State Feedback Control Law Design via Transformation to Control Canonical Form

- **Why transform to control canonical form?**
- **What is the state feedback control law for the original non-transformed state-space system?**
- **How to transform to control canonical form?**

Transformation to Control Canonical Form

Given $$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

- Many state-space representations

$$\mathbf{x}(k+1) = \Phi \mathbf{x}(k) + \Gamma u(k)$$
$$y(k) = \mathrm{H}\mathbf{x}(k) + Ju(k)$$

- Most "natural" state-space representation is one where states represent physical variables (position, velocity, temperature, etc.)

- A representation that allows state feedback design to be <u>very </u>easily done is the <u>control canonical form</u>.

$$\{\Phi, \Gamma, \mathrm{H}, J\} \xrightarrow{\mathbf{x} = \mathrm{T}\mathbf{z}} \{\Phi_c, \Gamma_c, \mathrm{H}_c, J_c\}$$

Control Canonical Form

$$\Phi_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix}, \qquad \Gamma_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\mathbf{H}_c = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix}, \qquad J_c = b_0$$

If strictly proper: $b_0 = 0$

$$\boxed{\det(z\mathbf{I} - \Phi_c) = z^n + a_1 z^{n-1} + \cdots + a_n = 0}$$

State Feedback Control of a System in Control Canonical Form

Closed-loop system matrix is $\Phi_c - \Gamma_c K_c$

$$\Gamma_c K_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} k_{1c} & k_{2c} & \cdots & k_{nc} \end{bmatrix} = \begin{bmatrix} k_{1c} & k_{2c} & \cdots & k_{nc} \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

$$\Phi_c - \Gamma_c K_c = \begin{bmatrix} -a_1 - k_{1c} & -a_2 - k_{2c} & \cdots & -a_n - k_{nc} \\ 1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix}$$

$$\det(zI - \Phi_c + \Gamma_c K_c) = z^n + (a_1 + k_{1c}) z^{n-1} + \cdots + (a_n + k_{nc})$$

Actual closed-loop characteristic equation:

$$\det(z\mathbf{I} - \Phi_c + \Gamma_c \mathbf{K}_c) = z^n + (a_1 + k_{1c})z^{n-1} + \cdots + (a_n + k_{nc})$$

Desired characteristic equation:

$$\alpha_c(z) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n$$

Equating coefficients gives:

Summary of State Feedback Control Law Design via Transformation to Control Canonical Form

Now we have the feedback gain vector for control canonical form, but we want $\mathbf{K}$ for the original state representation.

1. Find $\mathbf{T}$: $\{\Phi, \Gamma, \mathbf{H}, J\} \xrightarrow{\mathbf{x} = \mathbf{T}\mathbf{z}} \{\Phi_c, \Gamma_c, \mathbf{H}_c, J_c\}$

2. Feedback gain vector for control canonical form:

$$\mathbf{K}_c^T = \boldsymbol{\alpha} - \mathbf{a}$$

3. Transform it back: $\mathbf{K} = \mathbf{K}_c \mathbf{T}^{-1}$

$$\left\{\Phi, \Gamma, H, J\right\} \xrightarrow[\mathbf{z} = \mathbf{T}^{-1}\mathbf{x}]{\mathbf{x} = \mathbf{T}\mathbf{z}} \left\{\Phi_c, \Gamma_c, H_c, J_c\right\}$$

$$\mathbf{x}(k+1) = \Phi \mathbf{x}(k) + \Gamma u(k) \qquad \mathbf{z}(k+1) = \Phi_c \mathbf{z}(k) + \Gamma_c u(k)$$

$$y(k) = H\mathbf{x}(k) + Ju(k) \qquad y(k) = H_c \mathbf{z}(k) + J_c u(k)$$

$$\boxed{\begin{aligned} \Phi_c &= \mathbf{T}^{-1}\Phi\mathbf{T} \\ \Gamma_c &= \mathbf{T}^{-1}\Gamma \\ H_c &= H\mathbf{T} \\ J_c &= J \end{aligned}}$$

$$u = -K_c \mathbf{z}$$

How do we determine the $\mathbf{T}$ to transform the system to control canonical form?

First, does the transform affect controllability of a system?

$$\left\{\Phi_1, \Gamma_1, H_1, J_1\right\} \xleftrightarrow{\mathbf{x}_1 = \mathbf{T}\mathbf{x}_2} \left\{\Phi_2, \Gamma_2, H_2, J_2\right\}$$

$$\mathcal{C}_1 = \begin{bmatrix} \Gamma_1 & \Phi_1\Gamma_1 & \cdots & \Phi_1^{n-1}\Gamma_1 \end{bmatrix}$$

$$\mathcal{C}_2 = \begin{bmatrix} \Gamma_2 & \Phi_2\Gamma_2 & \cdots & \Phi_2^{n-1}\Gamma_2 \end{bmatrix}$$

$$\boxed{\begin{aligned} \Phi_2 &= \mathbf{T}^{-1}\Phi_1\mathbf{T} \\ \Gamma_2 &= \mathbf{T}^{-1}\Gamma_1 \\ H_2 &= H_1\mathbf{T} \\ J_2 &= J_1 \end{aligned}}$$

$$\boxed{\mathcal{C}_2 = \mathbf{T}^{-1}\mathcal{C}_1}$$

So for

$$\{\Phi, \Gamma, H, J\} \xleftrightarrow{\;T\;} \{\Phi_c, \Gamma_c, H_c, J_c\}$$

$$\Longrightarrow \mathcal{C}_c = T^{-1}\mathcal{C}$$

$$\boxed{K = (\alpha - a)^T \mathcal{C}_c \mathcal{C}^{-1}}$$

Controllability is a continuous property:

While we have talked about systems as being either controllable or uncontrollable, controllability is not only a binary property. Some systems are <u>more</u> controllable than others. Some <u>modes</u> of a system are more controllable than others.