1.

Consider two sequences of sets $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$ in a probability space.

(a) If a sequence if such that $A_k \subset A_{k+1}, k = 1, 2, 3, \ldots, \lim_{k\to\infty} A_k$ is defined as the union $A_1 \cup A_2 \cup A_3 \cup \ldots$ (i) Is the sequence non-decreasing or non-increasing? (ii) Find $\lim_{k\to\infty} A_k$ for

$$A_k = \{x : \frac{1}{k} \leq x \leq 3 - \frac{1}{k}\}, k = 1, 2, 3, \ldots$$

(i) Non-decreasing because set becomes a subset of the following set. In other words, if we keep going, the last set will be large enough to contain all the previous sets; all the sets before the last set become its subsets.

(ii)

$$\begin{aligned}\lim_{k\to\infty} A_k &= A_1 \cup A_2 \cup A_3 \cup \ldots \\ &= \{x : 0 < x < 3\}\end{aligned}$$

(b) If the sequence is such that $B_k \supset B_{k+1}, k = 1, 2, 3, \ldots, \lim_{k\to\infty} B_k$ is defined as the intersection $B_1 \cap B_2 \cap B_3 \cap \ldots$ (i) Is the sequence non-decreasing or non-increasing? (ii) Find $\lim_{k\to\infty} A_k$ for

$$B_k = \{x : 2 - \frac{1}{k} < x \leq 2\}, k = 1, 2, 3, \ldots$$

(i) Non-increasing because the current set is always larger than all the following sets. In other words, the first set is large enough to contain all the sets that follow it.

(ii)

$$\begin{aligned}\lim_{k\to\infty} B_k = B_1 \cap B_2 \cap B_3 \cap \ldots &= \Big(1, 2\Big] \cap \Big(3/2, 2\Big] \cap \Big(5/3, 2\Big] \cap \ldots \\ &= \{x : x = 2\}\end{aligned}$$

2.

Suppose that we have two events A and B in the sample space S.

(b) Suppose that two events A and B are statistically independent. Prove that A^C and B^C are independent or not.

Answer: If A and B are statistically independent, then the following axioms are true:

$$(i) P(A|B) = P(A) \qquad (ii) P(A \cap B) = P(A)P(B)$$

then,

$$
\begin{aligned}
P(A' \cap B') &= P\Big((A \cup B)'\Big) \\
&= 1 - P(A \cup B) \\
&= 1 - \Big(P(A) + P(B) - P(A \cap B)\Big) \\
&= 1 - \Big(P(A) + P(B) - P(A)P(B)\Big) \\
&= 1 - \Big(P(A) - P(A)P(B) + P(B)\Big) \\
&= 1 - P(A) + P(A)P(B) - P(B) \\
&= 1 - P(A) - P(B) + P(A)P(B) \\
&= \Big(1 - P(A)\Big) - P(B)\Big(1 - P(A)\Big) \\
&= \Big(1 - P(A)\Big)\Big(1 - P(B)\Big) \\
&= P(A')P(B')
\end{aligned}
$$

3.

A fleet of nine taxis is to be dispatched to three airports in such a way that three go to airport A, five go to airport B, and one goes to airport C. Assume that taxis are allocated to airports at random.

(a) If exactly one of the taxis is in need of repair, what is the probability that it is dispatched to airport C?

Answer: If we assume equally-likely chance, then the probability of exactly one taxi being dispatched to airport C is

$$P(C) = \frac{1}{9}$$

(b) If exactly three of the taxis are in need of repair, what is the probability that every airport receives one of the taxis requiring repairs?

Answer:

$$
\begin{aligned}
&N(Each\ Group\ Has\ Exactly\ One) = 3\binom{6}{2}\binom{4}{4} = 45 \\
&N(Nine\ Taxis\ Partitioned\ Into\ Three\ Groups) = \frac{9!}{3!5!1!} = 504 \\
&P(Every\ Airport\ Receives\ One) = \frac{45}{504}
\end{aligned}
$$

4.

Suppose that Y has the binomial distribution with parameters (n, p) Define $X = n - Y$.

(a) Using the moment generating function (MGF) of Y, find the MGF of X. (No Derivation required.)

Answer: (Reference): since Y has a binomial distribution, its MGF will be

$$M_y(t) = E\left(e^{tY}\right) = (pe^t + q)^n$$

. Therefore,

$$\begin{aligned} M_x(t) &= E\left(e^{tX}\right) \\ &= E\left(e^{t(n-Y)}\right) \\ &= E\left(e^{tn-tY}\right) \\ &= e^{tn} E\left(e^{-tY}\right) \\ &= e^{tn} M_y(-t) \\ &= e^{tn}\left(pe^{-t} + (1-p)\right)^n \\ &= \left(p + (1-p)e^t\right)^n \end{aligned}$$

$(b)[Undergraduate]$: Find $E(X)$ using the MGF of X in (a).

Answer: Find the derivative of the MGF then evaluate it at $t = 0$.

$$\begin{aligned} M_x'(t) &= \frac{d}{dt} E\left(e^{nt-tY}\right) \\ &= E\left(\frac{d}{dt} e^{nt-tY}\right) \\ &= E\left(e^{nt-tY}(n-Y)\right) \end{aligned}$$

So,

$$
\begin{aligned}
E(X) = M_x'(0) &= E\left(e^{0-0}(n-Y)\right) \\
&= E\left(n-Y\right) \\
&= E(n) - E(Y) \\
&= n - E(Y) \\
&= n - np \\
&= n(1-p) \\
&= nq
\end{aligned}
$$

$(b)[Graduate]$: Find $E(X)$ using the definition of expectation.

Answer: Recall that

$$E(X) = n - E(Y)$$

We need to derive $E(Y)$ then substitute it for the above equation.

$$
\begin{aligned}
E(Y) &= \sum_{y=0}^{n} y \frac{n!}{y!(n-y)!} p^y q^{n-y} \\
&= np \sum_{y=0}^{n} \frac{(n-1)!}{(y-1)!(n-y)!} p^{y-1} q^{n-y} \\
&= np \sum_{y=0}^{n} \frac{(n-1)!}{(y-1)!\left((n-1)-(y-1)\right)!} p^{y-1} q^{(n-1)-(y-1)}
\end{aligned}
$$

Letting $z = y - 1$,

$$
\begin{aligned}
E(Y) = np \sum_{z=0}^{n-1} \frac{(n-1)!}{z!(n-1-z)!} p^z q^{n-1-z} \\
= np(p+q)^{n-1} \\
= np(p+1-p)^{n-1} \\
= np(1)^{n-1} \\
= np
\end{aligned}
$$

because we applied the formula

$$(p+q)^n = \sum_{y=0}^{n} \binom{n}{y} p^y q^{n-y}$$

So,

$$E(X) = n - E(Y) = n - np = n(1-p) = nq$$

5.

Let X be a random variable with MGF given by

$$\frac{pe^t}{1-qe^t}$$

$p + q = 1.$

(a) (i) Identify the distribution of X and (ii) write down the probability function of X, $p(x)$ with specifying (iii) the domain of X, and (iv) the parameter space.

Answer:

$$
\begin{aligned}
&(i)\ Geometric\ Distribution\\
&(ii)\ p(x) = p(1-p)^{x-1}\\
&(iii)\ Domain: x = 1, 2, 3, \ldots\\
&(iv)\ Geometric(p);\ 0 < p < 1
\end{aligned}
$$

$(b)[Undergraduate]$: Derive $E(X)$ using the definition of expectation.

Answer: Let the pdf for the Geometric Distribution be

$$p(x) = p(1-p)^{x-1}$$

Then the expected value of X is

$$
\begin{aligned}
E(X) &= \sum_{x=1}^{\infty} xp(1-p)^{x-1}\\
&= p\sum_{x=1}^{\infty} xq^{x-1}\\
&= p\sum_{x=1}^{\infty} \frac{d}{dq}(q^x)\\
&= p\frac{d}{dq}\sum_{x=1}^{\infty} q^x\\
&= p\frac{d}{dq}\left(\frac{q}{1-q}\right)\\
&= p\left(\frac{1-q+q}{(1-q)^2}\right)\\
&= \frac{p}{(1-q)^2}\\
&= \frac{p}{p^2}\\
&= \frac{1}{p}
\end{aligned}
$$

$(b)[Graduate]$: Derive $Var(X)$ by using the definition of expectation.

Answer: Find $E(X^2)$ first. Recall the formula

$$E\Big[X(X-1)\Big] = E(X^2 - X) = E(X^2) - E(X)$$

We have to find this expectation first because it is easier than finding $E(X^2)$. Therefore,

$$
\begin{aligned}
E\Bigg(X(X-1)\Bigg) &= \sum_{x=1}^{\infty} x(x-1)pq^{x-1} \\
&= p\sum_{x=1}^{\infty} x(x-1)q^{x-1} \\
&= p\frac{d}{dq}\Bigg(\sum_{x=1}^{\infty}(x-1)q^{x}\Bigg) \\
&= p\frac{d}{dq}\Bigg(q^2\sum_{x=1}^{\infty}(x-1)q^{x-2}\Bigg) \\
&= p\frac{d}{dq}\Bigg(q^2\sum_{x=2}^{\infty}(x-1)q^{x-2}\Bigg) \\
&= p\frac{d}{dq}\Bigg(q^2\frac{d}{dq}\sum_{x=2}^{\infty}q^{x-1}\Bigg) \\
&= p\frac{d}{dq}\Bigg(q^2\frac{d}{dq}\sum_{x=1}^{\infty}q^{x}\Bigg) \\
&= p\frac{d}{dq}\Bigg(q^2\frac{d}{dq}\Bigg(\frac{q}{1-q}\Bigg)\Bigg) \\
&= p\frac{d}{dq}\Bigg(q^2\Bigg(\frac{1-q+q}{(1-q)^2}\Bigg)\Bigg) \\
&= p\Bigg(2\Bigg(\frac{q}{1-q}\Bigg)\Bigg(\frac{1-q+q}{(1-q)^2}\Bigg)\Bigg) \\
&= \frac{2pq}{p(p)^2} \\
&= \frac{2q}{p^2} = \frac{2(1-p)}{p^2}
\end{aligned}
$$

So,

$$E\Bigg[X(X-1)\Bigg] = \frac{2(1-p)}{p^2} = E(X^2) - E(x)$$

Solving for $E(X^2)$, we get

$$E(X^2) = \frac{2(1-p)}{p^2} + E(X) = \frac{2-p}{p^2}$$

Therefore,

$$
\begin{aligned}
Var(X) &= E(X^2) - \Bigg(E(X)\Bigg)^2 \\
&= \frac{2-p}{p^2} - \frac{1}{p^2} \\
&= \frac{2-p-1}{p^2} \\
&= \frac{1-p}{p^2}
\end{aligned}
$$

6.

Let Y be a random variable with $\mu = 11$ and $\sigma^2 = 9$. Using Tchebysheff's theorem, find

(a) the value of C such that $P(|X - 11| \geq C) \leq 0.09$.

Answer: Given $\mu = 11$, $\sigma = 3$,

$$P(|X - 11| \leq C) \geq 0.91 \implies P(-C \leq X - 11 \leq C) \geq 0.91$$

So

$$k\sigma = C \implies k = \frac{C}{\sigma} = \frac{C}{3}$$

Next, we get the following:

$$0.91 = 1 - \frac{1}{(C^2/9)} = 1 - \frac{9}{C^2} \implies 0.91C^2 = C^2 - 9$$

Solving this for C, we should get

$$C = 10$$

and therefore

$$k = \frac{C}{3} = \frac{10}{3}$$

(b) an interval that gives the lower bound 64%.

Answer: We want

$$1 - \frac{1}{k^2} = 0.64$$

Solving this for k, we get

$$k = \frac{10}{6}$$

and finally

$$Interval = (11 - 3k, 11 + 3k) = (6, 16)$$

7.

Explain the following explicitly.

(a) Probability space and power set of $S = \{\omega_1, \omega_2, \omega_3\}$.

Answer: The probability space is $(S, \mathcal{F}, P)$ where S is the sample space, $\mathcal{F}$ is the family of all possible subsets, and P is the probability(measure). The power set of $S = \{\omega_1, \omega_2, \omega_3\}$ is a set containing all the combinations of elements of S and whose number of elements is equal to $2^3 = 8$.

(b) Poisson process (or counting process).

Answer: Suppose $N(t)$ is the total number of events that have occurred up to time t. Then we have

$$P\{N(t+s)-N(s)\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad for\ n = 0, 1, 2, \ldots$$

where

$$E[N(t)] = \lambda t$$

and $N(0) = 0$. The Poisson distribution gives the probability of the number of events in an interval generated by a Poisson process.

(c) Conditions for being Poisson from the binomial distribution.

Answer: Bin(n, p) $\implies$ Poisson(λ)

$$Condition\ 1: \ n \to \infty$$
$$Condition\ 2: \ P\ becomes\ very\ small (P << 1)$$