Analysis of Discrete-Time Systems

- Linear difference equations
  - Solving linear constant coefficient difference equations
- Z-transform
  - Properties
- Inverse Z-transform
  - Long-division
  - Partial fraction expansion
  - Inverse Z-transform integral

Linear Difference Equations

- **Consider treatment of signals inside a digital computer**

- **Suppose output at time k depends on input signal up to time k and output signal up to time k - 1.**

- **If <u>linear</u>, and assume dependence on <u>finite</u> number of past input and output values**

Linear constant difference equation

Solving Linear Constant Coefficient Difference Equations

- **Given equation and initial conditions (I.C.s), how do we solve?**

- **Consider difference equation approximation to compute integral:**

- **Assume have approximation for integral from time 0 to time $(k-1)T$**

- **Approximate area under curve over each interval using a trapezoid.**

- **Area of trapezoid**

- **Recursive formula/solution**

Example

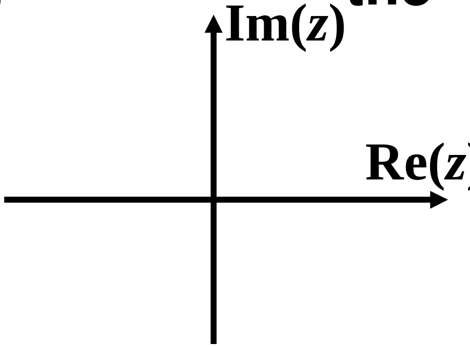
- Suppose $e(t) = t$

- What is u_k ?

- In this case, u_k is the exact integral since each trapezoid is the exact area of each increment.
- In this case, not too difficult to solve for u_k absolutely.
- In general, more difficult though . . .
- Often easier to solve by Z-transform.
- Similar to solving linear constant coefficient ordinary differential equations using Laplace transforms.

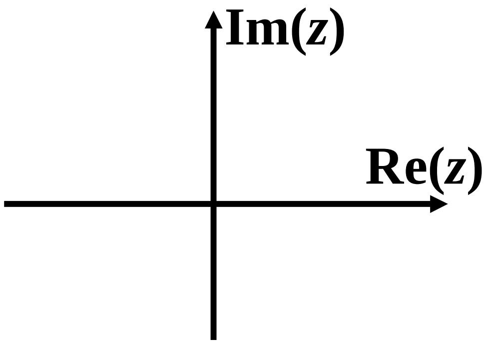
Z-Transform

If a signal has discrete values $e_0, e_1, \ldots, e_k, \ldots,$ the Z-transform of the signal is



$$E(z) = Z\{e_k\} = \sum_{k=-\infty}^{\infty} e_k z^{-k}, \qquad r_0 < |z| < R_0$$

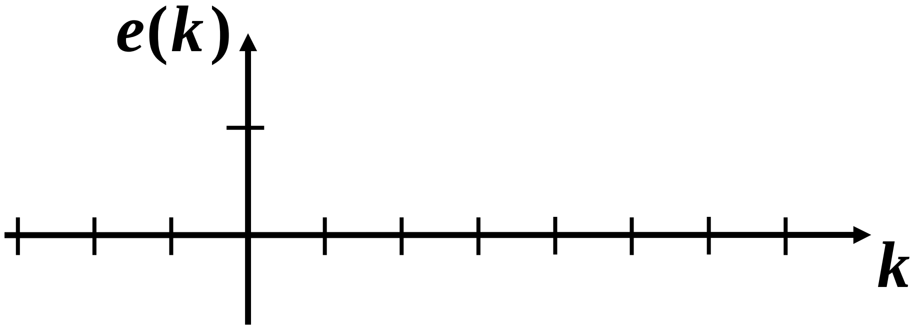
where r_0 and R_0 are bounds on $|z|$ such that the series converges.

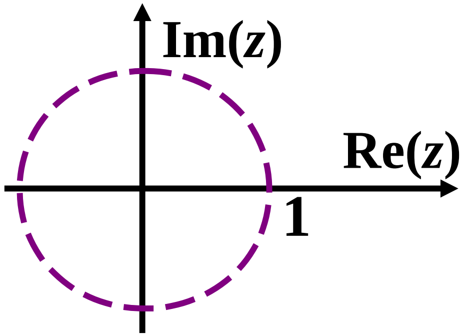


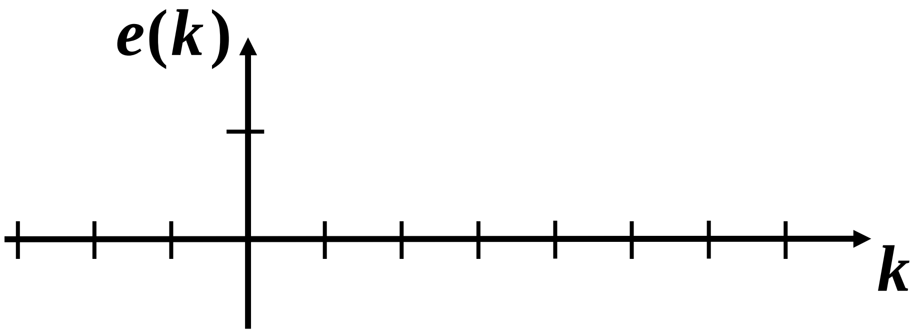
Example: $e_k = e(k) = r^k 1(k)$

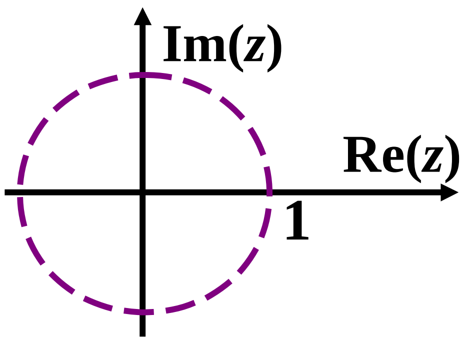
where $1(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$ is the unit step function.

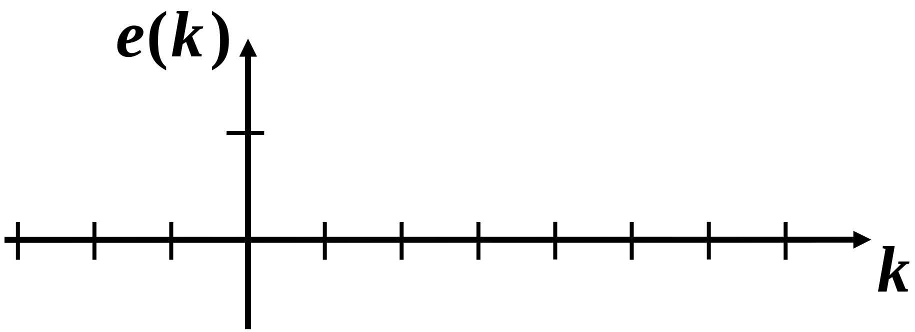
Examples of $e_k = e(k) = r^k 1(k)$

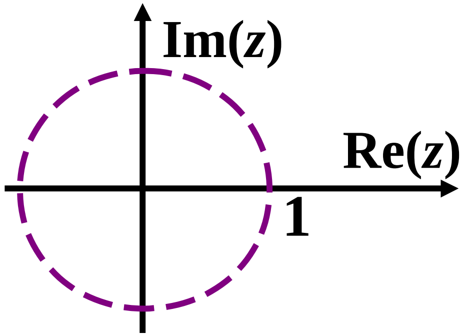












Properties of the Z-Transform

- **Linearity**

If $f_1(k) \xleftrightarrow{Z} F_1(z)$

and $f_2(k) \xleftrightarrow{Z} F_2(z)$

then $Z\{\alpha f_1(k) + \beta f_2(k)\} = \alpha F_1(z) + \beta F_2(z)$

- **Convolution of time sequences**

If $f_1(k) \xleftrightarrow{Z} F_1(z)$

and $f_2(k) \xleftrightarrow{Z} F_2(z)$

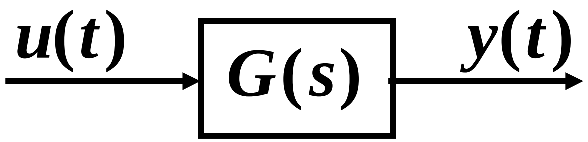
then $$Z\left[\sum_{\ell=-\infty}^{\infty} f_1(\ell)\, f_2(k-\ell)\right] = F_1(z)F_2(z)$$

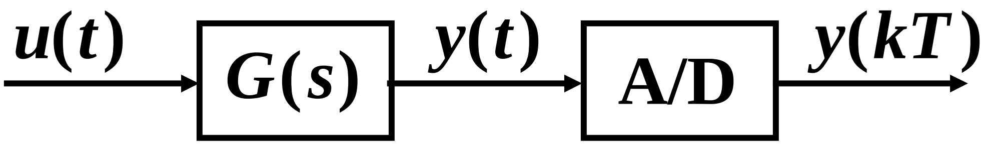
- **Time shift**

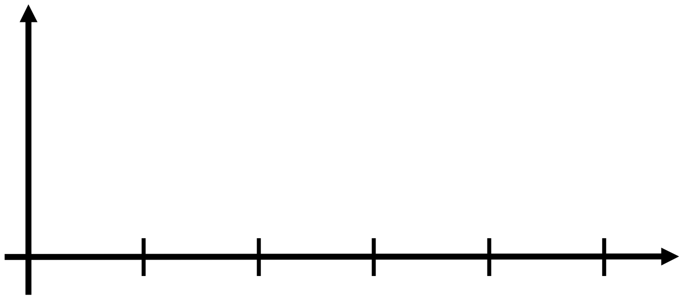
  If $f(k) \xleftrightarrow{Z} F(z)$

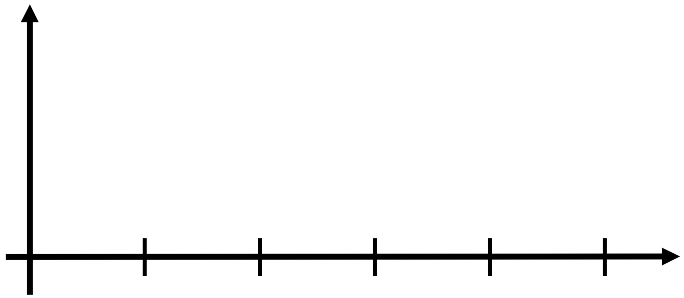
  then $Z\{f(k+n)\} = z^n F(z)$ for any integer n.

If $G(s)$ is time invariant, is a sampled-data system with $G(s)$ time-invariant?





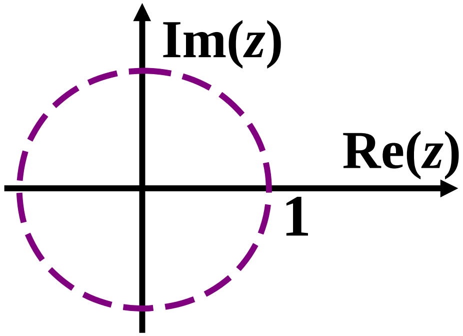




- **Scaling in the z-plane**

  If $\quad f(k) \xleftrightarrow{Z} F(z)$

  then $\quad Z\{r^{-k} f(k)\} = F(rz)$



- **Final Value Theorem**

  If $F(z)$ converges for $|z|>1$ and all poles of $(z - 1)F(z)$ are inside the unit circle, then

$$\lim_{k \to \infty} f(k) = \lim_{z \to 1} (z - 1)F(z)$$

Conditions on $F(z)$ → only pole not strictly inside the unit circle is a simple pole at $z = 1$, which is removed in $(z - 1)F(z)$.

The fact that $F(z)$ converges as the magnitude of z gets arbitrarily large ensures that $f(k)=0$ for $k<0$.

All components of $f(k)$ → 0 as k → infinity, with the possible exception of a constant term due to a pole at $z = 1$.

Inverse Z-Transform

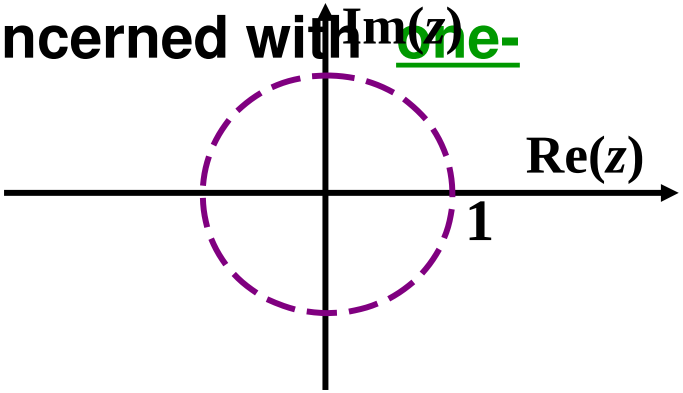
- Given the Z-transform $F(z)$, how do we compute the sequence $f(k)$ that $F(z)$ is the Z-transform of?

Two-sided Z-Transform:

$$F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k}, \qquad r_0 < |z| < R_0$$

In this course, we will mainly be concerned with one-sided Z-Transforms:

$$F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}, \qquad |z| > r_0$$



- Can find inverse Z-transform by a number of techniques:
  - Long-division
  - Partial fraction expansion
  - Inverse Z-transform integral

- Long-division
  - If $F(z)$ is a ratio of polynomials in z^{-1}, perform long-division of the numerator by the denominator. In the result, the coefficient of z^{-k} is the sequence value $f(k)$.
  - Example

- **Partial fraction expansion**
  - Compute and tabulate $F(z)$ for several basic $f(k)$'s.
  - Then when given a new $F(z)$, decompose $F(z)$ by partial fraction expansion and look-up the components of the sequence $f(k)$ in the previously prepared table.

In general:

$$F(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}$$

$$= \frac{z^{n-m}(b_0 z^m + b_1 z^{m-1} + \cdots + b_m)}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n}$$

– Assuming all poles are distinct and $m \leq n$

$$F(z) = c_0 + \frac{c_1 z}{z - p_1} + \frac{c_2 z}{z - p_2} + \cdots + \frac{c_n z}{z - p_n}$$

where $c_0 = F(z)\Big|_{z=0}$ and $c_i = (z - p_i)\dfrac{F(z)}{z}\Bigg|_{z=p_i}$

– **If there are repeated poles:**

Suppose p_1 is repeated 3 times:

$$F(z) = \frac{(\quad)(\quad)(\quad)\cdots}{(z-p_1)^3(z-p_4)(z-p_5)(\quad)\cdots}$$

$$= \frac{c_1 z}{z-p_1} + \frac{c_2 z}{(z-p_1)^2} + \frac{c_3 z}{(z-p_1)^3} + \frac{c_4 z}{z-p_4} + \frac{c_5 z}{z-p_5} + \cdots$$

where

$$c_i = (z - p_i)\frac{F(z)}{z}\bigg|_{z=p_i} \quad \text{for } i = 4, 5, \ldots \qquad \text{(distinct poles)}$$

$$c_3 = (z - p_1)^3\frac{F(z)}{z}\bigg|_{z=p_1}$$

$$c_2 = \frac{d}{dz}\left[(z - p_1)^3\frac{F(z)}{z}\right]\bigg|_{z=p_1}$$

$$c_1 = \frac{1}{2}\frac{d^2}{dz^2}\left[(z - p_1)^3\frac{F(z)}{z}\right]\bigg|_{z=p_1}$$

In general, if a pole p has multiplicity ℓ, then

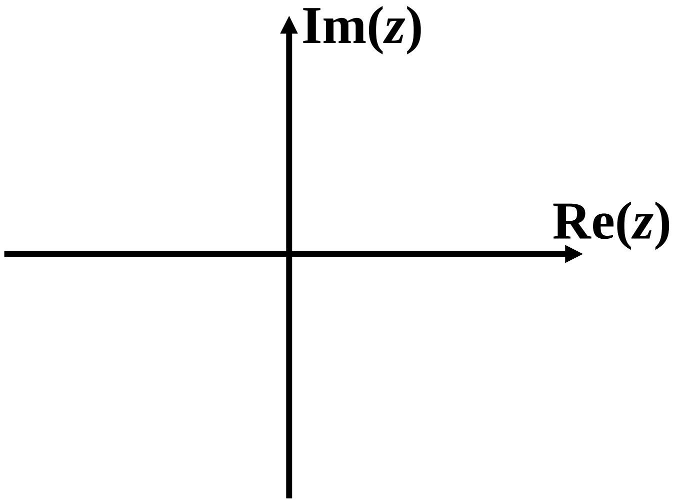
$$c_{\ell-i} = \frac{1}{i!}\frac{d^i}{dz^i}\left[(z-p)^{\ell}\frac{F(z)}{z}\right]\Bigg|_{z=p}, \qquad i = 0,\ 1,\ \ldots,\ \ell-1$$

are the coefficients associated with the pole p.

- **Inverse Z-transform integral**

  is the closed, complex integral:

$$f(k) = \frac{1}{2\pi j}\oint F(z) z^k \frac{dz}{z}$$



where the contour is a circle in the region of convergence of $F(z)$.

We will primarily use Partial Fraction Expansions in this course.