Discrete Controller Design

- **Discrete equivalent design example**
- **Direct design in the z-plane**
  - **Desired closed-loop pole locations in the z-plane**
  - **Example**

What is the "equivalent" discrete design?

$$D(s) = 150\frac{s+4}{s+20}$$

Use zero-pole mapping with $T = 0.075$ sec.

How many samples per rise time?

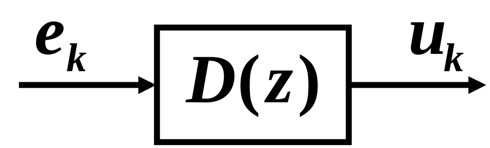
$\omega_n \geq 6$ rad/s $\qquad \omega_{n_{\min}} = 6$ rad/s

$$\omega_s = \frac{2\pi}{T} = 83.8 \text{ rad/s}$$

$$D(z) = K_d \frac{z - e^{-4T}}{z - e^{-20T}} = K_d \frac{z - 0.741}{z - 0.223}$$

$$D(s) = 150 \frac{s + 4}{s + 20}$$

Match DC gains:



$T = 0.075$ sec.

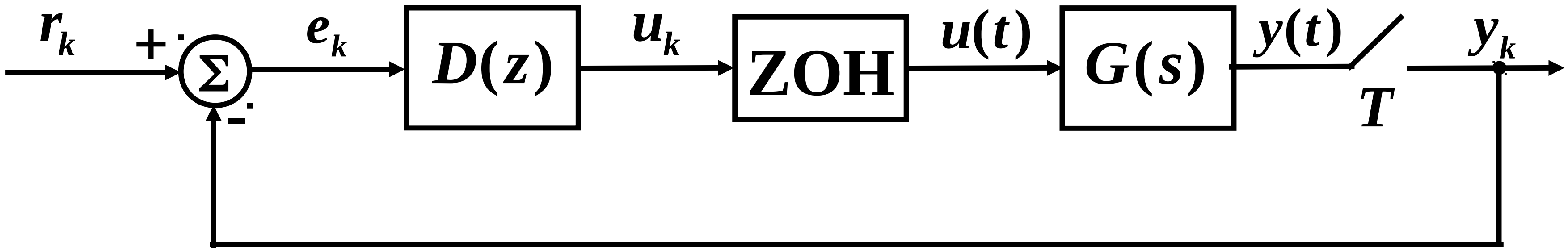
$$D(s)\Big|_{s=0}$$

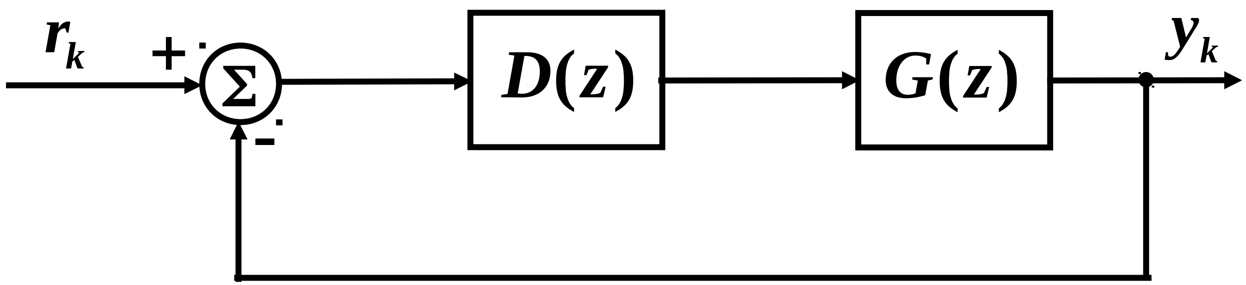
For $T = 0.075$ **sec.:**

$$D(z)\Big|_{z=1}$$

$$D(z) = 89.9 \frac{z - 0.741}{z - 0.223}$$

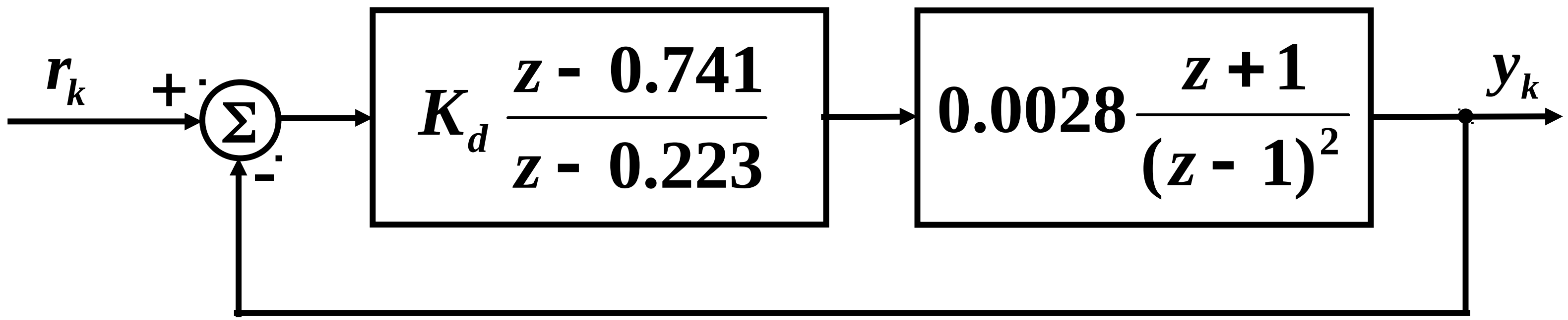
How does sampled-data system perform?





$$G(z) = \frac{z-1}{z} Z\left\{\frac{G(s)}{s}\right\} = \frac{z-1}{z} Z\left\{\frac{1}{s^3}\right\} =$$

For $T = 0.075$ **sec.:** $\quad G(z) = 0.0028 \dfrac{z+1}{(z-1)^2}$



$$\frac{Y(z)}{R(z)} = \frac{DG}{1+DG}$$

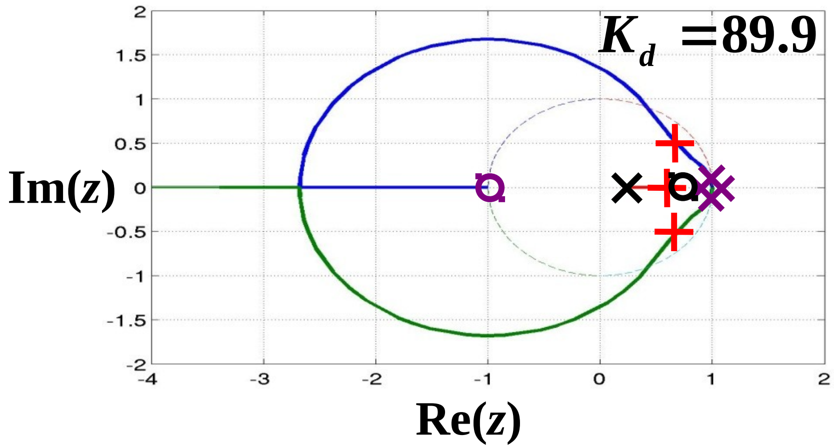
$$1+DG = 1+\frac{K_d T^2}{2}\left(\frac{z-0.741}{z-0.223}\right)\left(\frac{z+1}{(z-1)^2}\right) = 0$$

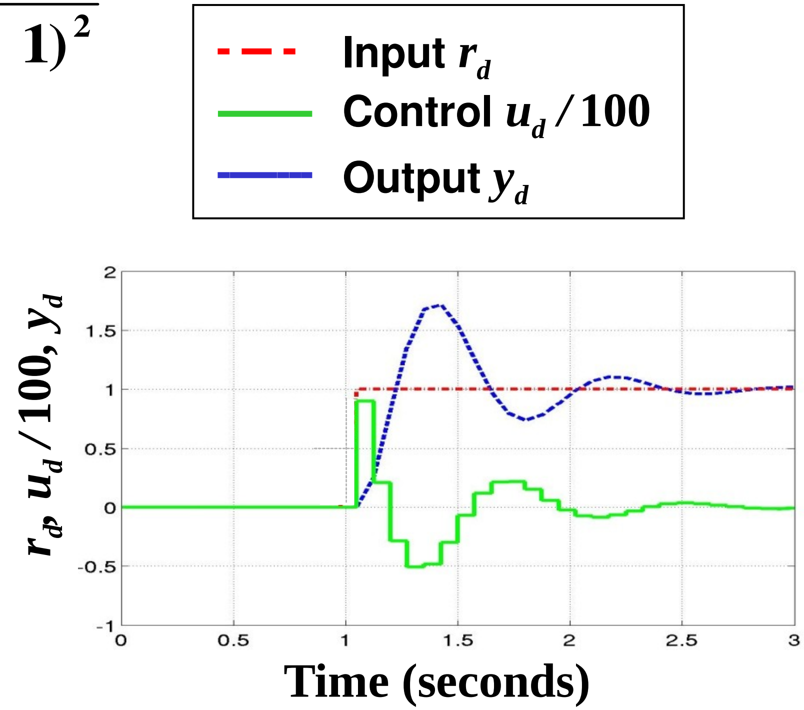
How is Root Locus drawn in z-plane?

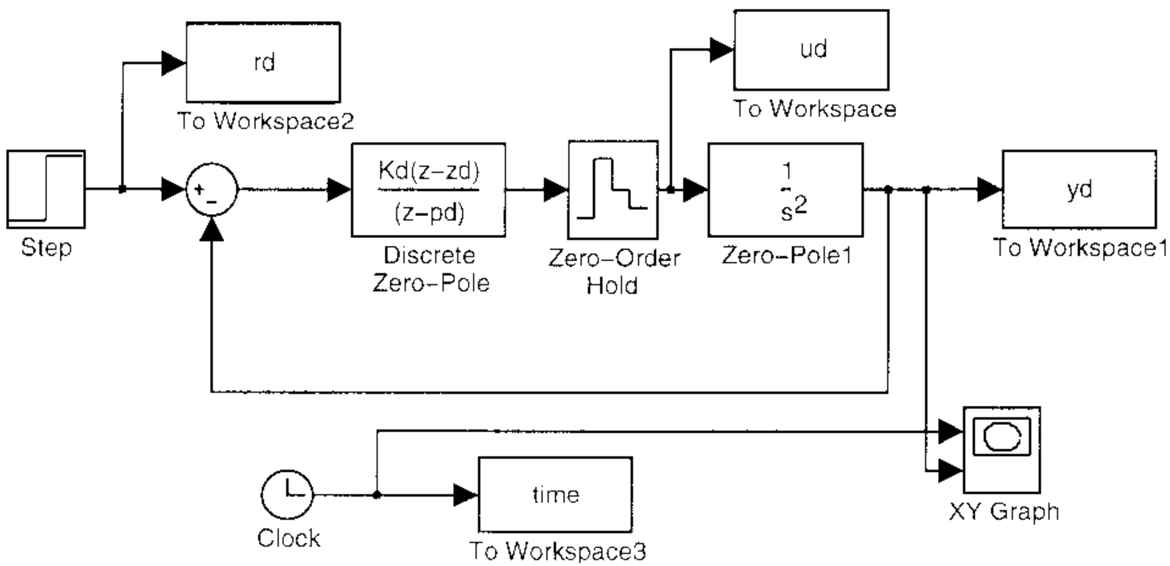
$$1+KH(z) = 0, \qquad K > 0$$

$$H(z) = -\frac{1}{K}$$

$$D(z) = 89.9\frac{z - 0.741}{z - 0.223},\ G(z) = 0.0028\frac{z + 1}{(z - 1)^2}$$







$$t_r = 0.15 \text{ sec} \le 0.3 \text{ sec}$$

$$M_p = 71.5\% \ge 20\%$$

$$t_s = 2.15 \text{ sec}$$

Useful MATLAB command: stairs

Further Comments on Root Locus

Continuous-time design:

$$1 + D(s)G(s) = 1 + K\left(\frac{s+z_0}{s+p}\right)\left(\frac{1}{s^2}\right) = 0$$

Fix K and z_0, and plot root locus vs. pole location p:

$$1 + p\frac{s^2}{s^3 + Ks + Kz_0} = 0$$

But for discrete equivalent design using pole-zero mapping:

$$1 + D(z)G(z) = 1 + K_d\left(\frac{z - e^{-z_0T}}{z - e^{-pT}}\right)\left(\cdots\right) = 0$$

Difficult to plot root locus vs. p or z_0.

Equation nonlinear in p and z_0.

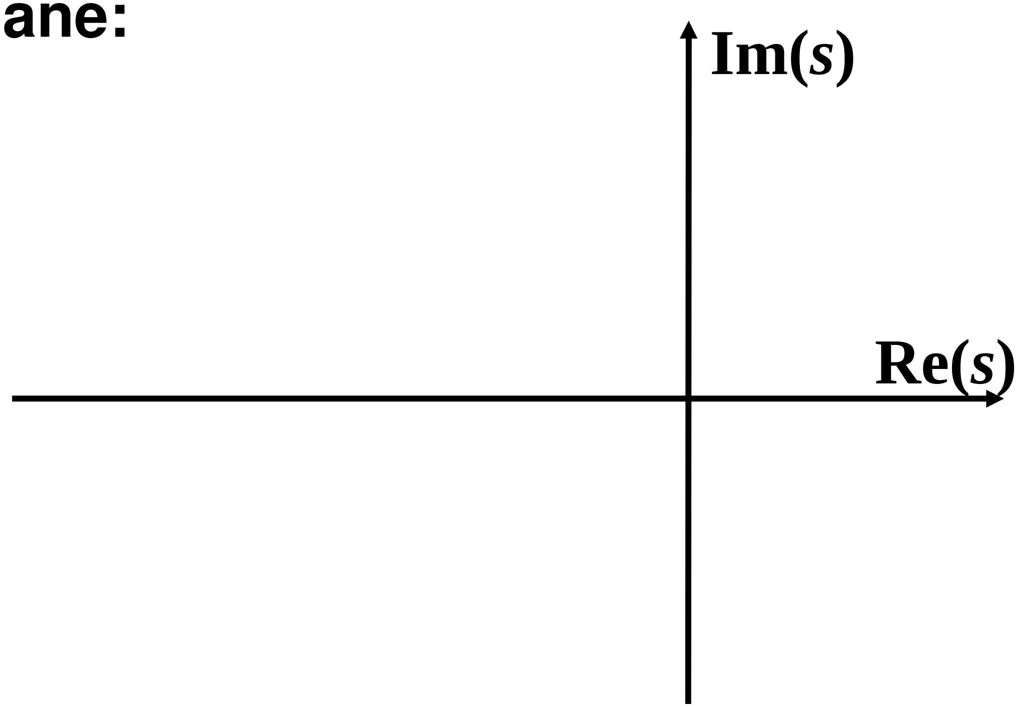
Direct Design in the z-Plane

$$1 + D(z)G(z) = 0$$

- Want to select the desired closed-loop poles of the system in the z-plane.
- Then design $D(z)$ so the actual closed-loop poles are (approximately) at the desired locations.
- To keep $D(z)$ causal, its denominator polynomial should be of equal or higher order than its numerator polynomial.

For $\left\{ \begin{array}{c} t_r \le t_{r_{desired}} \\ M_p \le M_{p_{desired}} \\ t_s \le t_{s_{desired}} \end{array} \right\}$ what is the desired region in the z-plane for the closed-loop poles of the system?

In the s-plane:



Convert desired regions in the *s*-plane to those in the *z*-plane using the mapping:

$$z = e^{sT} = e^{-\zeta\omega_n T} e^{\pm j\omega_n\sqrt{1-\zeta^2}T}$$

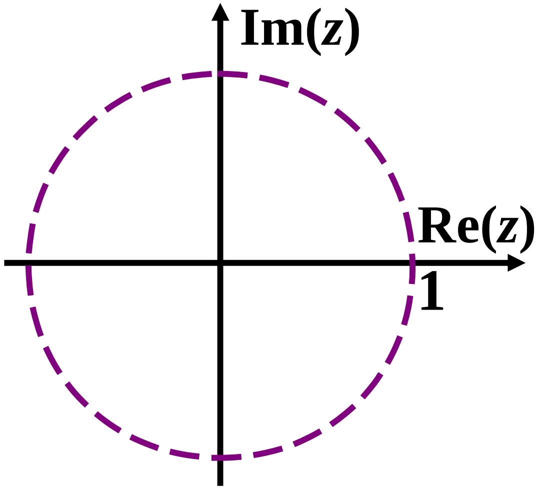
In the *s*-plane:

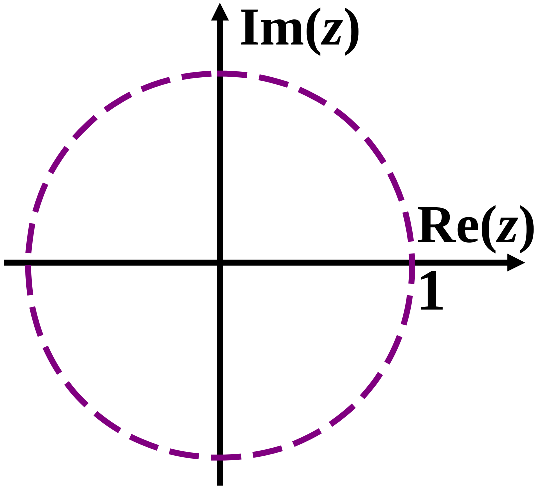
$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

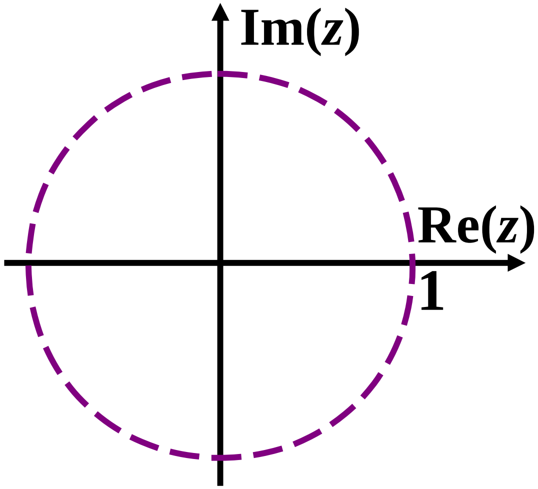
$$t_r \approx \frac{1.8}{\omega_n}$$

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \approx 1 - \frac{\zeta}{0.6}$$

$$t_s = \frac{4.6}{\zeta\omega_n}$$

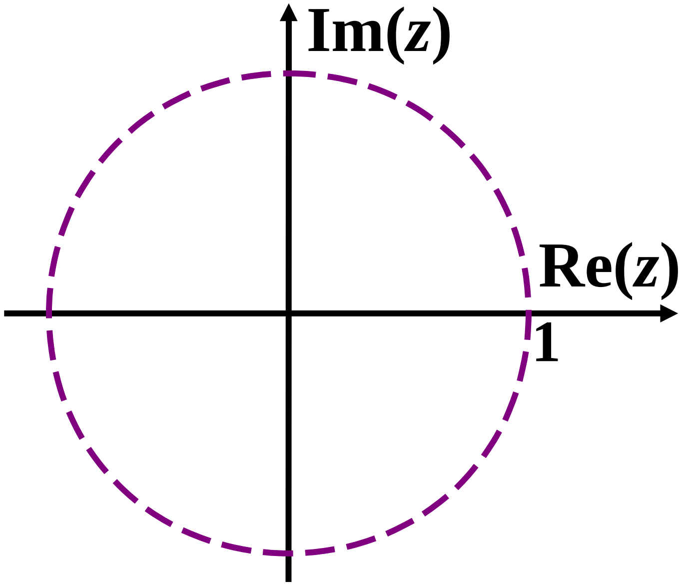






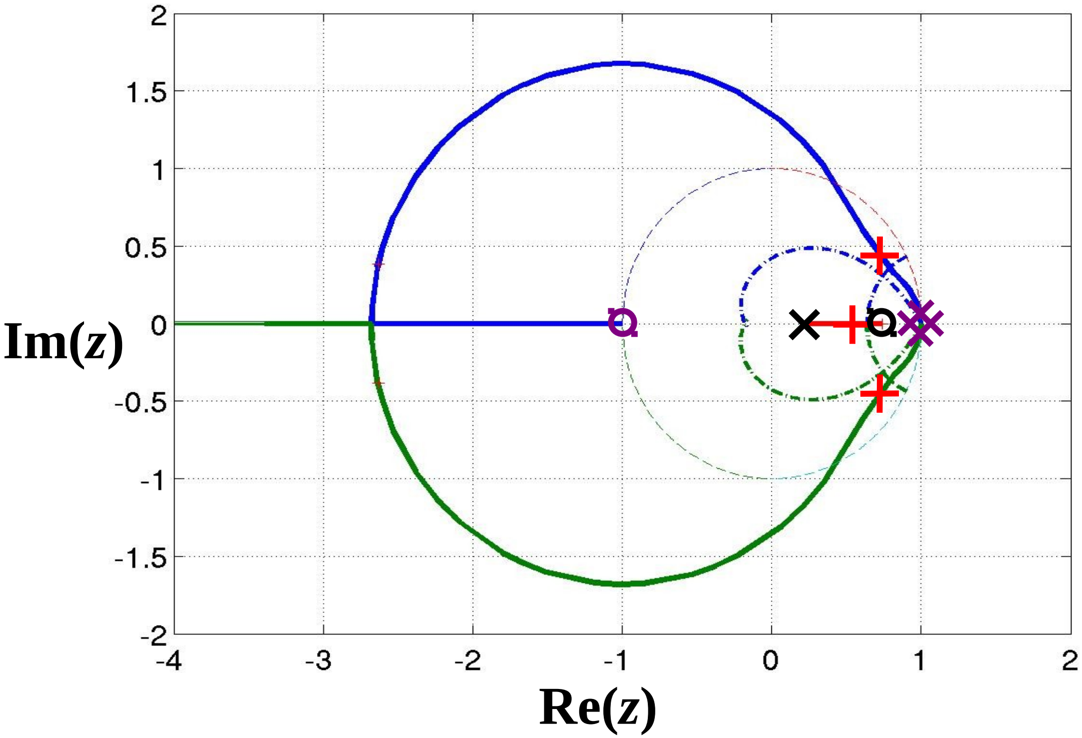
See Figure 4.24 of text.

- **Combining performance specifications on rise time, overshoot, and settling time:**

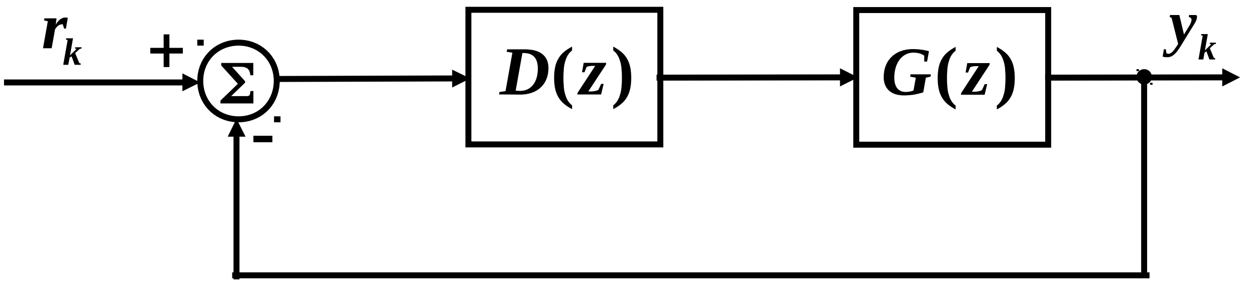


Taking another look at our discrete equivalent design in the example earlier:

$$t_r = 0.15 \ \text{sec} \leq 0.3 \ \text{sec}$$
$$M_p = 71.5\% \geq 20\%$$



Direct Design in *z*-Plane:



$$1 + D(z)G(z) = 0$$

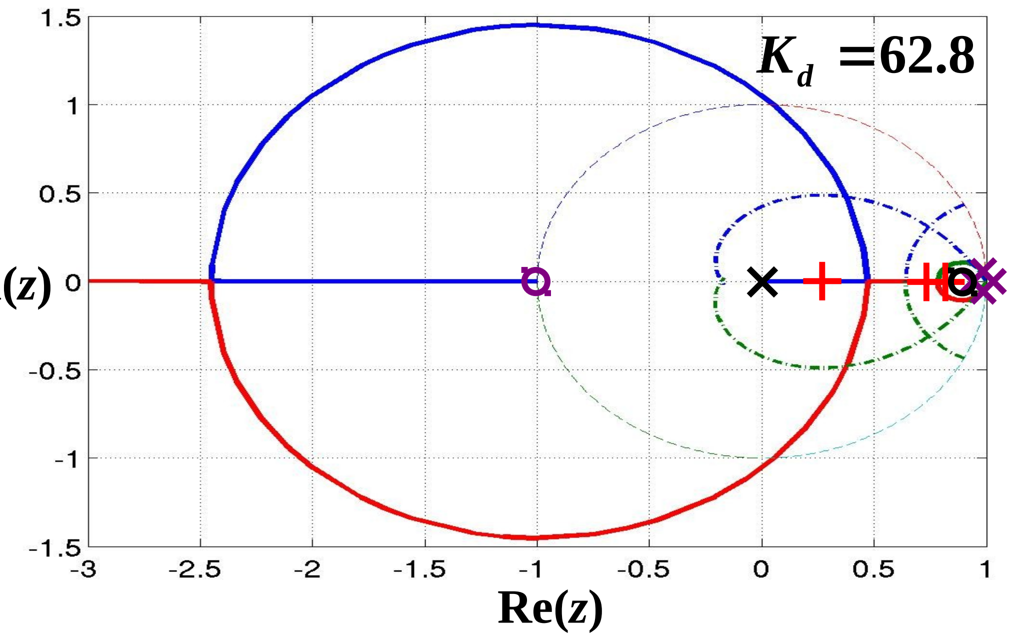
$$D(z) = K_d \frac{z - z_d}{z - p_d}$$

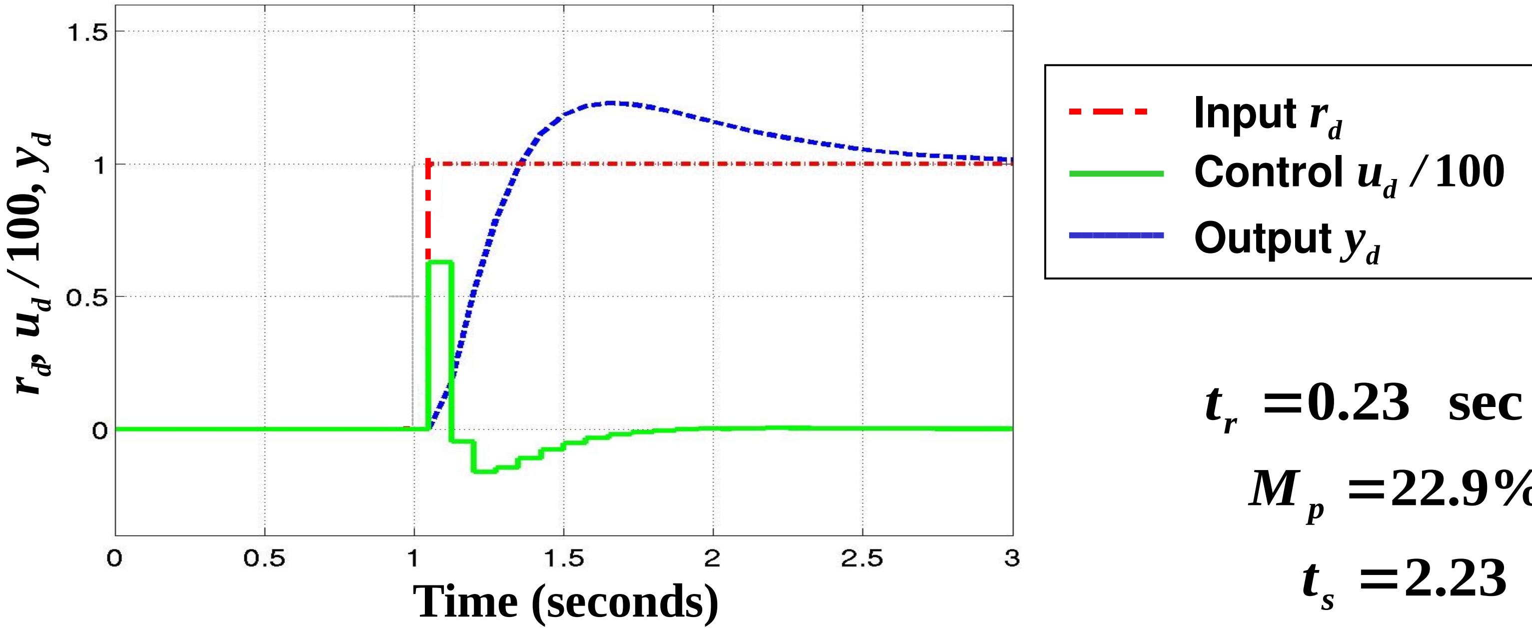
$$G(z) = \frac{T^2}{2} \frac{z+1}{(z-1)^2}$$

$$1 + D(z)G(z) = 1 + K_d \frac{T^2}{2} \left[\frac{z - z_d}{z - p_d} \right] \left[\frac{z+1}{(z-1)^2} \right] = 0$$

$$D(z) = K_d \frac{z - 0.9}{z}$$

$$G(z) = 0.0028 \frac{z + 1}{(z - 1)^2}$$





$$t_r = 0.23 \text{ sec} \le 0.3 \text{ sec}$$

$$M_p = 22.9\% \ge 20\%$$

$$t_s = 2.23 \text{ sec}$$

$$D(z) = 62.8\frac{z - 0.9}{z}$$

$$e_k \longrightarrow \boxed{D(z)} \xrightarrow{\ u_k\ }$$

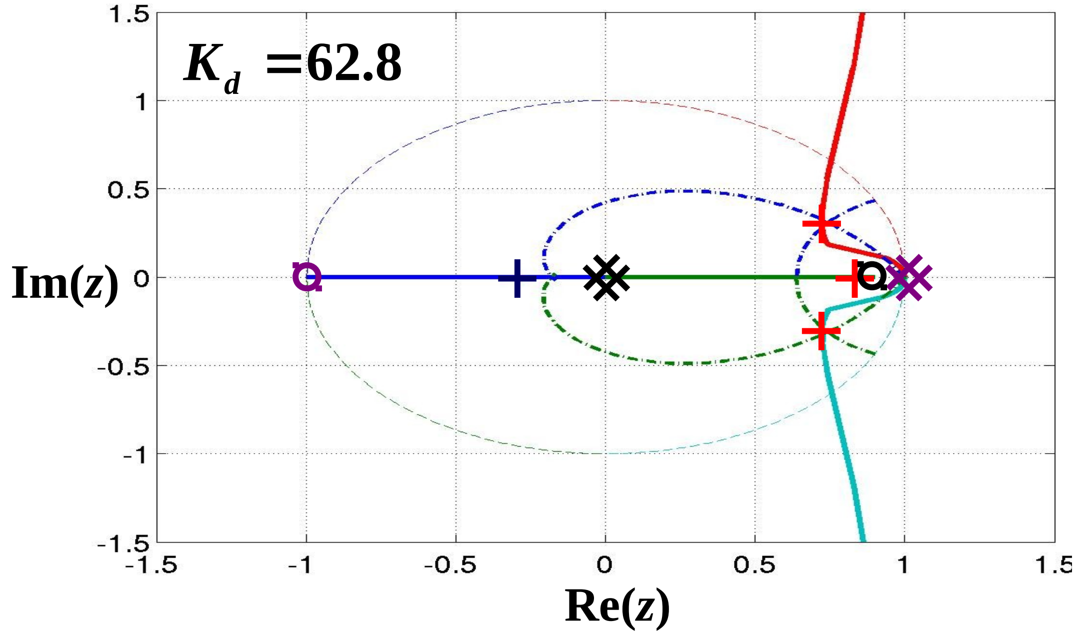
$$D(z) = K_d\frac{z - z_d}{z - p_d} \quad \Longrightarrow \quad \frac{U(z)}{E(z)} = K_d\frac{1 - z_d z^{-1}}{1 - p_d z^{-1}}$$

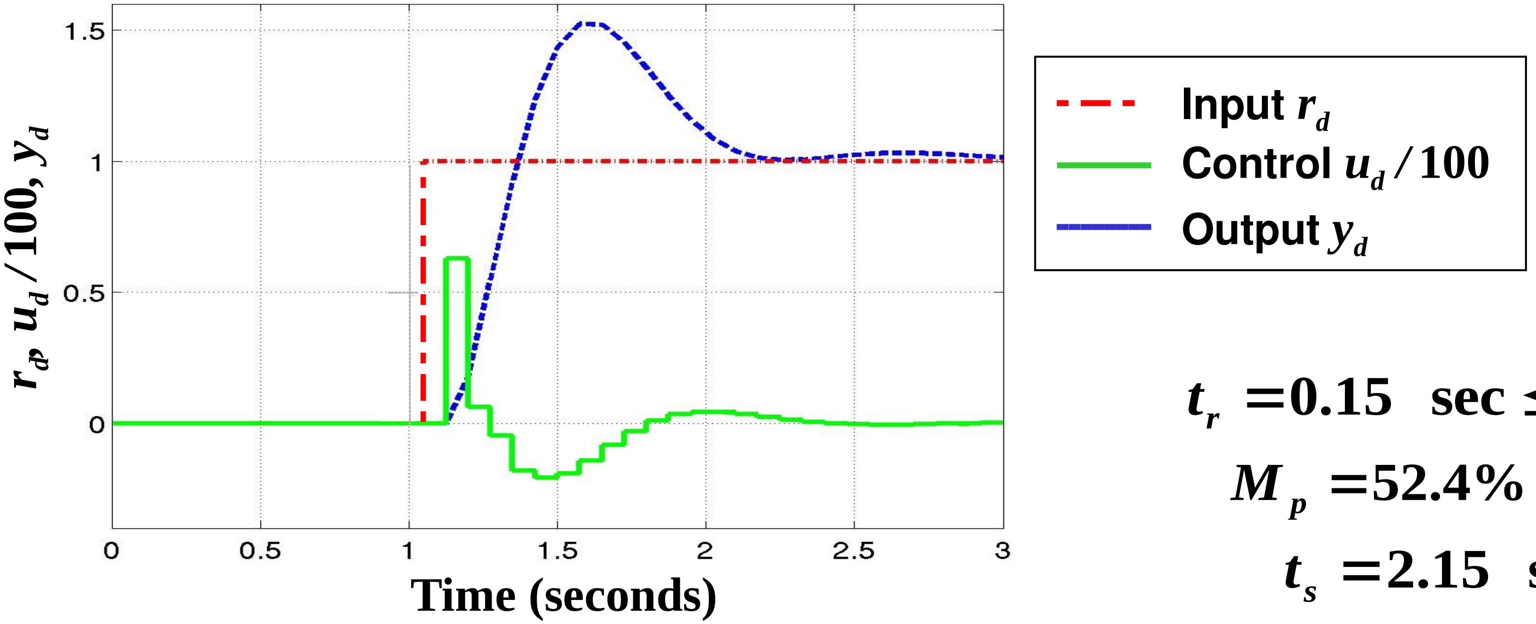
In practice, sometimes need to add a delay:

$$D(z) = K_d\frac{z - z_d}{(z - p_d)z} \quad \Longrightarrow \quad D(z) = 62.8\frac{z - 0.9}{z^2}$$

$$D(z) = K_d \frac{z - 0.9}{z^2}$$

$$G(z) = 0.0028 \frac{z+1}{(z-1)^2}$$





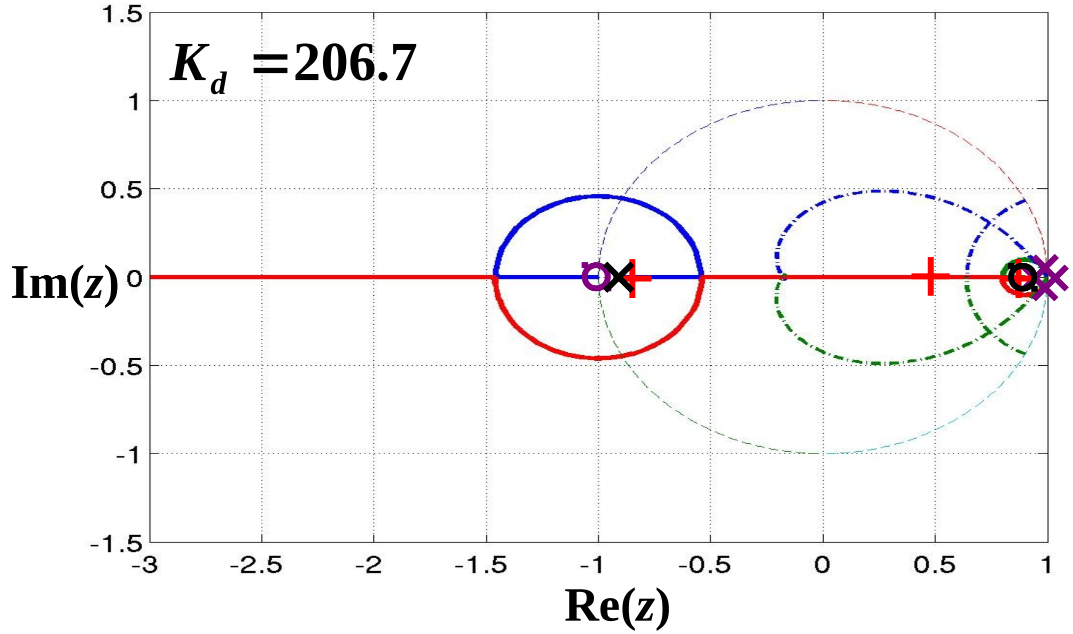
$$t_r = 0.15 \text{ sec} \le 0.3 \text{ sec}$$

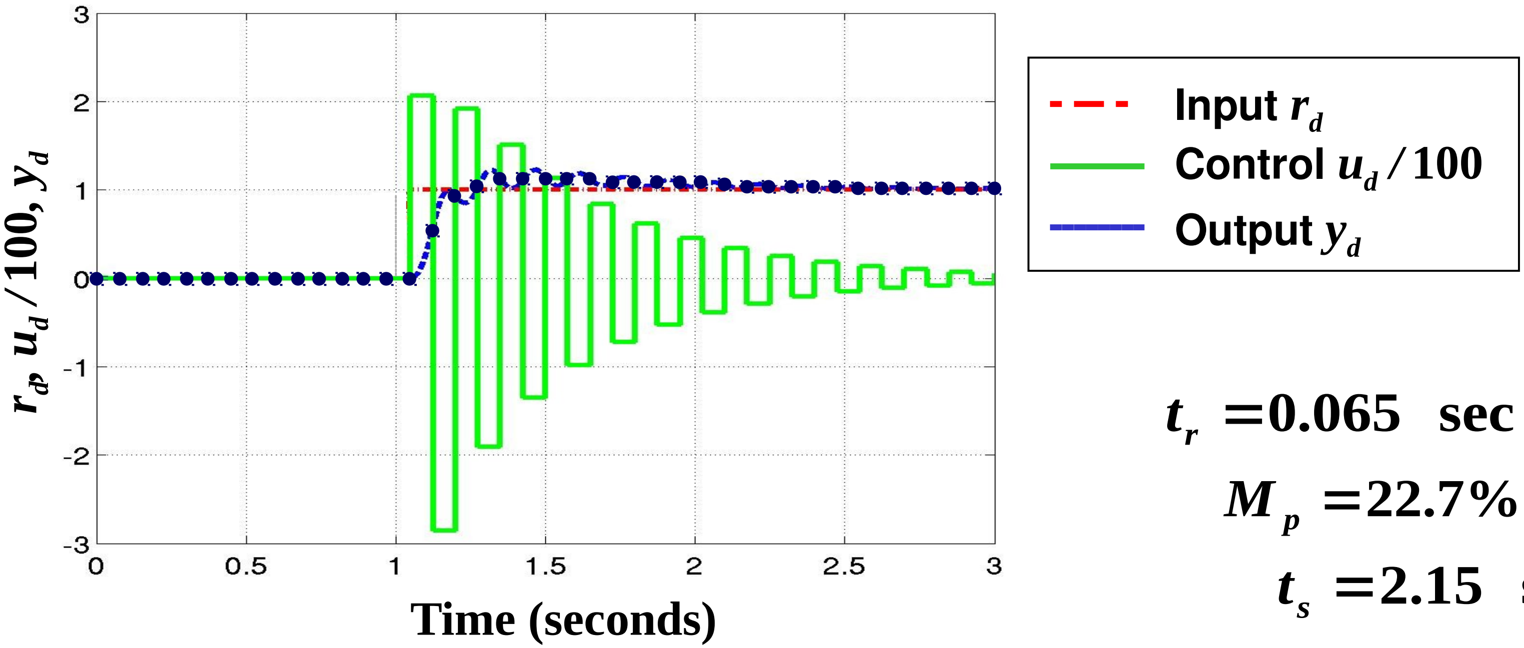
$$M_p = 52.4\% \ge 20\%$$

$$t_s = 2.15 \text{ sec}$$

$$D(z) = K_d \frac{z - 0.9}{z + 0.9}$$

$$G(z) = 0.0028 \frac{z + 1}{(z - 1)^2}$$





$$t_r = 0.065 \text{ sec} \leq 0.3 \text{ sec}$$

$$M_p = 22.7\% \geq 20\%$$

$$t_s = 2.15 \text{ sec}$$