Steady-State Analysis

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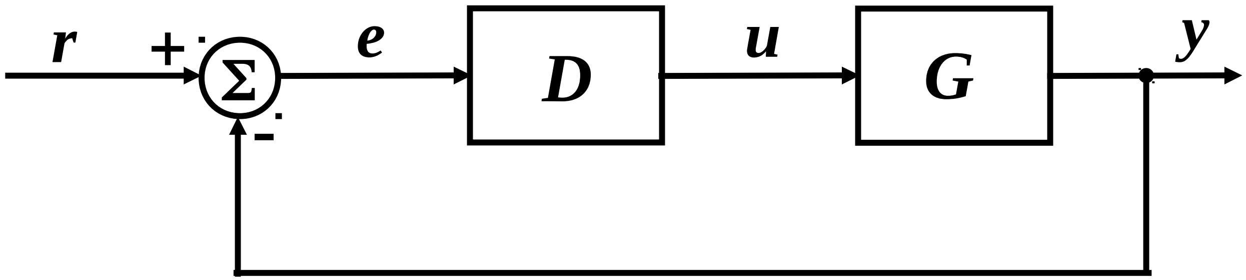
Steady-State Analysis



Error: $$e = r - y$$

$$E = R - Y = R - \left[\frac{DG}{1 + DG}\right] R = \frac{1}{1 + DG} R$$

Continuous-Time Systems

If a system is stable (except one simple pole at s = 0), the (continuous-time) final value theorem states:

$$e(\infty) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s}{1 + D(s)G(s)} R(s)$$

System Type for Continuous-Time Systems

We are often interested in what type of inputs a system can follow with finite error.

Type 0: System can follow a step (0^{th}-order polynomial) input with finite (nonzero) error.

$$R(s) = \frac{C_0}{s}$$

Type 1: System can follow a ramp (1^{st}-order polynomial) input with finite (nonzero) error.

Type ℓ: System can follow an ℓ^{th}-order polynomial input with finite (nonzero) error. A type ℓ system can follow polynomials of order < ℓ with zero steady-state error, and it cannot follow polynomials of order > ℓ.

Type 0 System with respect to $R(s)$

If $R(s)$ = step = $\dfrac{C_0}{s}$

$$e(\infty) = \text{finite and nonzero} = \lim_{s\to 0} \frac{s}{1+D(s)G(s)}\left[\frac{C_0}{s}\right] = \frac{C_0}{1+D(0)G(0)}$$

If $R(s)$ = ramp = $\dfrac{C_1}{s^2}$

$$E(s) = \frac{1}{1+D(s)G(s)}\left[\frac{C_1}{s^2}\right]$$

$$\lim_{t\to\infty} |e(t)| = \infty$$

Position Error Constant for a Type 0 Continuous System

$$K_p = \lim_{s \to 0} D(s)G(s) = \text{finite}$$

With respect to a reference input $r(t) = C_0 \, 1(t)$,

$$e(\infty) = \frac{C_0}{1 + K_p}$$

If want to design $D(s)$ such that the steady-state error to a unit step $< e_{0_{\max}}$

$$K_p > \frac{1 - e_{0_{\max}}}{e_{0_{\max}}}$$

Type 1 System with respect to $R(s)$

If $R(s)$ = step = $\dfrac{C_0}{s}$

$$e(\infty) = 0 = \lim_{s \to 0} \frac{s}{1 + D(s)G(s)} \left[\frac{C_0}{s} \right] = \frac{C_0}{1 + D(0)G(0)}$$

$\Longrightarrow$ $D(s)G(s)$ must have a pole at $s = 0$

If $R(s)$ = ramp = $\dfrac{C_1}{s^2}$

$$e(\infty) = \text{finite and nonzero} = \lim_{s \to 0} \frac{s}{1 + D(s)G(s)} \left[\frac{C_1}{s^2} \right]$$

Velocity Error Constant for a Type 1 Continuous System

$$K_v = \lim_{s \to 0} sD(s)G(s) = \text{finite and nonzero}$$

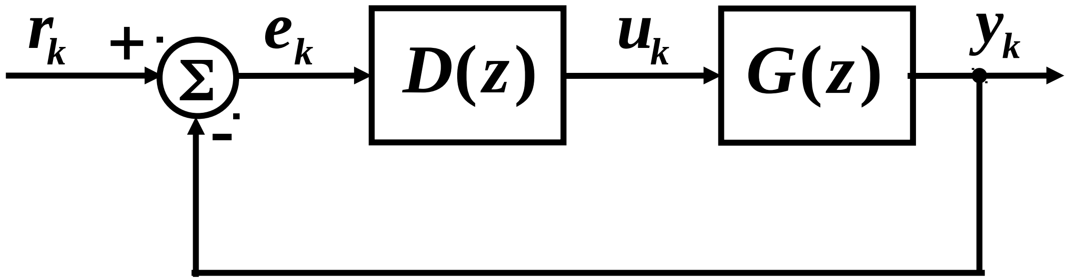
With respect to a reference input $r(t) = C_1\, t\, 1(t)$,

$$e(\infty) = \lim_{s \to 0} \frac{C_1}{sD(s)G(s)} = \frac{C_1}{K_v}$$

If want to design $D(s)$ such that the steady-state error to a unit ramp $< e_{1_{\max}}$:

$$K_v > \frac{1}{e_{1_{\max}}}$$

System Type for Discrete-Time Systems



$$G(z) = (1 - z^{-1}) Z\left[\frac{G(s)}{s}\right]$$

$$E(z) = \frac{1}{1 + D(z)G(z)} R(z)$$

If a system is stable (except one simple pole at $z = 1$), the (discrete-time) final value theorem states:

$$e(\infty) = \lim_{k \to \infty} e(k) = \lim_{z \to 1} (z - 1) \frac{1}{1 + D(z)G(z)} R(z)$$

A discrete-time system is:

Type 0: System can follow a discrete step input $1(k)$ with finite (nonzero) error.

Type ℓ: System can follow $r(k) = k^{\ell} 1(k)$ with finite (nonzero) error.

Type 0 System with respect to $R(z)$

If $R(z)$ = step = $\dfrac{C_0 z}{z-1}$

$$e(\infty) = \text{finite and nonzero} =$$

$$= \frac{C_0}{1 + D(1)G(1)}$$

$\Longrightarrow$ $D(1)G(1)$ must be finite.

If $R(z)$ = ramp = $\dfrac{C_1 T z}{(z-1)^2}$

$$E(z) = \frac{1}{1 + D(z)G(z)} \left[\frac{C_1 T z}{(z-1)^2} \right]$$

$$\lim_{k \to \infty} \left| e(k) \right| = \infty$$

$\Longrightarrow$ Infinite error in tracking ramp.

Position Error Constant for a Type 0 Discrete System

$$K_p = \lim_{z \to 1} D(z)G(z) = \text{finite}$$

With respect to a reference input $r(k) = C_0\, 1(k)$,

$$e(\infty) = \frac{C_0}{1 + K_p}$$

If want to design $D(z)$ such that the steady-state error to a unit step $< e_{0_{max}}$:

$$K_p > \frac{1 - e_{0_{max}}}{e_{0_{max}}}$$

Type 1 System with respect to $R(z)$

If $R(z)$ = ramp = $\dfrac{C_1 T z}{(z-1)^2}$

$$e(\infty) = \text{finite and nonzero}$$

$$= \lim_{z \to 1} \frac{C_1 T z}{(z-1) D(z) G(z)}$$

$\Longrightarrow$ $\lim_{z \to 1} (z-1) D(z) G(z)$ **must be finite and nonzero**

Velocity Error Constant for a Type 1 Discrete System

$$K_v = \lim_{z \to 1} \frac{(z-1)D(z)G(z)}{Tz}$$

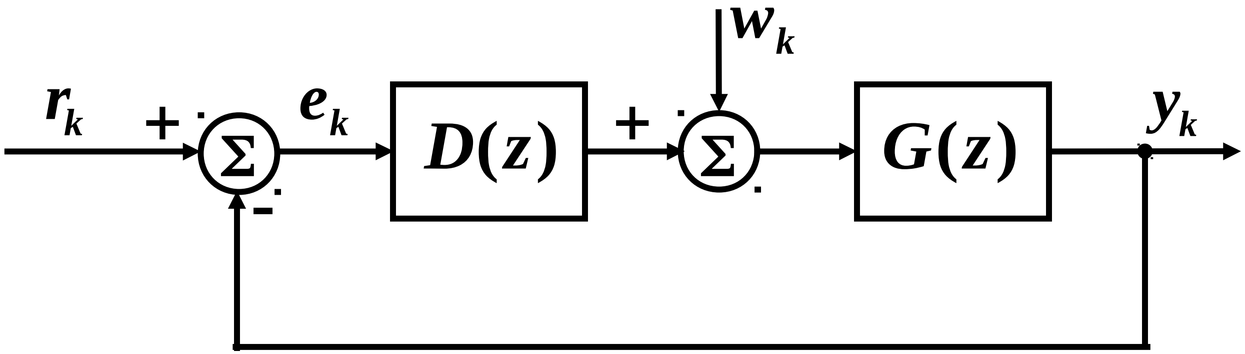
With respect to a reference input $r(k) = C_1 \, k \, 1(k)$,

$$e(\infty) = \frac{C_1}{K_v}$$

If want to design $D(z)$ such that the steady-state error to a unit ramp $< e_{1_{\max}}$

$$K_v > \frac{1}{e_{1_{\max}}}$$

System Type w.r.t. Disturbances



Assume $r_k = 0$ to evaluate effects of w_k

$$E(z) = \frac{-G(z)}{1 + D(z)G(z)} W(z)$$

If $|DG| >> 1$

Hence, if $|D| >> 1$