CSD2250/MAT250 Lecture 7 (Wed 1/6)

Topics covered:
- Linear independence
- Vector span

Recall that the rank of an $m \times n$ matrix A is the no. of pivot columns of the matrix A, and is denoted by the letter r.

In actuality, the columns of A corresponding to the pivot columns are actually linearly independent, and they form a basis for the column space $C(A)$.

We first give a motivation for linear independence:

Let A be the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

One observation here is that

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

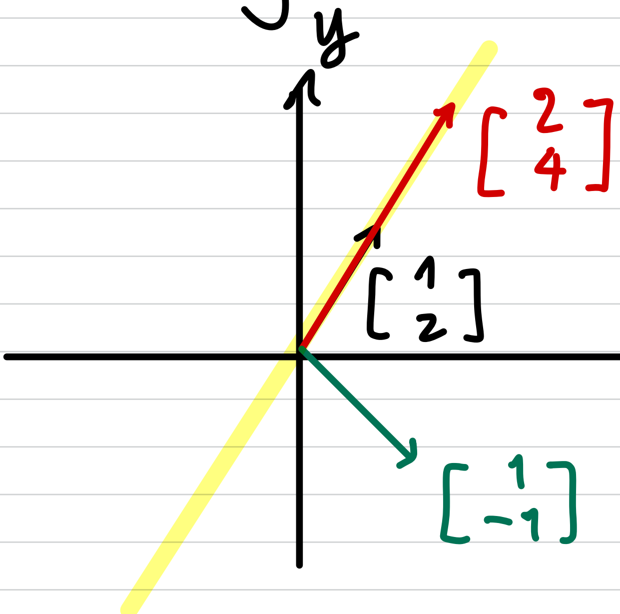
$$\Leftrightarrow \quad 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\text{nonzero vector}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

also special solution for $N(A)$

Geometrically, $\longrightarrow$ demonstrates.

$$2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



cannot be written as a multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

These two vectors are <u>parallel</u> to one another!

① In other words, since the two columns are parallel to each other, there is a <u>nonzero</u> solution to $A\underset{\sim}{x} = \underset{\sim}{0}$, namely $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

② Notice that in the diagram, any vector outside of the line cannot be written as a multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. In other words, no other vectors outside of this line can be <u>generated</u> by multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

* ① A set of vectors $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n \in \mathbb{R}^m$ are **linearly independent** if

$$x_1 \underset{\sim}{a}_1 + x_2 \underset{\sim}{a}_2 + \ldots + x_n \underset{\sim}{a}_n = \underset{\sim}{0}$$

only occurs when $x_1 = x_2 = \ldots = x_n = 0$.

** Another way of saying this is that the columns of an $m \times n$ matrix A whose columns are $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n$ are **linearly independent** if the only solution to $A\underset{\sim}{x} = \underset{\sim}{0}$ is $\underset{\sim}{x} = \underset{\sim}{0}$.
(i.e. $N(A)$ contains only $\underset{\sim}{0}$)

* If there are nonzero $\underset{\sim}{x}$ s.t $A\underset{\sim}{x} = \underset{\sim}{0}$ (i.e. $N(A)$ has at least one special solution), then we say that the columns of A are **linearly dependent**.

Notice in the previous example,

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has a nonzero solution $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Hence the columns of A are <u>linearly dependent</u>. In other words, linear dependence is a more general description of vectors compared to the word 'parallel'.

<u>Example</u> Prove that the columns of the following matrix A is linearly independent.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Convert this matrix into rref:

$$\xrightarrow[R_2 - 2R_1]{R_3 - R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{0}$ reduces to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2 = 0$$

which implies that $\underset{\sim}{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only solution to $A\underset{\sim}{x} = \underset{\sim}{0}$. Hence the columns of A are linearly independent.

You may have noticed that the rank of the matrix A in the previous example is 2, because it has two pivot columns. Notice that the rank of A is also equal to the no. of columns of A, and that the columns of A are linearly independent. This is of no coincidence!

* The columns of an $m \times n$ matrix A are linearly independent when $r = n$. In this case, A is said to have full column rank; there are n pivot columns and no free columns, i.e. $\underset{\sim}{x} = \underset{\sim}{0}$ is the only solution to $A\underset{\sim}{x} = \underset{\sim}{0}$.

* Another important observation is that the no. of pivots can never exceed the no. of rows nor the no. of columns. Thus, if there are more columns than rows, then

$$n > m \geq r \Rightarrow n > r$$

which means that the columns of A are linearly dependent!

Exercises

① Determine if the columns of A are linearly independent or dependent.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

② Determine if the following set of vectors are linearly independent or dependent.

(a) $\{(1,-3,2),(2,1,-3),(-3,2,1)\}$

(b) $\{(0,1,1),(1,1,0),(0,0,0)\}$

Span

English def$^{\underline{n}}$: The full extent of something from end to end; the amount of space that something covers.

★ A set of vectors **span** a space if all possible linear combinations of these vectors fill the space.

Eg. ① The columns of an $m \times n$ matrix A corresponding to the pivot columns span the column space $C(A)$.

② The vectors of size k

$$\begin{gathered}(1, 0, \ldots, 0)\\(0, 1, \ldots, 0)\\\vdots\\(0, 0, \ldots, 1)\end{gathered}$$

span $\mathbb{R}^k$.

→ If you put these vectors into columns of a matrix (in order left to right), you get the identity matrix I_k.

The **row space** of a matrix is the subspace of $\mathbb{R}^n$ spanned by the rows. In other words, it is spanned by the columns of A^T. Hence, the row space of A is the column space of A^T, i.e. $C(A^T)$.

E.g. Describe the row space of A, where

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix}. \qquad \overset{m}{3} \times \overset{n}{2}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \end{bmatrix}. \qquad \overset{m}{2} \times \overset{n}{3}$$

Note: $n > m$

The row space of A is the column space of A^T

$$= \left\{ x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 5 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$$

notice that the no. of columns of A^T is larger than the no. of rows of A^T, hence these vectors are linearly dependent: in actuality, one of them can be 'thrown out'. We will learn how to do it this Friday.