Combined Control Law & Estimator

- Separation principle
- What is $D(z)$?
- Example

Combined Control Law & Estimator

When all states are measured, the control law design is:

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma u(k)$$
$$y(k) = \mathrm{H}\mathbf{x}(k) + Ju(k)$$

$$\Longrightarrow \quad u(k) = -\mathrm{K}\mathbf{x}(k)$$

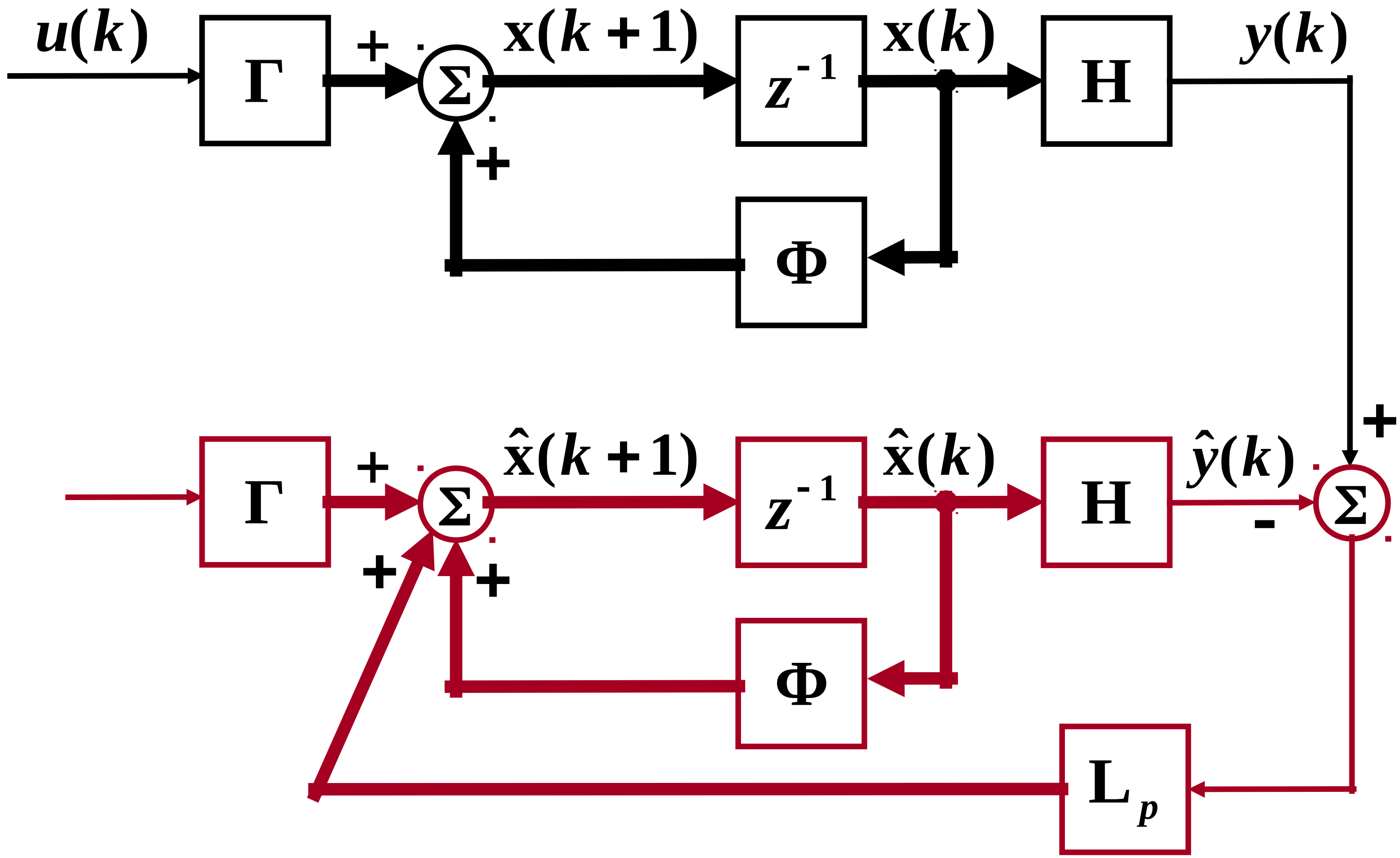
But if not all states are measured, or measurements are noisy, design a state estimator with desired estimator error dynamics:

$$\tilde{\mathbf{x}}(k+1) = \left(\Phi - \mathrm{L}_p\mathrm{H}\right)\tilde{\mathbf{x}}(k)$$

and then use: $u(k) = -\mathrm{K}\hat{\mathbf{x}}(k)$

- But state feedback was designed assuming $\mathbf{x}$ is fedback, not $\hat{\mathbf{x}}$.
- Is it really okay to feedback $\hat{\mathbf{x}}$?
- Or do we need to completely redesign K, taking into account now that we are feeding back $\mathbf{x}$ and not $\hat{\mathbf{x}}$?

Block Diagram of Combined Control & Estimator System



Analysis of the Combined Control & Estimator System with $\hat{\mathbf{x}}$ Fedback

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma u(k) = \Phi\mathbf{x}(k) - \Gamma\mathbf{K}\hat{\mathbf{x}}(k)$$

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) - \Gamma\mathbf{K}\left(\mathbf{x}(k) - \tilde{\mathbf{x}}(k)\right)$$

along with estimator error dynamics equation:

$$\tilde{\mathbf{x}}(k+1) = \left(\Phi - \mathbf{L}_p\mathbf{H}\right)\tilde{\mathbf{x}}(k)$$

yields the overall system dynamics (control and estimator):

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \tilde{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} \Phi - \Gamma\mathbf{K} & \Gamma\mathbf{K} \\ 0 & \Phi - \mathbf{L}_p\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \tilde{\mathbf{x}}(k) \end{bmatrix}$$

Overall Closed-Loop Characteristic Equation

$$\det\left(zI - F\right) = 0$$

$$\det\begin{bmatrix} zI - \Phi + \Gamma K & -\Gamma K \\ 0 & zI - \Phi + L_p H \end{bmatrix} = 0$$

$$\det\left(zI - \Phi + \Gamma K\right)\ \det\left(zI - \Phi + L_p H\right) = 0$$

Poles of the overall closed-loop system:

desired control poles + desired estimator poles

Separation Principle

Controller design and estimator design can be carried out independently.

What is $D(z)$?

$$\hat{\mathrm{x}}(k+1) = \Phi\hat{\mathrm{x}}(k) + \Gamma u(k) + \mathrm{L}_p\left(y(k) - \hat{y}(k)\right)$$

$$\hat{\mathrm{x}}(k+1) = \left(\Phi - \Gamma\mathrm{K} - \mathrm{L}_p\mathrm{H}\right)\hat{\mathrm{x}}(k) + \mathrm{L}_p y(k)$$

$$u(k) = -\mathrm{K}\hat{\mathrm{x}}(k)$$

Compare with standard state-space equation:

$$\mathrm{x}(k+1) = \Phi\mathrm{x}(k) + \Gamma u(k)$$

$$y(k) = \mathrm{H}\mathrm{x}(k) + Ju(k)$$

$$G(z) = \frac{b(z)}{a(z)} = \mathrm{H}\left(z\mathrm{I} - \Phi\right)^{-1}\Gamma + J$$

Compensator transfer function is then:

$$D(z) = \frac{U(z)}{Y(z)} = -\mathrm{K}\left(z\mathrm{I} - \Phi + \Gamma\mathrm{K} + \mathrm{L}_p\mathrm{H}\right)^{-1}\mathrm{L}_p$$

Comments

- Poles and zeros of $D(z)$ are never specified in the state-space compensator (controller & estimator) design.
- The compensator $D(z)$ itself may in fact be unstable!
  - Overall closed-loop system of plant and compensator is stable for stable controller and estimator designs, but the compensator itself may have unstable poles.
- The transfer function $D(z)$ from the state-space controller/estimator design has similarities with compensators designed using Root Locus or Frequency Response/Bode methods.

Example

$$G(s) = \frac{1}{s(s+1)} = \frac{Y(s)}{U(s)} \xrightarrow{T = 2\sec} G(z) = 1.135\frac{z+0.523}{(z-1)(z-0.135)}$$

$$\Phi = \begin{bmatrix} 1 & 1-e^{-T} \\ 0 & e^{-T} \end{bmatrix} = \begin{bmatrix} 1 & 0.865 \\ 0 & 0.135 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} T + e^{-T} - 1 \\ 1 - e^{-T} \end{bmatrix} = \begin{bmatrix} 1.135 \\ 0.865 \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad J = 0$$

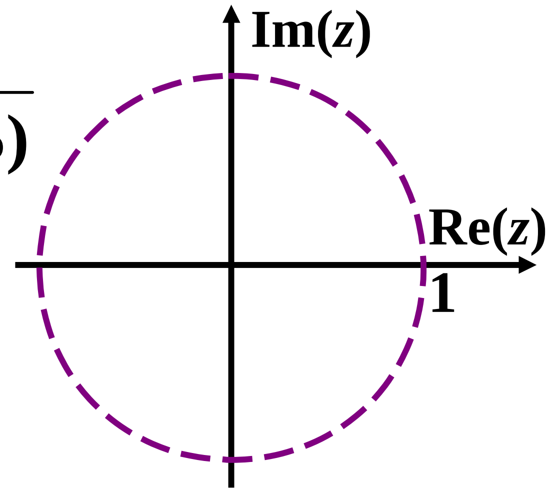
State feedback: $\zeta = 0.3, \quad \omega_n = 1 \text{ rad/sec}$

$$\Rightarrow \mathbf{K} = \begin{bmatrix} 0.9623 & 0.4693 \end{bmatrix}$$

Estimator: $\zeta = 0.5, \quad \omega_n = 1.5 \text{ rad/sec} \Rightarrow \mathbf{L}_p = \begin{bmatrix} 1.5173 \\ 0.1385 \end{bmatrix}$

What is the compensator transfer function $D(z)$?

$$D(z) = \frac{-1.53z + 0.147}{z^2 + 1.88z + 0.76} = \frac{-1.53(z - 0.096)}{(z + 1.296)(z + 0.585)}$$



$$\left| D\left(e^{j\omega T}\right) \right|$$

$$\angle D\left(e^{j\omega T}\right)$$