset{\sim}{b}$$

The red arrows denote scalar multiplication: multiplying each vector by a number is multiplying each component of that vector by this number.

The green arrows denote vector addition: adding two vectors (of the same size) is adding their respective components together.

2. Row form

first row of A

dot product

$$A\underset{\sim}{x} = \begin{bmatrix}(1,-2)\cdot(3,1)\\(3,2)\cdot(3,1)\end{bmatrix} = \begin{bmatrix}1\cdot 3+(-2)\cdot 1\\3\cdot 3+2\cdot 1\end{bmatrix}$$

Second row of A

$$= \begin{bmatrix}1\\11\end{bmatrix} = \underset{\sim}{b}$$

② System of 3 equations $\bar{w}$ 3 unknowns

Eg.
$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 5y + 2z &= 4 \\ 6x - 3y + z &= 2 \end{aligned}$$

has a matrix equation/form

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}.$$

<u>Check</u>: $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ is a solution to the system above using the <u>column form</u> of this matrix equation.

* Important matrix: Identity matrix

The identity matrix I_n is the $\overset{\text{rows}}{n} \times \overset{\text{columns}}{n}$ matrix consisting of ones on its main diagonal and zeros elsewhere.

main diagonal of a matrix

Eg. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Why is this matrix important?

If you multiply I_3 with a size 3 vector, you get back the same vector:

using column form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

In general, if I_n multiplied by a size n vector, we get back the same vector.

★Method of Elimination

The <u>standard method</u> to solve systems of linear equations.

① System of 2 equations

E.g.
$$x - 2y = 1$$
$$3x + 2y = 11$$

<u>Step 1</u> Identify the coefficient of the first variable in the first equation.

coefficient is <u>1</u>. We call this a <u>pivot</u>.

Note: This pivot has to be <u>nonzero</u>!

There are systems where this pivot is zero:

E.g.
$$-2y = 2$$
$$3x + y = 11$$

pivot is 0 here.

What we can do is to perform a <u>row swap</u>:

$$\Rightarrow \quad 3x + y = 11$$
$$-2y = 2$$

Now the first pivot is <u>3</u>.

Step 2 Our next goal is to eliminate the first variable x from all other equations.

- Let's eliminate x from the second equation!

We multiply 3 to the first equation:

$$3x - 6y = 3,$$

and subtract this from the second equation:

$$3x + 2y - \underbrace{(3x - 6y)}_{\text{first equation}} = 11 - \underbrace{3}_{\text{first equation}}$$

we get

$$2y + 6y = 8$$

$$\Rightarrow \quad 8y = 8.$$

Now the system becomes

$$\begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned} \iff \begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}.$$

upper triangular matrix

This is what we are aiming for!

Solving this system becomes easy now:

$8y = 8 \Rightarrow y = 1 \Rightarrow x = 1 + 2 = 3$

This process is called back-substitution.

This is the second and last pivot for this system of 2 equations.

* At the end of the elimination process,
  - the pivots will lie on the main diagonal of the matrix.
  - if the number of pivots = no of equations, then the solution to the system is unique.

* Note: Not every system of equations has a unique solution.

Example 1
$$x - 2y = 1$$
$$2x - 4y = 3$$

The first pivot here is 1, and we need to multiply the first equation by 2, then subtract it from the second equation. We get

$$x - 2y = 1$$
$$0y = 1.$$

But $0 \neq 1$, so this system has no solution.

Example 2

$$x - 2y = 1$$
$$4x - 8y = 4$$

First pivot $= 1$, we need to multiply the first equation by $\frac{4}{1} = 4$, then subtract it from the second equation. We get

$$x - 2y = 1$$
$$0y = 0.$$

Now, every $y \in \mathbb{R}$ (real numbers) satisfies this equation, hence this system has infinitely many solutions.

* Notice that in the 2 examples, after elimination there are zeros in the main diagonal? This is not a coincidence. If there are zeros on the main diagonal after elimination, then either the system will have no solutions OR infinitely many solutions.

* In the second example (and the previous examples), how did we know what to multiply the first equation with?

Answer: The multiplier l_{ij} where

$$l_{ij} = \frac{\text{coefficient of variable in row } i}{\text{pivot in row } j}$$

Check: The multipliers we used in the previous examples follow this formula.

Let's try this on a system of 3 equations:

② System of 3 equations.

$$2x + 4y - 2z = 2$$
$$4x + 9y - 3z = 8$$
$$-2x - 3y + 7z = 10$$

Step 1 The first pivot is 2.

Step 2 We aim to eliminate x from the second and third equation.

Substep 1 Eliminate x from second equation.

Multiplier $l_{21} = \frac{4}{2} = 2$. Multiply first equation by 2 and then subtract from the second equation. We get

$$2x + 4y - 2z = 2$$
$$y + z = 4$$
$$-2x - 3y + 7z = 10$$

Substep 2 Eliminate x from the third equation.

Multiplier $\ell_{31} = \frac{-2}{2} = -1$. Multiply first equation by -1, then subtract it from the third equation.

→ This is just adding the first equation to the third.

We get
$$2x + 4y - 2z = 2$$
$$y + z = 4$$
$$y + 5z = 12$$

Step 3 Identify second pivot = 1.

Step 4 Eliminate y from the third equation.

Multiplier $= \ell_{32} = \frac{1}{1} = 1$. Subtract second equation from third to get

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ y + z &= 4 \\ 4z &= 8. \end{aligned}$$

Elimination method stops as the system has become upper triangular.

The third pivot is 4. Hence this system has a unique solution.

Check: Back substitution yields

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$