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MATH 473
Test 2A

1.

(a)

Let $f_1(x)$ be the pdf for the washers and $f_2(x)$ be the pdf for the dryers. Let X be the repair cost random variable. Let the weights or probabilities of each individual machines be

$$w_1 = 0.60$$
$$w_2 = 0.40$$

Define the both the pdf to be

$$f_1(x) = \begin{cases} \frac{1}{500} & 0 \le x \le 500 \\ 0 & \text{elsewhere} \end{cases}$$

and

$$f_2(x) = \begin{cases} \frac{1}{200} e^{-x/200} & x \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$

because for the exponential distribution,

$$f_2(x) = \begin{cases} \frac{1}{\mu} e^{-x/\mu} & x \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $\mu = 200$. Let X be the random variable for the repair cost. Then the pdf for the repair cost is

$$f_X(x) = w_1 f_1(x) + w_2 f_2(x) = \frac{3}{2500} + \frac{1}{1500} e^{-x/200}$$

It will be easy if we can find a general formula for the kth moment of X which we will call $E(X^k)$. Therefore,

$$\begin{aligned} E(X^k) &= \int_0^{500} \frac{3}{2500} x^k dx + \int_0^{\infty} \frac{1}{500} x^k e^{-x/200} dx \\ &= A + B \end{aligned}$$

$$\begin{aligned} A &= \int_0^{500} \frac{3}{2500} x^k dx = \frac{3}{2500} * \frac{x^{k+1}}{k+1} \Bigg|_{x=0}^{x=500} \\ &= \frac{3}{2500} \frac{(500)^{k+1}}{k+1} \\ &= \frac{3(500)^{k+1}}{2500(k+1)} \end{aligned}$$

$B = \int_0^{\infty} \frac{1}{500} x^k e^{-x/200} dx$. Turn this into a Gamma function by u-substitution.

$$\begin{aligned} u = \frac{x}{200} &\implies du = \frac{1}{200} dx \\ x = 200u &\implies dx = 200du \end{aligned}$$

$$\begin{aligned} B &= \int_0^{\infty} \frac{1}{500} (200u)^k e^{-u} (200) du \\ &= \frac{1}{500} (200)(200)^k \int_0^{\infty} u^k e^{-u} du \\ &= \frac{(200)^{k+1}}{500} \Gamma(k+1) \\ &= \frac{(200)^{k+1}}{500} k! \end{aligned}$$

Therefore, $E(X^k) = A + B = \dfrac{3(500)^{k+1}}{2500(k+1)} + \dfrac{(200)^{k+1}}{500}k!$

When $k = 1$,

$$\mu_x = E(X^1) = 230$$

$$\implies Var(X) = E(X^2) - [E(X)]^2 = 29{,}100$$

(b)

$$P(X < 300) = \int_0^{300} f_X(x)dx = \int_0^{300} \left(\frac{3}{2500} + \frac{1}{500}e^{-x/200}\right)dx$$
$$= 0.670747935941$$

(c) Use part (a) and part (b). Let $P(\text{Claim}) = 0.03$.

$\implies$

$$\begin{aligned} E(\text{Loss Per Machine}) &= P(\text{Claim})P(X < 300)E(X) \\ &= (0.03)(0.670747935941)(230) \\ &= 4.62816075799 \end{aligned}$$

$$\begin{aligned} V(\text{Loss Per Machine}) &= P(\text{Claim})P(X < 300)Var(X) \\ &= (0.03)(0.670747935941)(29100) \\ &= 585.562948076 \end{aligned}$$

2. We are being asked to find $P(X \geq 80 | X > 65)$. Also, we are given the following:

$(i)\ \sigma_1 = 6$
$(ii)\ \sigma_2 = 8$
$(iii)\ \mu = 70$
$(iv)\ \sigma^2 = \sigma_1^2 + \sigma_2^2 = 36 + 64 = 100$
$\implies \sigma = 10$

Therefore,

$$P(X \geq 80 \,|\, X > 65) = \frac{P\left(\frac{X-70}{10} \geq \frac{80-70}{10}\right)}{P\left(\frac{X-70}{10} > \frac{65-70}{10}\right)}$$
$$= \frac{P(Z \geq 1)}{P(Z > -0.5)}$$
$$= \frac{1 - P(Z < 1)}{1 - P(Z \leq -0.5)}$$
$$= 0.2294488$$

3. Let X be the severity distribution random variable with mean 1000. That is, we are given the following

$$E(X) = 1000$$
$$d = 250$$

The pdf of the uniform distribution is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$$

For these problems, we assume $a = 0$. Thus,

$$E(X) = \frac{a+b}{2} = 1000 \implies b = 2000$$

$$\implies f_X(x) = \begin{cases} \frac{1}{2000} & 0 \le x \le 2000 \\ 0 & \text{elsewhere} \end{cases}$$

As usual, the random variable Y is defined as

$$Y = (X-d)_+ = \begin{cases} 0 & x \le d \\ X-d & x > d \end{cases} = \begin{cases} 0 & x \le 250 \\ X-250 & x > 250 \end{cases}$$

$\implies$

$$\begin{aligned} E(Y) = E[(X-d)_+] &= \int_d^\infty (x-d) f_X(x)dx \\ &= \int_{250}^{2000} (x-250)\frac{1}{2000}dx \\ &= \frac{1}{2000}\int_{250}^{2000}(x-250)dx \\ &= \frac{6125}{8} \end{aligned}$$

and

$$\begin{aligned} E(Y^2) &= \int_d^\infty (x-d)^2 f_X(x)dx \\ &= \int_{250}^{2000} (x-250)^2\frac{1}{2000}dx \\ &= \frac{1}{2000}\int_{250}^{2000}(x-250)^2dx \\ &= \frac{5359375}{6} \end{aligned}$$

Thus,

$$V(Y) = E(Y^2) - [E(Y)]^2 = \frac{58953125}{192} = 307047.526042$$

4.

(a)

We are given the following:

$$(i)\ S_X(x) = \left(\frac{\theta}{x+\theta}\right)^{\alpha}$$

$$(ii)\ E[(X-d)_+] = \frac{\theta^{\alpha}}{(\alpha-1)(d+\theta)^{\alpha-1}}$$

Therefore,

$$\begin{aligned}
E(X-d\,|\,X>d) &= \frac{E[(X-d)_+]}{P(X>d)} = \frac{E(X)-E(X\wedge d)}{1-F_X(d)} \\
&= \frac{E(X)-E(X\wedge d)}{S_X(d)} = \frac{E[(X-d)_+]}{S_X(d)} \\
&= \frac{\frac{\theta^{\alpha}}{(\alpha-1)(d+\theta)^{\alpha-1}}}{\left(\frac{\theta}{d+\theta}\right)^{\alpha}} = \frac{\frac{\theta^{\alpha}}{(\alpha-1)(d+\theta)^{\alpha-1}}}{\frac{\theta^{\alpha}}{(d+\theta)^{\alpha}}} \\
&= \frac{\theta^{\alpha}}{(\alpha-1)(d+\theta)^{\alpha-1}} * \frac{(d+\theta)^{\alpha}}{\theta^{\alpha}} \\
&= \frac{(d+\theta)^{\alpha}}{(\alpha-1)(d+\theta)^{\alpha}(d+\theta)^{-1}} \\
&= \frac{d+\theta}{\alpha-1}
\end{aligned}$$

(b)

We are given the following

$$E[X - 150 | X > 150] = \frac{7}{4} E[X - 75 | X > 75]$$

Just plug things into the formula that we found in part (a) starting with the left side.

$$E[X - 150 | X > 150] = \frac{150 + \theta}{\alpha - 1}$$

$$E[X - 75 | X > 75] = \frac{75 + \theta}{\alpha - 1}$$

Plugging these into the formula above, we get

$$\frac{150 + \theta}{\alpha - 1} = \frac{7}{4}\left(\frac{75 + \theta}{\alpha - 1}\right)$$

Solving this for θ, we get $\theta = 25$.

5. The formula for the $Loss\ Elimination\ Ratio (LER)$ is

$$LER = \frac{E[X \wedge d]}{E[X]}$$

We are also given the following:

$$\mu = E(X) = 1000$$
$$d = 250$$

Note that

$$E[X \wedge d] = \int_0^d S_X(x)dx = \int_0^d [1 - F_X(x)]dx$$

where

$$f_X(x) = \begin{cases} \frac{1}{\mu} e^{-x/\mu} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

and

$$F_X(x) = \int_0^x f_X(t)dt = \int_0^x \frac{1}{\mu} e^{-t/\mu} dt = 1 - e^{-x/\mu}$$

$$\implies S_X(x) = 1 - F_X(x) = e^{-x/\mu}$$

$$\implies E[X \wedge d] = \int_0^d e^{-x/\mu} dx = \mu(1 - e^{-d/\mu})$$

Therefore,

$$\begin{aligned} LER &= \frac{\mu(1 - e^{-d/\mu})}{E[X]} = \frac{\mu(1 - e^{-d/\mu})}{\mu} \\ &= \frac{1000(1 - e^{-250/1000})}{1000} = 1 - e^{-1/4} \\ &= 0.221199 \end{aligned}$$

6. Let α be the coinsurance factor. Let $Payment_{max}$ be the maximum payment for each loss, d be the deductible, and u be the maximum covered loss. They are given to be

$$\begin{aligned} &(i)\ \alpha = 0.80 \\ &(ii)\ d = 500 \\ &(iii)\ Payment_{max} = 23600 \end{aligned}$$

We are being asked to find

$$E[Y|X > d] = \frac{E[Y]}{1 - F_X(d)} = \frac{E[Y]}{1 - F_X(500)}$$

Y is the random variable of interest and it is defined as

$$Y = \alpha[(X \wedge u) - (X \wedge d)] = \alpha[(X \wedge u) - (X \wedge 500)]$$

We have to find u, the maximum covered loss. The payment is

$$Payment = \alpha(x - d) = 0.80(x - 500)$$

We need $Payment = Payment_{max}$ and solve this equation for x. We then get a special value of x, which is the maximum covered loss u. So

$$0.80(x - 500) = 23600 \implies u = 30000$$

Substitute u and α above

$$Y = 0.80[(X \wedge 30000) - (X \wedge 500)]$$

Take the expected value of both sides

$$E[Y] = 0.80[E(X \wedge 30000) - E(X \wedge 500)]$$

Recall the following formula

$$E[X \wedge v] = \int_0^v S_X(x)dx = \int_0^v [1 - F_X(x)]dx$$

where v is just a constant. We need to find $F_X(x)$. Note that the severity random variable X is uniformly distributed on the interval $(0,\ 32000)$. This means the pdf of X is

$$f_X(x) = \begin{cases} \frac{1}{32000} & 0 \le x \le 32000 \\ 0 & \text{elsewhere} \end{cases}$$

$$\implies F_X(x) = \int_0^x f_X(t)dt = \int_0^x \frac{1}{32000}dt = \frac{x}{32000}$$

$$\implies S_X(x) = 1 - \frac{x}{32000}$$

$$E(X \wedge 30000) = \int_0^{30000} \left(1 - \frac{x}{32000}\right)dx = \frac{31875}{2}$$

$$E(X \wedge 500) = \int_0^{500} \left(1 - \frac{x}{32000}\right)dx = \frac{15875}{32}$$

$$\implies E(Y) = \frac{98825}{8} = 12353.125$$

Finally

$$E[Y|X > 500] = \frac{E[Y]}{1 - F_X(500)} = \frac{12353.125}{1 - \frac{500}{32000}} = 12549.2063492$$

7. If the deductible $d = 3$ is to be replaced with a coinsurance factor θ and we want $E(Y)$ to be unchanged, then we are looking at the following

$$E(Y) = \theta E(X) \implies \theta = \frac{E(Y)}{E(X)}$$

where

$$E(X) = \lambda = 5$$

The pdf of the Poisson distribution is

$$p_X(x) = \frac{e^{-\lambda}(\lambda)^x}{x!} = \frac{e^{-5}(5)^x}{x!} \text{ for } x = 0,1,2,...$$

Recall the definition of the random variable Y to be

$$Y = (X-d)_+ = \begin{cases} 0 & x \le d \\ X-d & x > d \end{cases} = \begin{cases} 0 & x \le 3 \\ x-3 & x > 3 \end{cases}$$

We have to find a way to express $E(Y)$ in terms of $E(X)$ and $p_X(x)$ using pure Algebra.

$$\begin{aligned}
E(Y) = E(X-3) &= \sum_{x>d}(x-d)p_X(x) = \sum_{x=4}^{\infty}(x-3)p_X(x) \\
&= 1p(4) + 2p(5) + 3p(6) + \ldots \\
&= 1p(1) + 2p(2) + 3p(3) + 4p(4) + \ldots [-1p(1) - 2p(2) - 3p(3) - 3p(4) - \ldots] \\
&= \sum_{x=1}^{\infty} xp_X(x) - p(1) - 2p(2) - 3[p(3) + p(4) + p(5) + \ldots] \\
&= E(X) - p(1) - 2p(2) - 3\left[\sum_{x=3}^{\infty} p_X(x)\right] \\
&= 5 - p(1) - 2p(2) - 3[1 - P(X<3)] \\
&= 5 - p(1) - 2p(2) - 3\left[1 - \sum_{x=0}^{2} p_X(x)\right] \\
&= 2 + \frac{27}{2}e^{-3}
\end{aligned}$$

Thus,

$$\theta = \frac{E(Y)}{E(X)} = \frac{1}{5}\left(2 + \frac{27}{2}e^{-3}\right) = 0.534425084593$$