Introducing a Reference Input

- **Designing $\mathbf{M}$ and N**
  - Design so that estimator performance is independent of reference input
  - General design of choosing overall system zeros

- **Output error command structure**
  - Difference between transfer function and state-space compensator design approaches

Introducing a Reference Input

Thus far, we have assumed that the reference input is zero.

That is, we want $y = 0$, thus in steady state, $\mathbf{x} = \mathbf{0}$ and $u = 0$.

If there are disturbances that cause $\mathbf{x} \neq \mathbf{0}$ and $y \neq 0$, then we have $u = -\mathbf{K}\mathbf{x}$ to reject the disturbances to bring $\mathbf{x} \approx \mathbf{0}$, $y \approx 0$.

Now suppose we want $r \neq 0$.

What is the best way to introduce r into the system?

Controller Equations

$$\hat{\mathbf{x}}(k+1) = \left(\Phi - \Gamma\mathbf{K} - \mathbf{L}_p\mathbf{H}\right)\hat{\mathbf{x}}(k) + \mathbf{L}_p y(k) + \mathbf{M}r(k)$$

$$u(k) = -\mathbf{K}\hat{\mathbf{x}}(k) + Nr(k)$$

Since r is an external signal, $\mathbf{M}$ and N do <u>not</u> affect the closed-loop characteristic equation. That is, the poles of the system are fixed, hence <u>$\mathbf{M}$ and N do not affect stability</u>.

$\mathbf{M}$ and N affect the locations of zeros, which affect the transient response of the system.

$\mathbf{M}$ and N also affect steady-state error.

Designing $\mathbf{M}$ and N

The most common way is to select $\mathbf{M}$ and N such that

$$\tilde{\mathbf{x}} \perp r$$

The idea here is that if the estimator is good, it should be good independent of any external excitation.

This is not the most general way of designing $\mathbf{M}$ and N though. The most general way is to design n zeros, then solve for $\mathbf{M}$ and N to yield those zeros.

Estimator Error Dynamics Equation

$$\tilde{\mathbf{x}}(k+1) = \mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1)$$

$$\tilde{\mathbf{x}}(k+1) = \left(\Phi - \mathbf{L}_p \mathbf{H}\right)\tilde{\mathbf{x}}(k) + \left(\Gamma N - \mathbf{M}\right) r(k)$$

If r does not affect estimator performance, then

$$\boxed{\mathbf{M} = \Gamma N}$$

Controller Equations

Originally: $$\hat{\mathbf{x}}(k+1) = \left(\Phi - \Gamma\mathbf{K} - \mathbf{L}_p\mathbf{H}\right)\hat{\mathbf{x}}(k) + \mathbf{L}_p y(k) + \mathbf{M}r(k)$$

$$u(k) = -\mathbf{K}\hat{\mathbf{x}}(k) + Nr(k)$$

Now: $$\hat{\mathbf{x}}(k+1) = \left(\Phi - \mathbf{L}_p\mathbf{H}\right)\hat{\mathbf{x}}(k) - \Gamma\mathbf{K}\hat{\mathbf{x}}(k) + \mathbf{M}r(k) + \mathbf{L}_p y(k)$$

$$\hat{\mathbf{x}}(k+1) = \left(\Phi - \mathbf{L}_p\mathbf{H}\right)\hat{\mathbf{x}}(k) + \Gamma u(k) + \mathbf{L}_p y(k)$$

$$u(k) = -\mathbf{K}\hat{\mathbf{x}}(k) + Nr(k)$$

Designing N

For a constant r, we want $e_{ss} = 0$ so that $y_{ss} = r_{ss} = r$

Plant: $\mathbf{x}(k+1) = \Phi \mathbf{x}(k) + \Gamma u(k)$

$$y(k) = \mathbf{H}\mathbf{x}(k) + J u(k)$$

In steady-state: $\mathbf{x}(k+1) = \mathbf{x}(k) = \mathbf{x}_{ss}$

$$\mathbf{x}_{ss} = \Phi \mathbf{x}_{ss} + \Gamma u_{ss}$$

$$y_{ss} = \mathbf{H}\mathbf{x}_{ss} + J u_{ss}$$

We want $y_{ss} = r_{ss}$, then:

u_{ss} is some constant scalar: $u_{ss} = N_u r_{ss}$

$\mathbf{x}_{ss}$ is some constant vector: $\mathbf{x}_{ss} = \mathbf{N}_x r_{ss}$

$$\boxed{\begin{bmatrix} \mathbf{N}_x \\ N_u \end{bmatrix} = \begin{bmatrix} \Phi - \mathbf{I} & \Gamma \\ \mathbf{H} & J \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}}$$

Now, we can write

$$u = -K\hat{x} + Nr$$

In steady-state,

$$u_{ss} = -Kx_{ss} + Nr$$

Thus,

$$u = -K\hat{x} + Kx_{ss} + u_{ss}$$

$$u = -\mathbf{K}\hat{\mathbf{x}} + \left(N_u + \mathbf{K}\mathbf{N}_\mathbf{x}\right) r$$

Overall System State Equations with Non-Zero Reference Input

Plant $G(z)$:

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma u(k)$$

$$y(k) = \mathbf{H}\mathbf{x}(k) + Ju(k)$$

Compensator $D(z)$:

$$\hat{\mathbf{x}}(k+1) = \left(\Phi - \mathbf{L}_p\mathbf{H}\right)\hat{\mathbf{x}}(k) + \Gamma u(k) + \mathbf{L}_p y(k)$$

$$u(k) = -\mathbf{K}\hat{\mathbf{x}}(k) + Nr(k)$$

$$\mathbf{M} = \Gamma N$$

$$\tilde{\mathbf{x}}(k+1) = \left(\Phi - \mathbf{L}_p\mathbf{H}\right)\tilde{\mathbf{x}}(k)$$

$$N = N_u + \mathbf{K}\mathbf{N}_\mathbf{x}$$

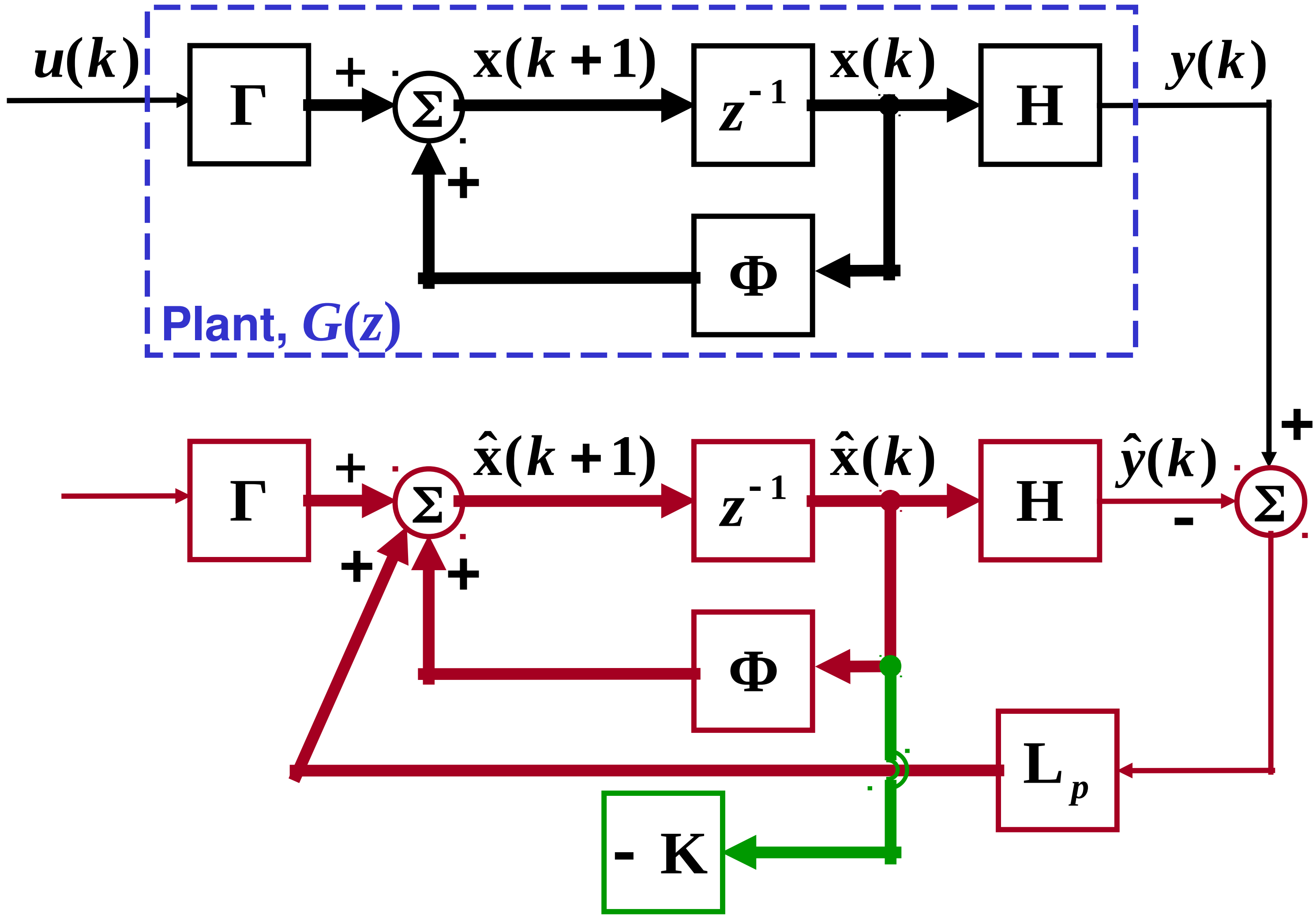
$$\begin{bmatrix} \mathbf{N}_\mathbf{x} \\ N_u \end{bmatrix} = \begin{bmatrix} \Phi - \mathbf{I} & \Gamma \\ \mathbf{H} & J \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

$$\mathbf{K}: \quad \det\left(z\mathbf{I} - \Phi + \Gamma\mathbf{K}\right) = \alpha_c(z)$$

$$\mathbf{L}_p: \quad \det\left(z\mathbf{I} - \Phi + \mathbf{L}_p\mathbf{H}\right) = \alpha_e(z)$$

State-Command Structure

Drives the plant model in the estimator with the <u>same</u> inputs that are applied to the actual plant.



General Case Where Zeros Are Designed

$$\hat{\mathbf{x}}(k+1) = \left(\Phi - \Gamma\mathbf{K} - \mathbf{L}_p\mathbf{H}\right)\hat{\mathbf{x}}(k) + \mathbf{L}_p y(k) + \mathbf{M}r(k)$$

$$u(k) = -\mathbf{K}\hat{\mathbf{x}}(k) + Nr(k)$$

1. Consider the case of only state feedback with no estimator, but with a non-zero reference input:

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma u(k)$$

$$y(k) = \mathbf{H}\mathbf{x}(k)$$

What are the zeros?

$$\begin{bmatrix} z\mathbf{I}-\Phi+\Gamma\mathbf{K} & -\Gamma \\ \mathbf{H} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}(z) \\ NR(z) \end{bmatrix} = 0$$

$$\det \begin{bmatrix} z\mathbf{I}-\Phi & -\Gamma \\ \mathbf{H} & 0 \end{bmatrix} = 0$$

2. Consider the case of the combined controller <u>and</u> estimator with non-zero reference input:

$$\hat{\mathrm{x}}(k+1) = \left(\Phi - \Gamma\mathrm{K} - \mathrm{L}_p\mathrm{H}\right)\hat{\mathrm{x}}(k) + \mathrm{L}_p y(k) + \mathrm{M}r(k)$$

$$u(k) = -\mathrm{K}\hat{\mathrm{x}}(k) + Nr(k)$$

What are the zeros from $r(k)$ to $u(k)$?

Without loss of generality, assume $y(k) = 0$.

$$\begin{bmatrix} z\mathrm{I} - \Phi + \Gamma\mathrm{K} + \mathrm{L}_p\mathrm{H} & -\dfrac{\mathrm{M}}{N} \\ -\mathrm{K} & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathrm{X}}(z) \\ NR(z) \end{bmatrix} = 0$$

$$\det\left[z\mathbf{I} - \boldsymbol{\Phi} + \boldsymbol{\Gamma}\mathbf{K} + \mathbf{L}_p\mathbf{H} - \frac{\mathbf{M}}{N}\mathbf{K} \right] = \gamma(z) = 0$$

$$\det\left[z\mathbf{I} - \mathbf{A} + \frac{\mathbf{M}}{N}(-\mathbf{K}) \right] = \gamma(z) = 0$$

Compare with estimator problem: $\det\left(z\mathbf{I} - \boldsymbol{\Phi} + \mathbf{L}\mathbf{H} \right) = 0$

We can choose zeros of compensator $D(z)$, and hence

$$H(z) = \frac{Y(z)}{R(z)}$$

If $(\mathrm{A}, \quad -\mathrm{K})$ is observable, then can choose desired zeros, then form $\gamma(z) = 0$ and solve for M/N.

$$\begin{bmatrix} -\mathrm{K} \\ -\mathrm{KA} \\ \vdots \\ -\mathrm{KA}^{n-1} \end{bmatrix}$$

$$\frac{Y(z)}{R(z)} = \eta \frac{\gamma(z) b(z)}{\alpha_e(z) \alpha_c(z)}$$

Special case:

$$\mathbf{M} = \Gamma N \qquad\qquad \tilde{\mathbf{x}} \perp r$$

What are the zeros from $r(k)$ to $u(k)$?

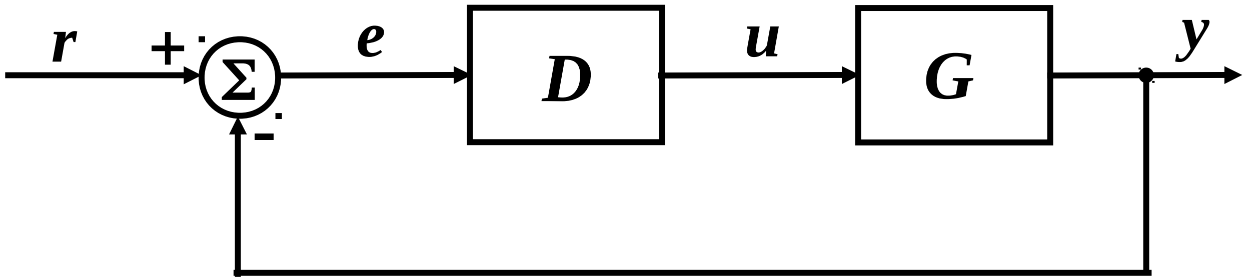
$$\det\left[z\mathbf{I} - \Phi + \Gamma\mathbf{K} + \mathbf{L}_p\mathbf{H} - \frac{\mathbf{M}}{N}\mathbf{K}\right] = \gamma(z) = 0$$

$$\gamma(z) = \det\left(z\mathbf{I} - \Phi + \mathbf{L}_p\mathbf{H}\right)$$

Overall poles of the system are: $\alpha_e(z)\alpha_c(z) = 0$

$$\frac{Y(z)}{R(z)} = \eta\frac{b(z)}{\alpha_c(z)}$$

Output Error Command Structure



The compensator equations:

$$\hat{\mathbf{x}}(k+1) = \left(\Phi - \Gamma\mathbf{K} - \mathbf{L}_p\mathbf{H}\right)\hat{\mathbf{x}}(k) + \mathbf{L}_p y(k) + \mathbf{M}r(k)$$

$$u(k) = -\mathbf{K}\hat{\mathbf{x}}(k) + Nr(k)$$

Since we need all r and y terms only in terms of $(r - y)$, we must have $\mathbf{M} = -\mathbf{L}_p, \quad N = 0$ and the compensator equations become:

$$\hat{\mathbf{x}}(k+1) = \left(\Phi - \Gamma\mathbf{K} - \mathbf{L}_p\mathbf{H}\right)\hat{\mathbf{x}}(k) - \mathbf{L}_p\left(r(k) - y(k)\right)$$

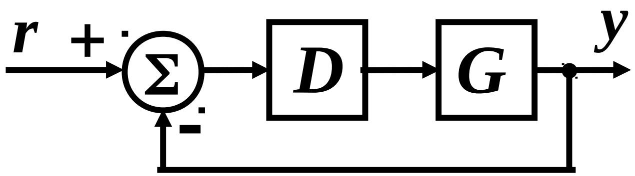
$$u(k) = -\mathbf{K}\hat{\mathbf{x}}(k)$$

What are the zeros of $D(z)$?

$$\begin{bmatrix} z\mathbf{I} - \Phi + \Gamma\mathbf{K} + \mathbf{L}_p\mathbf{H} & \mathbf{L}_p \\ -\mathbf{K} & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{X}}(z) \\ E(z) \end{bmatrix} = 0$$

$$\det\begin{bmatrix} zI - \Phi & L_p \\ 0 & K(zI - \Phi)^{-1}L_p \end{bmatrix} = 0$$

$$\det(zI - \Phi)K(zI - \Phi)^{-1}L_p = 0$$



These are zeros of the compensator, and they will be the zeros of the overall system unless they are cancelled by plant poles.

$$H(z) = \frac{Y(z)}{R(z)} = \frac{\gamma(z)b(z)}{\alpha_e(z)\alpha_c(z)}$$

Difference Between Transfer Function and State-Space Compensator Design Approaches

Drawback of transfer function approach:

Zeros of the compensator $D(z)$ become zeros of the overall system $H(z)$.

Advantage of state-space compensator with $\mathbf{M} = \Gamma N$:

Overall transfer function of the system only has plant zeros. The compensator does not add any additional zeros. This makes it often easier to predict the response.