

Ralph Jordan Zapitan  
 STAT 467  
 Test 2  
 Part II  
 Take Home

**1.**

$$f(y_1, y_2) = \begin{cases} \frac{1}{8} y_1 e^{-(y_1+y_2)/2} & y_1 > 0, y_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

**a)**

$$\begin{aligned} f_2(y_2) &= \int_0^\infty f(y_1, y_2) dy_1 = \int_0^\infty \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_1 \\ &= \frac{1}{8} e^{-y_2/2} \int_0^\infty y_1 e^{-y_1/2} dy_1 \\ u &= \frac{y_1}{2} \implies du = \frac{1}{2} dy_1 \\ y_1 &= 2u \implies dy_1 = 2du \end{aligned}$$

So

$$\begin{aligned} f_2(y_2) &= \frac{1}{8} e^{-y_2/2} \int_0^\infty (2u) e^{-u} (2) du \\ &= \frac{1}{2} e^{-y_2/2} \int_0^\infty u e^{-u} du \\ &= \frac{1}{2} e^{-y_2/2} \Gamma(2) \end{aligned}$$

So the marginal pdf of  $Y_2$  is

$$f_2(y_2) = \begin{cases} \frac{1}{2} e^{-y_2/2} & y_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

**b)**

The survival function of  $Y_2$  is

$$S_2(y_2) = 1 - F_2(y_2)$$

$$\begin{aligned}
 F_2(y_2) &= \int_0^{y_2} \frac{1}{2} e^{-t/2} dt \\
 &= \frac{1}{2} \int_0^{y_2} e^{-t/2} dt \\
 &= \frac{1}{2} (-2) e^{-t/2} \Big|_{t=0}^{t=y_2} = e^{-t/2} \Big|_{t=y_2}^{t=0} \\
 &= 1 - e^{-y_2/2}
 \end{aligned}$$

So

$$S_2(y_2) = \begin{cases} e^{-y_2/2} & y_2 \leq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Lastly,

$$P(Y_2 > 2) = S_2(2) = e^{-1} = 0.367879$$

To find the median life time,  $\phi_{0.50}$ , do the following:

$$\begin{aligned}
 F_2(\phi_{0.50}) &= 0.50 \\
 1 - e^{-\phi_{0.50}/2} &= 0.50 \implies \phi_{0.50} = -2 \ln(0.5) = 1.3862944
 \end{aligned}$$

We have to compare this with  $E(Y_2)$ .

$$\begin{aligned}
 E(Y_2) &= \int_0^{\infty} y_2 f_2(y_2) dy_2 \\
 &= \int_0^{\infty} y_2 \frac{1}{2} e^{-y_2/2} dy_2 \\
 &= \frac{1}{2} \int_0^{\infty} y_2 e^{-y_2/2} dy_2 \\
 u = \frac{y_2}{2} &\implies du = \frac{1}{2} dy_2 \\
 y_2 = 2u &\implies dy_2 = 2du \\
 E(Y_2) &= \frac{1}{2} \int_0^{\infty} (2u) e^{-u} (2) du \\
 &= 2 \int_0^{\infty} u e^{-u} du \\
 &= 2 \Gamma(2) \\
 &= 2
 \end{aligned}$$

So the mean is greater than the median. This would mean that the distribution is skewed to the right.

## 2.

We are given the following

$$Y_1 \sim N(6, 1)$$

$$Y_2 \sim N(7, 1)$$

$$\text{Cov}(Y_1, Y_2) = 0$$

So this means that

$$E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = 6 - 7 = -1$$

$$V(Y_1 - Y_2) = V(Y_1) + V(Y_2) = 1 + 1 = 2$$

$$\sigma_{Y_1 - Y_2} = \sqrt{2}$$

Therefore

$$\begin{aligned} P(Y_1 > Y_2) &= P(Y_1 - Y_2 > 0) \\ &= P\left[Y_1 - Y_2 - E(Y_1 - Y_2) > 0 - E(Y_1 - Y_2)\right] \\ &= P\left[\frac{Y_1 - Y_2 - E(Y_1 - Y_2)}{\sigma_{Y_1 - Y_2}} > \frac{-E(Y_1 - Y_2)}{\sigma_{Y_1 - Y_2}}\right] \\ &= P\left(Z > \frac{1}{\sqrt{2}}\right) \\ &= 1 - P\left(Z \leq \frac{1}{\sqrt{2}}\right) \\ &= 0.2397 \end{aligned}$$

### 3.

a)

For a gamma distribution with parameters  $\alpha$  and  $\beta$ , let the pdf be the following:

$$f(y) = \frac{1}{\Gamma(\alpha) \beta^\alpha} y^{\alpha-1} e^{-y/\beta} \text{ for } y \geq 0$$

Therefore,

$$\begin{aligned} E(Y^k) &= \int_0^\infty \frac{1}{\Gamma(\alpha) \beta^\alpha} (y^k) y^{\alpha-1} e^{-y/\beta} dy \\ &= \frac{1}{\Gamma(\alpha) \beta^\alpha} \int_0^\infty y^{(\alpha+k)-1} e^{-y/\beta} dy \end{aligned}$$

Using  $u$  substitution,

$$\begin{aligned} u &= \frac{y}{\beta} \implies du = \frac{1}{\beta} dy \\ y &= \beta u \implies dy = \beta du \end{aligned}$$

So

$$\begin{aligned} E(Y^k) &= \frac{1}{\Gamma(\alpha) \beta^\alpha} \int_0^\infty (\beta u)^{(\alpha+k)-1} e^{-u} (\beta) du \\ &= \frac{\beta^{(\alpha+k)-1} \beta}{\Gamma(\alpha) \beta^\alpha} \int_0^\infty u^{(\alpha+k)-1} e^{-u} du \\ &= \frac{\beta^k}{\Gamma(\alpha)} \Gamma(\alpha + k) \end{aligned}$$

b)

if  $k = 1$ ,

$$\begin{aligned}
 E(Y^1) &= \frac{\beta^1 \Gamma(\alpha + 1)}{\Gamma(\alpha)} \\
 &= \frac{\beta \alpha!}{(\alpha - 1)!} \\
 &= \frac{\beta \alpha (\alpha - 1)!}{(\alpha - 1)!} \\
 &= \alpha \beta
 \end{aligned}$$

**c)**

Note that

$$E\left(\frac{1}{Y}\right) = E(Y^{-1}) \implies k = -1$$

So,

$$\begin{aligned}
 E\left(\frac{1}{Y}\right) &= \frac{\beta^{-1}}{\Gamma(\alpha)} \Gamma(\alpha - 1) \\
 &= \frac{1}{\beta (\alpha - 1)!} (\alpha - 2)! \\
 &= \frac{1}{\beta (\alpha - 1)(\alpha - 2)!} (\alpha - 2)! \\
 &= \frac{1}{\beta (\alpha - 1)}
 \end{aligned}$$

**4.**

If  $Y_1$  and  $Y_2$  are two uncorrelated random variables, then the covariance between them must be zero. Let's list the following for our reference:

- (i)  $\text{Cov}(Y_1, Y_2) = 0$
- (ii)  $E(Y_i) = \mu_i$  and  $\text{Var}(Y_i) = \sigma_i^2$  for  $i = 1, 2$
- (iii)  $U = Y_1 + Y_2$
- (iv)  $W = Y_1 - Y_2$
- (vi)  $\text{Var}(U) = \text{Var}(Y_1) + \text{Var}(Y_2)$
- (vii)  $\text{Var}(W) = \text{Var}(Y_1) + \text{Var}(Y_2)$

**a)**

$$\begin{aligned}
Cov(U, W) &= E\left(\left[U - E(U)\right]\left[W - E(W)\right]\right) \\
&= E\left([Y_1 + Y_2 - (\mu_1 + \mu_2)][Y_1 - Y_2 - (\mu_1 - \mu_2)]\right) \\
&= E\left[(Y_1 + Y_2 - \mu_1 - \mu_2)(Y_1 - Y_2 - \mu_1 + \mu_2)\right] \\
&= E(\mu_1^2 - 2\mu_1 Y_1 - \mu_2^2 + 2\mu_2 Y_2 + Y_1^2 - Y_2^2) \\
&= \mu_1^2 - 2\mu_1 E(Y_1) - \mu_2^2 + 2\mu_2 E(Y_2) + E(Y_1^2) - E(Y_2^2) \\
&= \mu_1^2 - 2\mu_1^2 - \mu_2^2 + 2\mu_2^2 + E(Y_1^2) - E(Y_2^2) \\
&= E(Y_1^2) - \mu_1^2 - E(Y_2^2) + \mu_2^2 \\
&= E(Y_1^2) - \mu_1^2 - [E(Y_2^2) - \mu_2^2] \\
&= Var(Y_1) - Var(Y_2)
\end{aligned}$$

b)

$$\begin{aligned}
\rho_{U,W} &= \frac{Cov(U, W)}{\sqrt{Var(U)Var(W)}} \\
&= \frac{Var(Y_1) - Var(Y_2)}{\sqrt{[Var(Y_1) + Var(Y_2)][Var(Y_1) + Var(Y_2)]}} \\
&= \frac{Var(Y_1) - Var(Y_2)}{Var(Y_1) + Var(Y_2)}
\end{aligned}$$

c)

$$\rho_{U,W} < 0 \implies \frac{Var(Y_1) - Var(Y_2)}{Var(Y_1) + Var(Y_2)} < 0$$

Multiplying both sides of the inequality by the denominator,

$$\begin{aligned}
Var(Y_1) - Var(Y_2) &< 0 \\
Var(Y_1) &< Var(Y_2)
\end{aligned}$$

So when  $Var(Y_1) < Var(Y_2)$ , or when the standard deviation of  $Y_1$  is less than the standard deviation of  $Y_2$ , the correlation coefficient  $\rho_{U,W} < 0$ . The correlation coefficient  $\rho_{U,W}$  measures the strength of linearity between  $U$  and  $W$ .

## 4.

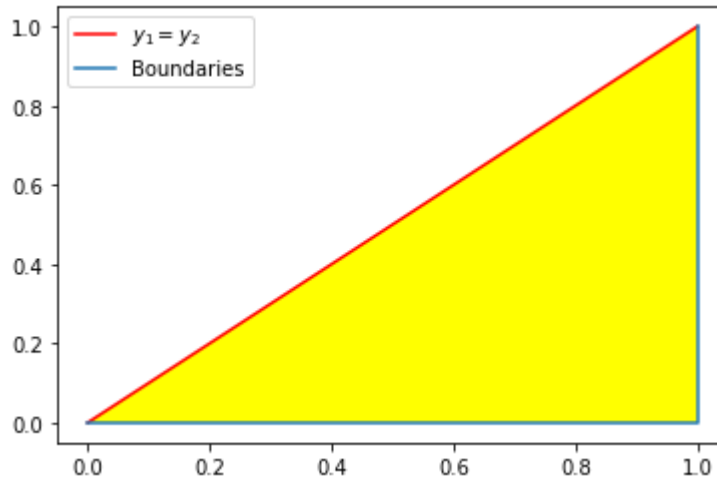
Let  $Y_1$  and  $Y_2$  have the joint pdf given by

$$f(y_1, y_2) = \begin{cases} k(1 - y_2) & 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

```

In [8]: 1 import matplotlib.pyplot as plt
        2 %matplotlib inline
        3 import numpy as np
        4
        5 f = lambda x: x
        6 x, y = [0, 1], [0, 1]
        7 plt.plot(x, y, 'r', label = '$ y_1 = y_2 $')
        8 plt.plot([0, 1, 1], [0, 0, 1], label = 'Boundaries')
        9 plt.fill_between([0, 1], [0, 1], facecolor = 'yellow')
       10 plt.legend()
       11 plt.show()

```



The yellow area is the overall region of integration. We always have to integrate within this area at all times and cannot go out of it.

a)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1, y_2) dy_1 dy_2 = 1$$

This means that

$$\begin{aligned}
 \int_0^1 \int_0^{y_1} k(1 - y_2) dy_2 dy_1 &= k \int_0^1 \int_0^{y_1} (1 - y_2) dy_2 dy_1 \\
 &= k \int_0^1 \left( y_2 - \frac{y_2^2}{2} \right) \Big|_{y_2=0}^{y_2=y_1} dy_1 \\
 &= k \int_0^1 \left( y_1 - \frac{1}{2} y_1^2 \right) dy_1 \\
 &= \frac{k}{3}
 \end{aligned}$$

Setting this equal to 1 and solving for  $k$ , we get

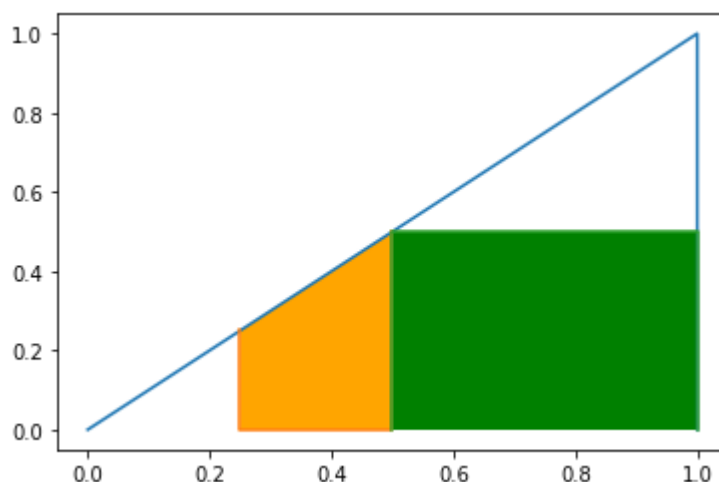
$$\frac{k}{3} = 1 \implies k = 3$$

so our pdf becomes

$$f(y_1, y_2) = \begin{cases} 3(1 - y_2) & 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

b)

```
In [20]: 1 plt.plot([0, 1, 1], [0, 1, 0])
2 plt.plot([1/4, 1/4, 1/2, 1/2], [1/4, 0, 0, 1/2])
3 plt.fill_between([1/4, 1/2], [1/4, 1/2], facecolor = 'orange')
4 plt.plot([1, 1, 1/2, 1/2], [0, 1/2, 1/2, 0])
5 plt.fill_between([1/2, 1], [1/2, 1/2], facecolor = 'green')
6 plt.show()
```



We have to find

$$P\left(Y_1 \geq \frac{1}{4}, Y_2 \leq \frac{1}{2}\right) = P(\text{Orange}) + P(\text{Green})$$

$$P(\text{Orange}) = \int_{1/4}^{1/2} \int_0^{y_1} 3(1 - y_2) dy_2 dy_1$$

$$= 3 \int_{1/4}^{1/2} \left( y_2 - \frac{1}{2} y_2^2 \right) \Big|_{y_2=0}^{y_2=y_1} dy_1$$

$$= 3 \int_{1/4}^{1/2} \left( y_1 - \frac{1}{2} y_1^2 \right) dy_1$$

$$= 3 \left( \frac{1}{2} y_1^2 - \frac{1}{6} y_1^3 \right) \Big|_{y_1=1/4}^{y_1=1/2}$$

$$= \frac{29}{128}$$

$$P(\text{Green}) = \int_0^{1/2} \int_{1/2}^1 3(1 - y_2) dy_1 dy_2$$

$$= \frac{9}{16}$$

So

$$P\left(Y_1 \geq \frac{1}{4}, Y_2 \leq \frac{1}{2}\right) = \frac{29}{128} + \frac{9}{16} = \frac{101}{128}$$

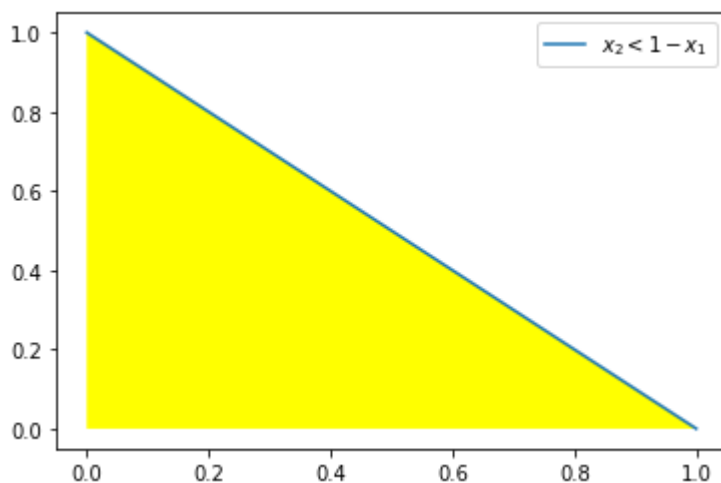
## 5.

Our joint density function in this problem is

$$f(x_1, x_2) = \begin{cases} 2 & 0 < x_1 + x_2 < 1, x_1 > 0, x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

a)

```
In [23]: 1 x, y = [0, 1], [1, 0]
2 plt.plot(x, y, label = '$x_2 < 1 - x_1$')
3 plt.fill_between([0, 1], [1, 0], facecolor = 'yellow')
4 plt.legend()
5 plt.show()
```



Our limits of integration are restricted to the first quadrant, particularly the yellow area. We can only integrate our joint density function within the yellow area. The conditional mean of  $X_1$  given  $X_2 = x_2$  is therefore

$$\begin{aligned} E(X_1 | X_2 = x_2) &= \int_0^{1-x_2} x_1 f(x_1 | x_2) dx_1 \\ &= \int_0^{1-x_2} x_1 \frac{f(x_1, x_2)}{f_2(x_2)} dx_1 \\ f_2(x_2) &= \int_0^{1-x_2} f(x_1, x_2) dx_1 \\ &= \int_0^{1-x_2} 2 dx_1 \\ &= 2(1 - x_2) \end{aligned}$$



So the marginal pdf of  $X_2$  is

$$\begin{aligned}
 f_2(x_2) &= \begin{cases} 2(1-x_2) & x_2 > 0 \\ 0 & \text{elsewhere} \end{cases} \\
 E(X_1|X_2 = x_2) &= \int_0^{1-x_2} x_1 \frac{2}{2(1-x_2)} dx_1 \\
 &= \int_0^{1-x_2} \frac{x_1}{1-x_2} dx_1 \\
 &= \frac{1}{1-x_2} \int_0^{1-x_2} x_1 dx_1 \\
 &= \frac{1}{2(1-x_2)} x_1^2 \Big|_{x_1=0}^{x_1=1-x_2} \\
 &= \frac{1}{2(1-x_2)} (1-x_2)^2 \\
 &= \frac{1}{2} (1-x_2)
 \end{aligned}$$

**b)**

We need to find

$$V(X_1|X_2 = x_2) = E(X_1^2|X_2 = x_2) - [E(X_1|X_2 = x_2)]^2$$

The integration is straightforward by now. I am tired so I am gonna be skipping some steps. It's just a matter of substitution from here once we have everything we need.

$$\begin{aligned}
 E(X_1^2|X_2 = x_2) &= \int_0^{1-x_2} x_1^2 f(x_1|x_2) dx_1 \\
 &= \frac{1}{1-x_2} \int_0^{1-x_2} x_1^2 dx_1 \\
 &= \frac{1}{3} (x_2 - 1)^2 \\
 V(X_1|X_2 = x_2) &= \frac{1}{3} (x_2 - 1)^2 - \frac{1}{4} (1-x_2)^2 = \frac{1}{12} (x_2 - 1)^2
 \end{aligned}$$

**c)**

$$\begin{aligned}
E[V(X_1|X_2 = x_2)] &= E\left(\frac{1}{12}(X_2 - 1)^2\right) \\
&= \int_0^1 \int_0^{1-x_1} \frac{1}{12}(x_2 - 1)^2 f(x_1, x_2) dx_2 dx_1 \\
&= \frac{1}{6} \int_0^1 \int_0^{1-x_1} (x_2 - 1)^2 dx_2 dx_1 \\
&= \frac{1}{6} \int_0^1 \int_0^{1-x_1} (x_2^2 - 2x_2 + 1) dx_2 dx_1 \\
&= \frac{1}{6} \int_0^1 \left( \frac{1}{3}x_2^3 - x_2^2 + x_2 \right) \Big|_{x_2=0}^{x_2=1-x_1} dx_1 \\
&= \frac{1}{6} \int_0^1 (1 - x_1) \left[ \frac{1}{3}(1 - x_1)^2 - (1 - x_1) + 1 \right] dx_1
\end{aligned}$$

This will simplify to

$$= \frac{1}{24}$$

**d)**

Objective 1: Find  $V[E(X_1|X_2)]$

$$V[E(X_1|X_2)] = E\left[\left(E(X_1|X_2)\right)^2\right] - \left(E[E(X_1|X_2)]\right)^2$$

$$\begin{aligned}
E\left[\left(E(X_1|X_2)\right)^2\right] &= E\left[\left(\frac{1}{2}(1 - x_2)\right)^2\right] \\
&= E\left[\frac{1}{4}(1 - x_2)^2\right] \\
&= \int_0^1 \int_0^{1-x_1} \frac{1}{4}(1 - x_2)^2 dx_2 dx_1 \\
&= \frac{1}{2} \int_0^1 \int_0^{1-x_1} (1 - 2x_2 + x_2^2) dx_2 dx_1 \\
&= \frac{1}{2} \int_0^1 \left( x_2 - x_2^2 + \frac{1}{3}x_2^3 \right) \Big|_{x_2=0}^{x_2=1-x_1} dx_1 \\
&= \frac{1}{2} \int_0^1 \left( \frac{1}{3} - \frac{1}{3}x_1^3 \right) dx_1 \\
&= \frac{1}{6} \int_0^1 (1 - x_1^3) dx_1 = \frac{1}{6} \left( x_1 - \frac{1}{4}x_1^4 \right) \Big|_{x_1=0}^{x_1=1} \\
&= \frac{1}{6} \left( 1 - \frac{1}{4} \right) \\
&= \frac{1}{8}
\end{aligned}$$

$$\begin{aligned}
 E[E(X_1|X_2)] &= E\left[\frac{1}{2}(1-x_2)\right] \\
 &= \int_0^1 \int_0^{1-x_1} (1-x_2) dx_2 dx_1 \\
 &= \int_0^1 \left(x_2 - \frac{1}{2}x_2^2\right) \Big|_{x_2=0}^{x_2=1-x_1} dx_1 \\
 &= \int_0^1 \left[(1-x_1) - \frac{1}{2}(1-x_1)^2\right] dx_1 \\
 &= \int_0^1 \left(\frac{1}{2} - \frac{x_1^2}{2}\right) dx_1 \\
 &= \frac{1}{2} \int_0^1 (1-x_1^2) dx_1 \\
 &= \frac{1}{2} \left(x_1 - \frac{1}{3}x_1^3\right) \Big|_{x_1=0}^{x_1=1} \\
 &= \frac{1}{2} \left(1 - \frac{1}{3}\right) \\
 &= \frac{1}{3}
 \end{aligned}$$

Therefore,  $V[E(X_1|X_2)] = \frac{1}{8} - \left(\frac{1}{3}\right)^2 = \frac{1}{72}$

Objective 2: Find  $V(X_1)$

$$V(X_1) = E(X_1^2) - [E(X_1)]^2$$

To make things a little less painful, I am going to make a general rule for the  $k$ th moment of  $X_1$ ,  $E(X_1^k)$ .

$$\begin{aligned}
E(X_1^k) &= \int \int_A x_1^k f(x_1, x_2) dA \\
&= \int_0^1 \int_0^{1-x_1} 2x_1^k dx_2 dx_1 \\
&= \int_0^1 2x_1^k \int_0^{1-x_1} dx_2 dx_1 \\
&= \int_0^1 2x_1^k x_2 \Big|_{x_2=0}^{x_2=1-x_1} dx_1 \\
&= \int_0^1 2x_1^k (1-x_1) dx_1 \\
&= 2 \int_0^1 \left( x_1^k - x_1^{k+1} \right) dx_1 \\
&= 2 \left[ \frac{x_1^{k+1}}{k+1} - \frac{x_1^{k+2}}{k+2} \right]_{x_1=0}^{x_1=1} \\
&= 2 \left[ \frac{1}{k+1} - \frac{1}{k+2} \right]
\end{aligned}$$

When  $k = 2$ ,

$$E(X_1^2) = 2 \left( \frac{1}{3} - \frac{1}{4} \right) = \frac{1}{6}$$

When  $k = 1$ ,

$$E(X_1) = 2 \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{3}$$

So the variance of  $X_1$  is

$$V(X_1) = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

Is  $V(X_1) = E[V(X_1|X_2)] + V[E(X_1|X_2)]$ ?

$$E[V(X_1|X_2)] + V[E(X_1|X_2)] = \frac{1}{24} + \frac{1}{72} = \frac{1}{18}$$

So yes, they are equal.