CSD2250 Linear Algebra Week 4 Homework

3rd June 2022

You are given until 10th of June 2022, 2359 HRS to submit this homework.

Question 1 (Linear Independence)

Check that the columns of the following matrix

$$A = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

are linearly independent by showing that the only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$.

Suggested Solution:
The reduced row echelon form for the matrix A is (working required by students)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The system $A\boldsymbol{x} = \mathbf{0}$ then reduces to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which implies that $x_1 = x_2 = 0$. Thus the only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$, and hence the columns of the matrix A are linearly independent.

Grading policy (for graders):

(1) An alternative solution constitutes expressing $A\boldsymbol{x} = \mathbf{0}$ as a system of linear equations:

$$\begin{aligned} 7x_1 + 8x_2 &= 0 \\ 9x_1 + 10x_2 &= 0 \\ 11x_1 + 12x_2 &= 0 \end{aligned}$$

then using elimination and back-substitution to get $x_1 = x_2 = 0$.

(2) Another alternative solution constitutes students recognizing that the rref of A has 2 pivot columns, which is equal to the number of columns of this matrix. Hence the only solution to $A\boldsymbol{x} = \mathbf{0}$ is the zero vector.

(3) How you grade this question is entirely up to you.

Question 2

Determine if the sets of vectors below are linearly independent or dependent.

(a) $B_1 = \{(1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1)\}$.

(b) $B_2 = \{(1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1), (0, 1, -1, 0)\}$.

(c) $B_3 = \{(2, 0, 0, 0), (3, 6, 0, 0), (4, 7, 0, 0), (1, 0, 9, 2)\}$.

(d) $B_4 = \{(2, 0, 0, 4), (3, 6, 0, 6), (4, 7, 1, 8), (0, -1, 2, 3)\}$.

(e) $B_5 = \{(2, 0, 0, 4), (3, 6, 0, 6), (4, 7, 1, 8), (0, -1, 2, 3), (1, -1, 0, 0)\}$

Suggested Solution:
The main methods are the solutions from part (a) and (b), but as there are many methods to solve these questions, they will be presented in the remaining parts, so that you can have a good grasp of them too.

(a) Putting these vectors into columns of a matrix

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The rref of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix has 3 pivot columns, equal to its number of columns. Hence the only solution to $A_1\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$, and thus the vectors in B_1 are linearly independent.

(b) Putting these vectors into columns of a matrix

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

The rref of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which has a free column (last column), hence there is a nonzero solution to $A_2\boldsymbol{x} = \mathbf{0}$. Thus, the vectors in B_2 are linearly dependent.

(c) Putting these vectors into columns of a matrix

$$A_3 = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

One can observe that row 4 of this matrix is a multiple of row 3. Thus the determinant of this matrix must be 0, and hence, there must be a nonzero solution to $A_3\boldsymbol{x} = \mathbf{0}$. Thus the vectors in B_3 are linearly dependent.

(d) Putting these vectors into columns of a matrix

$$A_4 = \begin{bmatrix} 2 & 3 & 4 & 0 \\ 0 & 6 & 7 & -1 \\ 0 & 0 & 1 & 2 \\ 4 & 6 & 8 & 3 \end{bmatrix}.$$

Using cofactor expansion (students must provide the working), we see that $\det(A_4) = 36 \neq 0$. Thus the inverse of A_4 exists. We left-multiply $A_4\boldsymbol{x} = \mathbf{0}$ by its inverse A_4^{-1} to get

$$\boldsymbol{x} = A_4^{-1}A_4\boldsymbol{x} = A_4^{-1}\mathbf{0} = \mathbf{0}.$$

Thus the only solution to $A_4\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$. Thus the vectors in B_4 are linearly dependent.

(e) Putting these vectors into columns of a matrix

$$A_5 = \begin{bmatrix} 2 & 3 & 4 & 0 & 1 \\ 0 & 6 & 7 & -1 & -1 \\ 0 & 0 & 1 & 2 & 0 \\ 4 & 6 & 8 & 3 & 0 \end{bmatrix}.$$

We see that in this matrix, the number of columns is greater than the number of rows. Hence the rref of this matrix will always have free columns, as the number of pivot columns can never exceed the number of rows, nor the number of columns (as mentioned in the lecture notes). Hence, there is a nonzero solution to $A_5\boldsymbol{x} = \mathbf{0}$. Therefore, the vectors in B_5 are linearly dependent.

Grading policy (for graders):

(1) 2 marks for each part.

(2) Students may use any of the methods mentioned in the solutions, for any part.

(3) You have full discretion on the mark distribution within each part.

Question 3 (Basis)

For each of the sets of vectors in Question 2, are they bases for $\mathbb{R}^4$? Explain your answer.

Suggested Solution:

(a) There are only 3 vectors in B_1, and thus they cannot span the whole of $\mathbb{R}^4$, hence B_1 cannot be a basis for $\mathbb{R}^4$.

(b) There may be 4 vectors in B_2, but they are linearly dependent, hence B_2 cannot be a basis for $\mathbb{R}^4$.

(c) Same answer as part (b).

(d) There are 4 vectors in B_4 and the determinant of the matrix A_4 (constructed treating the vectors as columns of a 4 by 4 matrix) is invertible. Hence B_4 is a basis for $\mathbb{R}^4$.

(e) The vectors in B_5 are linearly dependent, and hence cannot be a basis for $\mathbb{R}^4$. Another possible explanation is that the number of vectors in $B_5 > 4$ hence it cannot be a basis for $\mathbb{R}^4$.

Grading policy (for graders):

(1) 2 marks for each part.

(2) The explanation is key here, if there is no sufficient explanation for their answers, 0 marks for that part.

(3) You have full discretion on the mark distribution for each part.

Question 4 (Finding bases for $C(A)$ and $C(A^T)$)

Let A be the matrix

$$\begin{bmatrix} -1 & 2 & 4 & 3 \\ 4 & 2 & 3 & 9 \end{bmatrix}.$$

(a) Find a basis for the column space $C(A)$.

(b) Find a basis for the row space $C(A^T)$.

(c) Previously, if asked to describe the column space of A, we would write

$$C(A) = \left\{ a \begin{bmatrix} -1 \\ 4 \end{bmatrix} + b \begin{bmatrix} 2 \\ 2 \end{bmatrix} + c \begin{bmatrix} 4 \\ 3 \end{bmatrix} + d \begin{bmatrix} 3 \\ 9 \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Now, it should be clear to you that some of these columns in this description are redundant. Using the basis for $C(A)$ we found in part (a), describe $C(A)$.

Suggested Solution:

The rref of the matrix A is (workings are required)

$$\begin{bmatrix} 1 & 0 & -0.2 & 1.2 \\ 0 & 1 & 1.9 & 2.1 \end{bmatrix}.$$

(a) A basis for the column space $C(A)$ are the columns of A which correspond to the pivot columns of A, thus a basis for $C(A)$ (notice that $C(A)$ is a subspace of $\mathbb{R}^2$) is

$$\left\{ \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}.$$

Alternatively, a basis for $C(A)$ can also be written as

$$\{(-1, 4), (2, 2)\},$$

whichever notation the students prefer. Note that this is also a basis for $\mathbb{R}^2$.

(b) A basis for the row space $C(A^T)$ are the rows corresponding to the pivot rows of A, thus a basis for $C(A^T)$ is

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \\ 9 \end{bmatrix} \right\}.$$

Like part (a), students may also write it in this form

$$\{(-1, 2, 4, 3), (4, 2, 3, 9)\}.$$

(c) A 'new' description for $C(A)$ is

$$C(A) = \left\{ a \begin{bmatrix} -1 \\ 4 \end{bmatrix} + b \begin{bmatrix} 2 \\ 2 \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Grading policy (for graders):

(1) 4 marks each for part (a) and (b), 2 marks for part (c).

(2) You have full discretion on the mark distribution for each part.

Question 5

Using the matrix A found in Question 4,

(a) Find a basis for the nullspace $N(A)$.

(b) **Using the information you obtained in this question and in Question 4**, find a basis for $\mathbb{R}^4$.

Suggested Solution:

(a) We need to find the special solutions, as they constitute the basis vectors for $N(A)$. From the rref of A, we notice that there are two free columns, which correspond to the variables x_3 and x_4. We consider two different cases of these variables, which correspond to two different special solutions:

(i) $x_3 = 1$, $x_4 = 0$,

(ii) $x_3 = 0$, $x_4 = 1$.

The system corresponding to the rref of A is

$$\begin{aligned} x_1 - 0.2x_3 + 1.2x_4 &= 0 \\ x_2 + 1.9x_3 + 2.1x_4 &= 0. \end{aligned}$$

For the first case, we have $x_3 = 1$ and $x_4 = 0$, hence $x_1 = 0.2$ and $x_2 = 1.9$. Hence the first special solution is

$$\begin{bmatrix} 0.2 \\ -1.9 \\ 1 \\ 0 \end{bmatrix}.$$

For the second case, we have $x_3 = 0$ and $x_4 = 1$, hence $x_1 = -1.2$ and $x_2 = -2.1$. Hence the second special solution is

$$\begin{bmatrix} -1.2 \\ -2.1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, a basis for $N(A)$ is

$$\left\{ \begin{bmatrix} 0.2 \\ -1.9 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1.2 \\ -2.1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

This basis can also be written as

$$\{(0.2, -1.9, 1, 0), (-1.2, -2.1, 0, 1)\}.$$

(b) We combine the basis for $C(A^T)$ and $N(A)$ to form a basis for $\mathbb{R}^4$, thus a basis for $\mathbb{R}^4$ is

$$\{(-1, 2, 4, 3), (4, 2, 3, 9), (0.2, -1.9, 1, 0), (-1.2, -2.1, 0, 1)\}.$$

Students *may* verify that this is in fact, a basis for $\mathbb{R}^4$ by showing that the matrix containing these vectors as columns is invertible.

Grading policy (for graders):

(1) 5 marks each for both parts.

(2) You have full discretion on the mark distribution for each part.