Digital Filtering

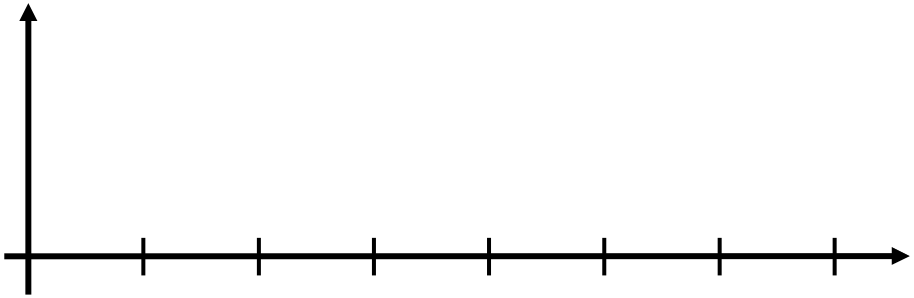
- Numerical integration
  - Forward rectangular rule
  - Backward rectangular rule
  - Trapezoid (or bilinear) rule
  - Bilinear rule with pre-warping

Digital Filtering

- One method of digital control design is called Emulation
  - Design a continuous compensator first, and then map that to a digital controller that emulates the continuous one.
    - A continuous compensator $D(s)$ is essentially a filter.
    - Goal is to determine a discrete controller $D(z)$ that approximates the behavior of $D(s)$.
- Methods of approximating a continuous transfer function $H(s)$ with a discrete one $H(z)$:
  - Numerical integration
  - Pole-zero mapping
  - Hold equivalents

Numerical Integration

- How do we numerically integrate a continuous function accurately?



- Numerically, only feasible to evaluate $e(t)$ at a <u>finite</u> number of points (that may be evenly or unevenly spaced).
- Assuming we have evenly spaced samples of $e(t)$:

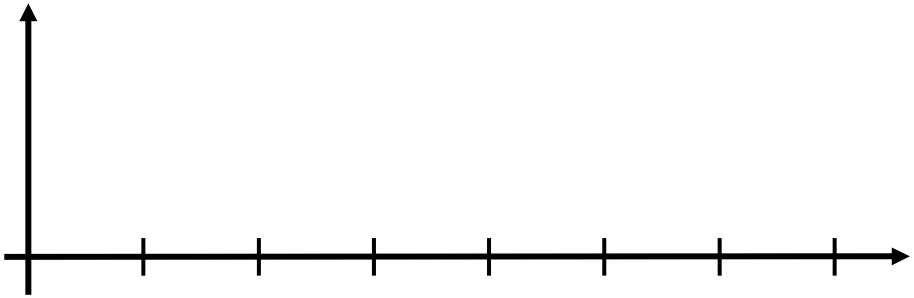
$$u(kT) = a\int_0^{kT} e(\tau)d\tau$$

$$= u(kT - T) + aA$$

- How do we approximate A ?
  - Forward rectangular rule (Euler's rule)
  - Backward rectangular rule
  - Trapezoid rule (Tustin's rule) (bilinear transform)

Forward Rectangular Rule

- Approximate A by looking "forward" from $kT - T$.



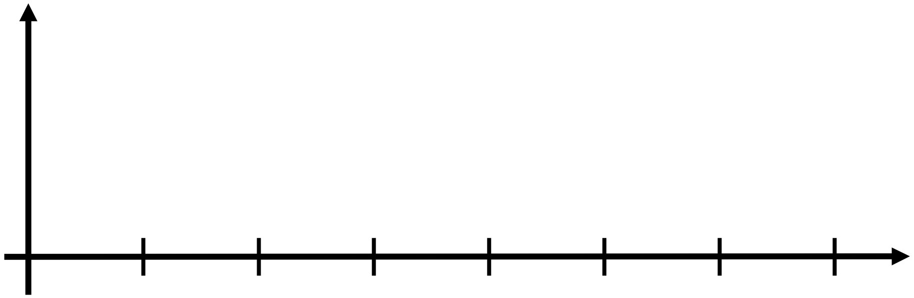
$$H_F(z) = \frac{a}{\left(\dfrac{z-1}{T}\right)}$$

vs.

$$H(s) = \frac{a}{s}$$

Backward Rectangular Rule

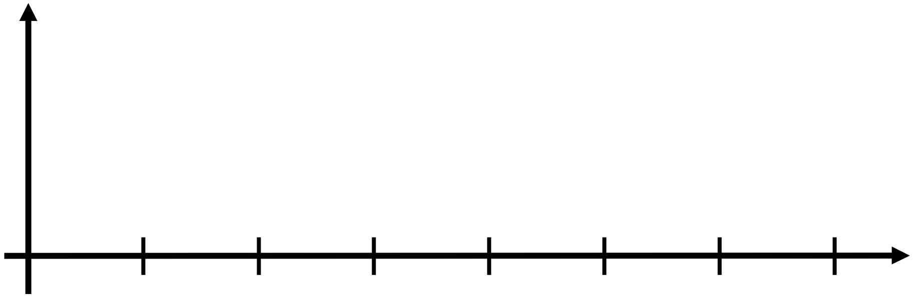
- Approximate A by looking “backward” from kT.



$$H_B(z) = \frac{a}{\left(\frac{z-1}{Tz}\right)} \qquad \text{vs.} \qquad H(s) = \frac{a}{s}$$

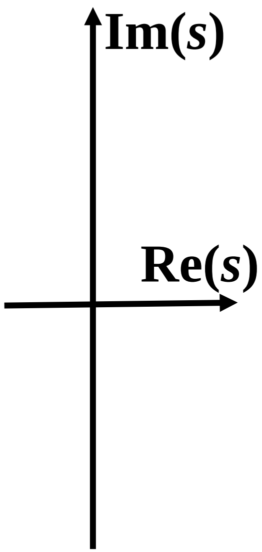
Trapezoid Rule

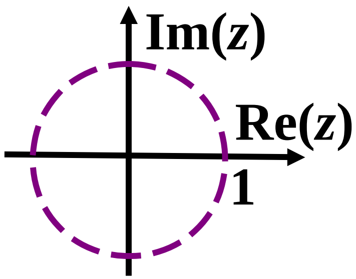
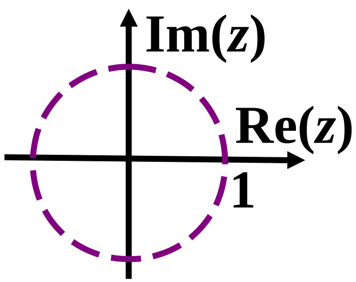
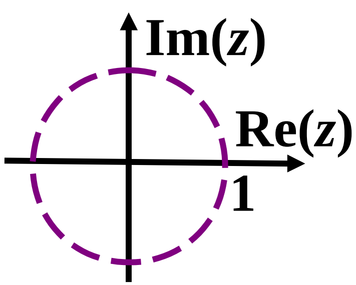
- **Approximate A by the area of the trapezoid formed by $e(kT-T)$ and $e(kT)$.**



$$H_T(z) = \frac{a}{\frac{2}{T}\left(\frac{z-1}{z+1}\right)} \quad \text{vs.} \quad H(s) = \frac{a}{s}$$

Summary of Numerical Integration



$H(s)$		$H(z)$	
$\left.\dfrac{a}{s}\right\rvert_{s=\frac{z-1}{T}}$	Forward rule	$\dfrac{a}{\left(\dfrac{z-1}{T}\right)}$	
$\left.\dfrac{a}{s}\right\rvert_{s=\frac{z-1}{Tz}}$	Backward rule	$\dfrac{a}{\left(\dfrac{z-1}{Tz}\right)}$	
$\left.\dfrac{a}{s}\right\rvert_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$	Trapezoid rule	$\dfrac{a}{\dfrac{2}{T}\left(\dfrac{z-1}{z+1}\right)}$	

- If we derive $H(z)$ similarly for other (more complex) $H(s)$, we will see that for each rule, we can substitute the same expressions above for s to get $H(z)$.

- Note that we do not need to know the poles and zeros of $H(s)$ to find $H(z)$.

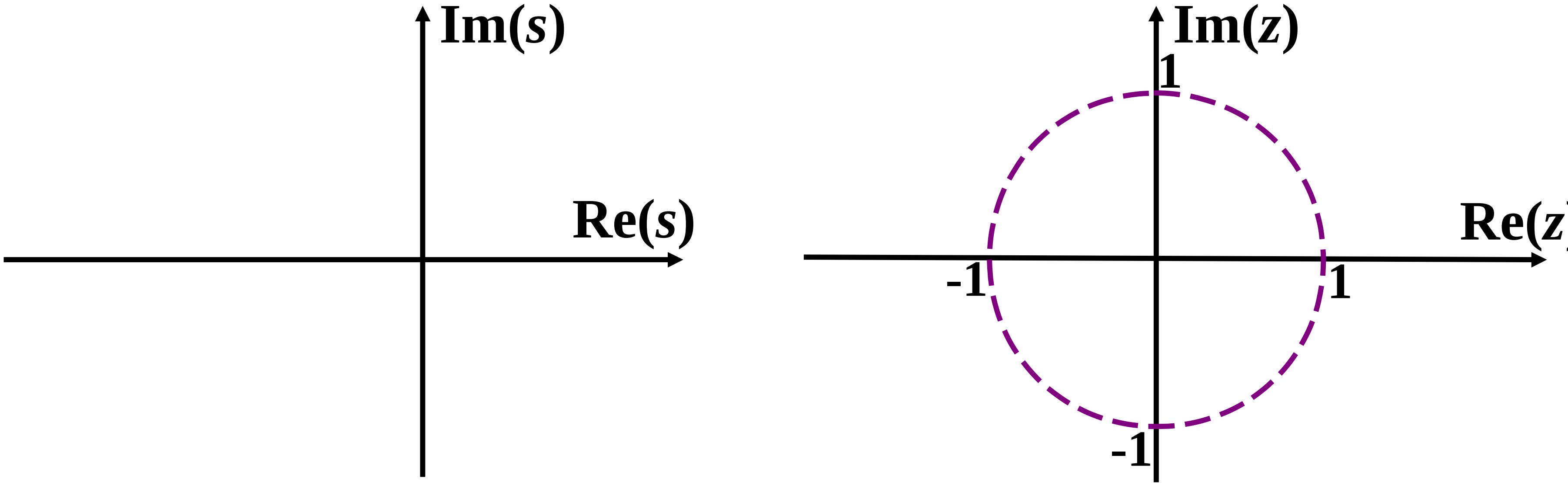
- These substitutions for s allow us to map an $H(s)$ to an $H(z)$. Note that all the substitution expressions depend on T.

- If $H(s)$ is stable (poles in LHP of s–plane), is $H(z)$ stable (poles inside UC of z–plane) under these maps?

Forward Rectangular Rule

$$H(s)\Big|_{s=\frac{z-1}{T}} = H(z) \qquad \Rightarrow \qquad z = 1 + Ts$$

If $H(s)$ has a pole at s_p , then $H(z)$ has a pole at $z_p = 1 + T\,s_p$.



If $H(s)$ is stable, $H(z)$ is <u>not</u> guaranteed to be stable!

Example

$$H(s) = \frac{1}{s+100}$$

$$H(z) =$$

Backward Rectangular Rule

$$H(s)\Big|_{s=\frac{z-1}{Tz}} = H(z)$$

If $H(s)$ has a pole at s_p , then $H(z)$ has a pole at $z_p = \dfrac{1}{1-Ts_p}$

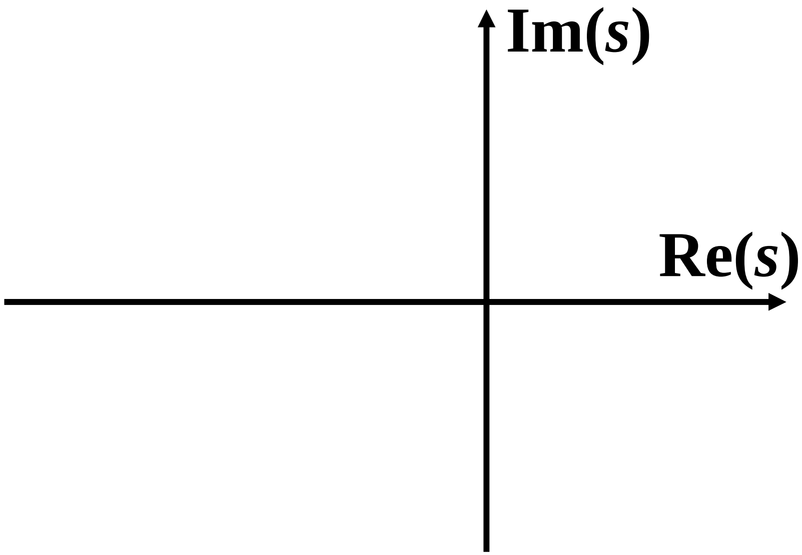
If $s_p = j\omega$ (on stability boundary), then $\left| z_p - \dfrac{1}{2} \right| = \dfrac{1}{2}\left| \dfrac{1+jT\omega}{1-jT\omega} \right|$

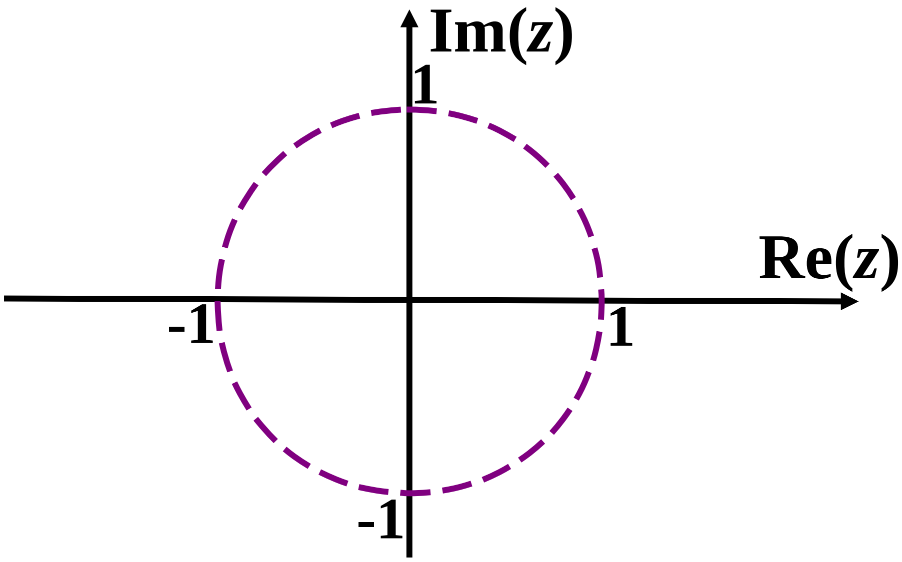
If s_p is in the LHP, say

$$s_p = -\alpha \pm j\beta, \qquad \alpha > 0$$

then

$$\left| z_p - \frac{1}{2} \right| = \frac{1}{2} \left| \frac{1 + T(-\alpha \pm j\beta)}{1 - T(-\alpha \pm j\beta)} \right|$$





If $H(s)$ is stable, $H(z)$ is guaranteed to be stable.

Trapezoid or Bilinear Rule

$$H(s)\Big|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)} = H(z)$$

If $H(s)$ has a pole at s_p , then $H(z)$ has a pole at $z_p = \dfrac{1+T s_p/2}{1-Ts_p/2}$

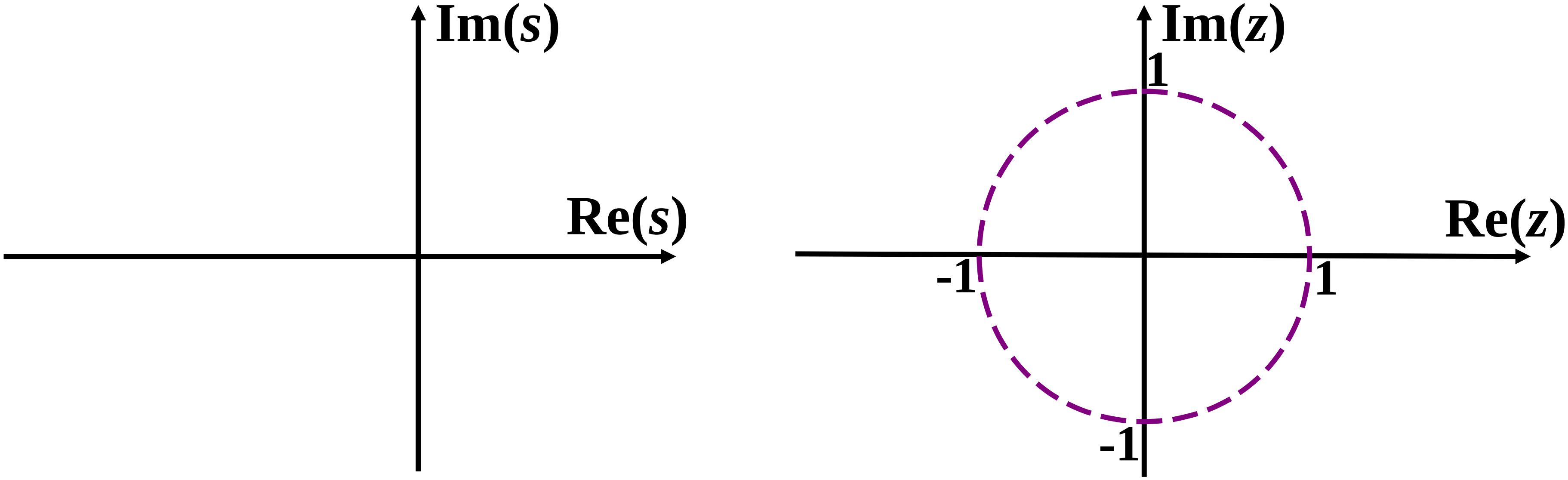
If $s_p = j\omega$ (on stability boundary), then $\left|z_p\right| = \left|\dfrac{1+jT\omega/2}{1-jT\omega/2}\right|$

If s_p is in the LHP, say

$$s_p = -\alpha \pm j\beta, \qquad \alpha > 0$$

then

$$\left|z_p\right| = \left|\frac{1+\frac{T}{2}(-\alpha \pm j\beta)}{1-\frac{T}{2}(-\alpha \pm j\beta)}\right|$$



If $H(s)$ is stable, $H(z)$ is guaranteed to be stable.

- **Stability region maps exactly from s-plane to z-plane, but behavior and characteristics of $H(z)$ are usually still different from those of $H(s)$.**

- **For example, apply a sinusoid of frequency ω_1 to both $H(s)$ and $H(z)$, will the output signal have the same magnitude?**

$$\longrightarrow \boxed{H(s)} \longrightarrow \qquad\qquad \longrightarrow \boxed{H(z)} \longrightarrow$$

Consider $H(s) = \dfrac{a}{s+a}$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

$$H\left(e^{jaT}\right) = \frac{a}{\frac{2}{T}\,\frac{e^{jaT}-1}{e^{jaT}+1}+a}$$

$$= \frac{a}{\frac{2}{T}\,j\tan\frac{aT}{2}+a}$$

Bilinear Rule with Pre-Warping

"Pre-warp" $H(s)$ before applying $s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$

so that final $H(z)$ has same half power point as original $H(s)$.

- **Re-write $H(s)$ in the form** $H\left(\frac{s}{\omega_1}\right)$ **where ω_1 is the critical frequency**

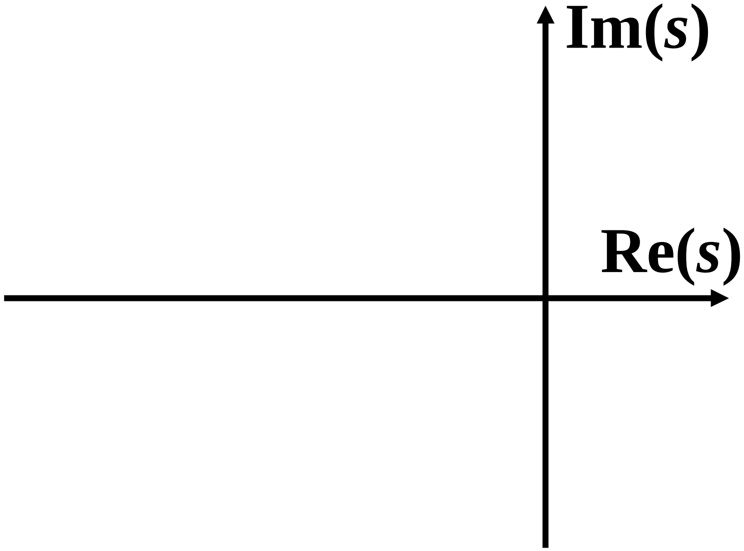
  where we want $|H(z)|$ to match $|H(s)|$.

- Replace ω_1 by a where $a = \dfrac{2}{T}\tan\dfrac{\omega_1 T}{2}$

- Map to $H_p(z)$ using $s = \dfrac{2}{T}\left(\dfrac{z-1}{z+1}\right)$

- **Overall frequency substitution rule is**

$$H_p(z) = H\left(\frac{s}{\omega_1}\right)\Bigg|_{\frac{s}{\omega_1} = \frac{1}{\tan(\omega_1 T/2)}\left(\frac{z-1}{z+1}\right)}$$



- **Consider Example 6.1 of text:**

$$H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

3rd-order Butterworth filter with unity bandwidth ω_p=1.

Find $H(z)$ to match $H(s)$ at ω_p=1. See Figure 6.4 of text.