Homework 5: Digital Control (ECEN 5458)

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Problem 1

(a)

Use the infinite series expansion to compute $\mathbf{\Phi} = e^{\boldsymbol{A}T}$. Where:

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

First find A^2 and A^3:

$$\boldsymbol{A}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\boldsymbol{A}^3 = \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix}$$

Thus, we find $\mathbf{\Phi}$.

$$\mathbf{\Phi} = I + \begin{bmatrix} -T & 0 \\ 0 & -2T \end{bmatrix} + \begin{bmatrix} \frac{T}{2!} & 0 \\ 0 & \frac{4T}{2!} \end{bmatrix} + \begin{bmatrix} \frac{-T}{3!} & 0 \\ 0 & \frac{-8T}{3!} \end{bmatrix} + \cdots = \begin{bmatrix} \sum_{i=0}^{\infty} \frac{(-1)^i T^i}{i!} & 0 \\ 0 & \sum_{i=0}^{\infty} \frac{(-2)^i T^i}{i!} \end{bmatrix}$$

The simple form of this is of course:

$$\mathbf{\Phi} = \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix}$$

(b)

Here we want to show that if $\boldsymbol{F} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}$ for some non-singular transformation $\boldsymbol{T}$ then:

$$e^{\boldsymbol{F}T} = \boldsymbol{T}e^{\boldsymbol{A}T}\boldsymbol{T}^{-1}$$

This property comes fairly quickly from the Taylor series.

$$e^{\boldsymbol{F}T} = \boldsymbol{I} + \boldsymbol{F}T + \frac{\boldsymbol{F}^2T^2}{2!} + \cdots = \boldsymbol{T}\boldsymbol{I}\boldsymbol{T}^{-1} + \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}T + \frac{\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}T^2}{2!} + \dots$$

Since all of the inner $\boldsymbol{T}^{-1}\boldsymbol{T}$ terms reduce to $\boldsymbol{I}$ all $\boldsymbol{F}^n$ terms will reduce to $\boldsymbol{T}A^n\boldsymbol{T}^{-1}$. Then we factor out to get:

$$e^{\boldsymbol{F}T} = \boldsymbol{T}\left(\boldsymbol{I} + \boldsymbol{A}T + \frac{\boldsymbol{A}^2T^2}{2!} + dots\right)\boldsymbol{T}^{-1} = \boldsymbol{T}e^{\boldsymbol{A}T}\boldsymbol{T}^{-1}$$

(c)

Show that if:

$$\boldsymbol{F} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$$

there exists a $\boldsymbol{T}$ such that $\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \boldsymbol{F}$.
This can be done by expanding both sides with $\boldsymbol{T}\boldsymbol{A} = \boldsymbol{F}\boldsymbol{T}$. $\boldsymbol{T}$ is an arbitrary matrix:

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then we get:

$$-1a = -3a + b \quad -1b = -2a \quad -2c = -3c + d \quad -2d = -2c$$

This gives that $c = d$ and $b = 2a$, which are the only constraints on the matrix T.

(d)

Using the property in part b we just need a transformation matrix:

$$\boldsymbol{T} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

The inverse of this is:

$$\boldsymbol{T}^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

Finally we get:

$$e^{\boldsymbol{F}T} = \boldsymbol{T}e^{\boldsymbol{A}T}\boldsymbol{T}^{-1} = \boldsymbol{T}\begin{bmatrix} -e^{-T} & e^{-T} \\ 2e^{-2T} & -e^{-2T} \end{bmatrix} = \begin{bmatrix} -e^{-T} + 2e^{T} & e^{-T} - e^{-2T} \\ -2e^{-T} + 2e^{-2T} & 2e^{-T} - e^{-2T} \end{bmatrix}$$

Problem 2

Given the rigid body plant:

$$G_1(s) = \frac{y(s)}{u(s)} = \frac{C}{s^2}$$

where $C = \frac{1}{21}$.

(a)

Convert the system to a discret-time state-space form with $T = 0.2$. Use the state vector $\boldsymbol{x} = \begin{bmatrix} \dot{y} = x_1 & y = x_2 \end{bmatrix}^T$ for the state representation of the continuous-time $G_1(s)$.
The differential equation is:

$$y\ddot{(t)} = Cu(t)$$

since $\dot{x_1} = \ddot{y} = Cu$ and $\dot{x_2} = x_1$ we get:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x_1} = \ddot{y} \\ \dot{x_2} = x_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{21} \\ 0 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now we find the discrete form. First, we realize that $\boldsymbol{F}^n$ for $n > 1$ is $\mathbf{0}$. Thus $\boldsymbol{\Phi}$ is:

$$\boldsymbol{\Phi} = \boldsymbol{I} + \boldsymbol{F}T = \begin{bmatrix} 1 & 0 \\ T = 0.2 & 1 \end{bmatrix}$$

Then, we need to find $\boldsymbol{\Gamma}$:

$$\boldsymbol{\Gamma} = \int_0^T e^{\boldsymbol{F}\eta} d\eta \boldsymbol{G} = \begin{bmatrix} T = 0.2 & 0 \\ T^2 = 0.04 & T = 0.2 \end{bmatrix} \begin{bmatrix} \frac{1}{21} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{105} \\ \frac{1}{21*25} \end{bmatrix}$$

(b)

Find the control law state-feedback gain $\boldsymbol{K}$ so that the poles of the full state-feedback system have a natural frequency of $\omega = 1.0$rad/sec and a damping coefficient $\zeta = 0.5$.
So, the closed-loop poles of our system need to be at:

$$p_1, p_2 = e^{sT}\Big|_{s=-\zeta\omega \pm j\omega\sqrt{1-\zeta^2}} = 0.8193 \pm 0.15594j$$

The closed loop characteristic equation is then:

$$\alpha_c(z) = (z - p_1)(z - p_2) = z^2 - 1.786z + 0.81873$$

Now, we need that $\det(z\boldsymbol{I} - \boldsymbol{\Phi} + \boldsymbol{\Gamma K}) = z^2 - 1.786z + 0.81873$. This gives:

$$\begin{bmatrix} z-1 & 0 \\ -0.2 & z-1 \end{bmatrix} + \begin{bmatrix} \frac{k_1}{105} & \frac{k_2}{105} \\ \frac{k_1}{21*25} & \frac{k_2}{21*25} \end{bmatrix} = \begin{bmatrix} z - 1 + \frac{k_1}{105} & \frac{k_2}{105} \\ -0.2 + \frac{k_1}{21*25} & z - 1 + \frac{k_2}{21*25} \end{bmatrix}$$

Then, the characteristic equation is:

$$\alpha_c(z) = z^2 + \left(-2 + \frac{k_1}{105} + \frac{k_2}{21*25}\right) + \left(\frac{k_2}{21*25} - \frac{k_1 k_2}{21^2 5^3} + 1 + \frac{k_1 k_2}{21^2 5^3} - \frac{k_1}{105} - \frac{k_2}{21*25}\right)$$

So we need to solve for $0.21740 = \frac{k_1}{105} + \frac{k_2}{21*25}$ and $-0.18127 = \frac{k_1}{105}$. This gives $k_1 = -19.033$ and $k_2 = 209.30$ and thus:

$$\boldsymbol{K} = \begin{bmatrix} -19.033 & 209.30 \end{bmatrix}$$

Problem 3

Consider the mass-spring-damper-mass plant:

$$G_2(s) = \frac{d(s)}{u(s)} = \frac{b}{Mm} \frac{s + \frac{k}{b}}{s^2(s^2 + (\frac{1}{m} + \frac{1}{M})(bs + k))}$$

Assume that $M = 20$kg, $m = 1$kg, $k = 32$N/m, $b = 0.3$ N-sec/m. Will be using matlab.

(a)

What is the damping coefficient and the oscillatory frequency of the system in Hz?
I have done this in Matlab with the damp function. We can also see that $\omega = \sqrt{(\frac{1}{m} + \frac{1}{M}) * k}$ and $2\omega\zeta = (\frac{1}{m} + \frac{1}{M}) * b$. This gives $\omega = 5.796$rad/s. In Hertz it will be $\omega_{Hz} = 0.9225$. The damping ratio is then $\zeta = 0.1707$

(b)

Convert the system to discrete-time state-space form with $T = 0.2$sec. Use the state vector $\boldsymbol{x} = \begin{bmatrix}\dot{y} y \dot{d} d\end{bmatrix}^T$ for the state space representation of the continuous system.

Unfortunately, Matlab won't put it in the required form, so I will do that by hand. We are given the equations:

$$\ddot{y} = \frac{u - b(\dot{y} - \dot{d}) - k(y - d)}{M} \quad \ddot{d} = \frac{-b(\dot{d} - \dot{y}) - k(d - y)}{m}$$

This gives the matrices:

$$A = \begin{bmatrix} -\frac{b}{M} & -\frac{k}{M} & \frac{b}{M} & \frac{k}{M} \\ 1 & 0 & 0 & 0 \\ \frac{b}{m} & \frac{k}{m} & -\frac{b}{m} & -\frac{k}{m} \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{M} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \quad D = 0$$

These are then put into matlab and the c2d function is used to put it into a discrete form. The code is in the appendix. The final matrices for the discrete form are:

$$\Phi = \begin{bmatrix} 0.3638 & -0.6436 & 0 & 0 \\ 1.226 & 0.4121 & 0 & 0 \\ 0.14 & 0.1558 & 1 & 0 \\ 0.009821 & 0.01788 & 0.2 & 1 \end{bmatrix} \quad \Gamma = \begin{bmatrix} 0.07662 \\ 0.06999 \\ 0.00491 \\ 0.0002519 \end{bmatrix} \quad H = \begin{bmatrix} 0 & 0 & 0.00375 & 0.4 \end{bmatrix} \quad J = 0$$

Code Appendix

```
%Zachary Vogel
%ecen 5458
%Problem 3 on homework 5

M1=20;
m2=1;
k=32;
b=0.3;

num=[b/(M1*m2) k/(M1*m2)];
den=[1 b*(1/M1+1/m2) k*(1/M1+1/m2) 0 0];
Gs=tf(num,den);
damp(Gs);
sys=ss(Gs);
sysd=c2d(sys,0.2);

A=[-b/M1 -k/M1 b/M1 k/M1;1 0 0 0;b/m2 k/m2 -b/m2 -k/m2;0 0 1 0];
B=[1/M1;0;0;0];
C=[0 0 0 1];
D=0;

sys1=ss(A,B,C,D)
sys1d=c2d(sys,0.2)
```