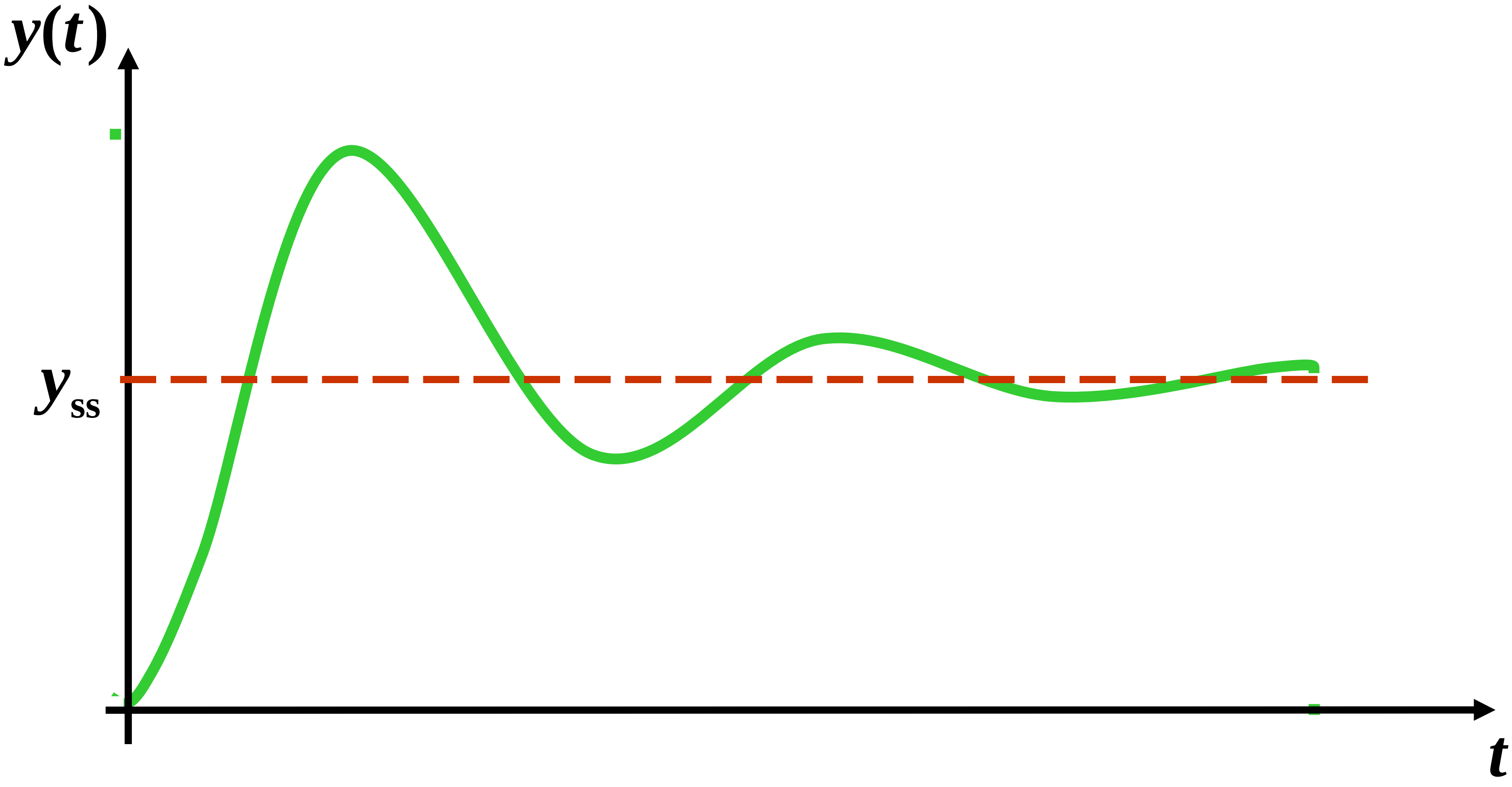
Performance Specifications & Root Locus Review

- Performance specifications
  - Continuous-time (review)
- Root Locus (review)
  - Design example

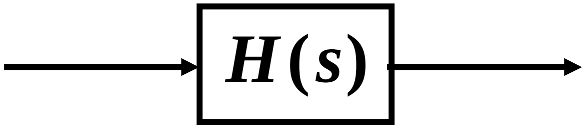
Performance Specifications

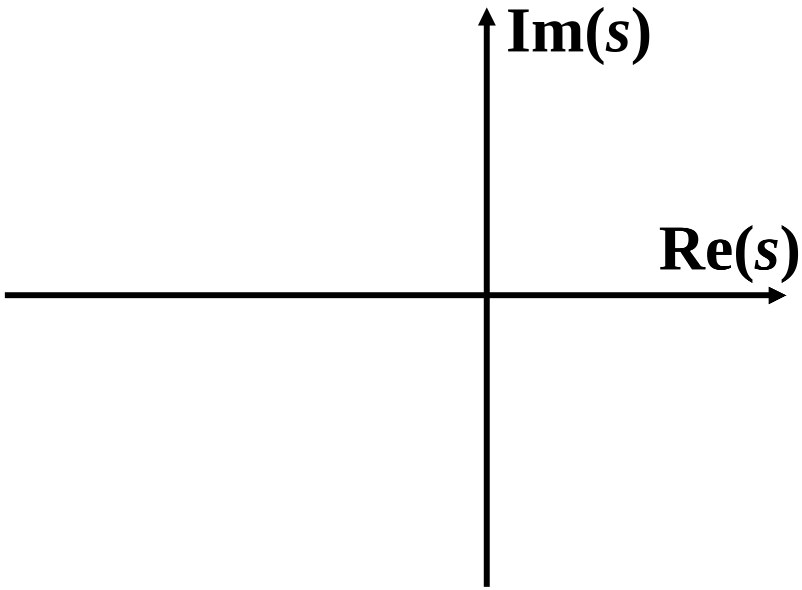
- Consider a continuous-time step response:



- **For a 2nd-order continuous-time system:**

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



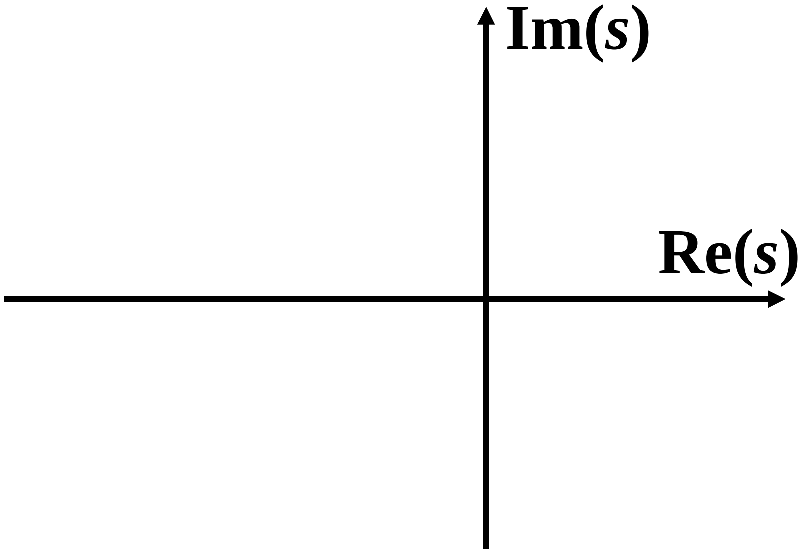


- **Step response:**

$$y(t) = 1 - e^{-\sigma t}\left[\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right], \qquad t \geq 0$$

See Figure 2.4 of text.

- **Rise time depends primarily on** ω_n **:** $t_r \approx \dfrac{1.8}{\omega_n}$

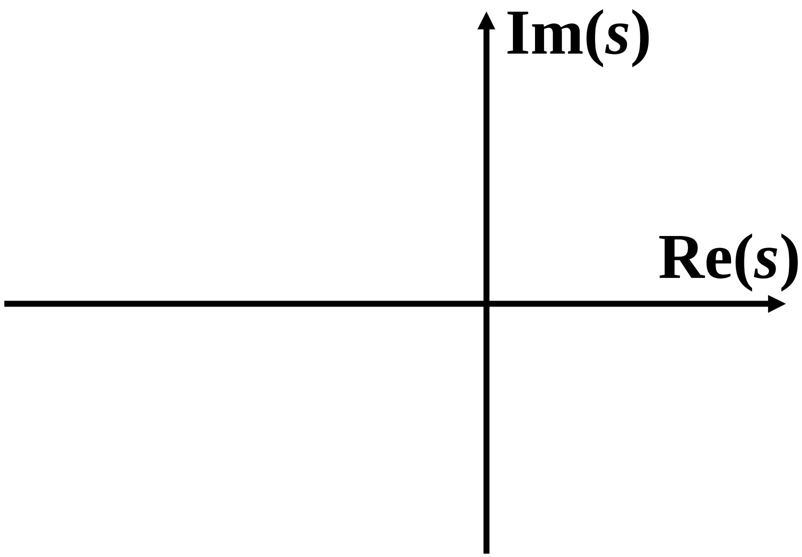


$$\omega_n \geq \frac{1.8}{t_{r_{\text{desired}}}}$$

- **Percent overshoot depends only on ζ :**

$$\% \ M_p \approx 100\left(1 - \frac{\zeta}{0.6}\right), \quad \text{for} \quad \zeta \leq 0.6$$

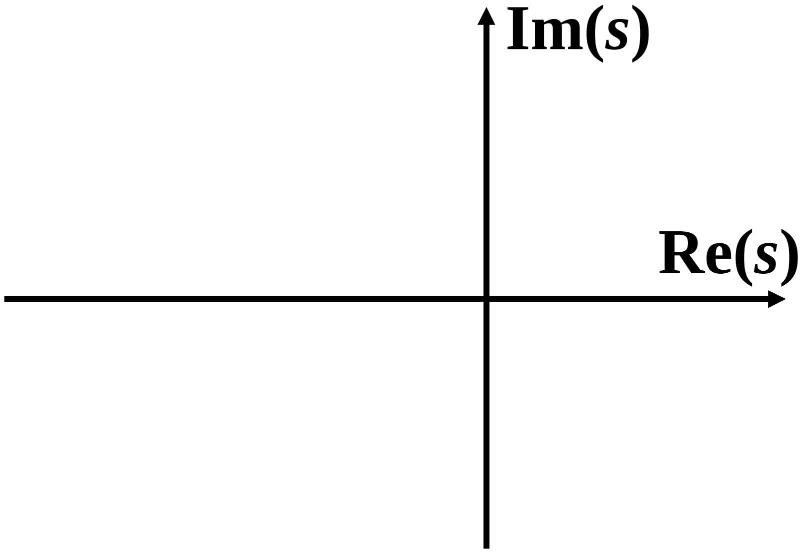
See Figure 2.7 of text.



$$\zeta \geq 0.6\left(1 - 0.01 M_{p_{\text{desired}}}\right)$$

- Can derive a more precise formula for M_p by determining $\dot{y}(t)$ and setting it to 0 to find the time t_p when M_p occurs.

$$y(t) = 1 - e^{-\sigma t}\left[\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right], \qquad t \geq 0$$



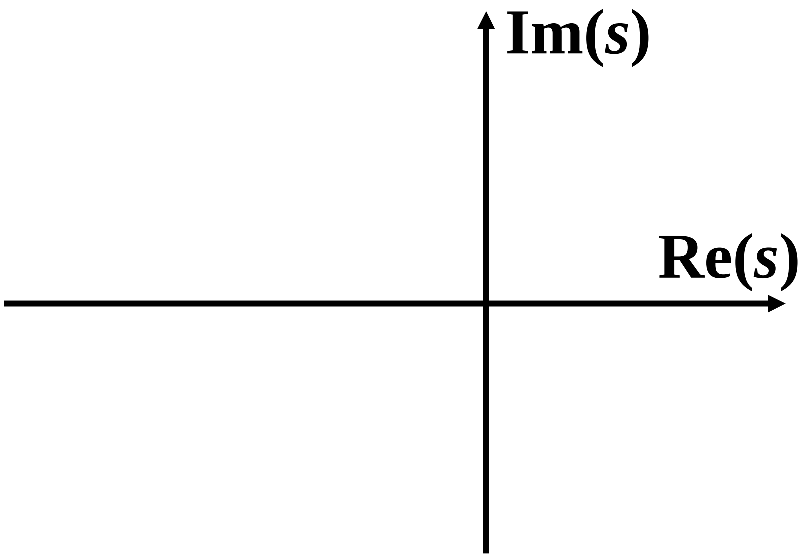
$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$$\zeta \geq \frac{\left|\ln M_{p_{\text{desired}}}\right|}{\sqrt{\pi^2 + \left(\ln M_{p_{\text{desired}}}\right)^2}}$$

- **Settling time depends only on σ :**

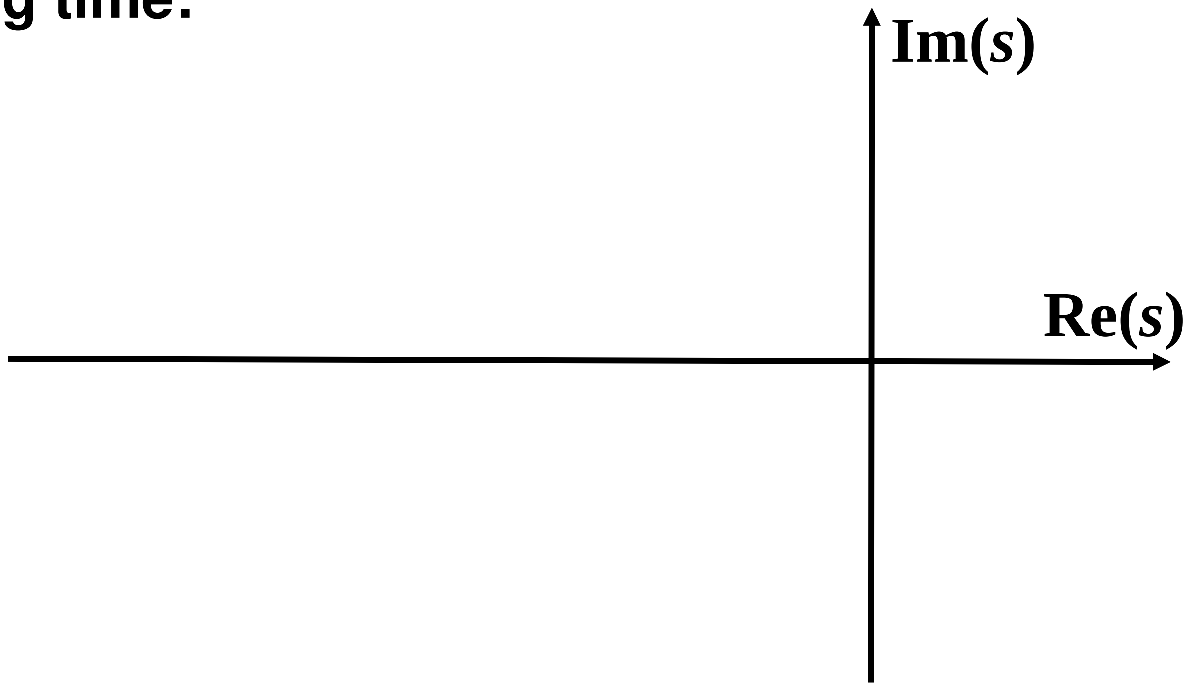
$$y(t) = 1 - e^{-\sigma t}\left[\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right], \qquad t \geq 0$$

$$t_s = \frac{4.6}{\zeta\omega_n}$$



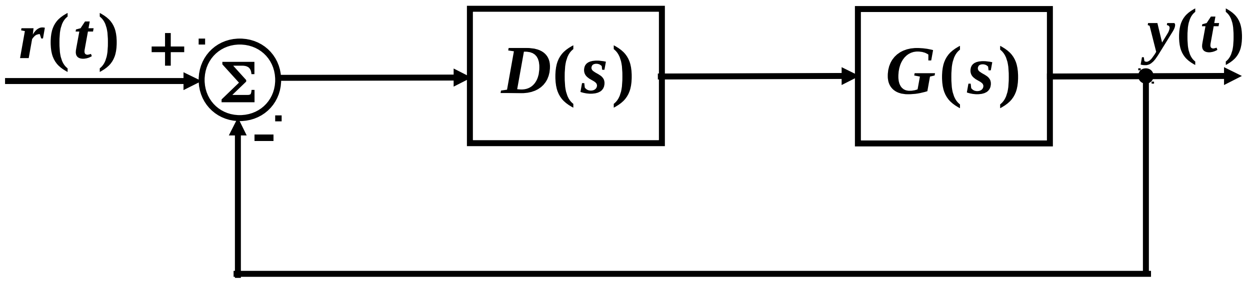
$$\sigma \geq \frac{4.6}{t_{s_{\text{desired}}}}$$

- **Combining performance specifications on rise time, overshoot, and settling time:**



- **If $H(s)$ has additional zeros or poles:**
  - **Additional zeros increase overshoot**
  - **Additional poles increase rise time**
- **This is similar to $H(z)$ having additional zeros or poles:**
  - **Additional zeros increase overshoot**
  - **Additional poles increase rise time**

Discrete Equivalent Design Example



- **Performance specifications:**

$$t_r \le 0.3 \quad \text{seconds}$$

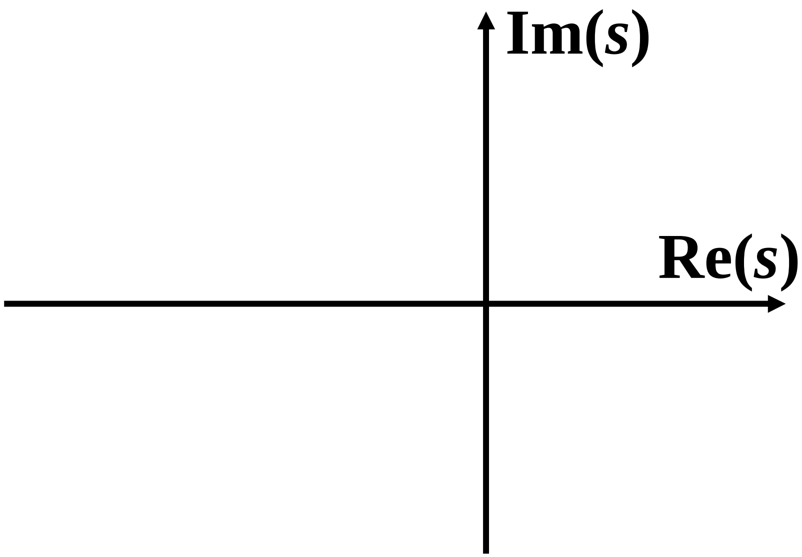
$$M_p \le 20\%$$

- **First design $D(s)$ to meet specifications.**
- **Then map over to a $D(z)$, with T = 0.075 seconds.**

- **Design a lead network to meet specifications:**

$$D(s) = K\frac{s+z}{s+p}$$

- **Want to design $D(s)$ such that closed-loop poles are in desired region:**



Root Locus Review

$$1 + D(s)G(s) = 1 + K\frac{s+z}{s+p}\cdot\frac{1}{s^2} = 0$$

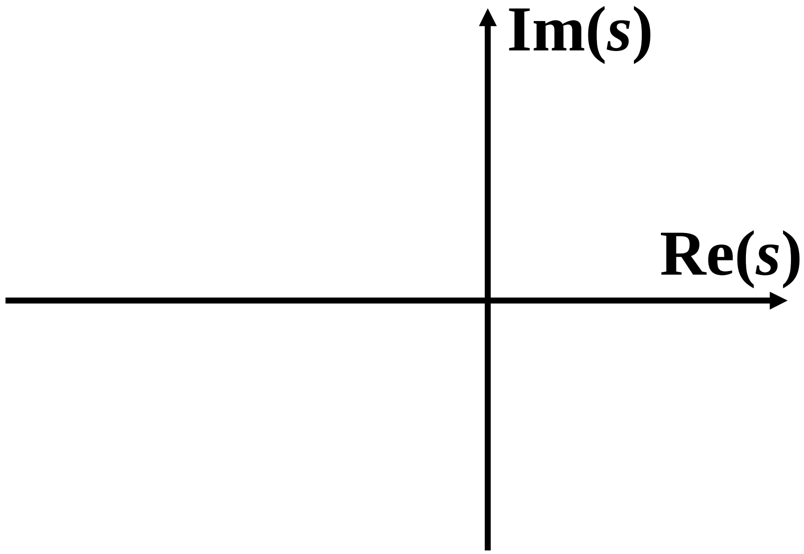
After choosing z and p, can just plot locus of roots of closed-loop equation as a function of K using root locus method:

$$1 + KH(s) = 0, \qquad K > 0$$

<u>180º Root Locus</u>

1. Mark poles and zeros of $H(s)$ in complex s-plane.

2. Segments to the left of an odd number of poles and zeros on the real axis are on the locus.



$$1+K\frac{s+z}{s+p}\cdot\frac{1}{s^2}=1+KH(s)=0, \qquad K>0$$

3. Sketch $(n-m)$ asymptotes for $K \rightarrow \infty$

$$\alpha=\frac{\sum p_i-\sum z_i}{n-m}$$

$$\phi_\ell=\frac{180^\circ+360^\circ(\ell-1)}{n-m}, \qquad \ell=1,\ 2,\ \ldots\ ,\ n-m$$

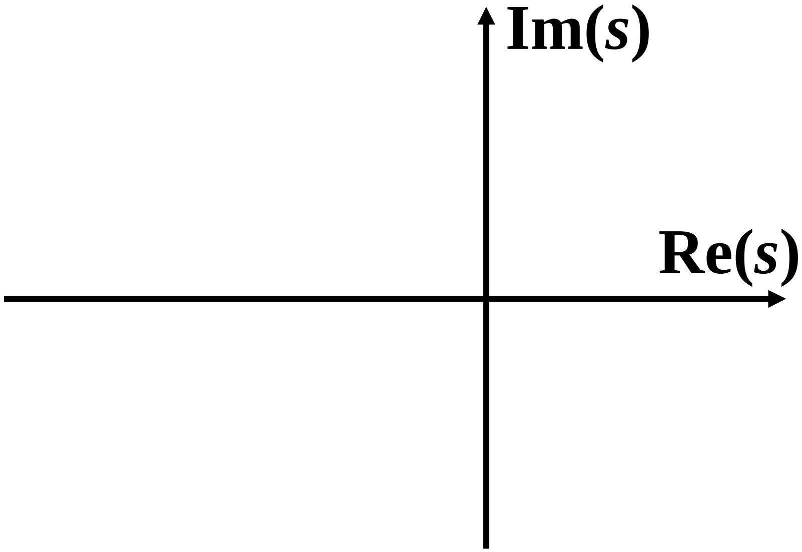
p_i : poles of $H(s)$

z_i : zeros of $H(s)$

n : number of poles of $H(s)$

m : number of zeros of $H(s)$

$n-m$: number of asymptotes



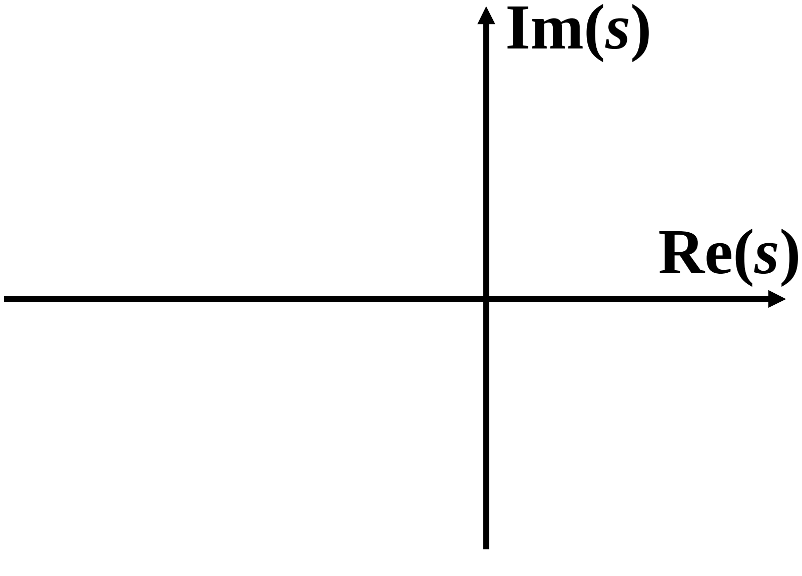
$$1+K\frac{s+5}{s+12}\cdot\frac{1}{s^2}=1+KH(s)=0, \qquad K>0$$

4. Determine departure angles from poles and arrival angles to zeros:

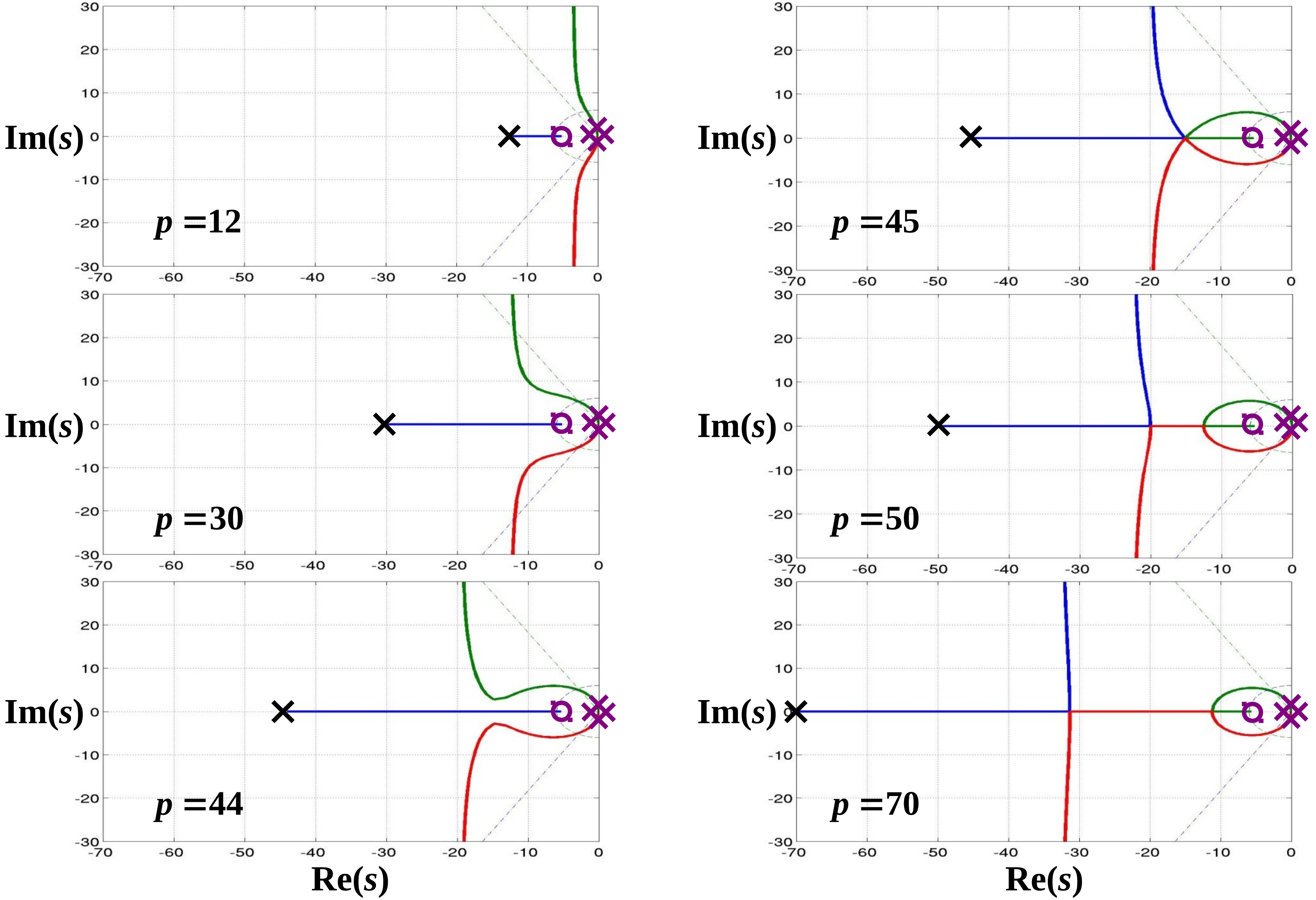
$$\phi_{dep}=\frac{1}{q}\left(\sum\psi_i-\sum\phi_i-180^\circ-360^\circ\ell\right), \qquad \ell=0,\ 1,\ \dots\ ,\ q-1$$

$$\psi_{arr}=\frac{1}{q}\left(\sum\phi_i-\sum\psi_i+180^\circ+360^\circ\ell\right), \qquad \ell=0,\ 1,\ \dots\ ,\ q-1$$

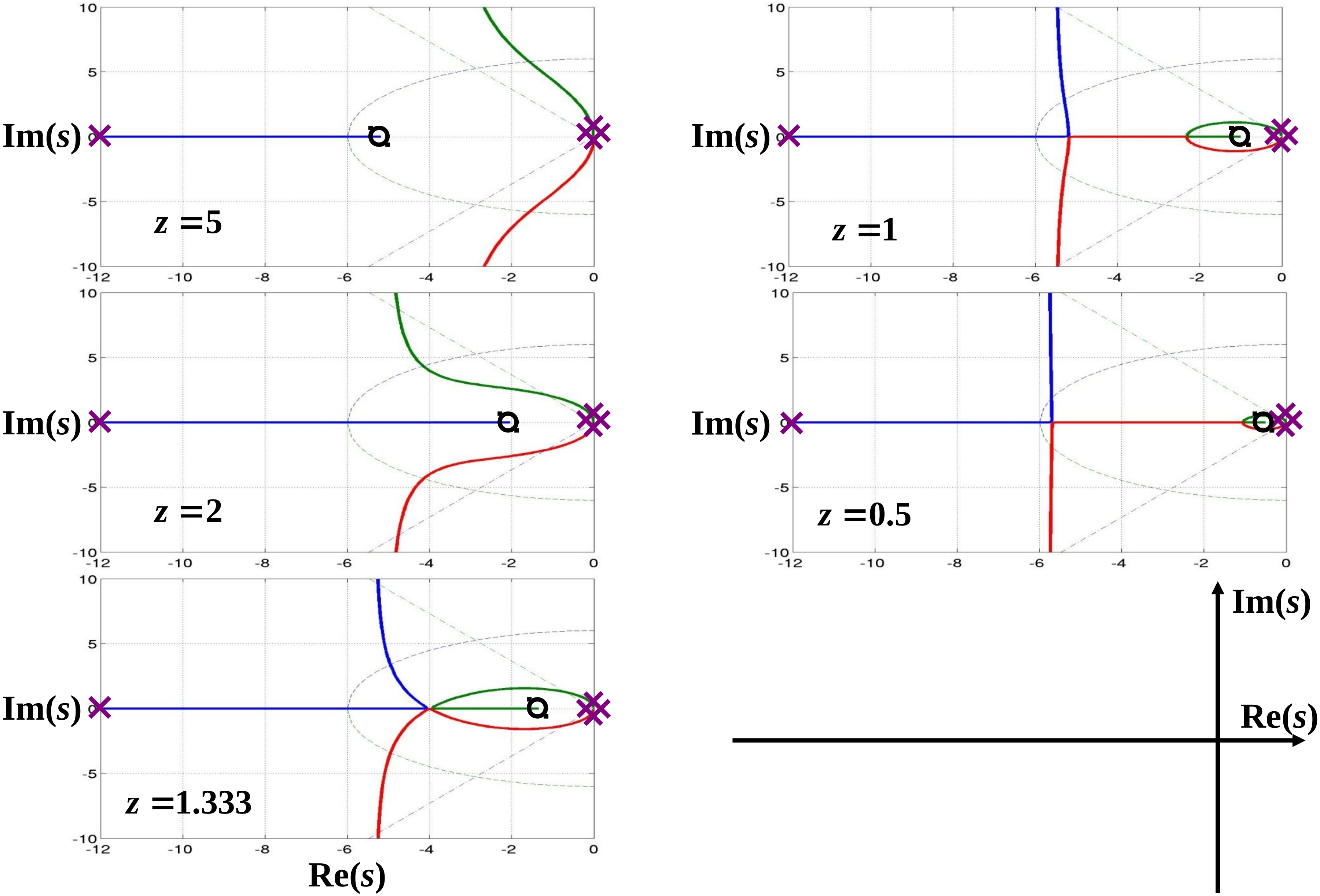
q : multiplicity of pole or zero



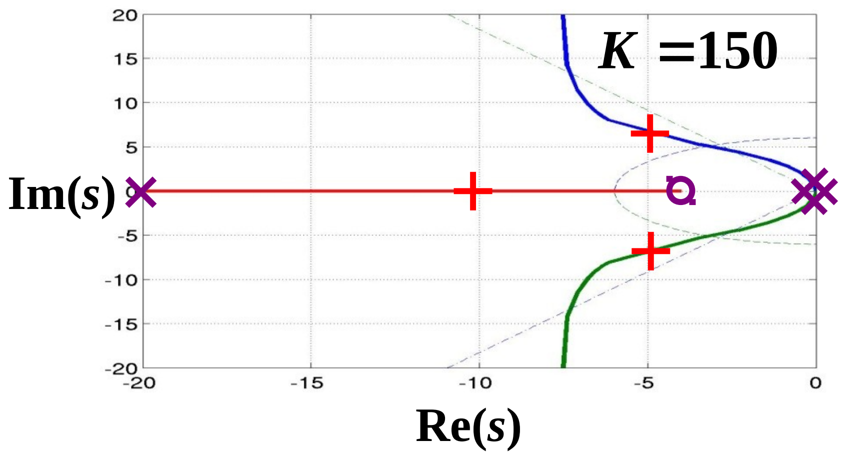
$$1 + K\frac{s+5}{s+p}\cdot\frac{1}{s^2} = 0, \qquad K > 0$$

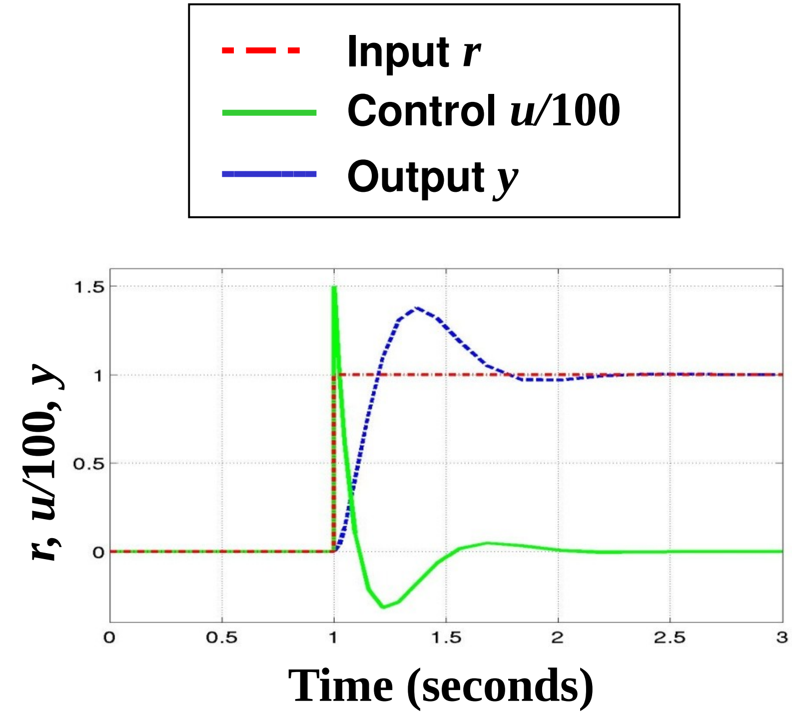


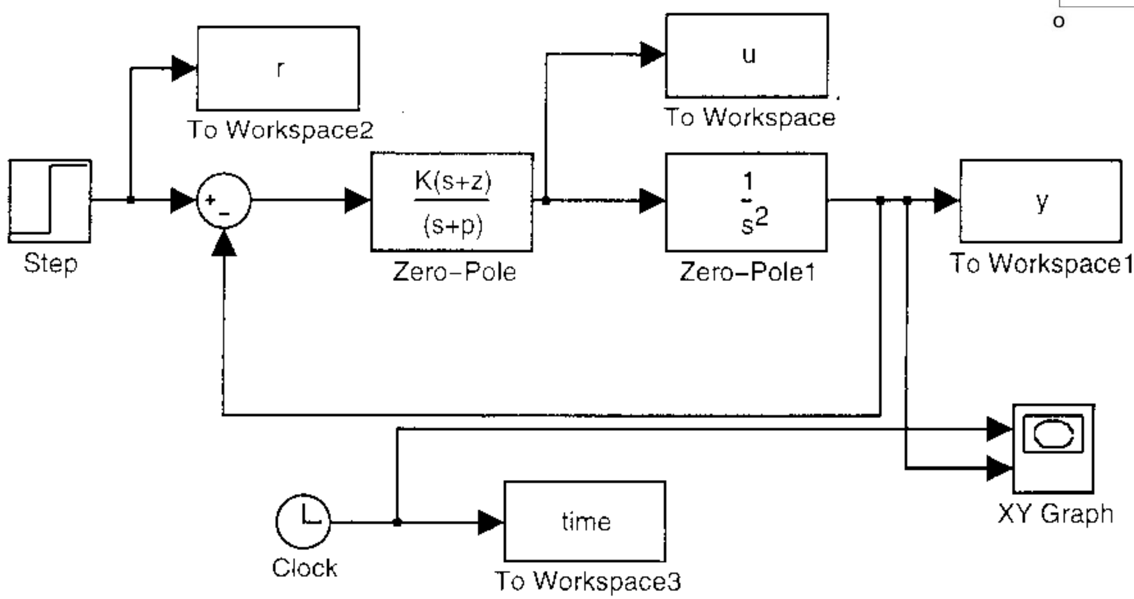
$$1 + K\frac{s+z}{s+12}\cdot\frac{1}{s^2} = 0, \qquad K > 0$$



Choose $D(s) = K\dfrac{s+4}{s+20}, \qquad K > 0$







$t_r = 0.17 \text{ sec} \leq 0.3 \text{ sec}$

$M_p = 37.8\% \geq 20\%$

$t_s = 1.19 \text{ sec}$

Useful MATLAB commands: rlocfind, sisotool