Digital Filtering

- Pole-zero mapping
- Hold equivalents
  - Zero-order hold
  - Triangle hold

Pole-Zero Mapping

- Apply $z = e^{sT}$ to poles and zeros of transfer function $H(s)$ to obtain an "equivalent" $H(z)$.

Given

$$H(s) = K_s \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}, \qquad m < n$$

then

$$H_{zp}(z) = K_d \frac{(z - e^{z_1 T})(z - e^{z_2 T}) \cdots (z - e^{z_m T})}{(z - e^{p_1 T})(z - e^{p_2 T}) \cdots (z - e^{p_n T})}$$

Need to know the poles and zeros of $H(s)$.

1. How do we determine K_d?
2. How do we map zeros at infinity?

Solutions:

1. Choose K_d to match gains at some frequency of interest.

2. How about mapping zeros at infinity to zeros at infinity?

Example: $$H(s) = \frac{K_s}{s^3 + 2s^2 + 2s + 1}$$

$$
\begin{tikzcd}
e_k \arrow[r] & \boxed{H_{zp}(z)} \arrow[r] & u_k
\end{tikzcd}
$$

$$U(z) = \frac{K_d z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} E(z)$$

Summary of Pole-Zero Mapping

1. Map finite poles and zeros of $H(s)$ with $z = e^{sT}$

2. Map all but one of the zeros of $H(s)$ at infinity to z = -1. Map one zero of $H(s)$ at infinity to infinity.

3. Determine K_d to match gains at some frequency of interest. (Often, DC gains are matched.)

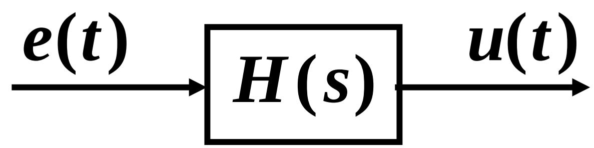
$$\left|H_{zp}(z)\right|\Big|_{z=e^{s_0 T}} = \left|H(s)\right|\Big|_{s=s_0}$$

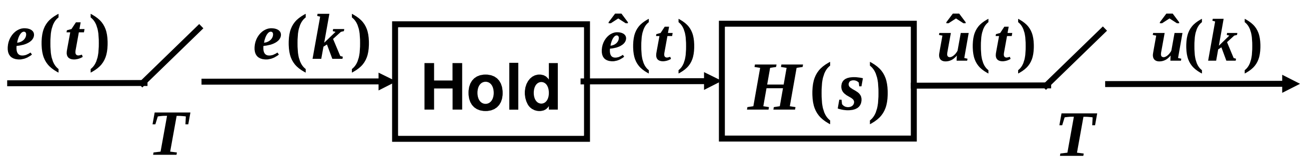
Example

$$H(s) = \frac{a}{s+a}$$

$$\boxed{H_{zp}(z) = \frac{1 - e^{-aT}}{z - e^{-aT}}}$$

Hold Equivalents





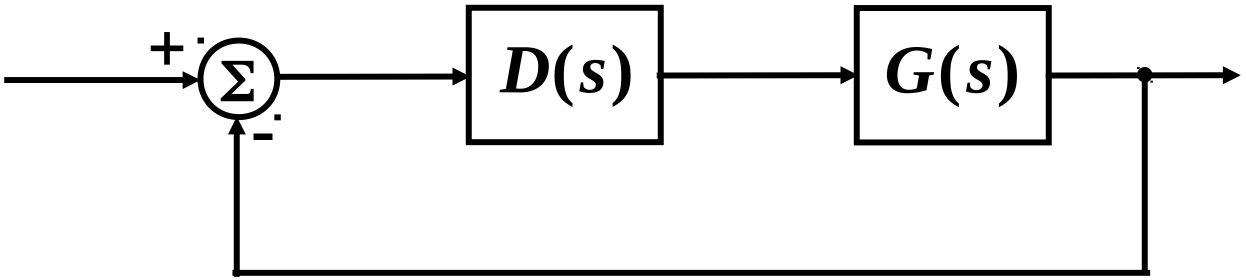
Want to approximate $H(s)$ with an $H(z)$ such that $\hat{u}(k)$ approximates $u(t)$ if $e(k)$ are samples of $e(t)$.

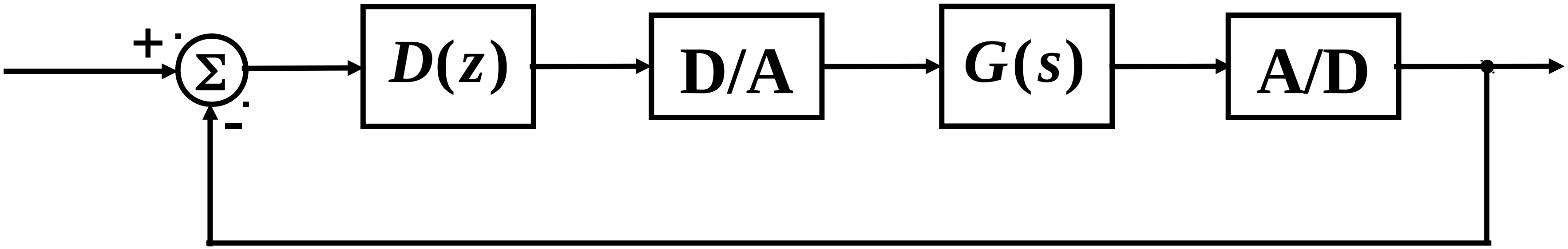
Idea of Hold Equivalents is to construct $\hat{e}(t) \approx e(t)$ from samples $e(k)$, then apply $\hat{e}(t)$ to the system $H(s)$ to get $\hat{u}(t)$, then sample to arrive at $\hat{u}(k)$, and we would like that

$$\hat{u}(k) = u(k).$$

If $e(t)$ is bandlimited:
can perfectly reconstruct $\hat{e}(t) = e(t)$ from samples $e(k)$.

Hold Equivalents to Find $D(z)$ to Approximate $D(s)$



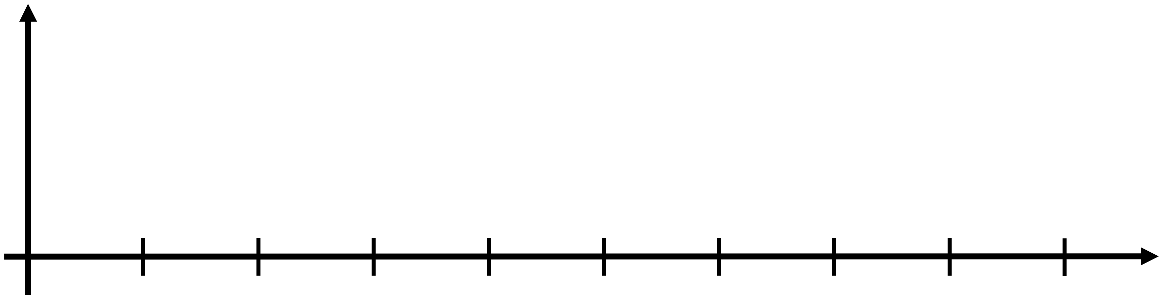


- "Hold" for D/A must be practically implementable.
- Here, hold equivalent is just a way of finding $D(z)$. If "hold" is non-causal but $D(z)$ is causal, that's fine!

Zero-Order Hold (ZOH)

- **Most common hold**
- **Determining $H(z)$ is exactly the same as we discussed earlier in the term:**

Triangle Hold



$$H_{\mathrm{TRI}}(z) = \mathcal{Z}\left\{ \mathrm{TRI}(s) H(s) \right\}$$

$$H_{\mathrm{TRI}}(z) = \frac{(z-1)^2}{Tz} \mathcal{Z}\left\{ \frac{H(s)}{s^2} \right\}$$

See Figure 6.9 of text.