Homework 7: Digital Control (ECEN 5458)

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Problem 1

Given the double integrator plant:

$$G_1(s) = \frac{1}{21s^2}$$

Design a prediction bias estimator (to null out constant disturbances at the input) by augmenting the system with the bias model. Place two off the estimator poles as before at $\omega = 4$rad/sec and $\zeta = 0.7$, and place the third pole at $z = 0.5$. What is the estimator gain $\boldsymbol{L_p}$ for the augmented system?
Section 8.5.2 of the book yields the state space model:

$$\begin{bmatrix} \boldsymbol{x}(k+1) \\ w(k+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{\Gamma}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(k) \\ w(k) \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Gamma} \\ 0 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} \boldsymbol{H} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(k) \\ w(k) \end{bmatrix}$$

We can fill this in with $\boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}$.

$$\begin{bmatrix} \boldsymbol{x}(k+1) \\ w(k+1)) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.0095 \\ 0.2 & 1 & 0.00095 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(k) \\ w(k) \end{bmatrix} + \begin{bmatrix} 0.0095 \\ 0.00095 \\ 0 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(k) & w(k) \end{bmatrix}$$

The feedback gain matrix should be:

$$K = \begin{bmatrix} 20.98 & 18.97 & 1 \end{bmatrix}$$

Then we find the estimator gain with:

$$|z\boldsymbol{I} - \boldsymbol{\Phi} + \boldsymbol{L}_p\boldsymbol{H}| = (z^2 - 0.96z + 0.33)(z - 0.5) = 0 = \begin{bmatrix} z-1 & l_1 & -0.0095 \\ -0.2 & z-1-l_2 & -0.00095 \\ 0 & l_3 & z-1 \end{bmatrix}$$

Thus:

$$z^3 - 1.46z + 0.81z - 0.165 = z^3 + z^2(-3 - l_2) + z(3 + 2l_2 + l_3 * 0.0019 + 0.2L_1 + 0.0095l_3) + 1 + l_2 + l_3 0.00285 + 0.2l_1$$

Solving yields $l_2 = -1.54$, and:

$$\begin{bmatrix} 0.89 \\ 0.705 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.00095 \\ 0.2 & 0.00285 \end{bmatrix} \begin{bmatrix} l_1 \\ l_3 \end{bmatrix}$$

which gives $l_1 = 4.9125$ and $l_3 = -97.3684$ and:

$$\boldsymbol{L_p} = \begin{bmatrix} 4.9125 \\ -1.54 \\ -97.3684 \end{bmatrix}$$

Problem 2

Consider the system:

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k), \quad y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{x}(k)$$

Suppose that the control signal $u(k)$ to the system must be quantized with m bits to the right of the fixed point and assume that round-off is used.

(a)

What is the transfer function $u(k)$ to $y(k)$.

$$G(z) = \begin{bmatrix} 1 & 1 \end{bmatrix} \left(\begin{bmatrix} z-0.1 & 0 \\ 0 & z-0.2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2(z-0.15)}{(z-0.1)(z-0.2)} = \frac{1}{z-0.1} + \frac{1}{z-0.2}$$

(b)

How many bits are needed to guarantee that the magnitude of the error $\tilde{y}(k)$ at the output due to the quantization fo the control signal $u(k)$ is always less than or equal to 0.1? That is, how many bits m are needed to guarantee that $|\tilde{y}(k)| \leq 0.1, \forall k$?
This means we want:

$$0.1/(2^{-m-1}) < \sum_{k=0}^{\infty} |h(k)|$$

We can find the inverse transform to be:

$$h(k) = 0.1^k + 0.2^k$$

I calculated this for 10000 trials of n in matlab really quickly (code not included cause it should be easy). That gave me that the bounded magnitude of $|h(k)|$ is very close to 2.3611. Thus,

$$2^{-m-1} = \frac{0.1}{2.3611}$$

and m is the ceiling of the resulting value which is 4.

(c)

The steady state worst case is:

$$2.3611 * 2^{-5} = 0.0738$$

Problem 3

11.7 from the text:
Consider a plant consisting of a diverging exponential, that is,

$$\frac{x(s)}{u(s)} = \frac{a}{s-a}$$

Controlled discretely with a ZOH, this yields a difference equation, namely:

$$x(k+1) = e^{aT} x(k) + (e^{aT} - 1) u(k)$$

Assume proportional feedback,

$$u(k) = -K x(k)$$

and compute the gain K that yields a z-plane root at $z = e^{-bT}$. Assume $a = 1\text{sec}^{-1}$ and $b = 2\text{sec}^{-1}$, and do the problem for $T = 0.1$, 1.0, 2, 5 sec. Is there an upper limit on the sample period that will stabilize this system? Compute the percent error in K that will result in an unstable system for $T = 2$ and 5 seconds. Do you judge that the case when $T = 5$ sec is practical?
Our equation becomes:

$$x(k+1) = (e^T - Ke^T + K)x(k)$$

The pole is at $e^T - Ke^T + K = e^{-2T}$ and thus $K = \dfrac{e^{-2T} - e^T}{1 - e^T}$. Below is a table of values for the various sample rates:

T(sec)	K
0.1	2.7236
1.0	1.5032
2	1.1537
5	1.0068

There is not an upper limit on a sample rate that will stabilize the system. The problem is that K becomse closer and closer to 1 as the sample rate goes up and thus the pole location will change rapidly with K. To compute the percent error of K that will lead to an unstable system we need to solve $e^T - Ke^T + K > 1$. That yields, $K = 1$. Thus the percent error for 2 seconds is 13% and for 5 seconds is 0.67%. This really isn't ideal for controller design because one wants to make a controller that is robust to minor changes in the system, but it could probabl be done.

Problem 4

Consider the continuous-time plant transfer function:

$$\mathfrak{G}(s) = \frac{Y(s)}{U(s)} = \frac{s}{s+1} = \frac{-1}{s+1} + 1$$

(a)

Determine the continuous-time state-space representation of the plant:

$$\dot{x}(t) = Fx(t) + Gu(t)$$
$$y(t) = Hx(t) + Ju(t)$$

we find it to be:

$$F = -1 \quad G = -1 \quad H = 1 \quad J = 1$$

(b)

First order hold state space equations.

(i)

Find $u(\tau)$ in the interval between kT and $T(k+1)$.
We find:

$$u(\tau) \approx u(kT) + \dot{u}(kT)(\tau - kT) \approx u(kT) + \frac{u(kT) - u(kT - T)}{T}(\tau - kT)$$

(ii)

Finding the equation in proper form

$$x(kT+T) = e^{FT}x(kT) + \int_{kT}^{kT+T} e^{F(kT+T-\tau)}G\left(u(kT) + \frac{u(kT)-u(kT-T)}{T}(\tau - kT)\right)d\tau$$

The part in the integral is equal to:

$$e^{F(kT+T-\tau)}G\left(u(kT)\left(1+\frac{\tau-kT}{T}\right) - u(kT-T)\left(\frac{\tau-kT}{T}\right)\right)$$

Thus our matrices are:

$$\Phi_F = e^{FT} \quad \Gamma_F = \int_{kT}^{kT+T} e^{F(kT+T-\tau)}G\left(1+\frac{\tau-kT}{T}\right)d\tau \quad \Gamma_{F1} = \int_{kT}^{kT+T} -e^{F(kT+T-\tau)}G\left(\frac{\tau-kT}{T}\right)d\tau$$

(iii)

Evaluating for the problem described in part (a) we get:

$$\Phi_F = \Phi = e^{-T}$$

Now for Γ_F:

$$\int_{kT}^{kT+T}(k-1)e^{(\tau-kT-T)} - \frac{\tau}{T}e^{(\tau-kT-T)}d\tau = (k-1)(1-e^{-T}) - \frac{1}{T}(kT+T-1) + \frac{e^{-T}}{T}(kT-1)$$

simplify a little:

$$k - ke^{-T} - 1 + e^{-T} - k - 1 + \frac{1}{T} + ke^{-T} - \frac{e^{-T}}{T} = e^{-T} - \frac{e^{-T}}{T} + \frac{1}{T} - 2$$

Which makes some sense because all the ks cancel.
Now for Γ_{F1}:

$$\Gamma_{F1} = \int_{kT}^{kT+T} e^{\tau-kT-T}\left(\frac{\tau-kT}{T}\right)d\tau = -ke^{\tau-kT-T} + \frac{1}{T}e^{\tau-kT-T}(\tau-1)\Big|_{kT}^{kT+T}$$

simplifying yields:

$$-k(1-e^{-T}) + \frac{1}{T}(kT+T-1) - \frac{1}{T}e^{-T}(kT-1) = ke^{-T} - k + k + 1 - \frac{1}{T} - ke^{-T} + \frac{1}{T} = 1$$

Similarly as we expect all the ks cancel.

iv

Take the Z-transform of the equations in defining the state space variables for a first order hold.
First I'm going to define $\alpha = e^{-T} - \frac{e^{-T}}{T} + \frac{1}{T} - 2$. Then the state space equations are:

$$\begin{aligned} x(kT+T) &= e^{-T}x(kT) + \alpha u(kT) + u(T(k-1)) \\ y(kT) &= x(kT) + u(kT) \end{aligned}$$

This yields:

$$\begin{aligned} X(z)z &= e^{-T}X(z) + U(z)(\alpha + z^{-1}) \\ Y(z) &= X(z) + U(z) \end{aligned}$$

Solving for the state:

$$X(z) = U(z)\frac{\alpha + z^{-1}}{z - e^{-T}}$$

Then we find the transfer function to be:

$$\frac{Y(z)}{U(z)} = \frac{\alpha + z^{-1}}{z - e^{-T}} + 1 = \frac{z^2 + z(\alpha - e^{-T}) + 1}{z(z - e^{-T})} = \frac{z^2 + z\left(\frac{1}{T} - \frac{e^{-T}}{T} - 2\right) + 1}{z(z - e^{-T})}$$

v

Now we want to find the transfer function for this problem using the method derived in homework 3.

$$\mathfrak{G}(z) = \frac{(z-1)^2}{Tz^2}\left(\mathcal{Z}\left(\frac{\mathfrak{G}(s)}{s^2}\right) + T\mathcal{Z}\left(\frac{\mathfrak{G}(s)}{s}\right)\right)$$

Solving directly:

$$\mathfrak{G}(z) = \frac{(z-1)^2}{Tz^2}\left(\frac{z(1-e^{-T})}{(z-1)(z-e^{-T})} + \frac{z}{z-e^{-T}}\right) = \frac{(z-1)}{Tz}\left(\frac{(1-e^{-T}) + (z-1)}{z-e^{-T}}\right)$$

One more:

$$\mathfrak{G}(z) = \frac{(z-1)}{Tz}$$

This isn't the same as what I derived, but then again I'm fairly sure the Γ_F I found was slightly wrong. I believe these two techniques should result in the exact same function. They both assume the same model of the system and use the same hold technique.