CSD2250/MAT250 Lecture 4 (Fri 20/5)

Topics covered:
- Symmetric matrices
- Determinants of matrices

Symmetric matrices

In a symmetric matrix, its ith row is the same as the ith column. This means:

★ A **symmetric matrix** A satisfies the equation $A^T = A$. This means that its entry on the ith row and jth column

$$a_{ij} = a_{ji}$$

is equal to its entry on the jth row and ith column

eg. $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

same entries about the main diagonal

main diagonal

* Recall that for any $m \times n$ matrix A, we can construct symmetric matrices by multiplying it with its transpose A^T:

① AA^T is an $m \times m$ symmetric matrix

② A^TA is an $n \times n$ symmetric matrix.

* These constructions are very useful later in our course, they are used in

(a) Least squares approximations (LSQR)

(b) Singular value decomposition (SVD)

which are the bedrock of this course.

I would even dare say that symmetric matrices are the most important matrices for this course.

We keep these symmetric matrices in view now, we will come back to them soon.

Determinants

The determinant of a square matrix is a single number that describes a ton of information of the matrix. In this part, we cover two things:

① How to calculate the determinant of a matrix using cofactor expansion.

② Some of the important properties and relations of the determinant of a matrix.

Cofactor expansion

We know how to calculate the determinant of a 2×2 matrix:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

★ Alternatively, the determinant of a matrix is written as $|A|$. In the case of a 2×2 matrix,

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The determinants of larger-sized matrix is based on the determinants of smaller-sized matrices. The determinant of a 3x3 matrix is based on the determinants of 2x2 matrices, the determinant of a 4x4 matrix is based on the determinant of 3x3 matrices and so on.

row column (of A)

* The ij-cofactor of a matrix A is the value

$$C_{ij} = \underbrace{(-1)^{i+j}}_{①} \det \underbrace{M_{ij}}_{②}$$

① i is the row number of A, j is the column number of A. If $i+j$ is even, then $(-1)^{i+j}$ is $+1$, and if $i+j$ is odd, then $(-1)^{i+j}$ is -1.

For a 3x3 matrix, the sign matrix is

$$(-1)^{1+1} = 1 \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}. \qquad (-1)^{2+3} = -1$$

(2) M_{ij} is the (sub)matrix left after removing the ith row and jth column of A.

Notice that the size of M_{ij} is one less than the size of the original matrix A.

Exercises

(1) Compute the sign matrix for a 4x4 matrix.

(2) Let

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}.$$

Write down M_{11}, M_{12} and M_{13}, and compute the determinants of these matrices.

We compute the determinant of an $n \times n$ matrix using cofactor expansion:

* The determinant of an $n \times n$ matrix A is

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in}$$

where i represents ANY row number of A.

This means that the determinant can be found by "expanding along" any row!

As one may expect, $\det A$ can also be found by expanding along any column:

* The determinant of an $n \times n$ matrix A is

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} C_{nj}$$

where j represents ANY column number of A.

* One of the tricks to reduce the tedious computation is to expand along a row/column with lots of zeros. I will leave you to find out why is that the case.

Formula for 3x3 matrix determinant

let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be any 3x3 matrix.

Expanding along the first row, we get

$$\det A = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23})$$

$$- a_{12}(a_{21}a_{33} - a_{31}a_{23})$$

$$+ a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Trick:

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

A

first two columns of A

Exercises

① Find the determinant of the following matrix

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$$

(a) expanding along the first row,

(b) expanding along the third column.

② Find the determinant of the following matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

by expanding along a row or column of your choice.

③ Show that the determinant of a 4×4 upper triangular matrix is the product of its entries on the main diagonal.

Properties of determinants

① The determinant of I_n is 1.

② If a row of A is multiplied by a constant c, then the determinant of the resulting matrix is $c \cdot \det(A)$.

③ If any two rows or any two columns of A are the same, then $\det A = 0$.

④ If A has a row of zeros, then $\det A = 0$.

⑤ If any row of A is a multiple of another row, then $\det A = 0$. This is also true for columns.

⑥ A is invertible if and only if $\det A \neq 0$.

⑦ If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det A}.$$

⑧ For any two square matrices A and B,

$$\det(A \cdot B) = \det(A) \cdot \det(B).$$

⑨ $\det(A^T) = \det(A)$

Try to explain this using cofactor expansion!

Exercises

① Find the determinant of

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 4 \\ 3 & 9 & 5 \end{bmatrix}.$$

② Show that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}$$

is invertible. Find $\det(A^{-1})$.

③ Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ a & -1 & -2 \\ 2 & 3 & 7 \end{bmatrix}$$

where a is an unknown constant. Find the values of a such that A is invertible.