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STAT 467
Test 2 REDO

1.

$$f(y_1, y_2) = \begin{cases} \frac{1}{8} y_1 e^{-(y_1+y_2)/2} & y_1 > 0, y_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

a)

$$\begin{aligned} f_2(y_2) = \int_0^\infty f(y_1, y_2) dy_1 &= \int_0^\infty \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_1 \\ &= \frac{1}{8} e^{-y_2/2} \int_0^\infty y_1 e^{-y_1/2} dy_1 \\ u = \frac{y_1}{2} &\implies du = \frac{1}{2} dy_1 \\ y_1 = 2u &\implies dy_1 = 2du \end{aligned}$$

So

$$\begin{aligned} f_2(y_2) &= \frac{1}{8} e^{-y_2/2} \int_0^\infty (2u) e^{-u} (2) du \\ &= \frac{1}{2} e^{-y_2/2} \int_0^\infty u e^{-u} du \\ &= \frac{1}{2} e^{-y_2/2} \Gamma(2) \end{aligned}$$

So the marginal pdf of Y_2 is

$$f_2(y_2) = \begin{cases} \frac{1}{2} e^{-y_2/2} & y_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

b)

The survival function of Y_2 is

$$\begin{aligned} S_2(y_2) &= 1 - F_2(y_2) \\ F_2(y_2) &= \int_0^{y_2} \frac{1}{2} e^{-t/2} dt \\ &= \frac{1}{2} \int_0^{y_2} e^{-t/2} dt \\ &= \frac{1}{2}(-2) e^{-t/2} \Big|_{t=0}^{t=y_2} = e^{-t/2} \Big|_{t=y_2}^{t=0} \\ &= 1 - e^{-y_2/2} \end{aligned}$$

So

$$S_2(y_2) = \begin{cases} e^{-y_2/2} & y_2 \leq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Lastly,

$$P(Y_2 > 200) = S_2(200) = 0$$

To find the median life time, $\phi_{0.50}$, do the following:

$$F_2(\phi_{0.50}) = 0.50$$
$$1 - e^{-\phi_{0.50}/2} = 0.50 \implies \phi_{0.50} = -2\ln(0.5) = 1.3862944$$

We have to compare this with $E(Y_2)$.

$$\begin{aligned} E(Y_2) &= \int_0^\infty y_2 f_2(y_2) dy_2 \\ &= \int_0^\infty y_2 \frac{1}{2} e^{-y_2/2} dy_2 \\ &= \frac{1}{2} \int_0^\infty y_2 e^{-y_2/2} dy_2 \end{aligned}$$

Using the same integration by parts that we did previously, this comes out to be

$$E(Y_2) = 2$$

So the mean is greater than the median. This would mean that the distribution is skewed to the right.

3.

a)

For a gamma distribution with parameters α and β, let the pdf be the following:

$$f(y) = \frac{1}{\Gamma(\alpha)\,\beta^\alpha} y^{\alpha-1} e^{-y/\beta} \text{ for } y \geq 0$$

Therefore,

$$\begin{aligned} E(Y^k) &= \int_0^\infty \frac{1}{\Gamma(\alpha)\,\beta^\alpha} (y^k) y^{\alpha-1} e^{-y/\beta} dy \\ &= \frac{1}{\Gamma(\alpha)\,\beta^\alpha} \int_0^\infty y^{(\alpha+k)-1} e^{-y/\beta} dy \end{aligned}$$

Using u substitution,

$$u = \frac{y}{\beta} \implies du = \frac{1}{\beta} dy$$
$$y = \beta u \implies dy = \beta du$$

So

$$\begin{aligned} E(Y^k) &= \frac{1}{\Gamma(\alpha)\,\beta^\alpha} \int_0^\infty (\beta u)^{(\alpha+k)-1} e^{-u} (\beta) du \\ &= \frac{\beta^{(\alpha+k)-1}\beta}{\Gamma(\alpha)\,\beta^\alpha} \int_0^\infty u^{(\alpha+k)-1} e^{-u} du \\ &= \frac{\beta^k}{\Gamma(\alpha)} \Gamma(\alpha + k) \end{aligned}$$

b)

if $k = 1$,

$$\begin{aligned} E(Y^1) &= \frac{\beta^1\ \Gamma(\alpha+1)}{\Gamma(\alpha)} \\ &= \frac{\beta\alpha!}{(\alpha-1)!} \\ &= \frac{\beta\alpha(\alpha-1)!}{(\alpha-1)!} \\ &= \alpha\beta \end{aligned}$$

c)

Note that

$$E\left(\frac{1}{Y}\right) = E(Y^{-1}) \implies k = -1$$

So,

$$\begin{aligned} E\left(\frac{1}{Y}\right) &= \frac{\beta^{-1}}{\Gamma(\alpha)}\Gamma(\alpha-1) \\ &= \frac{1}{\beta\,(\alpha-1)!}(\alpha-2)! \\ &= \frac{1}{\beta(\alpha-1)(\alpha-2)!}(\alpha-2)! \\ &= \frac{1}{\beta(\alpha-1)} \end{aligned}$$

4.

If Y_1 and Y_2 are two uncorrelated random variables, then the covariance between them must be zero. Let's list the following for our reference:

$$\begin{aligned} &(i)\ \text{Cov}(Y_1, Y_2) = 0 \\ &(ii)\ E(Y_i) = \mu_i \text{ and } Var(Y_i) = \sigma_i^2 \text{ for } i = 1, 2 \\ &(iii)\ U = Y_1 + Y_2 \\ &(iv)\ W = Y_1 - Y_2 \\ &(vi)\ Var(U) = Var(Y_1) + Var(Y_2) \\ &(vii)\ Var(W) = Var(Y_1) + Var(Y_2) \end{aligned}$$

a)

$$
\begin{aligned}
Cov(U,W) &= E\left(\left[U-E(U)\right]\left[W-E(W)\right]\right)\\
&= E\left([Y_1+Y_2-(\mu_1+\mu_2)][Y_1-Y_2-(\mu_1-\mu_2)]\right)\\
&= E\left[(Y_1+Y_2-\mu_1-\mu_2)(Y_1-Y_2-\mu_1+\mu_2)\right]\\
&= E(\mu_1^2-2\mu_1Y_1-\mu_2^2+2\mu_2Y_2+Y_1^2-Y_2^2)\\
&= \mu_1^2-2\mu_1E(Y_1)-\mu_2+2\mu_2E(Y_2)+E(Y_1^2)-E(Y_2^2)\\
&= \mu_1^2-2\mu_1^2-\mu_2+2\mu_2^2+E(Y_1^2)-E(Y_2^2)\\
&= E(Y_1^2)-\mu_1^2-E(Y_2^2)+\mu_2^2\\
&= E(Y_1^2)-\mu_1^2-[E(Y_2^2)-\mu_2^2]\\
&= Var(Y_1)-Var(Y_2)
\end{aligned}
$$

b)

$$
\begin{aligned}
\rho_{U,W} &= \frac{Cov(U,W)}{\sqrt{Var(U)Var(W)}}\\
&= \frac{Var(Y_1)-Var(Y_2)}{\sqrt{[Var(Y_1)+Var(Y_2)][Var(Y_1)+Var(Y_2)]}}\\
&= \frac{Var(Y_1)-Var(Y_2)}{Var(Y_1)+Var(Y_2)}
\end{aligned}
$$

c)

$$\rho_{U,W} < 0 \implies \frac{Var(Y_1)-Var(Y_2)}{Var(Y_1)+Var(Y_2)} < 0$$

Multiplying both sides of the inequality by the denominator,

$$
\begin{gathered}
Var(Y_1)-Var(Y_2) < 0\\
Var(Y_1) < Var(Y_2)
\end{gathered}
$$

So when $Var(Y_1) < Var(Y_2)$, or when the standard deviation of Y_1 is less than the standard deviation of Y_2, the correlation coefficient $\rho_{U,W} < 0$. The correlation coefficient $\rho_{U,W}$ measures the strength of linearity between U and W.

4.

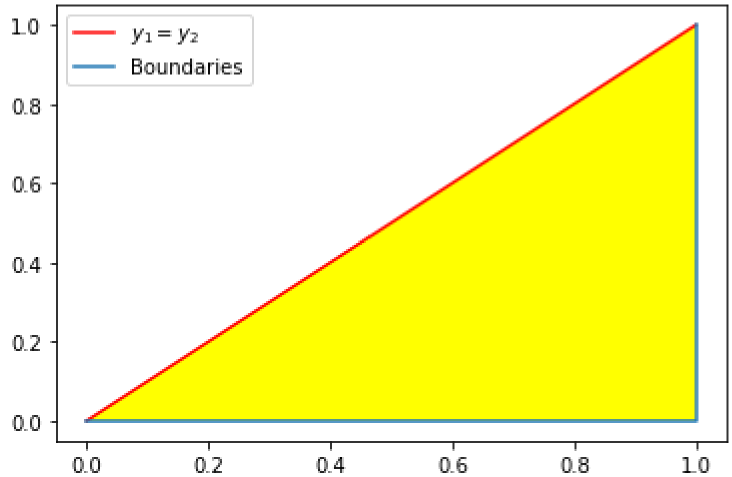
Let Y_1 and Y_2 have the joint pdf given by

$$f(y_1,y_2) = \begin{cases} k(1-y_2) & 0 \le y_2 \le y_1 \le 1\\ 0 & \text{elsewhere}\end{cases}$$

In [8]:

```
import matplotlib.pyplot as plt
%matplotlib inline
import numpy as np

f = lambda x: x
x, y = [0, 1], [0, 1]
plt.plot(x, y, 'r', label = '$ y_1 = y_2 $')
plt.plot([0, 1, 1], [0, 0, 1], label = 'Boundaries')
plt.fill_between([0, 1], [0, 1], facecolor = 'yellow')
plt.legend()
plt.show()
```



The yellow area is the overall region of integration. We always have to integrate within this area at all times and cannot go out of it.

a)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1, y_2) dy_1 dy_2 = 1$$

This means that

$$\begin{aligned}
\int_0^1 \int_0^{y_1} k(1 - y_2) dy_2 dy_1 &= k \int_0^1 \int_0^{y_1} (1 - y_2) dy_2 dy_1 \\
&= k \int_0^1 \left(y_2 - \frac{y_2^2}{2} \right) \Big|_{y_2=0}^{y_2=y_1} dy_1 \\
&= k \int_0^1 \left(y_1 - \frac{1}{2} y_1^2 \right) dy_1 \\
&= \frac{k}{3}
\end{aligned}$$

Setting this equal to 1 and solving for k, we get

$$\frac{k}{3} = 1 \implies k = 3$$

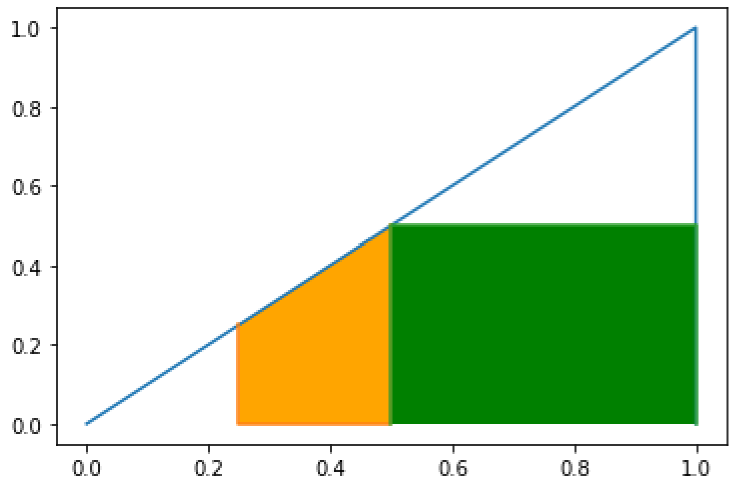
so our pdf becomes

$$f(y_1, y_2) = \begin{cases} 3(1 - y_2) & 0 \le y_2 \le y_1 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

b)

In [20]:

```
plt.plot([0, 1, 1], [0, 1, 0])
plt.plot([1/4, 1/4, 1/2, 1/2], [1/4, 0, 0, 1/2])
plt.fill_between([1/4, 1/2], [1/4, 1/2], facecolor = 'orange')
plt.plot([1, 1, 1/2, 1/2], [0, 1/2, 1/2, 0])
plt.fill_between([1/2, 1], [1/2, 1/2], facecolor = 'green')
plt.show()
```



We have to find

$$P\left(Y_1 \ge \frac{1}{4}, Y_2 \le \frac{1}{2}\right) = P(\text{Orange}) + P(\text{Green})$$

$$P(\text{Orange}) = \int_{1/4}^{1/2} \int_0^{y_1} 3(1 - y_2) dy_2 \, dy_1$$

$$= 3 \int_{1/4}^{1/2} \left(y_2 - \frac{1}{2} y_2^2\right)\Big|_{y_2=0}^{y_2=y_1} dy_1$$

$$= 3 \int_{1/4}^{1/2} \left(y_1 - \frac{1}{2} y_1^2\right) dy_1$$

$$= \frac{29}{128}$$

$$P(\text{Green}) = \int_0^{1/2} \int_{1/2}^{1} 3(1 - y_2) dy_1 \, dy_2$$

$$= \frac{9}{16}$$

So

$$P\left(Y_1 \geq \frac{1}{4}, Y_2 \leq \frac{1}{2}\right) = \frac{29}{128} + \frac{9}{16} = \frac{101}{128}$$

5.

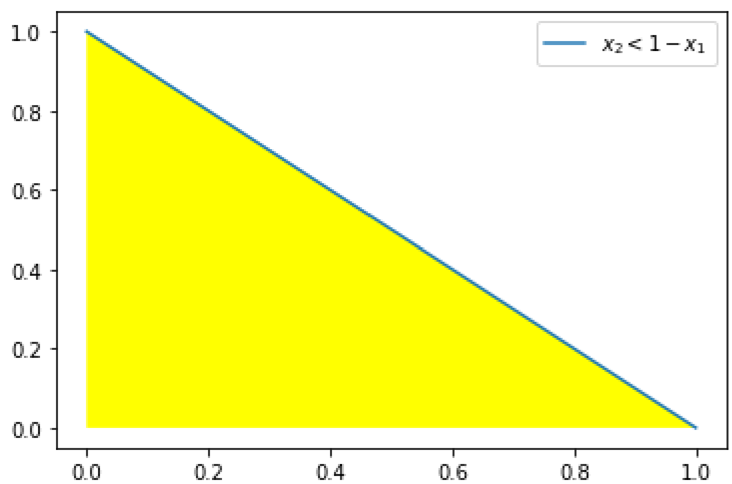
Our joint density function in this problem is

$$f(x_1, x_2) = \begin{cases} 2 & 0 < x_1 + x_2 < 1, x_1 > 0, x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

a)

In [23]:

```
x, y = [0, 1], [1, 0]
plt.plot(x, y, label = '$ x_2 < 1 - x_1 $')
plt.fill_between([0, 1], [1, 0], facecolor = 'yellow')
plt.legend()
plt.show()
```



Our limits of integration are restricted to the first quadrant, particularly the yellow area. We can only integrate our joint density function within the yellow area. The conditional mean of X_1 given $X_2 = x_2$ is therefore

$$\begin{aligned} E(X_1|X_2 = x_2) &= \int_0^{1-x_2} x_1 f(x_1|x_2) dx_1 \\ &= \int_0^{1-x_2} x_1 \frac{f(x_1, x_2)}{f_2(x_2)} dx_1 \end{aligned}$$

$$\begin{aligned} f_2(x_2) &= \int_0^{1-x_2} f(x_1, x_2) dx_1 \\ &= \int_0^{1-x_2} 2 dx_1 \\ &= 2(1 - x_2) \end{aligned}$$

So the marginal pdf of X_2 is

$$f_2(x_2) = \begin{cases} 2(1-x_2) & x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned}
E(X_1|X_2 = x_2) &= \int_0^{1-x_2} x_1 \frac{2}{2(1-x_2)} dx_1 \\
&= \int_0^{1-x_2} \frac{x_1}{1-x_2} dx_1 \\
&= \frac{1}{1-x_2} \int_0^{1-x_2} x_1 dx_1 \\
&= \frac{1}{2(1-x_2)} x_1^2 \Big|_{x_1=0}^{x_1=1-x_2} \\
&= \frac{1}{2(1-x_2)} (1-x_2)^2 \\
&= \frac{1}{2}(1-x_2)
\end{aligned}$$

b)

We need to find

$$V(X_1|X_2 = x_2) = E(X_1^2|X_2 = x_2) - [E(X_1|X_2 = x_2)]^2$$

The integration is straightforward by now. I am tired so I am gonna be skipping some steps. It's just a matter of substitution from here once we have everything we need.

$$\begin{aligned}
E(X_1^2|X_2 = x_2) &= \int_0^{1-x_2} x_1^2 f(x_1|x_2) dx_1 \\
&= \frac{1}{1-x_2} \int_0^{1-x_2} x_1^2 dx_1 \\
&= \frac{1}{3}(x_2-1)^2
\end{aligned}$$

$$V(X_1|X_2 = x_2) = \frac{1}{3}(x_2-1)^2 - \frac{1}{4}(1-x_2)^2 = \frac{1}{12}(x_2-1)^2$$

c)

$$
\begin{aligned}
E[V(X_1|X_2=x_2)] &= E\left(\frac{1}{12}(X_2-1)^2\right) \\
&= \int_0^1 \int_0^{1-x_1} \frac{1}{12}(x_2-1)^2 f(x_1,x_2)dx_2dx_1 \\
&= \frac{1}{6}\int_0^1 \int_0^{1-x_1} (x_2-1)^2 dx_2dx_1 \\
&= \frac{1}{6}\int_0^1 \int_0^{1-x_1} (x_2^2-2x_2+1) dx_2dx_1 \\
&= \frac{1}{6}\int_0^1 \left(\frac{1}{3}x_2^3 - x_2^2 + x_2\right)\Bigg|_{x_2=0}^{x_2=1-x_1} dx_1 \\
&= \frac{1}{6}\int_0^1 (1-x_1)\left[\frac{1}{3}(1-x_1)^2 - (1-x_1) + 1\right]dx_1
\end{aligned}
$$

This will simplify to

$$
= \frac{1}{24}
$$

d)

Objective 1: Find $V[E(X_1|X_2)]$

$$
V[E(X_1|X_2)] = E\left[\left(E(X_1|X_2)\right)^2\right] - \left(E[E(X_1|X_2)]\right)^2
$$

$$
\begin{aligned}
E\left[\left(E(X_1|X_2)\right)^2\right] &= E\left[\left(\frac{1}{2}(1-x_2)\right)^2\right] \\
&= E\left[\frac{1}{4}(1-x_2)^2\right] \\
&= \int_0^1 \int_0^{1-x_1} \frac{1}{2}(1-x_2)^2 dx_2dx_1 \\
&= \frac{1}{2}\int_0^1 \int_0^{1-x_1} (1-2x_2+x_2^2)dx_2dx_1 \\
&= \frac{1}{2}\int_0^1 (x_2 - x_2^2 + \frac{1}{3}x_2^3)\Bigg|_{x_2=0}^{x_2=1-x_1} dx_1 \\
&= \frac{1}{2}\int_0^1 \left(\frac{1}{3} - \frac{1}{3}x_1^3\right)dx_1 \\
&= \frac{1}{6}\int_0^1 (1-x_1^3)dx_1 = \frac{1}{6}(x_1 - \frac{1}{4}x_1^4)\Bigg|_{x_1=0}^{x_1=1} \\
&= \frac{1}{6}\left(1-\frac{1}{4}\right) \\
&= \frac{1}{8}
\end{aligned}
$$

$$\begin{aligned}
E[E(X_1|X_2)] &= E\left[\frac{1}{2}(1 - x_2)\right] \\
&= \int_0^1 \int_0^{1-x_1} (1 - x_2) dx_2 dx_1 \\
&= \int_0^1 \left(x_2 - \frac{1}{2}x_2^2\right)\Bigg|_{x_2=0}^{x_2=1-x_1} dx_1 \\
&= \int_0^1 \left[(1 - x_2) - \frac{1}{2}(1 - x_1)^2\right] dx_1 \\
&= \int_0^1 \left(\frac{1}{2} - \frac{x_1^2}{2}\right) dx_1 \\
&= \frac{1}{2}\int_0^1 (1 - x_1^2) dx_1 \\
&= \frac{1}{2}\left(x_1 - \frac{1}{3}x_1^3\right)\Bigg|_{x_1=0}^{x_1=1} \\
&= \frac{1}{2}\left(1 - \frac{1}{3}\right) \\
&= \frac{1}{3}
\end{aligned}$$

Therefore, $V[E(X_1|X_2)] = \frac{1}{8} - \left(\frac{1}{3}\right)^2 = \frac{1}{72}$

Objective 2: Find $V(X_1)$

$$V(X_1) = E(X_1^2) - [E(X_1)]^2$$

To make things a little less painful, I am going to make a general rule for the kth moment of X_1, $E(X_1^k)$.

$$\begin{aligned}
E(X_1^k) &= \int\int_A x_1^k f(x_1, x_2)dA \\
&= \int_0^1 \int_0^{1-x_1} 2x_1^k dx_2 dx_1 \\
&= \int_0^1 2x_1^k \int_0^{1-x_1} dx_2 dx_1 \\
&= \int_0^1 2x_1^k x_2 \Big|_{x_2=0}^{x_2=1-x_1} dx_1 \\
&= \int_0^1 2x_1^k (1-x_1) dx_1 \\
&= 2\int_0^1 \left(x_1^k - x_1^{k+1}\right) dx_1 \\
&= 2\left[\frac{x_1^{k+1}}{k+1} - \frac{x_1^{k+2}}{k+2}\right|_{x_1=0}^{x_1=1} \\
&= 2\left[\frac{1}{k+1} - \frac{1}{k+2}\right]
\end{aligned}$$

When $k = 2$,

$$E(X_1^2) = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}$$

When $k = 1$,

$$E(X_1) = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

So the variance of X_1 is

$$V(X_1) = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

Is $V(X_1) = E[V(X_1|X_2)] + V[E(X_1|X_2)]$?

$$E[V(X_1|X_2)] + V[E(X_1|X_2)] = \frac{1}{24} + \frac{1}{72} = \frac{1}{18}$$

So yes, they are equal.