6.3 The Method of Distribution Functions

Example 2

Suppose that $X \sim N(\mu, \sigma^2)$. Consider $Y = aX + b$. Then

$$V(Y) = a^2 V(X)$$

and

$$E(Y) = aE(X) + b$$

Consider

$$\begin{aligned} F_y(y) &= P(Y \leq y) = P(aX + b \leq y) \\ &= P\left(X \leq \frac{y-b}{a}\right) \\ &= F_x\left(\frac{y-b}{a}\right) \end{aligned}$$

Differentiate both sides with respect to y.

$$\begin{aligned} \frac{dF_Y(y)}{dy} &= \frac{d}{dy} F_x\left(\frac{y-b}{a}\right) \\ \implies \quad f_Y(y) &= \frac{1}{a} f_x\left(\frac{y-b}{a}\right) \end{aligned}$$

Notice

$$\begin{aligned} f(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \\ \implies \quad f_Y(y) &= \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-(b-a\mu)}{a\sigma}\right)^2} \\ \implies \quad & N(a\mu + b, a^2\sigma^2) \end{aligned}$$

Proposition: If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$, then $Y \sim N(a\mu + b, a^2 b^2)$.

Example 3

Suppose $U \sim \text{Uniform}(0, 1)$, $u \in (0, 1)$. Consider

$$V = \frac{1}{u}$$

. Then $f_V(v) = ?$ Note that $u \in (0, 1)$, $v \in (0, 1)$. Also,

$$F(u) = u \implies f(u) = 1$$

Now consider

$$
\begin{aligned}
F_V(v) &= P(V \leq v) \\
&= P\left(\frac{1}{U} \leq v\right) \\
&= P\left(U \geq \frac{1}{v}\right) \\
&= 1 - P\left(U \leq \frac{1}{v}\right) \\
&= 1 - F_U\left(\frac{1}{v}\right) \\
&= 1 - \frac{1}{v}
\end{aligned}
$$

So

$$
F_V(v) = \begin{cases} \frac{1}{v^2} & 1 \leq v \leq \infty \\ 0 & \text{elsewhere} \end{cases}
$$

Example 4

Suppose X is discrete. The PMF of X is

In [5]:

```
pd.DataFrame(D1)
```

Out[5]:

	x	p(x)
0	-1	0.2
1	0	0.4
2	1	0.4

In [7]:

```
pd.DataFrame(D2)
```

Out[7]:

	y	p(y)
0	-1	0.4
1	0	0.6

Not 1 to 1 but 2 to 1.

Example 5

Let

$$
f(y) = \begin{cases} \frac{y+1}{2} & -1 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}
$$

Find the PDF of $X = Y^2$. Solution: This is not 1 to 1 but 2 to 1.

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(Y^2 \leq x) \\ &= P(-\sqrt{x} \leq Y \leq \sqrt{x}) \end{aligned}$$

Since $f_X(x) = \frac{d}{dx} F_X(x)$,

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}}[f_Y(-\sqrt{x}) + f_Y(\sqrt{x})] \\ 0 \text{ elsewhere} \end{cases}$$