CSD2250 Linear Algebra Week 2 Homework

20th May 2022

You are given until 27th of May 2022, 2359 HRS to submit this homework.

Question 1 (Matrix multiplication and transpose)

Let A be a 3×4 matrix where $A_{ij} = 2i - 3j$, and B be a 4×3 matrix where

$$B_{ij} = \begin{cases} -1 & \text{if } i + j \text{ is odd} \\ 1 & \text{if } i + j \text{ is even,} \end{cases}$$

$$\text{and } C = \begin{bmatrix} 5 & -1 \\ 9 & 1 \\ 2 & 0 \end{bmatrix}.$$

Compute the following, whenever possible:

(a) CA,

(b) AB,

(c) BC,

(d) $C^T C$,

(e) CC^T.

Suggested Solution:

(a) CA is not possible, as the number of columns of $C = 2 \neq 3 =$ number of rows of A.

(b)

$$AB = \begin{bmatrix} 6 & -6 & 6 \\ 6 & -6 & 6 \\ 6 & -6 & 6 \end{bmatrix}.$$

(c)

$$BC = \begin{bmatrix} -2 & -2 \\ 2 & 2 \\ -2 & 2 \\ 2 & 2 \end{bmatrix}.$$

(d)

$$C^T C = \begin{bmatrix} 110 & 4 \\ 4 & 2 \end{bmatrix}.$$

(e)

$$CC^T = \begin{bmatrix} 26 & 44 & 10 \\ 44 & 82 & 18 \\ 10 & 18 & 4 \end{bmatrix}.$$

Note that in (d) and (e), the matrices obtained are **symmetric**.

Grading policy (for graders):

(a) 2 marks for each part.

(b) If you perceive that they are able to matrix multiply correctly (from any part), then any careless mistakes should only deduct one mark max from each part. Else, you can deduct the full two marks from each part.

Question 2 (Finding inverse through Gauss-Jordan)

Let

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

Apply the Gauss-Jordan algorithm to compute A^{-1}. You are expected to explain every step of the process clearly.

Suggested Solution:
We start off with the matrix

$$\begin{bmatrix} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{bmatrix}$$

and we aim to bring this matrix to a *reduced row echelon form (rref)*. We first subtract $\dfrac{1}{3}$ of row 2 from row 3, and this yields

$$\begin{bmatrix} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & 0 & \frac{8}{3} & 0 & -\frac{1}{3} & 1 \end{bmatrix}.$$

We then multiply row 3 by $\dfrac{3}{8}$, and we get

$$\begin{bmatrix} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{8} & \frac{3}{8} \end{bmatrix}.$$

We subtract row 3 from row 2, yielding

$$\begin{bmatrix} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & \frac{9}{8} & -\frac{3}{8} \\ 0 & 0 & 1 & 0 & -\frac{1}{8} & \frac{3}{8} \end{bmatrix}.$$

Dividing row 2 by 3 yields

$$\begin{bmatrix} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & 0 & -\frac{1}{8} & \frac{3}{8} \end{bmatrix}.$$

Adding row 2 and row 3 to row 1 gives

$$\begin{bmatrix} 2 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{3}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & 0 & -\frac{1}{8} & \frac{3}{8} \end{bmatrix}.$$

Dividing row 1 by 2 yields then yields the rref

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{8} & \frac{1}{8} \\ 0 & 1 & 0 & 0 & \frac{3}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & 0 & -\frac{1}{8} & \frac{3}{8} \end{bmatrix}.$$

Then the inverse matrix A^{-1} of A is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{3}{8} & -\frac{1}{8} \\ 0 & -\frac{1}{8} & \frac{3}{8} \end{bmatrix}.$$

Grading policy (for graders):

(a) If students did not use Gauss-Jordan, -10 marks.

(b) If you perceive that a small calculation errors/mistakes have caused their final answer to be different, -1 or -2 marks (up to you).

Question 3

Let A be the matrix in Question 2. Using the A^{-1} you have computed in Question 2, solve the system $A\boldsymbol{x} = \boldsymbol{b}$ where

$$\boldsymbol{b} = \begin{bmatrix} 8 \\ 16 \\ 24 \end{bmatrix}.$$

Suggested Solution:
Left-multiplying both sides of the equation $A\boldsymbol{x} = \boldsymbol{b}$ with A^{-1} gives the solution to the system:

$$\begin{aligned}\boldsymbol{x} = I_3\,\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}&\\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{3}{8} & -\frac{1}{8} \\ 0 & -\frac{1}{8} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} 8 \\ 16 \\ 24 \end{bmatrix} \\ &= \begin{bmatrix} 9 \\ 3 \\ 7 \end{bmatrix}.\end{aligned}$$

Grading policy (for graders):

(a) If students did not use the method mandated by the question, -10 marks.

(b) If students calculated A^{-1} wrongly in Question 2, but the method is consistent with the solution, give full marks.

(c) For any minor calculation error, at your discretion, you may subtract up to 3 marks.

Question 4 (Determinants and invertibility)

Let

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 6 & 3 & 0 \\ 4 & 2 & 0 \end{bmatrix}.$$

Without using cofactor expansion, explain why $\det(A) = 0$.

Suggested Solution:
The first column of the matrix is 2 times the second column of the matrix, hence $\det(A) = 0$.

Grading policy (for graders):

(a) If students did not use the method mandated by the question, -10 marks.

(b) Any other valid reasoning, e.g.

   (a) Row 2 $= \frac{3}{2} \times$ Row 3,

   (b) The equation $A\boldsymbol{x} = \mathbf{0}$ has a nonzero solution

$$\boldsymbol{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

   (c) The reduced row echelon form of A has a zero row, etc.

Question 5

Using cofactor expansion, by expanding along a row or column of your choice, compute $\det(A)$ where

$$A = \begin{bmatrix} 4 & 0 & -3 & 1 \\ 5 & 1 & 0 & -8 \\ 2 & -5 & 9 & -1 \\ 0 & 3 & -1 & 0 \end{bmatrix}.$$

(a) Is A invertible? Explain.

(b) If A is invertible in part (a), without explicitly computing A^{-1}, determine $\det(A^{-1})$. If A is not invertible, you may ignore this part.

Suggested Solution:

(a) We expand along the 4th row, where there are the most number of zeros.

$$\begin{aligned}
\det(A) &= 3 \cdot \begin{vmatrix} 4 & -3 & 1 \\ 5 & 0 & -8 \\ 2 & 9 & -1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 4 & 0 & 1 \\ 5 & 1 & -8 \\ 2 & -5 & -1 \end{vmatrix} \\
&= 3\left(-5 \cdot \begin{vmatrix} -3 & 1 \\ 9 & -1 \end{vmatrix} - (-8) \cdot \begin{vmatrix} 4 & -3 \\ 2 & 9 \end{vmatrix}\right) + \left(4 \cdot \begin{vmatrix} 1 & -8 \\ -5 & -1 \end{vmatrix} + \begin{vmatrix} 5 & 1 \\ 2 & -5 \end{vmatrix}\right) \\
&= 3(-5(-6) + 8 \cdot 42) + 4(-41) - 27 \\
&= 1098 - 191 = 907.
\end{aligned}$$

Since $\det(A) \neq 0$, therefore A is invertible.

(b) We know that for every pair of square matrices A and B of the same size,

$$\det(AB) = \det(A) \cdot \det(B).$$

Hence,

$$\det(AA^{-1}) = \det(A) \cdot \det(A^{-1}).$$

Since, the determinant of the identity matrix is 1, we have

$$1 = \det(I_n) = \det(AA^{-1}) = \det(A) \cdot \det(A^{-1}).$$

and hence

$$\begin{aligned} \det(A^{-1}) &= \frac{1}{\det(A)} \\ &= \frac{1}{907}. \end{aligned}$$

Grading policy (for graders):

(a) If students did not use the method mandated by the question to compute $\det(A)$, -4 marks.

(b) If students computed A^{-1} explicitly in (b) to derive $\det(A^{-1})$, -3 marks.

(c) For part (a), the students may use any valid justification for A invertible (though I believe it will be much more tedious, so I expect students to use the $\det(A) \neq 0$ method).

(d) Any minor calculation errors, at your discretion, -1 to -3 marks.