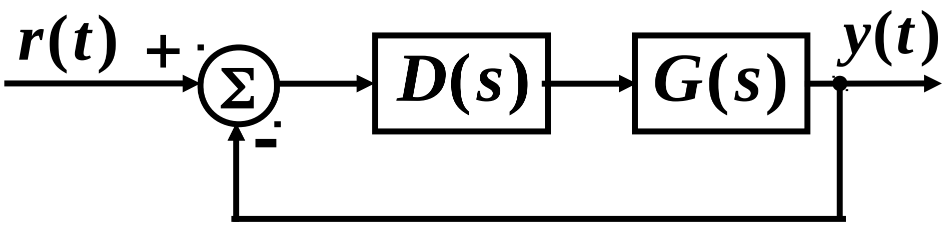
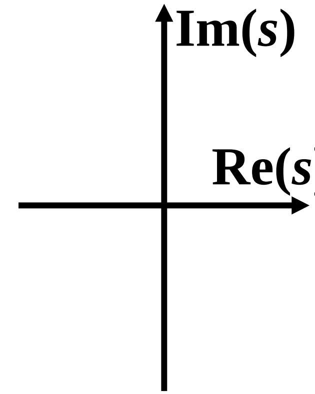
Frequency Response Methods

- Frequency response methods
  - Advantages
  - Continuous-time systems (review)
  - Discrete-time systems
    - Position error constants
    - Phase lag due to sampling
    - Gain and phase margins

Frequency Response Methods



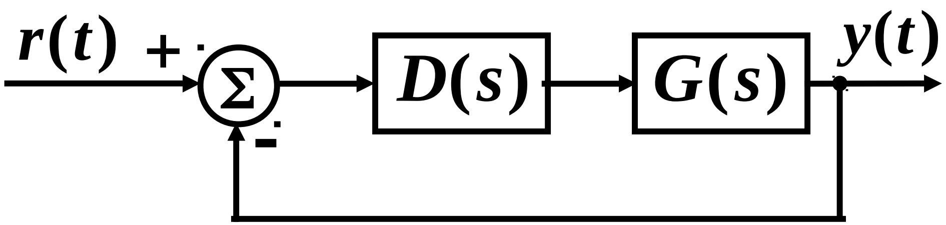


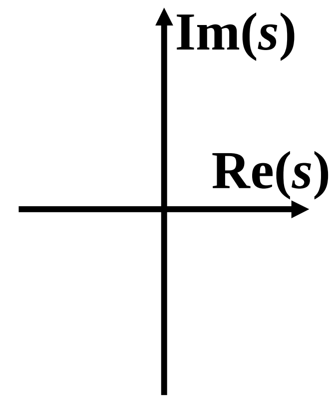
$$G(s) = K \frac{(s+z_1)(s+z_2) \cdots (s+z_m)}{(s+p_1)(s+p_2) \cdots (s+p_n)}, \qquad m \le n$$

$$G(s) = K_0 \frac{(s/z_1+1)(s/z_2+1) \cdots (s/z_m+1)}{(s/p_1+1)(s/p_2+1) \cdots (s/p_n+1)}$$

- **Bode plots: using gain margin (GM) and phase margin (PM) to design compensators**
- **Nyquist plots: for stability analysis**

Advantages of Frequency Response Methods





$$\left|G(s)\right| \Big|_{s=j\omega}$$

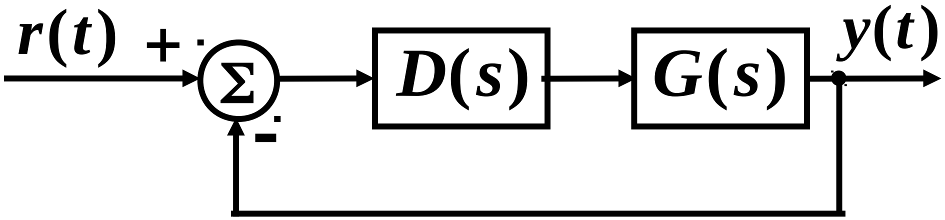
$$\angle\, G(s) \Big|_{s=j\omega}$$

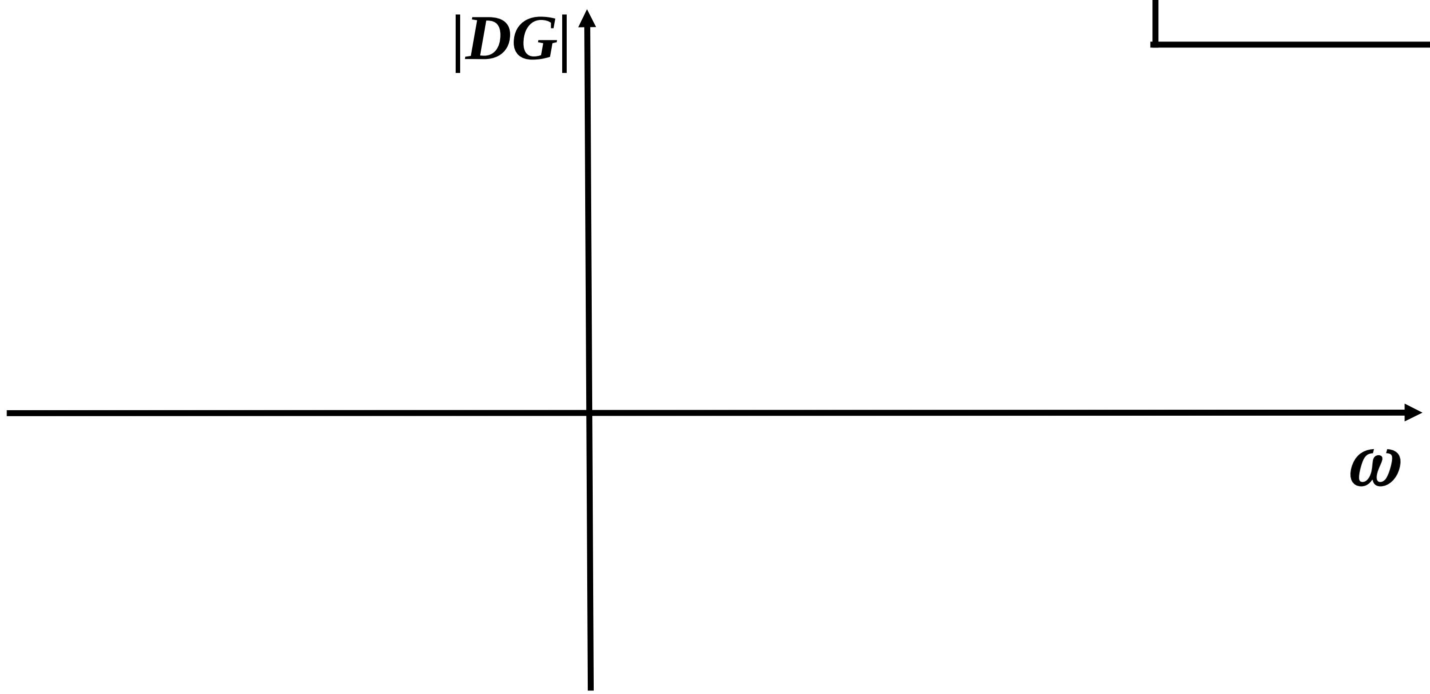
1. Sketching Bode gain/phase plots is easy.

2. Effect of $D(s)$ is easily sketched.

3. If minimum phase (all poles and zeros in LHP), then by Bode's gain-phase relationship, only gain plot is needed.

4. Can determine system type.





$$K_p = \lim_{s\to 0} D(s)G(s)$$

Steady-state error to unit step: $$e(\infty) = \frac{1}{1+K_p}$$

$$K_v = \lim_{s \to 0} sD(s)G(s)$$

$$D(s)G(s) = \frac{K}{s} \frac{(s+z_1)(s+z_2) \cdots (s+z_m)}{(s+p_1)(s+p_2) \cdots (s+p_n)}$$

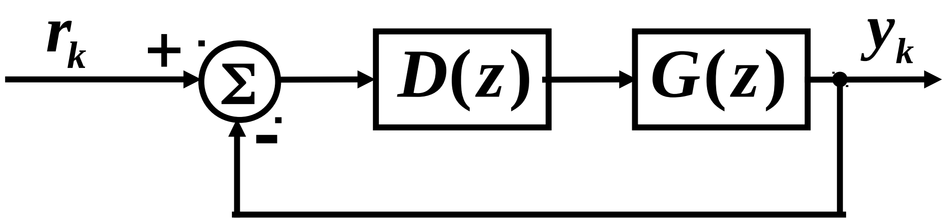
$$\lim_{s \to 0} D(s)G(s) =$$

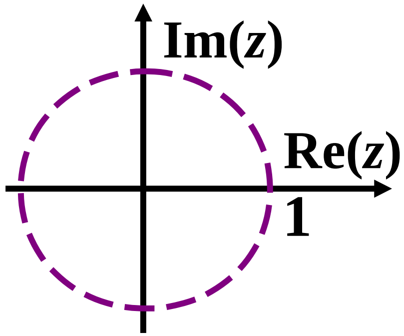
$$\approx \lim_{s \to 0} \frac{K_0}{s}$$

Steady-state error to unit ramp: $e(\infty) = \dfrac{1}{K_v}$

5. Easy to sketch Nyquist plot and apply stability criterion.

Frequency Response for Discrete Systems





$$G(z) = K\frac{(z+z_{d_1})(z+z_{d_2})\ \cdots\ (z+z_{d_m})}{(z+p_{d_1})(z+p_{d_2})\ \cdots\ (z+p_{d_n})}, \qquad m \le n$$

$$|G(z)|\,\Big|_{z=e^{j\omega T}} \qquad\qquad \angle G(z)\,\Big|_{z=e^{j\omega T}}$$

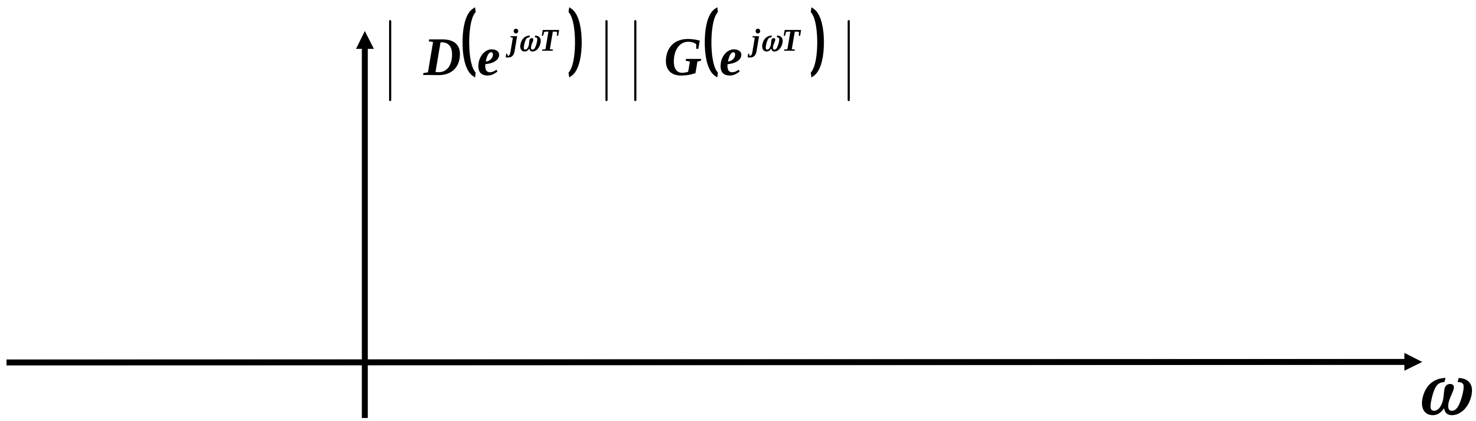
1. **Difficult to sketch by hand.**
2. **Difficult to see the effect of $D(z)$ on system.**

Generally use computers to plot discrete-time frequency response.

- **Frequency plots of $G(z)$ are for $z = e^{j\omega T}$ for $0 \le \omega T \le \pi$.**

- **Discuss Example 7.8 of text.**

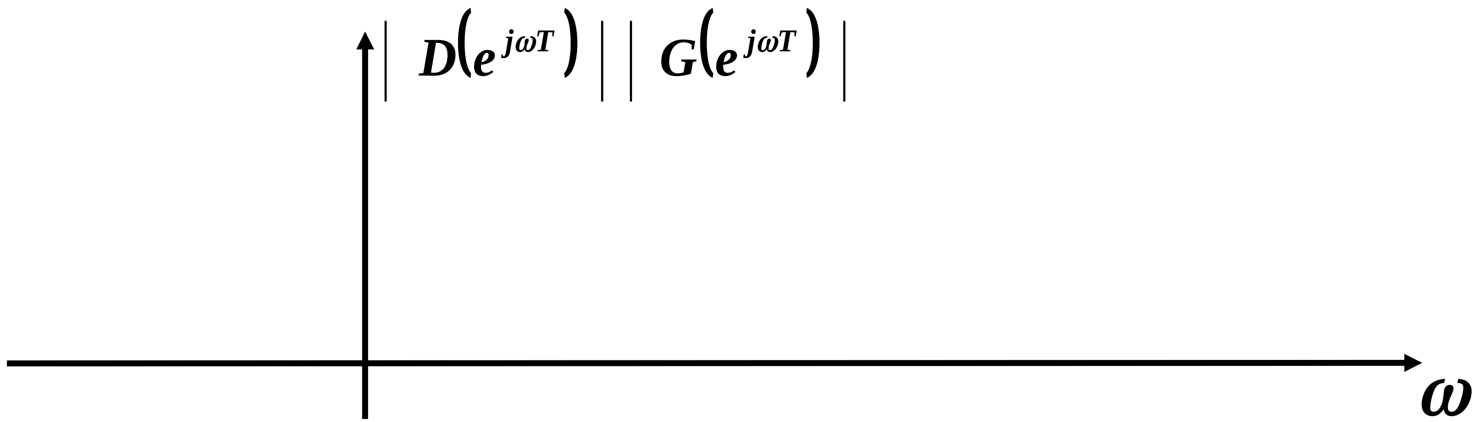
3. Can still easily determine system type, as well as error constants (K_p or K_v), from frequency response plots.



$$K_p = \lim_{z\to 1} D(z)G(z)$$

$$=$$

$$= \lim_{\omega\to 0} D(e^{j\omega T})G(e^{j\omega T})$$



$$K_v = \lim_{z \to 1} \frac{(z-1)D(z)G(z)}{Tz}$$

Now, $e^{j\omega T} \approx 1 + j\omega T$ **for** $\omega T \to 0$

$$K_v = \lim_{\omega \to 0} \frac{j\omega T \quad D(\cdot)G(\cdot)}{T(1 + j\omega T)}$$

4. Easy to sketch Nyquist plot and apply stability criterion.

Phase Lag Due to Sampling

$$\Delta\phi = \frac{\omega T}{2}$$

good approximation for $\omega T < \frac{\pi}{2} \approx 1.6$

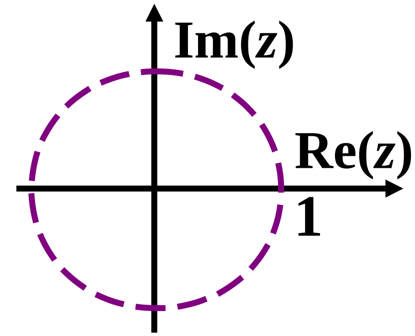
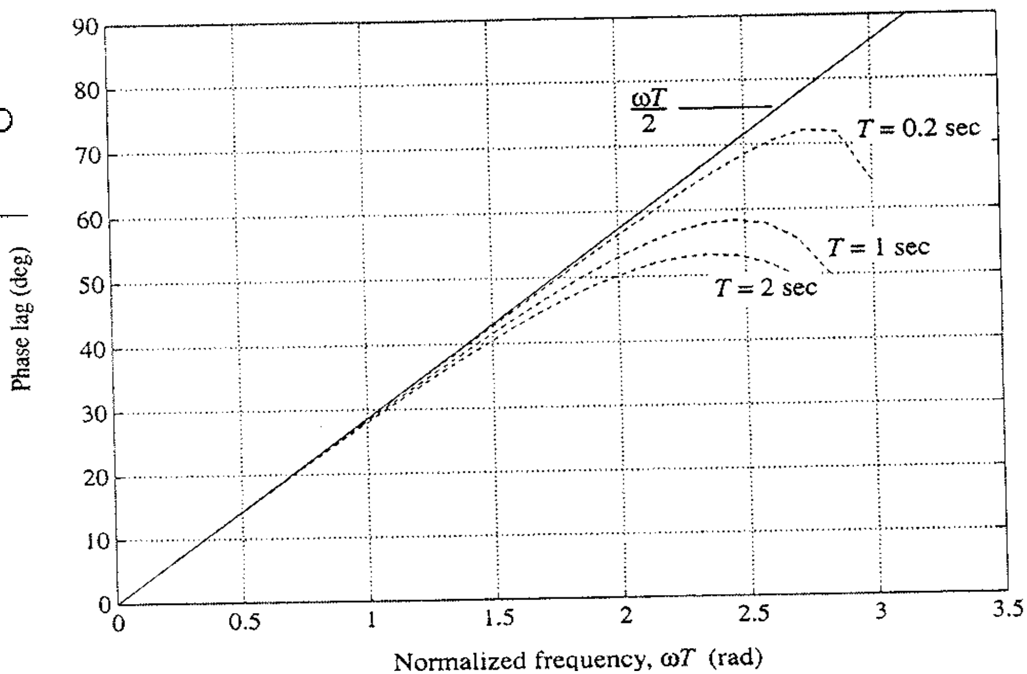


Figure 7.17
Phase lag due to sampling

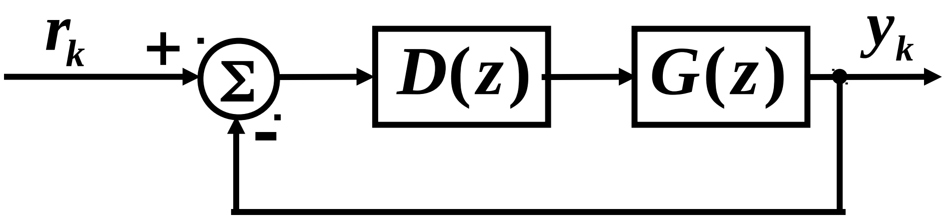


If have plotted $|G(j\omega)|$ **and** $\angle G(j\omega)$ **(easy to do by hand), then find** PM **of** $G(s)$**.**

Then: PM **of** $G(z) \approx PM$ **of** $G(s) - \frac{\omega T}{2}$

Gain and Phase Margins

Defined exactly the same way for discrete-time systems as for continuous-time systems.



Critical condition: $|DG| = 1 \,, \quad \angle DG = 180^\circ$

GM: **Factor by which can increase open-loop gain before system becomes unstable.**

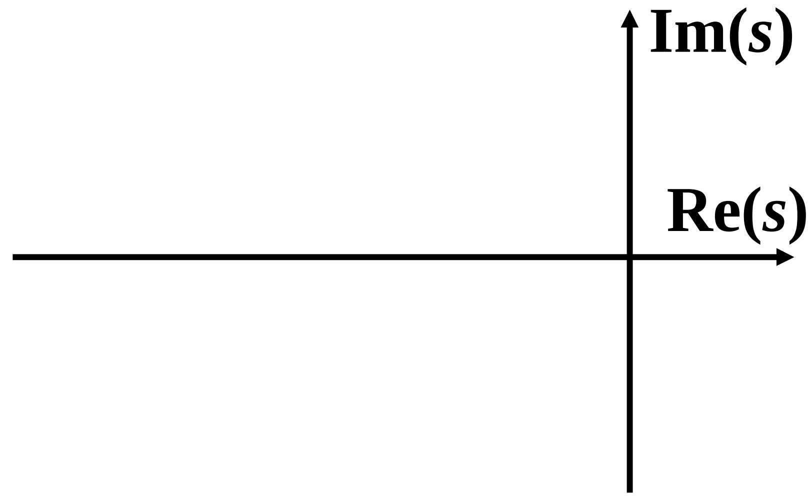
– find ω_{180} where $\angle DG = 180^\circ$ on the frequency response plot of the phase.

– then read off $|DG|$ at ω_{180}: $\quad GM = \left.\dfrac{1}{|DG|}\right|_{\omega_{180}}$

Revisit Example 7.8 of text:

$$G(s) = \frac{1}{s(s+1)}$$

infinite GM



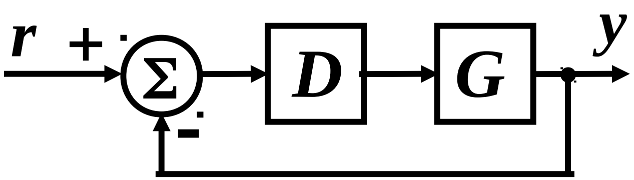
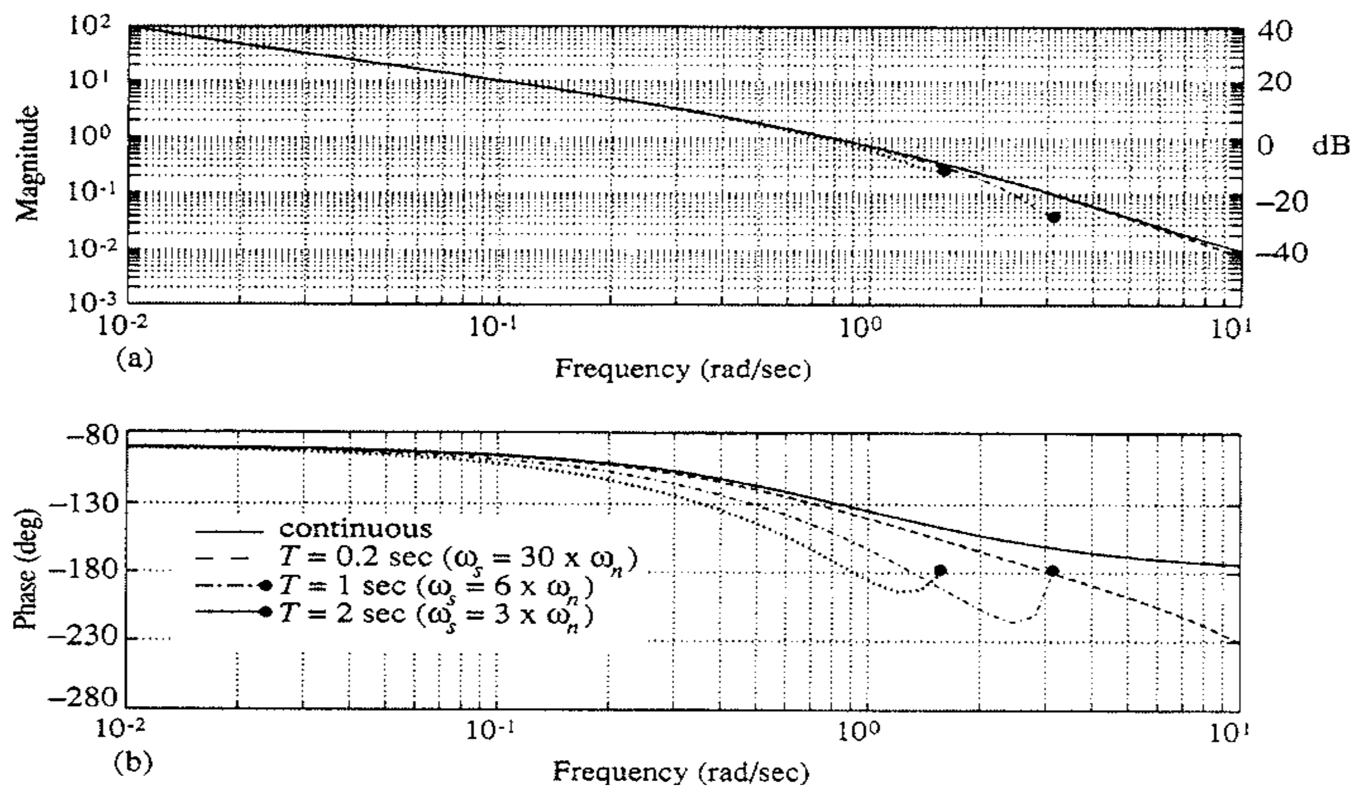


Figure 7.16 of text.

For $G(z)$ with $T = 0.2$ sec:

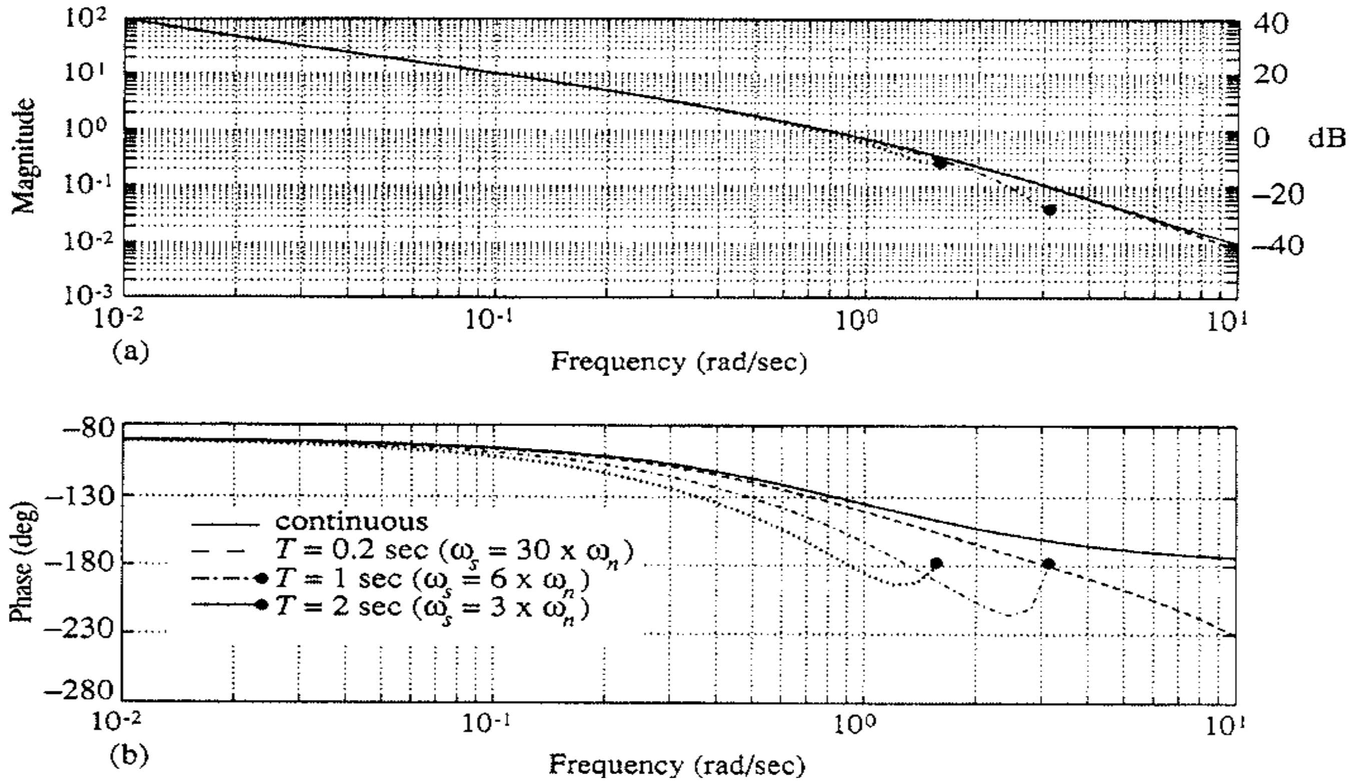
$\omega_{180} \approx$



T = 1 sec: $\omega_{180} \approx$

T = 2 sec: $\omega_{180} \approx$

Figure 7.16 of text.



As $T\uparrow$, $GM\downarrow$.

If stable, $GM > 1$.

***PM*:** Find ω_c where $|DG| = 1$.

$$PM = \angle DG\big|_{\omega_c} + 180^\circ$$

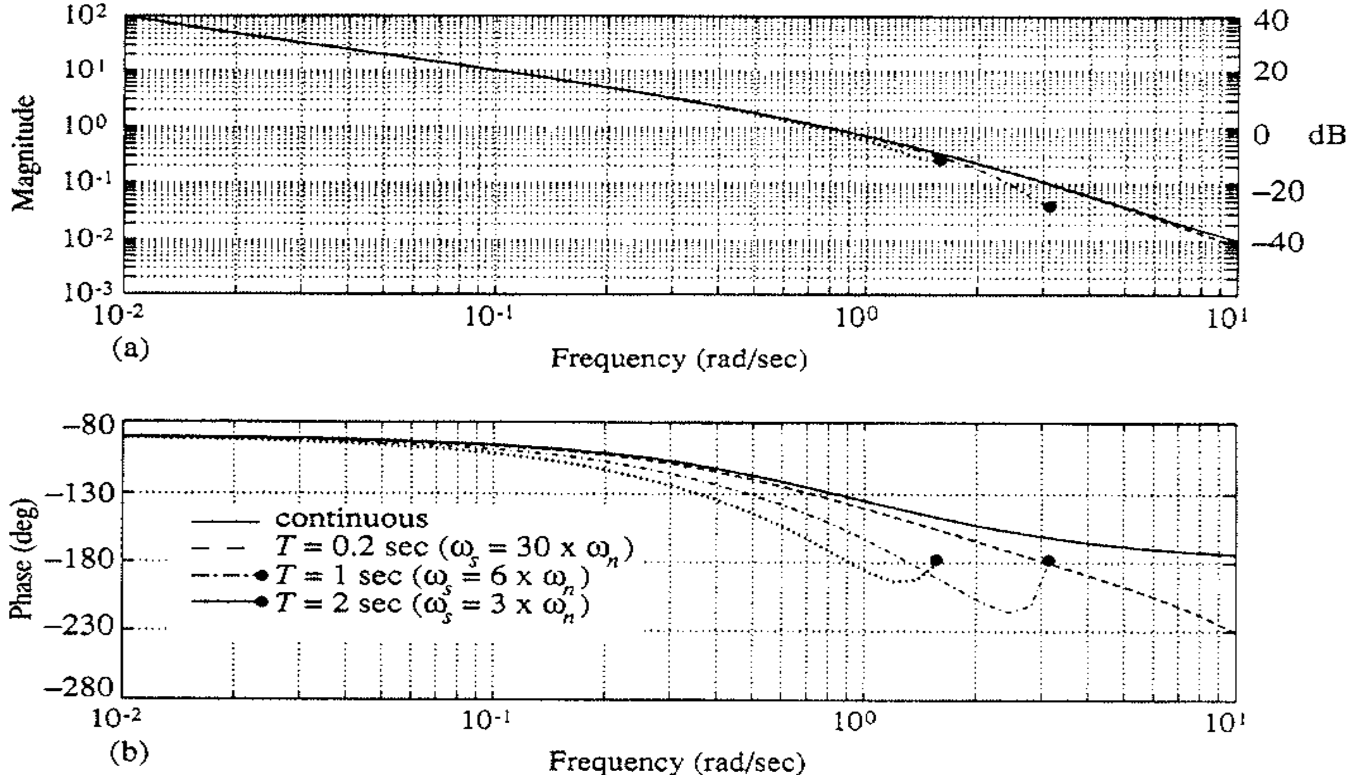
If stable, $PM > 0$.

$G(s)$: $\angle DG\big|_{\omega_c} \approx$

$PM =$

Revisit Example 7.8 of text:

Figure 7.16 of text.



$\omega_c \approx 0.7$ rad/s

$G(z)$: $T = 0.2$ sec: $\angle DG\big|_{\omega_c} \approx$ $PM =$

$T = 1$ sec: $\angle DG\big|_{\omega_c} \approx$ $PM =$

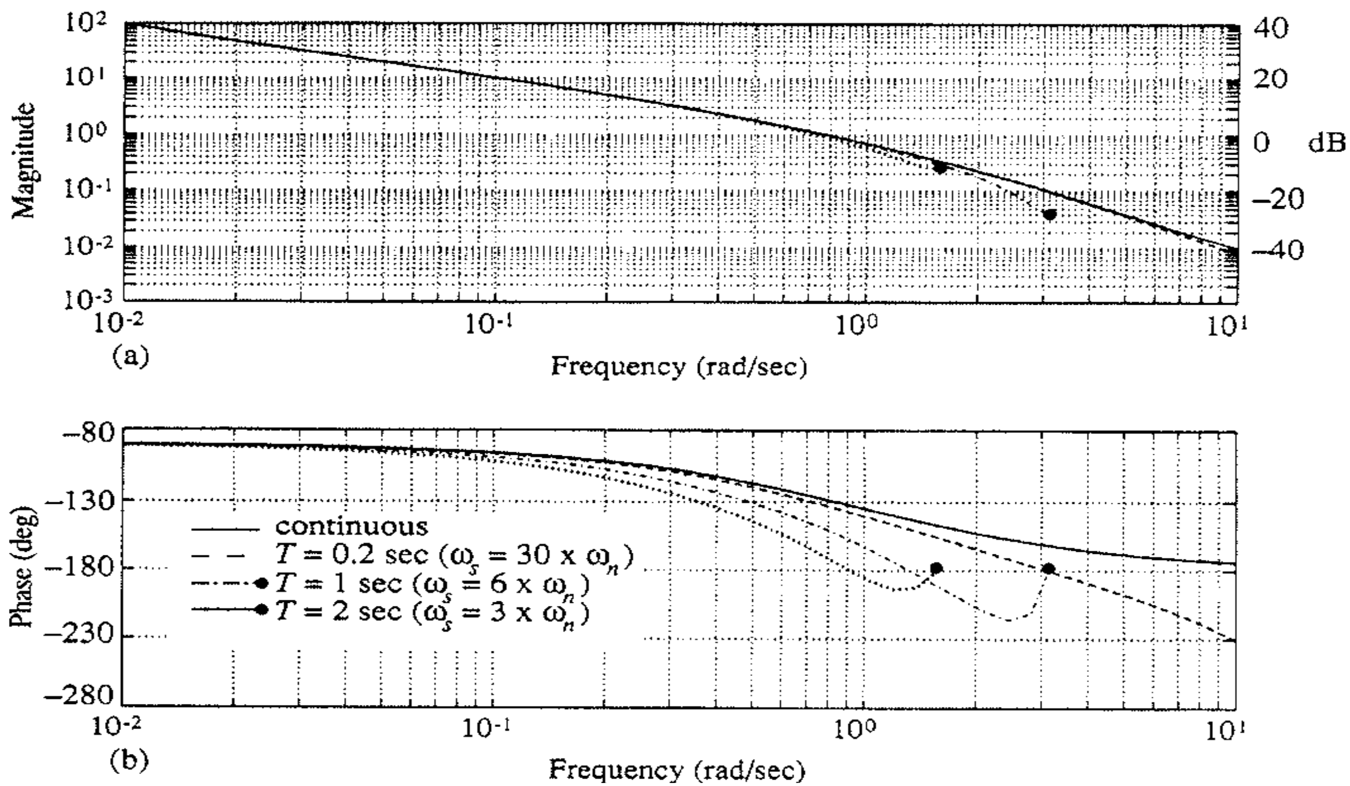
$T = 2$ sec: $\angle DG\big|_{\omega_c} \approx$ $PM =$

Figure 7.16 of text.

$\omega_c \approx 0.7$ rad/s

As $T\uparrow$, $PM\downarrow$.

If stable, $PM > 0$.



- **For system to be stable, need $PM > 0$ & $GM > 1$.**
- **Rule of thumb: $PM > 0$ & $GM > 1$ ➔ stable.**
  - **Works <u>most</u> of the time.**