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MATH 473
Test 2A

1.

(a)

Let $f_1(x)$ be the pdf for the washers and $f_2(x)$ be the pdf for the dryers.
Also, let the weights be

$$w_1 = 0.60$$

$$w_2 = 0.40$$

Define the both the pdf to be

$$f_1(x) = \begin{cases} \frac{1}{500} & 0 \leq x \leq 500 \\ 0 & \text{elsewhere} \end{cases}$$

and

$$f_2(x) = \begin{cases} \frac{1}{200} e^{-x/200} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

because for the exponential distribution,

$$f_2(x) = \begin{cases} \frac{1}{\mu} e^{-x/\mu} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $\mu = 200$. Let X be the random variable for the repair cost. Then the pdf for the repair cost is

$$f_X(x) = w_1 f_1(x) + w_2 f_2(x) = \frac{3}{2500} + \frac{1}{1500} e^{-x/200}$$

It will be easy if we can find a general formula for the k th moment of X which we will call $E(X^k)$. Therefore,

$$\begin{aligned} E(X^k) &= \int_0^{500} \frac{3}{2500} x^k dx + \int_0^{\infty} \frac{1}{500} x^k e^{-x/200} dx \\ &= A + B \end{aligned}$$

$$\begin{aligned} A &= \int_0^{500} \frac{3}{2500} x^k dx = \frac{3}{2500} * \frac{x^{k+1}}{k+1} \Bigg|_{x=0}^{x=500} \\ &= \frac{3}{2500} \frac{(500)^{k+1}}{k+1} \\ &= \frac{3(500)^{k+1}}{2500(k+1)} \end{aligned}$$

$B = \int_0^{\infty} \frac{1}{500} x^k e^{-x/200} dx$. Turn this into a Gamma function by u-substitution.

$$\begin{aligned} u &= \frac{x}{200} \implies du = \frac{1}{200} dx \\ x &= 200u \implies dx = 200du \end{aligned}$$

$$\begin{aligned} B &= \int_0^{\infty} \frac{1}{500} (200u)^k e^{-u} (200) du \\ &= \frac{1}{500} (200)(200)^k \int_0^{\infty} u^k e^{-u} du \\ &= \frac{(200)^{k+1}}{500} \Gamma(k+1) \\ &= \frac{(200)^{k+1}}{500} k! \end{aligned}$$

$$\text{Therefore, } E(X^k) = A + B = \frac{3(500)^{k+1}}{2500(k+1)} + \frac{(200)^{k+1}}{500}k!$$

When $k = 1$,

$$\mu_x = E(X^1) = 230$$

$$\implies \text{Var}(X) = E(X^2) - [E(X)]^2 = 29,100$$

(b)

$$\begin{aligned} P(X < 300) &= \int_0^{300} f_X(x)dx = \int_0^{300} \left(\frac{3}{2500} + \frac{1}{1500}e^{-x/200} \right) dx \\ &= 0.463583 \end{aligned}$$

2. We are being asked to find $P(X \geq 80 | X > 65)$. Also, we are given the following:

$$(i) \sigma_1 = 6$$

$$(ii) \sigma_2 = 8$$

$$(iii) \mu = 70$$

$$(iv) \sigma^2 = \sigma_1^2 + \sigma_2^2 = 36 + 64 = 100$$

$$\implies \sigma = 10$$

Therefore,

$$\begin{aligned}
P(X \geq 80 | X > 65) &= \frac{P\left(\frac{X-70}{10} \geq \frac{80-70}{10}\right)}{P\left(\frac{X-70}{10} > \frac{65-70}{10}\right)} \\
&= \frac{P(Z \geq 1)}{P(Z > -0.5)} \\
&= \frac{1 - P(Z < 1)}{1 - P(Z \leq -0.5)} \\
&= 0.2294488
\end{aligned}$$

3. Let X be the severity distribution random variable with mean 1000. That is, we are given the following

$$\begin{aligned}
E(X) &= 1000 \\
d &= 250
\end{aligned}$$

The pdf of the uniform distribution is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$$

For these problems, we assume $a = 0$. Thus,

$$E(X) = \frac{a+b}{2} = 1000 \implies b = 2000$$

$$\implies f_X(x) = \begin{cases} \frac{1}{2000} & 0 \leq x \leq 2000 \\ 0 & \text{elsewhere} \end{cases}$$

As usual, the random variable Y is defined as

$$Y = (X - d)_+ = \begin{cases} 0 & x \leq d \\ X - d & x > d \end{cases} = \begin{cases} 0 & x \leq 250 \\ X - 250 & x > 250 \end{cases}$$

\Rightarrow

$$\begin{aligned} E(Y) &= E[(X - d)_+] = \int_d^{\infty} (x - d)f_X(x)dx \\ &= \int_{250}^{2000} (x - 250)\frac{1}{2000}dx \\ &= \frac{1}{2000} \int_{250}^{2000} (x - 250)dx \\ &= \frac{6125}{8} \end{aligned}$$

and

$$\begin{aligned} E(Y^2) &= \int_d^{\infty} (x - d)^2 f_X(x)dx \\ &= \int_{250}^{2000} (x - 250)^2 \frac{1}{2000}dx \\ &= \frac{1}{2000} \int_{250}^{2000} (x - 250)^2 dx \\ &= \frac{5359375}{6} \end{aligned}$$

Thus,

$$V(Y) = E(Y^2) - [E(Y)]^2 = \frac{58953125}{192} = 307047.526042$$

4.

(a)

We are given the following:

$$(i) S_X(x) = \left(\frac{\theta}{x + \theta} \right)^\alpha$$

$$(ii) E[(X - d)_+] = \frac{\theta^\alpha}{(\alpha - 1)(d + \theta)^{\alpha-1}}$$

Therefore,

$$\begin{aligned} E(X - d | X > d) &= \frac{E[(X - d)_+]}{P(X > d)} = \frac{E(X) - E(X \wedge d)}{1 - F_X(d)} \\ &= \frac{E(X) - E(X \wedge d)}{S_X(d)} = \frac{E[(X - d)_+]}{S_X(d)} \\ &= \frac{\frac{\theta^\alpha}{(\alpha - 1)(d + \theta)^{\alpha-1}}}{\left(\frac{\theta}{d + \theta} \right)^\alpha} = \frac{\frac{\theta^\alpha}{(\alpha - 1)(d + \theta)^{\alpha-1}}}{\frac{\theta^\alpha}{(d + \theta)^\alpha}} \\ &= \frac{\theta^\alpha}{(\alpha - 1)(d + \theta)^{\alpha-1}} * \frac{(d + \theta)^\alpha}{\theta^\alpha} \\ &= \frac{(d + \theta)^\alpha}{(\alpha - 1)(d + \theta)^\alpha (d + \theta)^{-1}} \\ &= \frac{d + \theta}{\alpha - 1} \end{aligned}$$

(b)

We are given the following

$$E[X - 150 | X > 150] = \frac{7}{4} E[X - 75 | X > 75]$$

Just plug things into the formula that we found in part (a) starting with the left side.

$$E[X - 150 | X > 150] = \frac{150 + \theta}{\alpha - 1}$$

$$E[X - 75 | X > 75] = \frac{75 + \theta}{\alpha - 1}$$

Plugging these into the formula above, we get

$$\frac{150 + \theta}{\alpha - 1} = \frac{7}{4} \left(\frac{75 + \theta}{\alpha - 1} \right)$$

Solving this for θ , we get $\theta = 25$.

5. The formula for the *Loss Elimination Ratio*(LER) is

$$LER = \frac{E[X \wedge d]}{E[X]}$$

We are also given the following:

$$\begin{aligned}\mu &= E(X) = 1000 \\ d &= 250\end{aligned}$$

Note that

$$E[X \wedge d] = \int_0^d S_X(x) dx = \int_0^d [1 - F_X(x)] dx$$

where

$$f_X(x) = \begin{cases} \frac{1}{\mu} e^{-x/\mu} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

and

$$F_X(x) = \int_0^x f_X(t)dt = \int_0^x \frac{1}{\mu} e^{-t/\mu} dt = 1 - e^{-x/\mu}$$

$$\implies S_X(x) = 1 - F_X(x) = e^{-x/\mu}$$

$$\implies E[X \wedge d] = \int_0^d e^{-x/\mu} dx = \mu(1 - e^{-d/\mu})$$

Therefore,

$$\begin{aligned} LER &= \frac{\mu(1 - e^{-d/\mu})}{E[X]} = \frac{\mu(1 - e^{-d/\mu})}{\mu} \\ &= \frac{1000(1 - e^{-250/1000})}{1000} = 1 - e^{-1/4} \\ &= 0.221199 \end{aligned}$$

6. Let α be the coinsurance factor. Let $Payment_{max}$ be the maximum payment for each loss, d be the deductible, and u be the maximum covered loss. They are given to be

$$(i) \alpha = 0.80$$

$$(ii) d = 500$$

$$(iii) Payment_{max} = 23600$$

We are being asked to find

$$E[Y | X > d] = \frac{E[Y]}{1 - F_X(d)} = \frac{E[Y]}{1 - F_X(500)}$$

Y is the random variable of interest and it is defined as

$$Y = \alpha[(X \wedge u) - (X \wedge d)] = \alpha[(X \wedge u) - (X \wedge 500)]$$

We have to find u , the maximum covered loss. The payment is

$$Payment = \alpha(x - d) = 0.80(x - 500)$$

We need $Payment = Payment_{max}$ and solve this equation for x . We then get a special value of x , which is the maximum covered loss u . So

$$0.80(x - 500) = 23600 \implies u = 30000$$

Substitute u and α above

$$Y = 0.80[(X \wedge 30000) - (X \wedge 500)]$$

Take the expected value of both sides

$$E[Y] = 0.80[E(X \wedge 30000) - E(X \wedge 500)]$$

Recall the following formula

$$E[X \wedge v] = \int_0^v S_X(x)dx = \int_0^v [1 - F_X(x)]dx$$

where v is just a constant. We need to find $F_X(x)$. Note that the severity random variable X is uniformly distributed on the interval $(0, 32000)$. This means the pdf of X is

$$f_X(x) = \begin{cases} \frac{1}{32000} & 0 \leq x \leq 32000 \\ 0 & \text{elsewhere} \end{cases}$$

$$\implies F_X(x) = \int_0^x f_X(t)dt = \int_0^x \frac{1}{32000}dt = \frac{x}{32000}$$

$$\implies S_X(x) = 1 - \frac{x}{32000}$$

$$E(X \wedge 30000) = \int_0^{30000} \left(1 - \frac{x}{32000}\right)dx = \frac{31875}{2}$$

$$E(X \wedge 500) = \int_0^{500} \left(1 - \frac{x}{32000}\right) dx = \frac{15875}{32}$$

$$\implies E(Y) = \frac{98825}{8} = 12353.125$$

Finally

$$E[Y | X > 500] = \frac{E[Y]}{1 - F_X(500)} = \frac{12353.125}{1 - \frac{500}{32000}} = 12549.2063492$$

7. If the deductible $d = 3$ is to be replaced with a coinsurance factor θ and we want $E(Y)$ to be unchanged, then we are looking at the following

$$E(Y) = \theta E(X) \implies \theta = \frac{E(Y)}{E(X)}$$

where

$$E(X) = \lambda = 5$$

The pdf of the Poisson distribution is

$$p_X(x) = \frac{e^{-\lambda}(\lambda)^x}{x!} = \frac{e^{-5}(5)^x}{x!} \text{ for } x = 0, 1, 2, \dots$$

Recall the definition of the random variable Y to be

$$Y = (X - d)_+ = \begin{cases} 0 & x \leq d \\ X - d & x > d \end{cases} = \begin{cases} 0 & x \leq 3 \\ x - 3 & x > 3 \end{cases}$$

We have to find a way to express $E(Y)$ in terms of $E(X)$ and $p_X(x)$ using pure Algebra.

$$\begin{aligned}
E(Y) &= E(X - 3) = \sum_{x>d} (x - d)p_X(x) = \sum_{x=4}^{\infty} (x - 3)p_X(x) \\
&= 1p(4) + 2p(5) + 3p(6) + \dots \\
&= 1p(1) + 2p(2) + 3p(3) + 4p(4) + \dots [-1p(1) - 2p(2) - 3p(3) - 3p(4) - \dots] \\
&= \sum_{x=1}^{\infty} xp_X(x) - p(1) - 2p(2) - 3[p(3) + p(4) + p(5) + \dots] \\
&= E(X) - p(1) - 2p(2) - 3\left[\sum_{x=3}^{\infty} p_X(x)\right] \\
&= 5 - p(1) - 2p(2) - 3[1 - P(X < 3)] \\
&= 5 - p(1) - 2p(2) - 3\left[1 - \sum_{x=0}^2 p_X(x)\right] \\
&= 2 + \frac{27}{2}e^{-3}
\end{aligned}$$

Thus,

$$\theta = \frac{E(Y)}{E(X)} = \frac{1}{5}\left(2 + \frac{27}{2}e^{-3}\right) = 0.534425084593$$