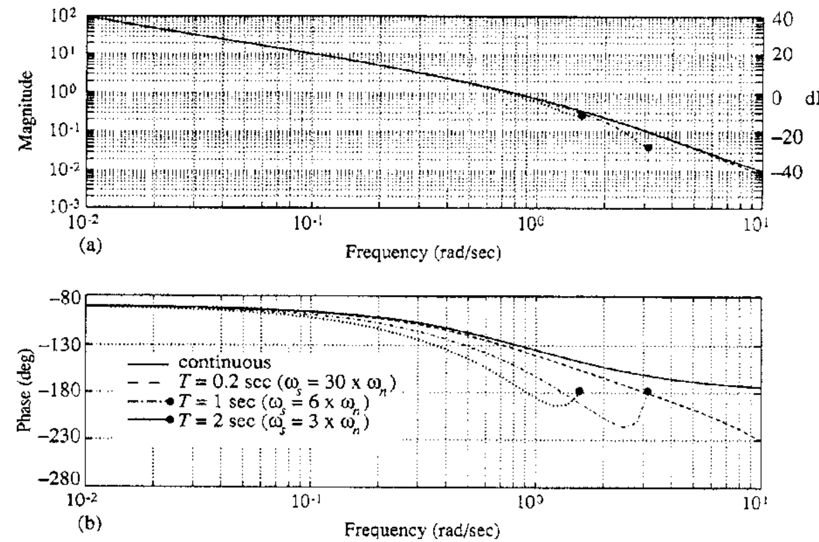
Bode Design and Lead, Lag, & PID Compensators

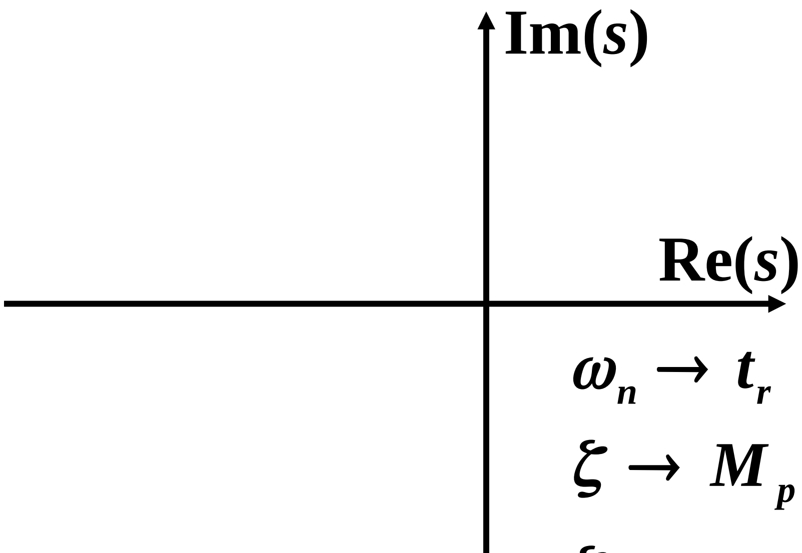
- Discrete-time compensator design using Bode plots
- Z-plane Bode plotting rule of thumb
- Lead and lag compensators
  - Example
- PID control
  - P, D, PD, I, PI control

Discrete-Time Compensator Design using Bode Plots

What is correspondence of previous design parameters with quantities determined from Bode plots?



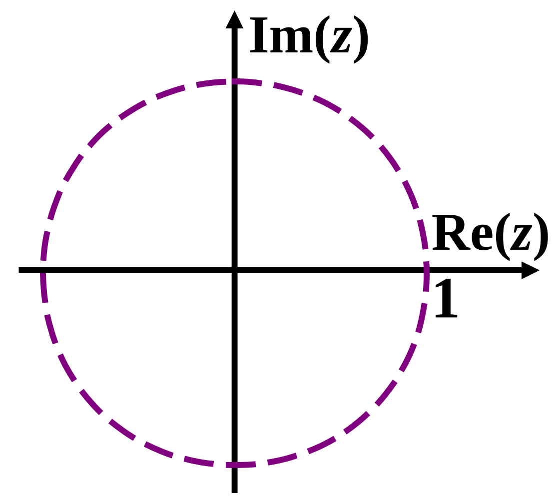
For a 2nd-order continuous-time system w/o finite zeros:



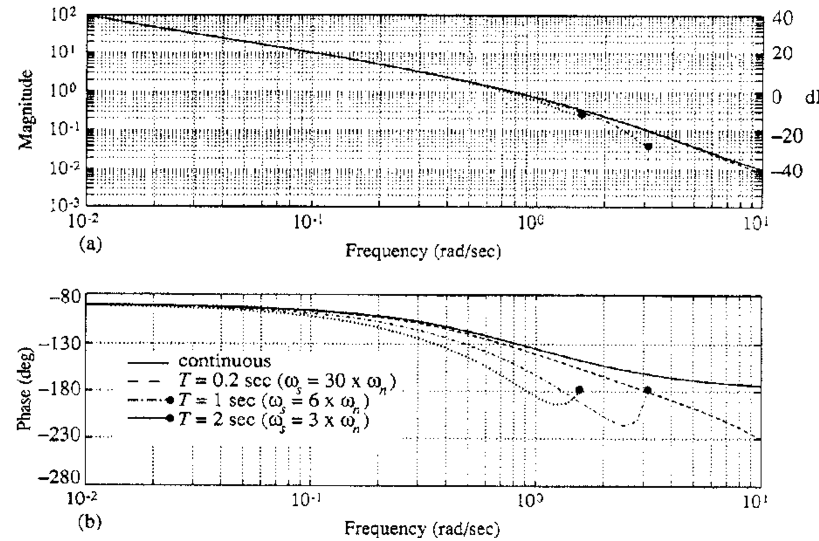
$$\omega_n \rightarrow t_r$$

$$\zeta \rightarrow M_p$$

$$\zeta\omega_n \rightarrow t_s$$



Revisiting Example 7.8 yet again:

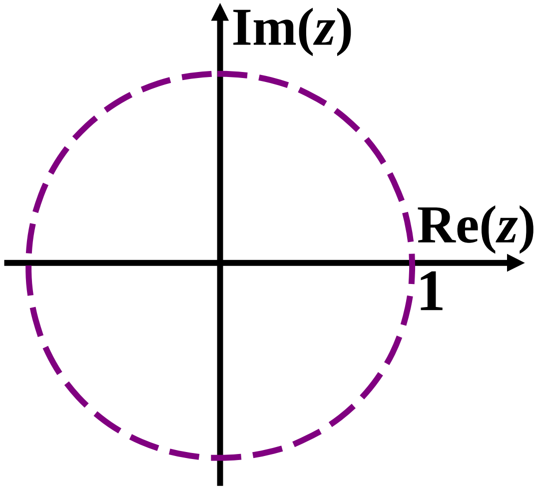


$$H_{CL} = \frac{\frac{1}{s(s+1)}}{1+\frac{1}{s(s+1)}} =$$

See Figure 7.22 of text.

Z-Plane Bode Plotting Rule of Thumb

- **Equivalent idea of "breakpoint" in plotting discrete Bode plots:**
  - **Magnitude will change slope at a frequency when ωT, the angular position on the unit circle in radians, has the same value as the fractional distance of the singularity on the real axis to $z = +1$**
    - **Rule of thumb accurate for ωT relatively small.**



For instance,

$$D(z) = K\frac{z - z_1}{z - p_1}$$

$$D\left(e^{j\omega T}\right) = K\frac{e^{j\omega T} - z_1}{e^{j\omega T} - p_1}$$

For ωT small $\Longrightarrow$

$$D\left(e^{j\omega T}\right) \approx$$

$$D\left(e^{j\omega T}\right) = K\frac{(1 - z_1) + j\omega T}{(1 - p_1) + j\omega T}$$

Breakpoints for numerator and denominator occur at ω where the real and imaginary parts are equal:

Numerator : $|1 - z_1| = \omega T$

Denominato r : $|1 - p_1| = \omega T$

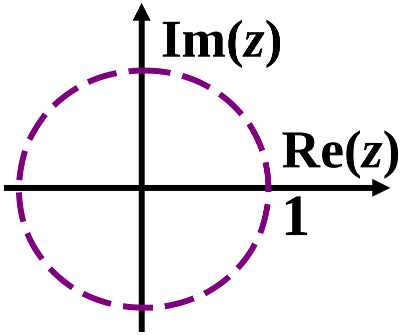
Very accurate for $\omega T \le 0.1$ rad

Moderately accurate for $\omega T \le 0.8$ rad

Lead and Lag Compensators

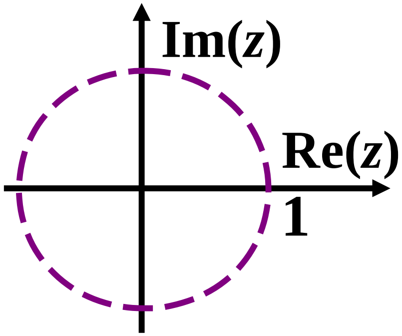
Lead Compensation

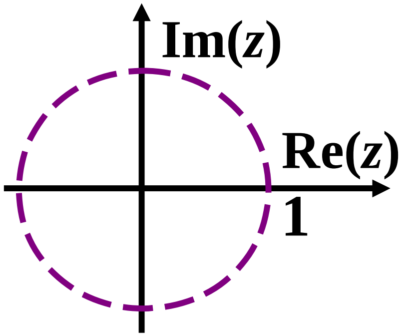
$$D(z) = K\frac{z - z_1}{z - p_1}, \qquad z_1 > p_1$$



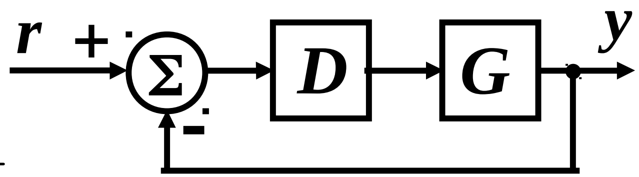
Lag Compensation

$$D(z) = K\frac{z - z_1}{z - p_1}, \qquad z_1 < p_1$$

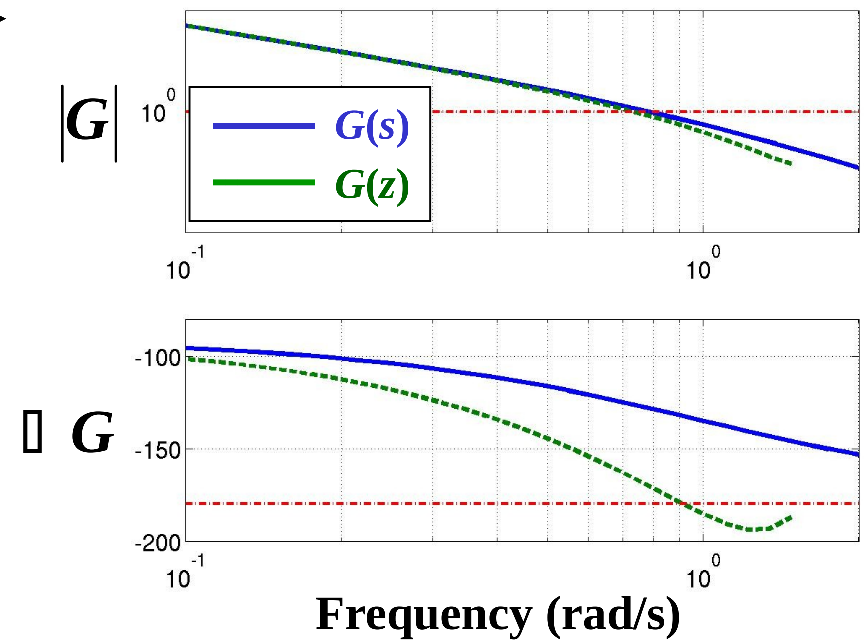




Example



$$G(s) = \frac{1}{s(s+1)}$$



$T = 2$ sec:

$$G(z) = 1.135\frac{z+0.523}{(z-1)(z-0.135)}$$

Uncompensated ($D(z) = 1$):

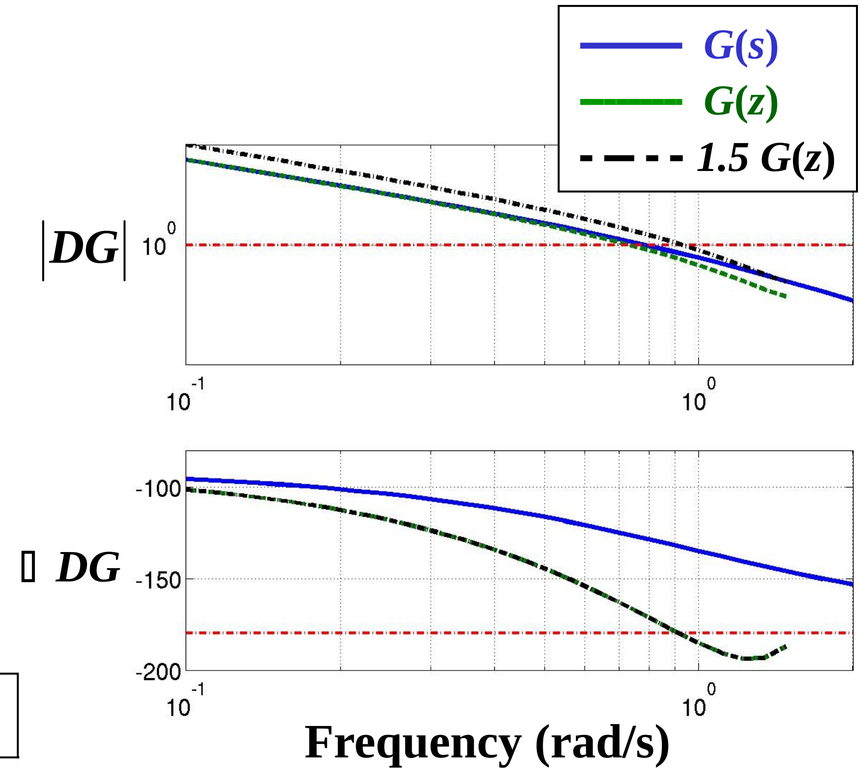
Type 1 system: $K_v = \lim_{z \to 1} \dfrac{(z-1)D(z)G(z)}{Tz} =$

$K_v \approx \qquad \Longrightarrow \qquad e_{ss}$ to unit ramp

Design a compensator so e_{ss} to unit ramp < 2/3.

If $D(z) = K = \frac{3}{2} \quad \Longrightarrow \quad K_v = \lim_{z\to 1} \frac{(z-1)D(z)G(z)}{Tz} =$

Are there any problems with doing this?



Useful MATLAB command: margin

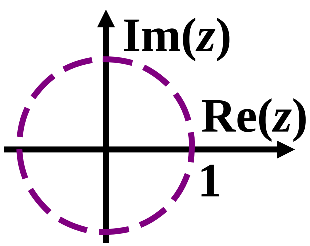
Design a lag compensator to achieve e_{ss} to unit ramp < 2/3.

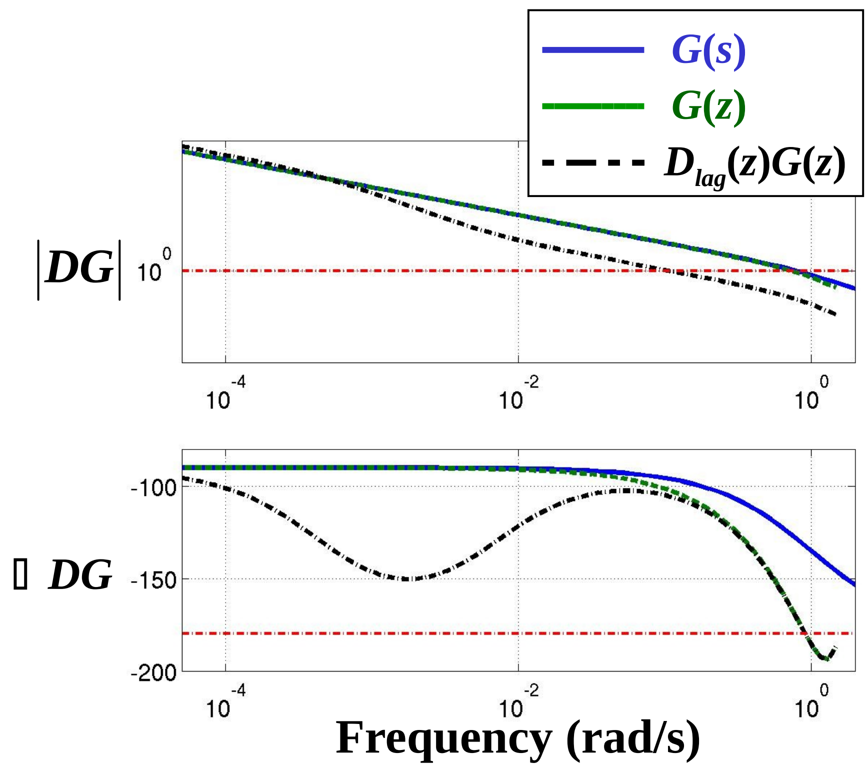
$$D_{lag}(z) = K\frac{z - z_1}{z - p_1}, \qquad z_1 < p_1$$

Choose pole at $1 - \omega T$ such that

Choose zero at $1 - \omega T$ such that

$$D_{lag}(z) = 0.1071\left(\frac{z - 0.9870}{z - 0.9991}\right)$$





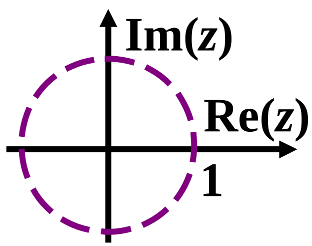
Design a lead compensator to achieve $t_r < 2$ sec and $M_p < 40\%$.

$$D_{lead}(z) = K\frac{z - z_1}{z - p_1}, \qquad z_1 > p_1$$

Choose pole at $1 - \omega T$ such that

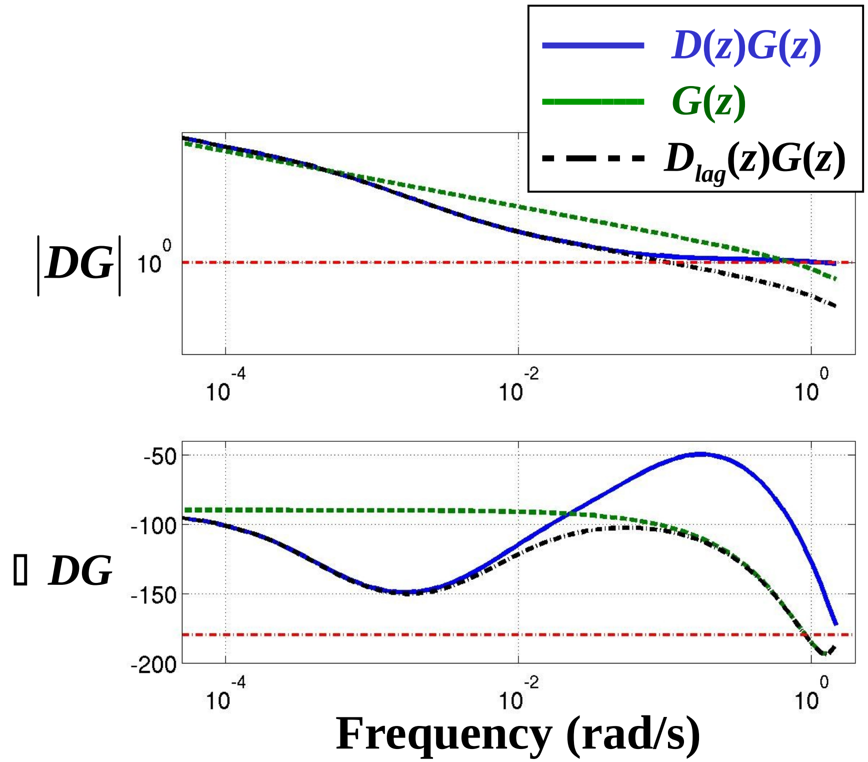
Choose zero at $1 - \omega T$ such that

$$D_{lead}(z) = 10\left(\frac{z - 0.8517}{z + 0.4830}\right)$$

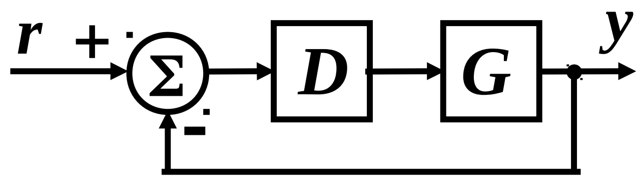


$$D(z) = D_{lag}(z)D_{lead}(z)$$

$$= 1.071\left(\frac{z - 0.9870}{z - 0.9991}\right)\left(\frac{z - 0.8517}{z + 0.4830}\right)$$



P and D Control

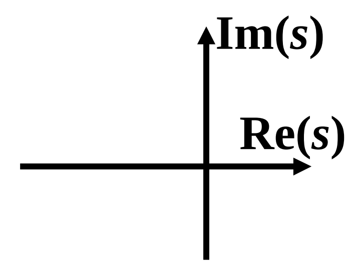


P: Proportional Control

Continuous: $u(t) = Ke(t) \Longrightarrow D(s) = K$

Discrete: $u(k) = Ke(k) \Longrightarrow D(z) = K$

Increases K_p or K_v (decreases e_{ss}), but can destabilize the system.

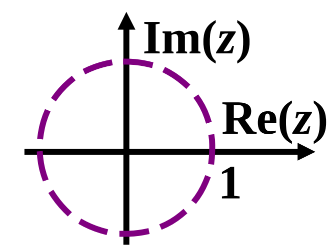


D: Derivative Control

Continuous: $u(t) = KT_D\dot{e}(t) \Longrightarrow D(s) = KT_D s$

Discrete: approximate derivative using 1st-order difference

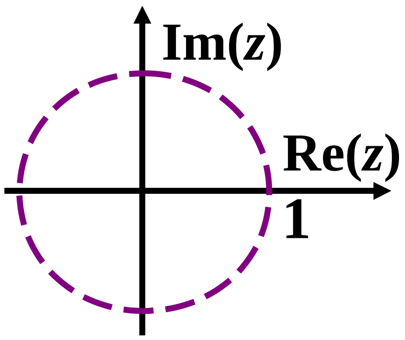
$$u(k) = KT_D\left[\frac{e_k - e_{k-1}}{T}\right] \Longrightarrow D(z) = KT_D\left[\frac{1 - z^{-1}}{T}\right]$$



Increases damping (decreases overshoot).

PD Control

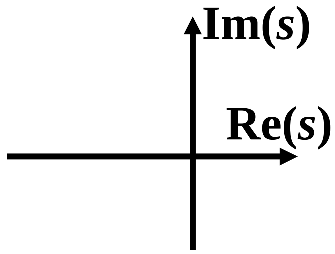
$$D(z) = K + KT_D \frac{z-1}{Tz}$$



$$D(z) = K_D \frac{z-\alpha}{z}, \qquad (0 < \alpha < 1)$$

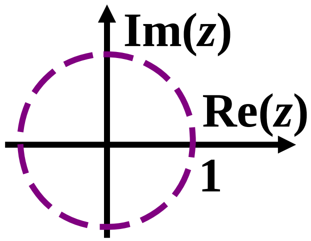
I: Integral Control

Continuous: $$u(t) = \frac{K}{T_I}\int_{t_0}^{t} e(\tau)\,d\tau \Longrightarrow D(s) = \frac{K}{T_I s}$$



Discrete: one way is to approximate integral using backward rectangular rule

$$u(k) = u(k-1) + \frac{K}{T_I}Te(k) \Longrightarrow D(z) = \frac{KT}{T_I(1-z^{-1})}$$



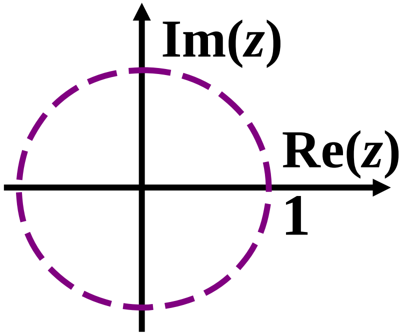
Decreases e_{ss}.

$$K_v = \lim_{z\to 1}\frac{(z-1)D(z)G(z)}{Tz}, \qquad e_{ss} = \frac{1}{K_v}$$

PI Control

$$D(z) = K\left[1 + \frac{Tz}{T_I(z-1)}\right]$$

$$D(z) = K_I \frac{z-\beta}{z-1}, \qquad (0 < \beta < 1)$$



PID Control

$$D(z) = K\left[1 + \frac{Tz}{T_I(z-1)} + T_D\frac{z-1}{Tz}\right]$$

In general:

$$T_I \downarrow \quad \Longrightarrow \quad e_{ss} \downarrow \quad (t_r \downarrow)$$

$$K \uparrow \quad \Longrightarrow \quad e_{ss} \downarrow \quad (t_r \downarrow)$$

$$T_D \uparrow \quad \Longrightarrow \quad M_p \downarrow \quad (t_r \downarrow)$$