Discrete Models of Sampled-Data Systems

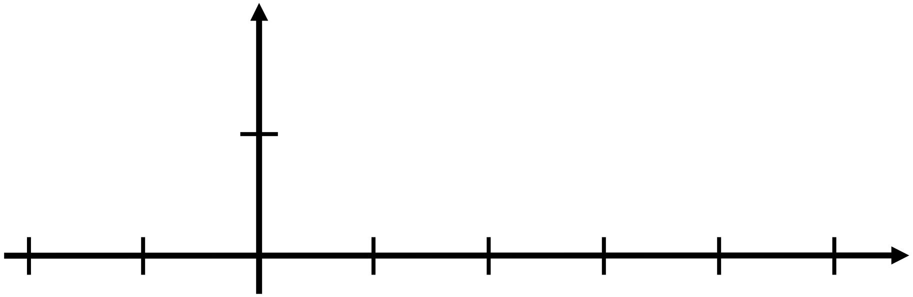
- Transfer function representations
- State-space representations
- Relation between state-space and transfer function representations
- Poles and zeros from state-space representations

Discrete Transfer Function Models of Sampled-Data Systems

- **How to compute discrete T.F. from $u(kT)$ to $y(kT)$?**
- **Assume D/A is a ZOH device:**
  - **Accepts a sample $u(kT)$ at $t = kT$ and then holds its output at this value until the next sample is accepted at $t = kT + T$.**
  - **Input $u(t)$ to plant $G(s)$ is a piecewise constant signal.**

- **From discussion before, $G(z)$ is just the Z-transform of the pulse response:**

$$u(kT) = \delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$



$$Y(s) = G(s)U(s) = \frac{1 - e^{-Ts}}{s} G(s)$$

$$\boxed{G(z) = (1 - z^{-1})\,Z\left\{\frac{G(s)}{s}\right\} = \left(\frac{z-1}{z}\right) Z\left\{\frac{G(s)}{s}\right\}}$$

Example

$$G(s) = \frac{1}{s^2}$$

What is $G(z)$?

$$\boxed{G(z) = \frac{T^2}{2}\frac{z+1}{(z-1)^2}}$$

Discrete State-Space Models of Sampled-Data Systems

- How to find discrete state-space model of system from $u(kT)$ to $y(kT)$?
- Let's solve the continuous-time state equations:
  - Two steps:
    - Homogeneous solution
    - Particular solution

Homogeneous Solution

$$\dot{\mathbf{x}}_h = \mathbf{F}\mathbf{x}_h, \qquad \mathbf{x}_h(t_0) = \mathbf{x}_0$$

$$\boxed{\mathbf{x}_h(t) = e^{\mathbf{F}(t-t_0)}\mathbf{x}_0}$$

It can be shown that the solution is unique.

Inverse of Matrix Exponential

$$\mathbf{x}_h(t_1) = e^{F(t_1 - t_0)}\mathbf{x}(t_0)$$

$$\mathbf{x}_h(t_2) = e^{F(t_2 - t_0)}\mathbf{x}(t_0)$$

Particular Solution

- **Variation of parameters method**

Given above, <u>guess</u> a solution of the form

$$\mathbf{x}_p(t) = e^{\mathbf{F}(t-t_0)}\mathbf{v}(t)$$

Substitute into the state equation:

$$\dot{\mathbf{v}}(t) = e^{-\mathbf{F}(t-t_0)}\mathbf{G}u(t)$$

$$\boxed{\mathbf{x}_p(t) = \int_{t_0}^{t} e^{\mathbf{F}(t-\tau)}\mathbf{G}u(\tau)d\tau}$$

- **Total solution is the sum of the homogeneous and particular solutions:**

$$\boxed{\mathbf{x}(t) = e^{\mathbf{F}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{F}(t-\tau)}\mathbf{G}u(\tau)d\tau}$$

- **Want to use this solution over one sample period to find discrete-time state-space difference equation:**

  - **This equation is independent of the type of hold used for $u(t)$, since $u(t)$ is represented in its general form.**

– **If we assume that a ZOH is used (as is commonly done):**

$$\boxed{\begin{aligned} \mathbf{x}(k+1) &= \Phi\mathbf{x}(k) + \Gamma u(k) \\ y(k) &= \mathbf{H}\mathbf{x}(k) + Ju(k) \end{aligned}}$$

where

$$\boxed{\begin{aligned} \Phi &= e^{\mathbf{F}T} \\ \Gamma &= \int_0^T e^{\mathbf{F}\eta} d\eta \ \mathbf{G} \end{aligned}}$$

Relation Between State-Space Difference Equation and Discrete-Time Transfer Function

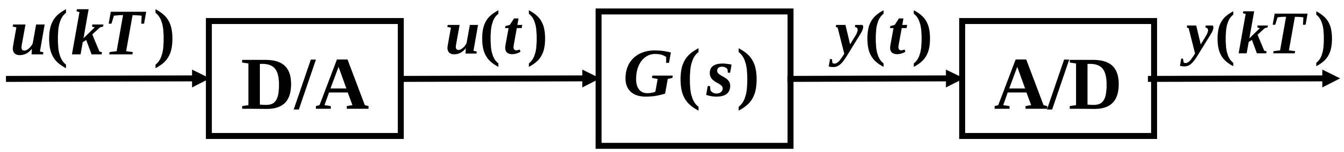
$$\begin{aligned} \mathbf{x}(k+1) &= \Phi \mathbf{x}(k) + \Gamma u(k) \\ y(k) &= \mathbf{H}\mathbf{x}(k) + J u(k) \end{aligned}$$

where

$$\begin{aligned} \Phi &= e^{\mathbf{F}T} \\ \Gamma &= \int_0^T e^{\mathbf{F}\eta} d\eta \, \mathbf{G} \end{aligned}$$

$$\boxed{\frac{Y(z)}{U(z)} = \mathbf{H}(z\mathbf{I} - \Phi)^{-1}\Gamma + J}$$

Example



$$G(s) = \frac{1}{s^2}$$

- Find discrete-time state-space representation of sampled data system.
- Find discrete-time transfer function from the state-space representation.

$$\Phi = e^{\mathbf{F}T}$$

$$\Gamma = \int_0^T e^{\mathbf{F}\eta} d\eta \, \mathbf{G}$$

$$\boxed{\Phi = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}}$$

$$G(z) = \frac{Y(z)}{U(z)} = \mathrm{H}(z\mathrm{I} - \Phi)^{-1}\Gamma + J$$

$$\boxed{G(z) = \frac{T^2}{2}\frac{z+1}{(z-1)^2}}$$

Poles and Zeros from State-Space Models

- A pole of a discrete-time system is a value of z such that the system (or difference equation) has a nontrivial solution when the forcing input $u = 0$.

$$
\begin{aligned}
\mathrm{x}(k+1) &= \Phi \mathrm{x}(k) + \Gamma u(k) \\
y(k) &= \mathrm{H}\mathrm{x}(k) + J u(k)
\end{aligned}
$$

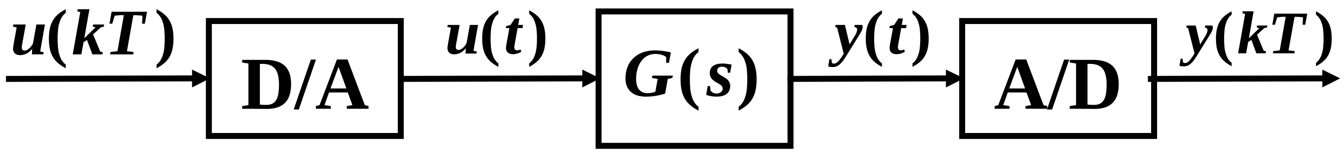
$$\boxed{\det(z\mathrm{I} - \Phi) = 0}$$

- A zero of a discrete-time system is a value of z such that the system output $y = 0$ even if the initial state $x(t_0)$ and the forcing input u are nonzero.

$$
\begin{aligned}
\mathbf{x}(k+1) &= \Phi\mathbf{x}(k) + \Gamma u(k) \\
y(k) &= \mathrm{H}\mathbf{x}(k) + Ju(k)
\end{aligned}
$$

$$
\boxed{\det\begin{bmatrix} z\mathrm{I} - \Phi & -\Gamma \\ \mathrm{H} & J \end{bmatrix} = 0}
$$

Example



$$G(s) = \frac{1}{s^2}$$

$$\boxed{G(z) = \frac{Y(z)}{U(z)} = H(zI - \Phi)^{-1}\Gamma + J = \frac{\det\begin{bmatrix} zI - \Phi & -\Gamma \\ H & J \end{bmatrix}}{\det(zI - \Phi)} = \frac{b(z)}{a(z)}}$$