# THE DISTRIBUTION OF EIGENVALUES OF DOUBLY CYCLIC $Z^{+}$-MATRICES*

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**Abstract.** It is shown that every matrix in a large class of $n$-by-$n$ doubly cyclic $Z^{+}$ matrices with negative determinant has exactly one eigenvalue in the closed left half-plane. This generalizes a result for $n = 4$ used in a recent analysis of cancer cell dynamics. A further conjecture is made based on computational evidence. All work relates to the inertia of certain doubly cyclic circulants.

**Key words.** doubly-cyclic $Z^{+}$ matrices, eigenvalue location

**AMS subject classifications.** 15A18, 15B99

## 1. Introduction.

We call a matrix $A \in M_n(\mathbb{R})$ *doubly cyclic* ($DC$) if all its nonzero entries are confined to the main diagonal, the super-diagonal, and the $n,1$ position. A matrix in $M_n(\mathbb{R})$ is called a $Z$-matrix ($Z^{+}$-matrix) if all its off-diagonal entries are non-positive (if, in addition, its diagonal entries are positive). For a matrix $B \in M_n(\mathbb{C})$, the *inertia*, $i(B) = (i_{+}(B), i_{-}(B), i_0(B))$ is just the count of the numbers of eigenvalues in the right half-plane (RHP), left half-plane (LHP), or imaginary axis, respectively, counting multiplicities. A $DC$ $Z^{+}$-matrix, $A \in M_n(\mathbb{R})$, with positive determinant, is necessarily an $M$-matrix [2] because its leading principal minors are positive and, so, has $i(A) = (n, 0, 0)$. If its determinant is negative, $i_{-}(A) \geq 1$.

We are interested in the possible inertias of $DC$ $Z^{+}$-matrices with negative determinant.

Note that our matrices have the form

$$A = \begin{bmatrix} a_1 & -b_1 & 0 & \cdots & 0 \\ 0 & a_2 & -b_2 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & \cdots & a_{n-1} & -b_{n-1} \\ -b_n & \cdots & \cdots & 0 & a_n \end{bmatrix} \tag{1.1}$$

in which $a_j, b_j > 0$, $j = 1, \ldots, n$ and that

$$\det A = (a_1 a_2 \cdots a_n) - (b_1 b_2 \cdots b_n).$$

Denote by $DC(a, b)$ the set of all matrices of the form (1.1) with given geometric means

$$a = (a_1 a_2 \cdots a_n)^{1/n} \text{ and } b = (b_1 b_2 \cdots b_n)^{1/n},$$

so that for $A \in DC(a, b)$, $\det A < 0$ if and only if $a < b$. Denote this subclass by $DC_{-}(a, b)$.

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For $P$ the basic circulant permutation matrix, the matrix $aI - bP \in DC(a,b)$, of course. The inertia of this matrix plays a special role in the possible inertias occurring within $DC(a,b)$.

In [3], it was shown that when $n = 4$, for $A \in DC(a, 2a)$, $i(A) = (3, 1, 0)$. (Smaller values of $n$ are straightforward to analyze and also exhibit just one eigenvalue in the LHP.) This fact was critically used to analyze a small cancer cell dynamics model. The ability to use more elaborate models would benefit from knowledge about inertias of matrices in $DC_{-}(a,b)$ when $n$ is larger. Note that, though the formal statement in [3] is less general than what we have said, it is equivalent to that by diagonal similarity and that the same proof also shows that all matrices in $DC_{-}(a,b)$ have inertia $(3, 1, 0)$ when $n = 4$; unlike the statement for arbitrary values of $n$, this case is unconditional with regard to the ratio $a/b$.

When $n > 4$ and $a < b$, the matrix $aI - bP$ may have more than one eigenvalue in the LHP, as long as $a$ is sufficiently small relative to $b$. More precisely, it will when $a < b\cos\frac{2\pi}{n}$, and, because of the theory of circulants, the exact inertia is easy to state trigonometrically [4]. We noticed empirically that, although $i_{-}(A) > 1$ can occur in $DC_{-}(a,b)$, it did not occur when $i_{-}(aI - bP) = 1$ (i.e. $\cos(2\pi/n) < a/b < 1$). The trigonometric condition stated above motivates our method of proof, as standard methods for stability analysis (e.g. Routh-Hurwitz theorem, Gersgorin discs) were unenlightening [1].

Our purpose here is to verify this important phenomenon as a theorem in the next section, then make an even stronger conjecture in section 3 and give some evidence for this stronger conjecture. An implication of the theorem is that much larger models, such as in [3], may be analyzed essentially qualitatively (i.e. with little explicit knowledge of model coefficients), which could be useful for studying the stability of biological systems under environmental stresses. The authors would like to thank Clark Jeffries for pointing out the biological importance of these matrices in analyzing "how huge systems, without central authorities, can possibly work."

## 2. Main result: Inertia of $DC_{-}(a,b)$ matrices with $b\cos\frac{2\pi}{n} < a$.

Note that the characteristic polynomial of the matrix (1.1) is

$$\prod_{j=1}^{n}(a_j - \lambda) - b^n, \tag{2.1}$$

and for $A = aI - bP$ in particular it is simply $(a - \lambda)^n - b^n$. Consequently,

$$\sigma(aI - bP) = \{a - be^{2\pi ij/n}, \quad j = 0, \ldots, n-1\}. \tag{2.2}$$

According to (2.1), the spectrum of a doubly-cyclic $Z^{+}$ matrix (1.1) does not depend on the individual values of $b_j$ but only on their geometric mean $b$. The same, of course, is not true for $a_j$. However, some properties of $\sigma(A)$ are the same for all matrixes in $DC(a,b)$.

THEOREM 2.1. *The following statements about $a, b > 0$ are equivalent:*

1. $i_{-}(aI - bP) = 1$,
2. $\cos(2\pi/n) < a/b < 1$,
3. $i_{-}(A) = 1$ *for all* $A \in DC(a,b)$.

*Proof.* The equivalence of *1.* and *2.* follows immediately from (2.2). The implication *3.* $\Longrightarrow$ *1.* is trivial. It remains to show that *2.* $\Longrightarrow$ *3.* To this end, recall that due to (2.1), $\sigma(A)$ for any matrix (1.1) is the same as for

$$\begin{bmatrix} a_1 & -b & 0 & \cdots & 0 \\ 0 & a_2 & -b & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & \cdots & a_{n-1} & -b \\ -b & \cdots & \cdots & 0 & a_n \end{bmatrix} \tag{2.3}$$

with the "averaged" off-diagonal entries $b_j$.

Observe now that the set of matrices (2.3) is path-connected. Since the eigenvalues of $A$ depend on $A$ continuously, in order to show that $i_-(A)$ is the same for all matrices (2.3) satisfying *2.* it suffices therefore to establish that such matrices do not have pure imaginary eigenvalues. Moreover, replacing $A$ with $A/b$ (which does not change the number of eigenvalues in the left or right half plane), we may without loss of generality suppose that $b = 1$ and

$$\cos \frac{2\pi}{n} < a < 1. \tag{2.4}$$

So, let $iy$ be a root of (2.1) with $b = 1$, that is,

$$\prod_{j=1}^{n} (a_j - iy) = 1$$

for some $y \in \mathbb{R}$. Equivalently,

$$\prod_{j=1}^{n} (a_j^2 + y^2) = 1 \tag{2.5}$$

and

$$\sum_{j=1}^{n} \arg(a_j + iy) = 0 \mod 2\pi. \tag{2.6}$$

Without loss of generality, $y > 0$ (since $-y$ is a root of (2.1) along with $iy$), and $\arg(a_j + iy) = \arctan y/a_j \in (0, \pi/2)$.

We will show that under the constraints (2.4), (2.5) the sum $\Phi(y) := \sum_{j=1}^{n} \arctan y/a_j$ never exceeds $2\pi$, thus contradicting (2.6). This is of course obvious for $n \leq 4$, in which case the result of the theorem was established in a different manner in [3]. The reasoning applicable to arbitrary $n$ proceeds as follows.

Introducing new variables $x_j = y/a_j$, rewrite (2.5) as

$$\prod_{j=1}^{n} (1 + x_j^2) = a^{-2n}. \tag{2.7}$$

Of course, $\sup \Phi(y)$ coincides with the maximum $\Theta$ of the continuous function

$$F_n(x_1, \ldots, x_n) = \sum_{j=1}^{n} \arctan x_j$$

on the (compact) set of non-negative $x_j$ satisfying (2.7). Let $(\xi_1, \dots, \xi_n)$ be a point at which this maximum is attained. Due to the symmetry of (2.4), (2.7) with respect to arbitrary permutations of the arguments, we may without loss of generality suppose that non-zero $\xi_j$ precede the ones equal to zero, that is,

$$\xi_1, \dots, \xi_m > 0, \quad \xi_{m+1} = \cdots = \xi_n = 0 \text{ for some } m \geq 1.$$

Observe that

$$\prod_{j=1}^{m} (1 + \xi_j^2) = a^{-2n} \tag{2.8}$$

and

$$F_n(\xi_1, \dots, \xi_n) = \sum_{j=1}^{m} \arctan \xi_j = F_m(\xi_1, \dots, \xi_m).$$

In other words, $\Theta$ coincides, for some $m \in \{1, \dots, n\}$, with a local maximum of the function $F_m$ on the portion of the surface (2.8) lying in the open first orthant of $\mathbb{R}^m$. Using Lagrange multipliers we find that at this local maximum

$$\frac{1}{1 + \xi_j^2} + 2t\xi_j = 0, \quad j = 1, \dots, m$$

for some $t \in \mathbb{R}$. From here and (2.8),

$$\xi_j = \sqrt{a^{-2n/m} - 1}, \quad j = 1, \dots, m,$$

and so

$$\Theta = m \arctan \sqrt{a^{-2n/m} - 1} = m \arccos a^{n/m}.$$

But due to (2.4),

$$\arccos a^{n/m} < \arccos \left( \cos \frac{2\pi}{n} \right)^{n/m}.$$

It therefore remains to show that

$$\arccos \left( \cos \frac{2\pi}{n} \right)^{n/m} \leq \frac{2\pi}{m} \tag{2.9}$$

for $m = 1, \dots, n$. This is obvious for $m = 1, 2, 3, 4$. Starting with $m = 5$, both angles involved in (2.9) lie in the first quadrant, and so we can apply cos and invert the inequality, which thus takes the form

$$\left( \cos \frac{2\pi}{n} \right)^n \geq \left( \cos \frac{2\pi}{m} \right)^m, \quad m = 5, \dots, n.$$

In its turn, this is a direct consequence of the decreasing behavior of the function $f(x) = (\cos x)^{1/x}$ on $(0, \pi/2)$. $\square$

**3. Conjecture: Possible inertias in $DC_-(a,b)$ with $a/b$ given.**

The essence of Theorem 2.1 is that for a certain range of ratios, $a/b$, the inertia of $A \in DC_-(a,b)$ is fully determined by that of $aI - bP$. This is also true when $a/b > 1$ ($M$-matrix case). For other ratios, we have observed some variation in inertia within $DC_-(a,b)$. However, we have been led to guess that the inertias in $DC_-(a,b)$ are limited by that of $aI - bP$. A number of experiments have led us to

**Conjecture.** *$A \in DC_-(a,b)$ implies $i_-(A) \le i_-(aI - bP)$.*

Moreover, $i_-(aI - bP)$ and $i_-(A)$ are odd, since both of their determinants are negative and, being real matrices, any complex eigenvalues must occur in conjugate pairs. The inertia $i_-(A)$ has been observed computationally to be any odd number up to $i_-(aI - bP)$ when $A \in DC_-(a,b)$. However, inertias other than $i_-(aI - bP)$ occur less frequently.

For $n = 20$, we computationally generated 10,000,000 matrices with "random" diagonal entries, given a fixed ratio of diagonal and off-diagonal products. For the particular ratio $a/b$, $i_-(aP - bP) = 7$. The frequency of occurrence of values of $i_-(A)$, averaged over two trials, are as follows:

| $i_-(A)$ | Frequency |
|---|---|
| $> 9$ | 0 |
| 7 | 0.6628059 |
| 5 | 0.3354701 |
| 3 | 0.00122155 |
| 1 | 0.00000245 |

So, not only were no counter-examples observed, it can be seen that all inertias less than our conjectured upper bound are obtained in practice.

In another set of trials, we generated 1,000,000 matrices for each of $n = 5, \dots, 65$ with random diagonal entries, admitting an expected inertia of $i_-(A) \le 3$ given the conjecture. No counter-examples, i.e. $i_-(A) > 3$, were observed. Our proof of (2.1) does not seem to generalize easily to prove this conjecture.

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