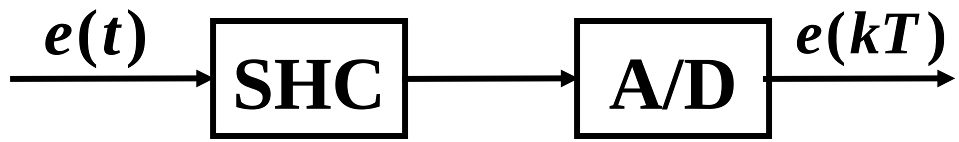
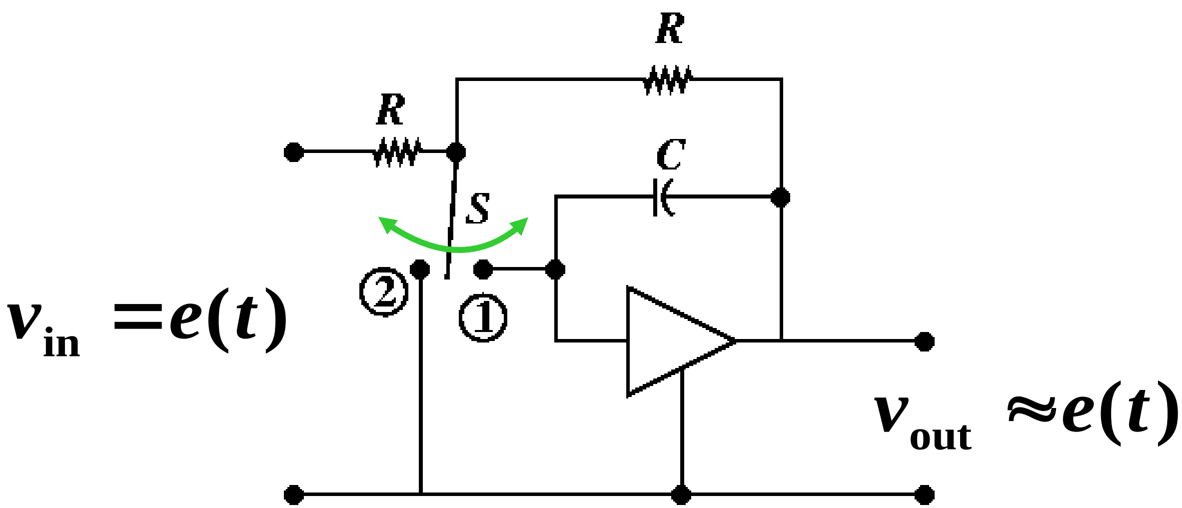
Sample and Hold and Block Diagram Analysis

- Sample and hold circuit
  - Mathematical representation
  - Zero-order hold
  - First-order hold
- Spectrum of a sampled signal
  - Sampling theorem
  - Frequency response of ZOH (and FOH)
- Block diagram analysis of sampled-data systems
  - Modeling digital control systems using only Laplace transforms

Sample and Hold Circuit (SHC)





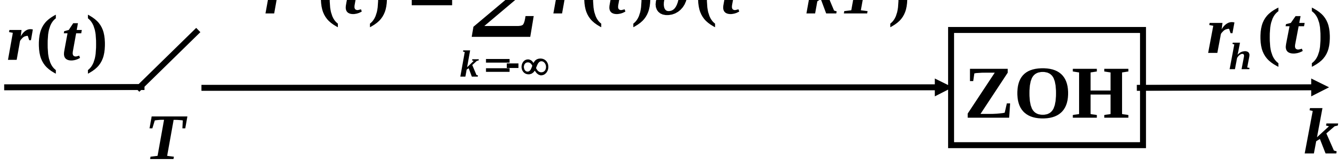
- When switch is in position 1, circuit is a low pass filter.

- **When switch is in position 2, the capacitor holds the output voltage equal to the input voltage long enough for the A/D to convert the signal to a digital number to feed to the digital computer or microprocessor.**

Mathematical Representation for Sample and Hold

- Sampling operation represented as impulse modulation
- Hold operation represented as a linear filter

$$r^*(t) = \sum_{k=-\infty}^{\infty} r(t)\delta(t - kT)$$



$$r_h(t) = r(kT), \quad kT \le t < kT + T$$

$$\mathcal{L}\left\{r^*(t)\right\} =$$

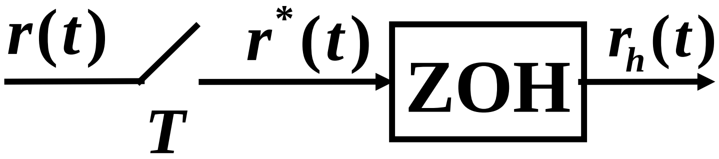
$$R^*(s) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} r(\tau)\delta(\tau - kT)e^{-s\tau}d\tau$$

Sifting property of impulse function: $\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$

Note similarity with Z-transform.

$$R(z) = \sum_{k=-\infty}^{\infty} r(kT)z^{-k}$$

Assuming a zero-order hold:



$$r_h(t) = r(kT), \qquad kT \le t < kT + T$$

What is $\mathcal{L}\left\{\mathrm{ZOH}\right\} = \mathrm{ZOH}(s)$?

For $r^*(t) = \delta(t)$, **what is** $r_h(t)$?

$$\mathrm{ZOH}(s) = \mathcal{L}\left\{r_h(t)\right\} =$$

First-Order Hold

- Extrapolates data between sampling periods by a first-order polynomial

- Unit impulse response

Spectrum of a Sampled Signal

Can we reconstruct $r(t)$ from $r^*(t)$ exactly?

$$r^*(t) = \sum_{k=-\infty}^{\infty} r(t)\delta(t-kT) = r(t)\sum_{k=-\infty}^{\infty}\delta(t-kT)$$

Can represent impulse train using a Fourier series:

$$\sum_{k=-\infty}^{\infty}\delta(t-kT) = \sum_{n=-\infty}^{\infty} c_n e^{j\left(\frac{2\pi n}{T}\right)t}$$

Fourier coefficients are found by integral over one period:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2}\sum_{k=-\infty}^{\infty}\delta(t-kT)e^{-jn\left(\frac{2\pi t}{T}\right)}dt$$

And we can write this as:

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j\left(\frac{2\pi n}{T}\right)t} = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

T	Sampling interval or sampling period (sec)
$\frac{1}{T}$	Sampling rate or sampling frequency (Hz)
$\omega_s = \frac{2\pi}{T}$	Sampling frequency in radians/s

So

$$r^*(t) = r(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) =$$

Now,
$$\mathcal{L}\left\{r^*(t)\right\} = \int_{-\infty}^{\infty} r(t)\left[\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}\right]e^{-st}dt$$

Integrate term by term:

$$R^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} R(s - jn\omega_s)$$

See Figure 5.4 of text.

<u>Aliasing</u> occurs when:

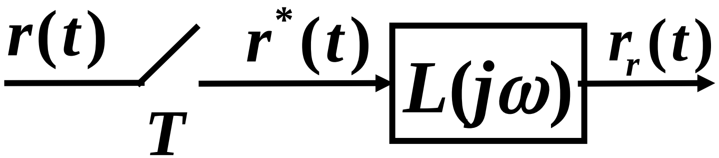
- In the frequency domain, components of spectrum of sampled signal overlap. See Figure 5.4 of text.
- In time domain, 2 different signals appear the <u>same</u> after sampling. See Figure 5.5 of text.

Sampling Theorem

- If the sampling frequency is at least twice the highest frequency in the signal, then it is possible to recover the original signal exactly from its samples.

- Low pass filter the sampled signal spectrum to obtain the original signal spectrum:

$$R_r(j\omega) = L(j\omega)R^*(j\omega)$$



$L(j\omega) =$ Ideal low-pass filter

What is $\ell(t)$?

$$\ell(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} L(j\omega) e^{j\omega t} d\omega$$

$$\ell(t) = \text{sinc}\left(\frac{\pi t}{T} \right)$$

$$r_r(t) = \ell(t) \;\; * \;\; r^*(t)$$

$$r_r(t) = \sum_{k=-\infty}^{\infty} r(kT)\ \mathrm{sinc}\left[\frac{\pi(t-kT)}{T}\right]$$

See Figure 5.7 of text for plot of $\ell(t)$.

Do you see any practical problem with this $\ell(t)$?

Frequency Response of ZOH

$$\mathrm{ZOH}(s) = \frac{1 - e^{-sT}}{s}$$

$$\mathrm{ZOH}(j\omega) = \left[e^{-j\frac{\omega T}{2}} \right] T \ \mathrm{sinc}\left(\frac{\omega T}{2} \right)$$

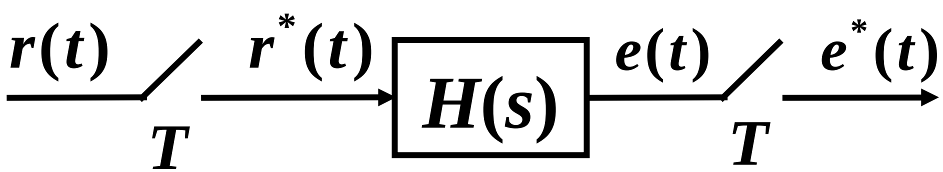
$$\mathrm{ZOH}(j\omega) = \left[e^{-j\frac{\omega T}{2}} \right] T \ \mathrm{sinc}\left(\frac{\omega T}{2} \right)$$

- **Compared to the ideal low-pass filter:**
  - **Magnitude multiplied by** $\left| \mathrm{sinc}\left(\frac{\omega T}{2} \right) \right|$
  - **Phase shift of** $\left(\frac{-\omega T}{2} \right)$

See text Fig. 5.8 for freq response plots for ZOH & FOH filters.

Block Diagram Analysis of Sampled-Data Systems

- How do we perform block diagram analysis when some of the signals are sampled?
  - Fairly straightforward . . . but a few difficulties can sometimes arise . . .



$$E(s) = H(s)R^*(s)$$

$$E^*(s) = \left(H(s)R^*(s)\right)^*$$

$$E^*(s) = \frac{1}{T}\sum_{m=-\infty}^{\infty} H(s - jm\omega_s) \; \frac{1}{T}\sum_{n=-\infty}^{\infty} R(s - jm\omega_s - jn\omega_s)$$

– **When sampling a product of a periodic transform (of a sampled signal) and a non-periodic one, the periodic transform gets factored out.**

$$E^*(s) = \left(H(s)R^*(s)\right)^* = H^*(s)R^*(s)$$

– But <u>only</u> periodic transforms get factored out. If

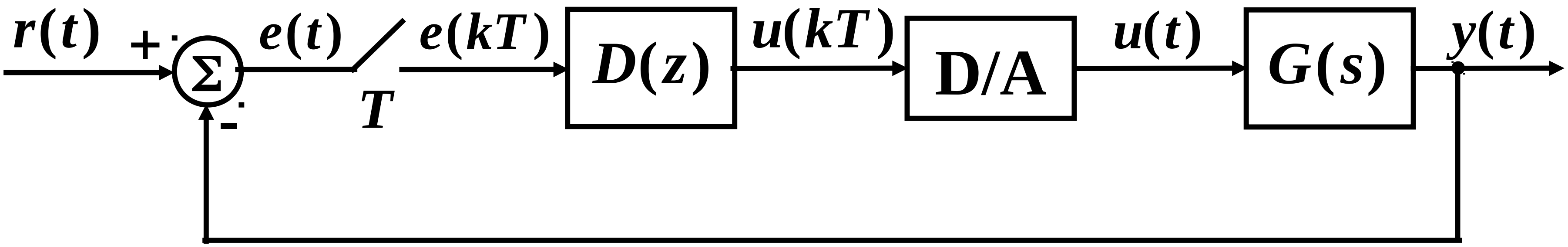
$$\overline{E}(s) = \overline{H}(s)\overline{R}(s)$$

then

$$\overline{E}^*(s) = \left(\overline{H}(s)\overline{R}(s)\right)^* \neq \overline{H}^*(s)\overline{R}^*(s)$$

- If we have the Laplace transform of an impulse-modulated continuous-time sampled signal $E^*(s)$, the Z-transform of the discrete-time sampled signal is

Modeling Digital Control Systems Using Only Laplace Transforms



- **In digital control design, we often want to convert everything to the Z-domain.**

- **Working with this "continuous-time" model of a digital control system, we can compute the transfer function:**

$\dfrac{Y^*(s)}{R^*(s)}$ **and then use the map** $z = e^{sT}$ **to get** $\dfrac{Y(z)}{R(z)}$

  - **In block diagram:**

$$E(s) = R(s) - Y(s)$$

$$U(s) = \frac{1 - e^{-sT}}{s} M^*(s)$$

$$Y^*(s) = \left(G(s)U(s)\right)^*$$

$$= \left(1 - e^{-sT}\right) D^*(s) \left(R^*(s) - Y^*(s)\right) \left[\frac{G(s)}{s}\right]^*$$

$$\frac{Y(z)}{R(z)} = \left.\frac{Y^*(s)}{R^*(s)}\right|_{s=\frac{\ln z}{T}} = \left.\frac{H^*(s)}{1+H^*(s)}\right|_{s=\frac{\ln z}{T}}$$

Now

$$Y(s) = G(s)U(s)$$

$$= G(s)\left[\frac{1-e^{-sT}}{s}\right]D^*(s)\left[\frac{1}{1+H^*(s)}\right]R^*(s)$$

Determining the inverse Laplace transform of $Y(s)$ allows you to see how $y(t)$ behaves between samples.