CSD2250/MAT250 Lecture 5 (Wed 25/5)

Topics covered:	
	• Vector spaces
	• Subspaces of a vector space
	• Column space $C(A)$

Vector spaces

Let's recall the set of real numbers $\mathbb{R}$.

Common numbers in $\mathbb{R}$ include

$$2,\ -30,\ \frac{1}{2},\ -\frac{3}{4},\ \sqrt{2},\ \pi.$$

natural numbers — integers — rational numbers (fractions) — irrational numbers (numbers which cannot be expressed as a fraction)

whole numbers

We already know about vector addition and scalar multiplication. These two operations are key to a vector space: they define what operations can be done within the space itself.

The vector addition is denoted by a '+' sign, and scalar multiplication a '$\cdot$' (sometimes the dot is omitted).

With the vector addition and scalar scalar multiplication defined, there are

* eight core properties of a vector space V:

For $\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z} \in V$ and c_1, c_2 constants:

① $\underset{\sim}{x} + \underset{\sim}{y} = \underset{\sim}{y} + \underset{\sim}{x}$

② $\underset{\sim}{x} + (\underset{\sim}{y} + \underset{\sim}{z}) = (\underset{\sim}{x} + \underset{\sim}{y}) + \underset{\sim}{z}$

* ③ There is a unique zero vector $\underset{\sim}{0}$ such that

$$\underset{\sim}{0} + \underset{\sim}{x} = \underset{\sim}{x} \quad \text{for every } \underset{\sim}{x} \in V.$$

* ④ For every $\underset{\sim}{x} \in V$, there is a $-\underset{\sim}{x}$ such that

$$\underset{\sim}{x} + (-\underset{\sim}{x}) = \underset{\sim}{0}$$

⑤ $1 \cdot \underset{\sim}{x} = \underset{\sim}{x}$ for all $\underset{\sim}{x} \in V$

⑥ $(c_1 c_2) \cdot \underset{\sim}{x} = c_1 (c_2 \underset{\sim}{x})$

⑦ $c_1 (\underset{\sim}{x} + \underset{\sim}{y}) = c_1 \underset{\sim}{x} + c_1 \underset{\sim}{y}$

⑧ $(c_1 + c_2) \underset{\sim}{x} = c_1 \underset{\sim}{x} + c_2 \underset{\sim}{x}$

★ Another two properties of a vector space V are:
(These are the key properties of a subspace too)

① If you add any two vectors $\underset{\sim}{x}, \underset{\sim}{y}$ in V the resulting $\underset{\sim}{x}+\underset{\sim}{y}$ is also in V.

② If you multiply a vector $\underset{\sim}{x}$ in V by any scalar c, the resulting $c\underset{\sim}{x}$ is also in V.

Examples of vector spaces

Note that $\mathbb{R}$ is also a vector space: numbers can be viewed as vectors with one component.

① $\mathbb{R}^k$, where $k = 1, 2, 3, \ldots$

they represent the usual vectors

$$(x_1, x_2, \ldots, x_k)$$

that we deal with.

Each of these components are real numbers.

Vector addition is defined as

$$(x_1, x_2, \ldots, x_k) + (x_1', x_2', \ldots, x_k')$$
$$= (x_1 + x_1', x_2 + x_2', \ldots, x_k + x_k')$$

Scalar multiplication is defined as

$$c(x_1, \ldots, x_k) = (cx_1, cx_2, \ldots, cx_k).$$

② $M_2(\mathbb{R})$ the space containing the 2×2 matrices with real entries

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad a,b,c,d \in \mathbb{R}$$

Vector addition

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$$

Scalar multiplication

$$\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$$

③ The space containing only the zero vector, i.e.

$$\{\underbrace{(0,0,\ldots,0)}_{k \text{ zeroes}}\}$$

Vector addition and scalar multiplication defined in the same way as the first example.

* This is the smallest vector space.

* Subspaces

Let's say we work within the confines of a vector space V, where vector addition and scalar multiplication have already been defined, and the 8 core properties (pg 2) hold, and most importantly, the vectors resulting from adding two vectors or scaling a vector stays within V.

Qn Since a vector space is large, is it possible that these properties hold for a 'smaller part' of the vector space?

subset of V

The answer is yes. For any subset of a vector space V, vector addition and scalar multiplication are the same. However, when we add two vectors within the subset or scale a vector in the subset, the resulting vector may not be in that subset!

Example

Let $V = \mathbb{R}^2$, and let S be the subset of V such that the 2 vector components are non negative, i.e.

S consists of (x_1, x_2) where $x_1, x_2 \geq 0$.

Then for example, $(1,1) \in S$ but scaling it by a negative number, e.g. $-\frac{1}{2}$

$$-\frac{1}{2}(1,1) = \left(-\frac{1}{2}, -\frac{1}{2}\right) \notin S$$

results in a vector that is not in S!

Thus, vector space operations cannot be performed 'within' S, unless we are okay with objects ending up outside of the vector space we are working with, which is unheard of.

* We seek to look for subsets of V with such properties. These are what we call subspaces of V. It is like a 'localized' vector space.

** A subset S of V is called a **subspace** of V if these three conditions are satisfied:

① The zero vector $\underset{\sim}{0} \in S$.

② For any $\underset{\sim}{v}, \underset{\sim}{w} \in S$, $\underset{\sim}{v} + \underset{\sim}{w} \in S$.

③ For any scalar c, and any $\underset{\sim}{v} \in S$,

$$c\underset{\sim}{v} \in S.$$

E.g.

(a) For $V = \mathbb{R}^2$, let S be the subset where (x_1, x_2) satisfies $x_2 = 2x_1$.

① $(0,0) \in S$ because $0 = 2 \cdot 0$

② Let (x_1', x_2') and $(x_1'', x_2'') \in S$, then

$$x_2' = 2x_1' \text{ and } x_2'' = 2x_1''$$

$$\Rightarrow \quad x_2' + x_2'' = 2x_1' + 2x_1''$$

$$= 2(x_1' + x_1'')$$

$$\Rightarrow \quad (x_1' + x_1'', x_2' + x_2'') \in S$$

③ Let c be a scalar, and $(x_1, x_2) \in S$.

Then $c(x_1, x_2) = (cx_1, cx_2)$.

Note that $c\underbrace{x_2} = c \cdot \underbrace{2x_1} = 2(cx_1)$

because $(x_1, x_2) \in S$

So $(cx_1, cx_2) \in S$.

Thus S is a subspace of $V = \mathbb{R}^2$.

(b) For $V = M_2(\mathbb{R})$, let S be the subset

where

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies $b = c = 0$.

(i.e S is the subset of diagonal matrices)

(a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ (obvious) ✓

(b) $\begin{bmatrix} a' & 0 \\ 0 & d' \end{bmatrix} + \begin{bmatrix} a'' & 0 \\ 0 & d'' \end{bmatrix}$

$= \begin{bmatrix} a'+a'' & 0 \\ 0 & d'+d'' \end{bmatrix}$ ✓

(c) $c\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} ca & 0 \\ 0 & cd \end{bmatrix}$ ✓

Therefore S is a subspace of $V = M_2(\mathbb{R})$.

Exercises

① Let $V = \mathbb{R}^3$. Let S be the subset of V where (x_1, x_2, x_3) satisfies $x_1 + 2x_2 - 5x_3 = 0$. Show that S is a subspace of V.

② Let $V = M_2(\mathbb{R})$. Let S be the subset of V where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies $c = 0$.

Show that S is a subspace of V.

③ Let S be a subspace of a vector space V. Show that for any $\underset{\sim}{v}, \underset{\sim}{w} \in S$, and a, b scalars, $\underbrace{a\underset{\sim}{v} + b\underset{\sim}{w}}_{\text{linear combination of } \underset{\sim}{v} \text{ and } \underset{\sim}{w}.} \in S$.

④ For the following subsets of $\mathbb{R}^3$, determine if they are subspaces of $\mathbb{R}^3$ or not. Give an example like in page 6 if it is not a subspace of $\mathbb{R}^3$.

1. The set of (x_1, x_2, x_3) where $x_2 = 2$.
2. The set of (x_1, x_2, x_3) where $x_1 x_2 x_3 = 0$.
3. The set of (x_1, x_2, x_3) where $x_1 \geq x_2 \geq x_3$.
4. The set of (x_1, x_2, x_3) where $5x_1^2 - 3x_2^2 + 6x_3^2 = 0$.

⑤ Replicate the same exercises as in ④ for these subsets of $M_2(\mathbb{R})$.

1. The set of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $ad - bc \neq 0$.
2. The set of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $b = c$.
3. The set of 2 by 2 elimination matrices.

Column space of a $m \times n$ matrix A, $C(A)$

Let A be a $m \times n$ matrix. Recall the matrix equation

$$A\underset{\sim}{x} = \underset{\sim}{b}.$$

* We say that this equation/system is solvable if there is a solution to this equation/system.

If A is invertible (this also means $m=n$), then this system is solvable for every $\underset{\sim}{b}$.

Otherwise, there are some $\underset{\sim}{b}$ for which this system is not solvable.

$A\underset{\sim}{x} = \underset{\sim}{b}$ solvable

Qn How to find out which $\underset{\sim}{b}$ works?

The answer lies in the column form of $A\underset{\sim}{x}$.

Let $\underset{\sim}{a_1}, \underset{\sim}{a_2}, \ldots, \underset{\sim}{a_n}$ be the columns of A, then

$$A\underset{\sim}{x} = x_1 \underset{\sim}{a_1} + x_2 \underset{\sim}{a_2} + \ldots + x_n \underset{\sim}{a_n}.$$

Here, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a $n \times 1$ matrix,

* $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

Each of the columns of A are $m \times 1$ matrices.
We can also treat them as m-dimensional vectors, i.e.

$$\underset{\sim}{a_1}, \underset{\sim}{a_2}, \dots, \underset{\sim}{a_n} \in \mathbb{R}^m$$

very easy to confuse with n!

as each $\underset{\sim}{a_i}$ has m entries!

$\bigstar$ A linear combination of vectors $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_n}$ is the form

$$c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + \dots + c_n \underset{\sim}{v_n}$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants in $\mathbb{R}$.

Therefore, $A\underset{\sim}{x} = \underset{\sim}{b}$ is solvable if $\underset{\sim}{b}$ is a linear combination of the columns of A:

$$\underset{\sim}{b} = x_1 \underset{\sim}{a_1} + \dots + x_n \underset{\sim}{a_n}.$$

From here, we define the column space of A, based on these linear combinations.

★ The column space of a $m \times n$ matrix A is the subspace of $\mathbb{R}^m$ which contains all linear combinations of the columns of A, i.e.

$$x_1 \underset{\sim}{a_1} + x_2 \underset{\sim}{a_2} + \dots + x_n \underset{\sim}{a_n}$$

where $x_1, x_2, \dots, x_n$ are any real numbers.

The column space is denoted as $C(A)$.

Here, we say that the subspace $C(A)$ is spanned by the columns of A.

take all linear combinations of

Eg.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Note A is invertible

$$\underset{\sim}{a_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \underset{\sim}{a_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$m=2$

$C(A)$ contains

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ for every } x_1 \text{ and } x_2.$$

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ for every } x_1, x_2$$

$\Rightarrow C(A) = \mathbb{R}^2$.

whole space $\mathbb{R}^m$

(b) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$C(A)$ contains

$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ for all $x_1, x_2 \in \mathbb{R}$

$= x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= x_1 + 2x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

kind of like in integration constants

$= c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ since x_1 and x_2 are arbitrary numbers, replace $x_1 + 2x_2$ as another arbitrary number c.

$\Rightarrow$ $C(A)$ consists of vectors that obey

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ 2c \end{bmatrix} \Rightarrow \begin{matrix} x = c \\ y = 2c \end{matrix} \Rightarrow y = 2x$$

$\Rightarrow$ $C(A)$ consists of all points on the line $y = 2x$.

Note $C(A)$ is not the whole space $\mathbb{R}^2$.

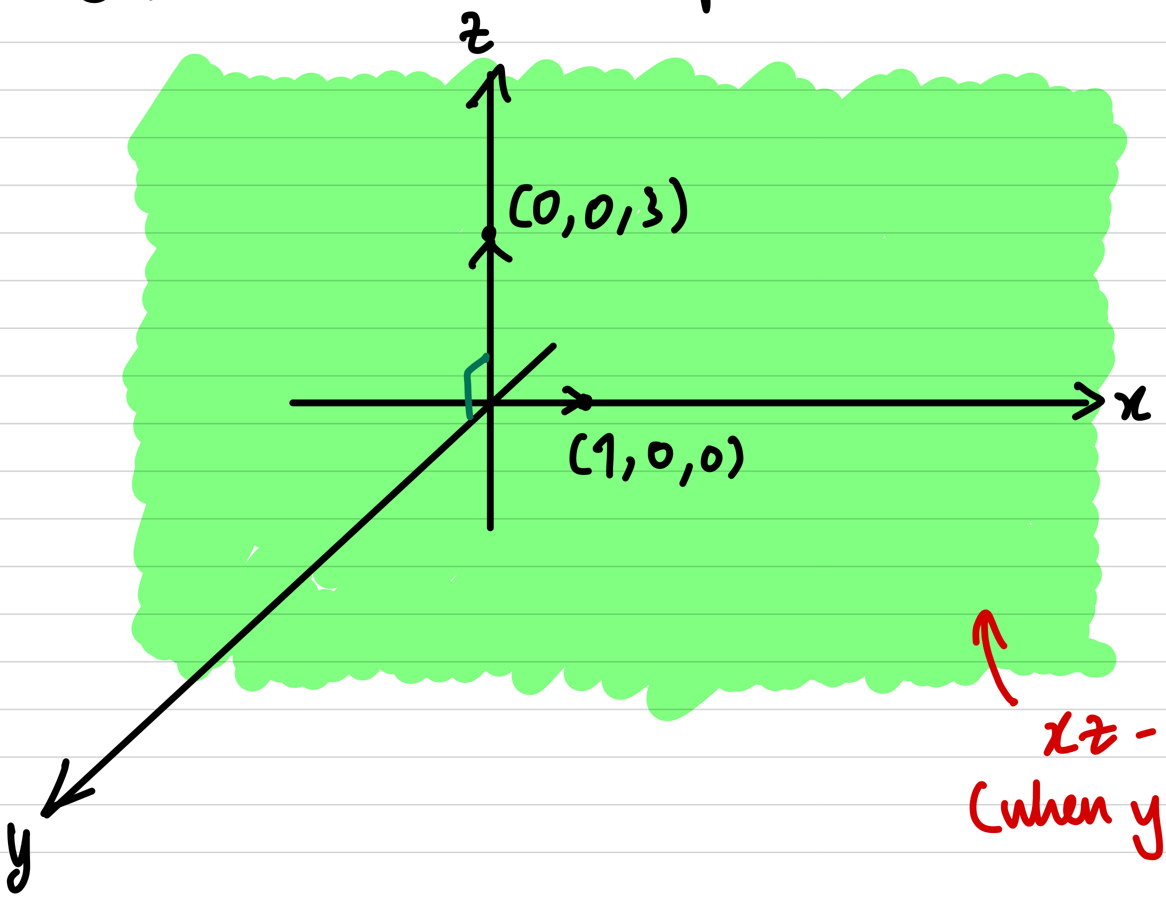
(c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 3 \end{bmatrix}$ $m=3$, $n=2$

$C(A)$ contains

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ x_2' \end{bmatrix}$$

$x_2' = 3x_2$
since x_2 is arbitrary

$\Rightarrow$ $C(A)$ is the xz-plane



xz-plane
(when $y=0$)

Note that $m > n$, so $C(A)$ can never be $\mathbb{R}^3$!
You need at least three columns for $C(A)$ to be $\mathbb{R}^3$.
↳ we will see later in bases and linear independence.

Exercises

① Describe $C(A)$ where

(a) $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$. (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 1 & 0 \end{bmatrix}$.

② What is the column space of a 7×7 invertible matrix A?

③ Find 5 different vectors in $C(A)$ for ①(b).