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STAT 467
Homework
Chapter 6a
1. b)

$$f(y) = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$F_2(u_2) = P(U_2 \le u_2) = P(1-2Y \le u_2) = P(2Y \le 1-u_2) = P\left(Y \ge \frac{1}{2} - \frac{1}{2}u_2\right)$$

$$= 1 - P\left(Y \le \frac{1}{2} - \frac{1}{2}u_2\right) = 1 - F_y\left(\frac{1}{2} - \frac{1}{2}u_2\right)$$

$$\implies \int_{\frac{1}{2}-\frac{1}{2}u_2}^{1} 2(1-y)dy = 2\left(y - \frac{1}{2}y_2^2\right)\Bigg|_{\frac{1}{2}-\frac{1}{2}u_2}^{1} = (2y - y^2)\Bigg|_{\frac{1}{2}-\frac{1}{2}u_2}^{1}$$

$$\implies 2 - 1 - \left[2\left(\frac{1}{2} - \frac{1}{2}u_2\right) - \left(\frac{1}{2} - \frac{1}{2}u_2\right)^2\right] = \frac{1}{2}u_2^2 + \frac{1}{2}u_2 + \frac{1}{4}$$

So

$$F_{U_2}(u_2) = \begin{cases} 0 & u < -1 \\ \frac{1}{4}u_2^2 + \frac{1}{2}u_2 + \frac{1}{4} & -1 \le u \le 1 \\ 1 & u \ge 1 \end{cases}$$

d)

$$f_{U_2}(u_2) = \frac{d}{du_2}F_{U_2}(u_2) = \begin{cases} 0 & \text{elsewhere} \\ \frac{1}{2}(u_2+1) & -1 \le u \le 1 \end{cases}$$

$$E(U_2) = \frac{1}{2}\int_{-1}^{1} u_2(u_2+1)du_2 = \frac{1}{2}\int_{-1}^{1} (u_2^2 + u_2)du_2$$

$$\implies \qquad = \frac{1}{2}\left(\frac{1}{3}u_2^3 + \frac{1}{2}u_2^2\right)\Bigg|_{-1}^{1}$$

$$= \frac{1}{3}$$

5.

$$f_Y(y) = \begin{cases} \frac{1}{4} & 1 \le y \le 5 \\ 0 & \text{elsewhere} \end{cases} \implies U = 2Y^2 + 3 \implies 5 \le u \le 53$$

So

$$\begin{aligned} P(U \le u) &= F_U(u) = P(2Y^2 + 3 \le u) \\ &= P\left(Y^2 \le \frac{u-3}{2}\right) = P\left(Y \le \sqrt{\frac{u-3}{2}}\right) = \frac{1}{4}\int_1^{\sqrt{\frac{u-3}{2}}} 1dy \\ &= \frac{1}{4}y\Big|_1^{\sqrt{\frac{u-3}{2}}} = \frac{1}{4}\sqrt{\frac{u-3}{2}} - \frac{1}{4} \end{aligned}$$

Therefore,

$$F_U(u) = \begin{cases} 0 & u < 5 \\ \frac{1}{4}\left[\sqrt{\frac{u-3}{2}} - 1\right] & 5 \le u \le 53 \\ 1 & u > 53 \end{cases}$$

So

$$f_U(u) = \begin{cases} \frac{1}{16}\left(\frac{2}{u-3}\right)^{1/2} & 5 \le u \le 53 \\ 0 & \text{elsewhere} \end{cases}$$

9.

a)

$$f(y_1, y_2) = \begin{cases} k & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1,\ 0 \le y_1 + y_2 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$0 \le y_1 + y_2 \le 1 \implies -y_1 \le y_2 \le 1 - y_1$$
$$U = Y_1 + Y_2 \implies Y_2 = U - Y_1$$

$$\int_0^1 \int_0^{1-y_1} k dy_2 dy_1 = k\int_0^1 y_2 \Big|_{y_2=0}^{y_2=1-y_1} dy_1 = 1$$

$$\implies k\int_0^1 (1-y_1)dy_1 = 1 \implies k = 2$$

So the joint density function is

$$f(y_1, y_2) = \begin{cases} 2 & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1,\ 0 \le y_1 + y_2 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned} F_U(u) &= P(U \le u) = P(Y_1 + Y_2 \le u) \\ &= P(Y_2 \le u - Y_1) = \int_0^u \int_0^{u-y_1} 2dy_2dy_1 \\ &= \int_0^u 2y_2\Big|_{y_2=0}^{y_2=u-y_1} dy_1 = \int_0^u 2(u-y_1)dy_1 \\ &= u^2 \end{aligned}$$

$$\implies F_U(u) = \begin{cases} 0 & u < 0 \\ u^2 & 0 \le u \le 1 \\ 1 & u > 1 \end{cases}$$

$$\implies f_U(u) = \begin{cases} 2u & 0 \le u \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

b)

$$\begin{aligned} E(U) &= \int_{-\infty}^{\infty} uf_U(u)du = \int_0^1 u(2u)du \\ &= \int_0^1 2u^2du = \frac{2}{3}u^3\Big|_0^1 \\ &= \frac{2}{3} \end{aligned}$$

c)

$$E(U) = E(Y_1) + E(Y_2)$$

$$E(Y_1) = \int_0^1 2y_1y_2 \Bigg|_{y_2=0}^{y_2=1-y_1} dy_1 = \int_0^1 2y_1(1-y_1)dy_1 = \frac{1}{3}$$

$$E(Y_2) = \int_0^1 y_2^2 \Bigg|_{y_2=0}^{y_2=1-y_1} dy_1 = \int_0^1 (1-y_1)^2 dy_1 = \frac{1}{3}$$

$$\implies E(U) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

13.

The joint density function for two independent exponential random variables Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} \frac{1}{\beta}e^{-(y_1+y_2)/\beta} & y_1 > 0,\ y_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Also note that $U = Y_1 + Y_2$ and $0 \leq u < \infty$. Then,

$$F_U(u) = P(U \leq u) = P(Y_1 + Y_2 \leq u) = P(Y_2 \leq u - Y_1) = \int_0^u \int_0^{u-y_1} f(y_1, y_2)dy_2dy_1$$

$$= \int_0^u \int_0^{u-y_1} \frac{1}{\beta^2}e^{-(y_1+y_2)/\beta}dy_2dy_1 = \frac{1}{\beta^2}\int_0^u e^{-\frac{y_1}{\beta}} \int_0^{u-y_1} e^{-y_2/\beta}dy_2dy_1$$

$$= \frac{1}{\beta^2}\int_0^u e^{-\frac{y_1}{\beta}}\beta\left[1 - e^{(y_1-u)\beta}\right]dy_1 = \frac{1}{\beta}\int_0^u \left(e^{-\frac{y_1}{\beta}} - e^{-\frac{u}{\beta}}\right)dy_1$$

$$= \frac{1}{\beta}\left[-\beta e^{-y_1/\beta} + \beta e^{-u/\beta}\right]\Bigg|_{y_1=0}^{y_1=u} = \left(e^{-u/\beta} - e^{-y_1/\beta}\right)\Bigg|_{y_1=0}^{y_1=u}$$

$$= \left(e^{-u/\beta} - e^{-u/\beta}\right) - \left(e^{-u/\beta} - 1\right) = 1 - e^{-u/\beta}$$

So the Cumulative Distribution Function is

$$F_U(u) = \begin{cases} 1 - e^{-u/\beta} & u > 0 \\ 0 & \text{elsewhere} \end{cases}$$

And the PDF is obtained by taking the derivative of this with respect to u,

$$f_U(u) = \begin{cases} \frac{1}{\beta} e^{-u/\beta} & u > 0 \\ 0 & \text{elsewhere} \end{cases}$$

31. The joint density function of the random variables Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} \frac{1}{8} e^{-(y_1+y_2)/2} & y_1 > 0,\ y_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Also, $U = \dfrac{Y_2}{Y_1} \implies y_2 = uy_1 = h^{-1}(u) \implies \dfrac{d}{du}h^{-1}(u) = y_1$. We want to redefine the joint density function above by plugging in this value of y_2 into it. Then,

$$g(y_1, u) = \begin{cases} \frac{1}{8} y_1 e^{-(y_1+uy_1)/2}(y_1) & y_1 > 0,\ uy_1 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\implies g(y_1, u) = \begin{cases} \frac{1}{8} y_1^2 e^{-y_1(1+u)/2} & y_1 > 0,\ u > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\implies f_U(u) = \int_0^\infty \frac{1}{8} y_1^2 e^{-y_1(1+u)/2} dy_1$$

Using u - substitution,

$$v = \frac{(u+1)y_1}{2} \implies dv = \frac{u+1}{2} dy_1$$
$$y_1 = \frac{2v}{u+1} \implies dy_1 = \frac{2}{u+1} dv$$

$$\implies \frac{1}{8}\int_0^\infty \frac{4}{(u+1)^2} v^2 e^{-v}\left(\frac{2}{u+1}\right) dv$$

$$\implies \frac{1}{(u+1)^3}\int_0^\infty v^2 e^{-v} dv = \frac{1}{(u+1)^3}\Gamma(3) = \frac{2}{(u+1)^3}$$

Therefore, the PDF of U is

$$f_U(u) = \begin{cases} \frac{2}{(u+1)^3} & u > 0 \\ 0 & \text{elsewhere} \end{cases}$$

35. Let Y_1 and Y_2 be two independent random variables that are uniformly distributed on the interval $0 \leq y \leq 1$. Then,

$$f_1(y_1) = \begin{cases} 1 & 0 \leq y_1 \leq 1 \\ 0 & \text{elsewhere} \end{cases} \text{ and } f_2(y_2) = \begin{cases} 1 & 0 \leq y_2 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Recall that $f(y_1, y_2) = f_1(y_1)f_2(y_2)$. This means that

$$f(y_1, y_2) = \begin{cases} 1 & 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Also,

$$U = Y_1 Y_2 \implies y_2 = h^{-1}(u) = \frac{u}{y_1} \implies \frac{d}{du}h^{-1}(u) = \frac{1}{y_1}$$

$$\implies g(y_1, u) = \begin{cases} 1 \times \frac{1}{y_1} & 0 \leq y_1 \leq 1,\ 0 \leq u \leq y_1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned} \implies f_U(u) &= \int_u^1 \frac{1}{y_1} dy_1 = \ln(y_1)\Big|_u^1 = 0 - \ln(u) \\ &= -\ln(u) \end{aligned}$$

Therefore,

$$f_U(u) = \begin{cases} -\ln(u) & 0 \le u \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

37. We are given the following

$$(i)\ 0 < p < 1$$
$$(ii)\ P(Y_i = 1) = p$$
$$(iii)\ P(Y_i = 0) = 1 - p = q$$

and

$$P(Y_1 = y) = \begin{cases} p^y(1-p)^{1-y} & y = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

a)

$$M_{Y_1}(t) = E[e^{tY_1}] = (1-p) \times 1 + pe^{ty_1}$$
$$= q + pe^t$$

b) We are given that $W = Y_1 + Y_2 + \ldots + Y_n = \sum_{i=1}^{n} Y_i$. So,

$$M_W(t) = E[e^{tW}] = E\left[\exp\left(t\sum_{i=1}^{n} Y_i\right)\right]$$
$$= E[e^{t(Y_1+Y_2+\ldots+Y_n)}] = E[e^{tY_1} \times e^{tY_2} \times \ldots e^{tY_n}]$$
$$= E(e^{tY_1})E(e^{tY_2})\ldots E(e^{tY_n}) = [M_{Y_1}(t)]^n$$
$$= (q + pe^t)^n$$

c) The distribution of W is Binomial(n, p).

51. We are given the following information

$$Y_1 \sim Binomial(n_1, p_1 = 0.2)$$
$$Y_2 \sim Binomial(n_2, p_2 = 0.8)$$
$$U = Y_1 + n_2 - Y_2$$

Also, the moment generating functions of Y_1 and Y_2 is

$$M_{Y_1}(t) = (0.2e^t + 0.8)^{n_1}$$
$$M_{Y_2}(t) = (0.8e^t + 0.2)^{n_2}$$

Given these information, we can derive the moment generating function of U.

$$\begin{aligned}
M_U(t) = E[e^{tU}] &= E[e^{t(Y_1+n_2-Y_2)}] \\
&= E[e^{tY_1}e^{tn_2}e^{-tY_2}] = e^{tn_2}E[e^{tY_1}]E[e^{-tY_2}] \\
&= e^{-tY_2}M_{Y_1}(t)M_{Y_2}(-t) \\
&= e^{tn_2}(0.2e^t + 0.8)^{n_1}(0.8e^{-t} + 0.2)^{n_2} \\
&= (0.2e^t + 0.8)^{n_1}[e^t(0.8e^{-t} + 0.2)]^{n_2} \\
&= (0.2e^t + 0.8)^{n_1}(0.8 + 0.2e^t)^{n_2} \\
&= (0.2e^t + 0.8)^{n_1+n_2}
\end{aligned}$$

59. We are given the following information

$$Y_1 \sim \chi^2(v_1)$$
$$Y_2 \sim \chi^2(v_2)$$
$$U = Y_1 + Y_2$$

Also, the their probability density functions are

$$f_{Y_1}(y_1) = \begin{cases} \frac{1}{\Gamma(v_1/2)2^{v_1/2}} y_1^{(v_1/2)-1} e^{-y_1/2} & y_1 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

and

$$f_{Y_2}(y_2) = \begin{cases} \frac{1}{\Gamma(v_2/2)2^{v_2/2}} y_2^{(v_2/2)-1} e^{-y_2/2} & y_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Then the moment generating function of U is

$$
\begin{aligned}
M_U(t) &= E[e^{tU}] = E[e^{t(Y_1+Y_2)}] \\
&= E[e^{tY_1}e^{tY_2}] = E[e^{tY_1}]E[e^{tY_2}] \\
&= M_{Y_1}(t)M_{Y_2}(t) \\
&= \left(\frac{1}{1-2t}\right)^{v_1/2}\left(\frac{1}{1-2t}\right)^{v_2/2} \\
&= \left(\frac{1}{1-2t}\right)^{(v_1+v_2)/2}
\end{aligned}
$$