Estimator Design

- Open-loop estimator
- Closed-loop estimator
- Comparison between state feedback and estimator design
- Observability
- Observer canonical form
- Relationship between control & observer canonical forms

Estimator Design

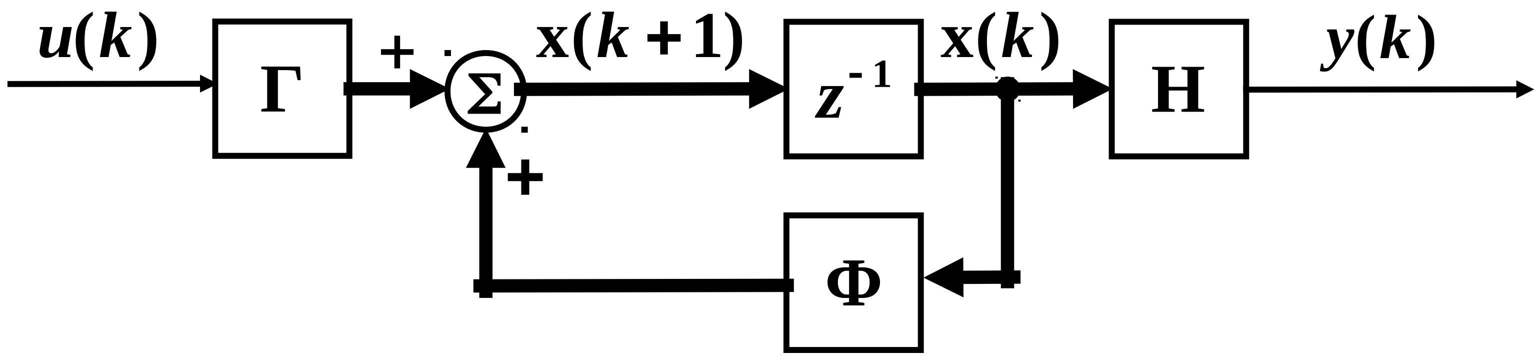
In most systems, not all the states are measured. Some states have to be estimated for state feedback control.

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma u(k)$$

$$y(k) = \mathrm{H}\mathbf{x}(k) + Ju(k)$$

One possibility: $\hat{\mathbf{x}}(k+1) = \Phi\hat{\mathbf{x}}(k) + \Gamma u(k)$

Assume $J = 0$:



Open-loop Estimator Problems

$$\hat{\mathbf{x}}(k+1) = \Phi\hat{\mathbf{x}}(k) + \Gamma u(k)$$

1. Do not know what $\mathbf{x}(0)$ is exactly.
2. Usually some modeling errors exist.
3. The initial error $\tilde{\mathbf{x}}(0) = \mathbf{x}(0) - \hat{\mathbf{x}}(0)$ may grow over time or go to zero too slowly to be useful.

Estimator error dynamics: $\tilde{\mathbf{x}}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$

$$\tilde{\mathbf{x}}(k+1) = \mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1)$$

$$\boxed{\tilde{\mathbf{x}}(k+1) = \Phi\tilde{\mathbf{x}}(k)}$$

Estimate error has same dynamics as uncompensated system!

Using Feedback

Can we alter the behavior or performance of the estimator through feedback?

$$\hat{\mathbf{x}}(k+1) = \Phi\hat{\mathbf{x}}(k) + \Gamma u(k) + \mathbf{L}\left(y(k) - \mathbf{H}\hat{\mathbf{x}}(k)\right)$$

where $\mathbf{L} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{bmatrix}$

Estimator error dynamics: $\tilde{\mathbf{x}}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$

$$\tilde{\mathbf{x}}(k+1) = \mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1)$$

$$\tilde{\mathbf{x}}(k+1) = (\Phi - \mathbf{L}\mathbf{H})\tilde{\mathbf{x}}(k)$$

Characteristic equation of estimate error:

$$\boxed{\det\left[z\mathbf{I} - (\Phi - \mathbf{L}\mathbf{H})\right] = 0}$$

Can choose $\mathbf{L}$ to change dynamics of the estimator.

If we can choose $\mathbf{L}$ such that $\Phi - \mathbf{L}\mathbf{H}$ has fast, stable eigenvalues, then the estimate error $\tilde{\mathbf{x}}(k) \to 0$ for all $\tilde{\mathbf{x}}(0)$ and $u(k)$.

Comparison Between State Feedback and Estimator Design

- Plant is a physical process.

- Controller design affects physical system directly.
  - Choice of actuators/motors.

- Estimator is a software algorithm.
  - Making estimator have fast poles does not affect the choice of motors.
  - If estimator poles are too fast, the measurement noise of sensors may become noticeable in the estimates.

Designing $\mathbf{L}$ is similar to designing $\mathbf{K}$

Determine desired estimator poles $p_{1e}, \quad \ldots \quad , \quad p_{ne}$.

Form desired estimator characteristic polynomial:

$$\alpha_e(z) = (z - p_{1e})(z - p_{2e}) \cdots (z - p_{ne})$$

Match coefficients with $\det\left[z\mathbf{I} - \Phi + \mathbf{L}\mathbf{H}\right]$.

Is it always possible to solve for $\mathbf{L}$ for any desired $\alpha_e(z)$ **?**

Observability

- It is possible to arbitrarily assign estimator poles if and only if the system is observable.
- If the system is not observable, it may be possible to solve for $\mathbf{L}$ to move some of the estimator poles, but can not move all the poles.

Definition: A system is observable if and only if the observability matrix is nonsingular, where the observability matrix is:

$$\mathcal{O} = \begin{bmatrix} \mathbf{H} \\ \mathbf{H\Phi} \\ \vdots \\ \mathbf{H\Phi}^{n-1} \end{bmatrix} \qquad \det \mathcal{O} \neq 0 \Longrightarrow \text{system is observable}$$

Observer Canonical Form

Consider third-order system for simplicity:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b(z)}{a(z)} = \frac{b_0 z^3 + b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}$$

Block Diagram of Observer Canonical Form

$$x_1(k+1) = -a_1 y(k) + x_2(k) + b_1 u(k)$$

$$x_2(k+1) = -a_2 y(k) + x_3(k) + b_2 u(k)$$

$$y(k) = x_1(k) + b_0 u(k)$$

$$x_3(k+1) = -a_3 y(k) + b_3 u(k)$$

We have:

$$x_1(k+1) = -a_1 x_1(k) + x_2(k) + (b_1 - a_1 b_0)u(k)$$

$$x_2(k+1) = -a_2 x_1(k) + x_3(k) + (b_2 - a_2 b_0)u(k)$$

$$x_3(k+1) = -a_3 x_1(k) + (b_3 - a_3 b_0)u(k)$$

$$y(k) = x_1(k) + b_0 u(k)$$

In state-variable matrix notation:

$$\mathbf{x}(k+1) = \Phi_o \mathbf{x}(k) + \Gamma_o u(k)$$

$$y(k) = \mathrm{H}_o \mathbf{x}(k) + J_o u(k)$$

where

$$\mathbf{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Observer Canonical Form in General

Given $$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

$$\Phi_o = \begin{bmatrix} -a_1 & 1 & 0 & 0 \\ -a_2 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ -a_n & 0 & \dots & 0 \end{bmatrix}, \qquad \Gamma_o = \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix},$$

$$H_o = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \qquad J_o = b_0$$

If strictly proper: $b_0 = 0$

$$\boxed{\det(zI - \Phi_o) = z^n + a_1 z^{n-1} + \cdots + a_n = 0}$$

Relationship Between Control & Observer Canonical Forms

Given $G(z) = \dfrac{Y(z)}{U(z)} = \dfrac{b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n}$

Observer canonical form:

$$\Phi_o = \begin{bmatrix} -a_1 & 1 & 0 & 0 \\ -a_2 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix}, \quad \Gamma_o = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathrm{H}_o = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \quad J_o = 0$$

Control canonical form:

$$\Phi_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Gamma_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathrm{H}_c = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}, \quad J_c = 0$$