Introduction to State-Space Methods

- Advantages
- Relationship between transfer functions and state equations (review)
- Control law design via state feedback

State-Space Methods

Continuous-Time

$$Y(s) = G(s)U(s)$$

$$\boxed{\begin{aligned}\dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}u \\ y &= \mathbf{H}\mathbf{x} + Ju\end{aligned}}$$

Discrete-Time

$$Y(z) = G(z)U(z)$$

$$\boxed{\begin{aligned}\mathbf{x}(k+1) &= \Phi\mathbf{x}(k) + \Gamma u(k) \\ y(k) &= \mathbf{H}\mathbf{x}(k) + Ju(k)\end{aligned}}$$

$$\boxed{\begin{aligned}\Phi &= e^{\mathbf{F}T} \\ \Gamma &= \int_0^T e^{\mathbf{F}\eta}d\eta\ \mathbf{G}\end{aligned}}$$

Advantages of state-space methods

1. Easier to solve numerically.
2. Transfer functions are only for LTI systems.
3. State-space methods are easily extendible to MIMO.

Relationship Between Transfer Functions and State Equations

$$G(z) = \frac{Y(z)}{U(z)} = \mathrm{H}(z\mathrm{I} - \Phi)^{-1}\Gamma + J$$

poles: $$\det(z\mathrm{I} - \Phi) = 0$$

zeros: $$\det\begin{bmatrix} z\mathrm{I} - \Phi & -\Gamma \\ \mathrm{H} & J \end{bmatrix} = 0$$

$$G(z) = \frac{\det\begin{bmatrix} z\mathrm{I} - \Phi & -\Gamma \\ \mathrm{H} & J \end{bmatrix}}{\det(z\mathrm{I} - \Phi)}$$

Control Law Design via State Feedback

For now, assume all states are measured and $r = 0$.

Suppose uncompensated system:

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma u(k)$$
$$y(k) = \mathrm{H}\mathbf{x}(k) + Ju(k)$$

does not perform satisfactorily.

Then: $u(k) = -\mathrm{K}\mathbf{x}(k)$

$$= -\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}$$

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma u(k)$$
$$y(k) = \mathbf{H}\mathbf{x}(k) + Ju(k)$$

$$u(k) = -\mathbf{K}\mathbf{x}(k)$$

$$\boxed{\mathbf{x}(k+1) = (\Phi - \Gamma\mathbf{K})\mathbf{x}(k)}$$

Characteristic equation: $\det(z\mathbf{I} - \Phi + \Gamma\mathbf{K}) = 0 \implies$ **compensated poles**

Determine desired poles $p_1, \quad \ldots \quad, \quad p_n$

Desired closed-loop characteristic equation:

- **Want to set n poles**
- **Have $k_1, \ldots, k_n$ gains to design**
- **n degrees of freedom**

Question: Is it always possible to choose $k_1, \ldots, k_n$ such that the closed-loop (compensated) poles are at any arbitrary locations?

Controllability

- **State feedback control can arbitrarily move poles to any desired locations if and only if the system is controllable.**
- **If the system is not controllable, state feedback control may be able to move some of the poles, but can not move all the poles.**

Definition: A system is controllable if and only if the controllability matrix is nonsingular, where the controllability matrix is:

$$\mathcal{C} = \begin{bmatrix} \Gamma & \Phi\Gamma & \cdots & \Phi^{n-1}\Gamma \end{bmatrix}$$

$$\det \mathcal{C} \neq 0$$

Example

$$G(s) = \frac{1}{s(s+1)} = \frac{Y(s)}{U(s)}$$

1. Find a state-space representation of the system.
2. Find the discrete state-space representation of the sampled system.
3. For $T = 2$ sec, design a state feedback control law such that $t_r < 2\text{sec}, \quad M_p < 40\%.$

1. $$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s}$$

$$x_1 = y \qquad \dot{x}_1 = x_2$$

$$x_2 = \dot{y} \qquad \dot{x}_2 = -x_2 + u$$

2. Find the discrete state-space representation of the sampled system.

$$\mathbf{x}(k+1) = \Phi \mathbf{x}(k) + \Gamma u(k)$$
$$y(k) = \mathbf{H}\mathbf{x}(k) + Ju(k)$$

$$\boxed{\begin{aligned} \Phi &= e^{\mathbf{F}T} \\ \Gamma &= \int_0^T e^{\mathbf{F}\eta}\, d\eta\, \mathbf{G} \end{aligned}}$$

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boxed{\Phi = \begin{bmatrix} 1 & 1 - e^{-T} \\ 0 & e^{-T} \end{bmatrix}}$$

$$\boxed{\Phi = e^{\mathbf{F}T} = \begin{bmatrix} 1 & 1 - e^{-T} \\ 0 & e^{-T} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$\Gamma = \int_0^T e^{\mathbf{F}\eta} d\eta \, \mathbf{G}$$

$$\boxed{\Gamma = \begin{bmatrix} T + e^{-T} - 1 \\ 1 - e^{-T} \end{bmatrix}}$$

3. For $T = 2$ sec, design a state feedback control law such that $t_r < 2\text{sec}, \quad M_p < 40\%$.

For $T = 2$ sec: $$\Phi = \begin{bmatrix} 1 & 0.865 \\ 0 & 0.135 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1.135 \\ 0.865 \end{bmatrix}$$

Where are uncompensated (open-loop) poles?

Where are desired compensated (closed-loop) poles?

Desired compensated characteristic polynomial:

What is the actual compensated characteristic polynomial as a function of the state feedback gains?

$$= \det\left[\begin{bmatrix} z-1 & -0.865 \\ 0 & z-0.135 \end{bmatrix} + \begin{bmatrix} 1.135k_1 & 1.135k_2 \\ 0.865k_1 & 0.865k_2 \end{bmatrix} \right]$$

$$= \det \begin{bmatrix} z - 1 + 1.135k_1 & -0.865 + 1.135k_2 \\ 0.865k_1 & z - 0.135 + 0.865k_2 \end{bmatrix}$$

$$\det(z\mathrm{I} - \Phi + \Gamma \mathrm{K}) = z^2 + (1.135k_1 + 0.865k_2 - 1.135)z$$
$$+ (0.595k_1 - 0.865k_2 + 0.135)$$

Coefficients of z should always be a linear function of the k_i's.

Match coefficients with desired compensated characteristic polynomial:

$$\alpha_c(z) = z^2 + 0.363z + 0.301$$

$$\det(z\mathrm{I} - \Phi + \Gamma\mathrm{K}) = z^2 + (1.135k_1 + 0.865k_2 - 1.135)z + (0.595k_1 - 0.865k_2 + 0.135)$$

$$\boxed{\mathrm{K} = \begin{bmatrix} 0.962 & 0.469 \end{bmatrix}}$$

Summary of Matching Coefficients Method

1. Determine and compute the desired compensated characteristic polynomial:

$$\alpha_c(z) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n .$$

2. Compute the actual compensated characteristic polynomial as a function of the state feedback gains:

$$\det(z\mathrm{I} - \Phi + \Gamma\mathrm{K}).$$

3. Match coefficients and solve for the k_i's.

This is a way of designing the state feedback control gains. May not always work though. If the system is uncontrollable, then it may be impossible to solve for k_i's to achieve the desired coefficients of $\alpha_c(z)$.