Analysis of Systems

- Transfer functions
- State-space representations
- Block diagram manipulation
- Relation of transfer function to pulse response
- BIBO stability

Transfer Functions

- The Z-transform has the same role in discrete-time systems that the Laplace transform has in the analysis of continuous-time systems.

- If $u(k)$ and $e(k)$ are related by a difference equation, the transfer function, $H(z)$, is defined as the ratio of the transform of the "output" $u(k)$ of the system to the transform of the "input" $e(k)$ of the system.

- **For a general difference equation:**

$$u_k = -a_1 u_{k-1} - a_2 u_{k-2} - \cdots - a_n u_{k-n} + b_0 e_k + b_1 e_{k-1} + \cdots + b_m e_{k-m}$$

- **Take Z-transform (and use the time-shift property):**

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}$$

$$H(z) = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n} = \frac{b(z)}{a(z)}$$

$$H(z) = \frac{z^{n-m}(b_0 z^m + b_1 z^{m-1} + \cdots + b_m)}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n} = \frac{b(z)}{a(z)}$$

- $H(z)$ is a rational function of a complex variable.
  - Solutions of $b(z) = 0$ are <u>zeros</u> of the transfer function.
  - Solutions of $a(z) = 0$ are <u>poles</u> of the transfer function.

- Special example

State-Space Representations

- You should know how to convert an nth-order ordinary differential equation (O.D.E.) into a set of n 1st-order O.D.E.'s.
- Similarly, an nth-order difference equation can be converted into a set of n 1st-order difference equations.
- Often done because set of 1st-order equations is easier to solve.
- Consider 3rd-order system as an example:

  – There are <u>many</u> state-space representations of this system.

Control Canonical Form

$$U(z) = H(z)E(z) = \frac{b(z)}{a(z)}E(z)$$

Let $\xi(z) = \dfrac{E(z)}{a(z)}$

then $U(z) = b(z)\xi(z)$ and $a(z)\xi(z) = E(z)$

$$\xi(k+3) = e(k) - a_1\xi(k+2) - a_2\xi(k+1) - a_3\xi(k)$$

$$u(k) = b_0\xi(k+3) + b_1\xi(k+2) + b_2\xi(k+1) + b_3\xi(k)$$

Block Diagrams of Systems

- **Can be drawn directly from state-space representations using only simple delay z^{-1} blocks, gains, and summers.**

$$
\begin{aligned}
x_1(k+1) &= -a_1 x_1(k) - a_2 x_2(k) - a_3 x_3(k) + e(k) \\
x_2(k+1) &= x_1(k) \\
x_3(k+1) &= x_2(k)
\end{aligned}
$$

- **Control Canonical Form (and Observer Canonical Form, which we will discuss much later . . .) is a <u>Direct Canonical</u> realization because all the gains in the block diagram are just the coefficients in the transfer function polynomials.**

- **Will discuss Control and Observer forms more later . . .**

- **General Control Canonical Form for *n*th-order system**

$$H(z) = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n}$$

$$A_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix}, \quad D_c = b_0$$

Non-Uniqueness of State-Space Representations

- **Any nonsingular transformation of state yields another state-space representation of the same transfer function.**

$$\mathbf{x}(k+1) = \mathrm{A}\mathbf{x}(k) + \mathrm{B}e(k)$$
$$u(k) = \mathrm{C}\mathbf{x}(k) + \mathrm{D}e(k)$$

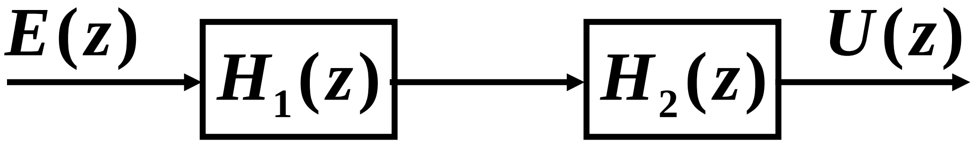
Let $\bar{\mathbf{x}}(k) = \mathrm{T}\mathbf{x}(k)$

$$\bar{\mathbf{x}}(k+1) = \mathrm{TAT}^{-1}\bar{\mathbf{x}}(k) + \mathrm{TB}e(k)$$
$$u(k) = \mathrm{CT}^{-1}\bar{\mathbf{x}}(k) + \mathrm{D}e(k)$$

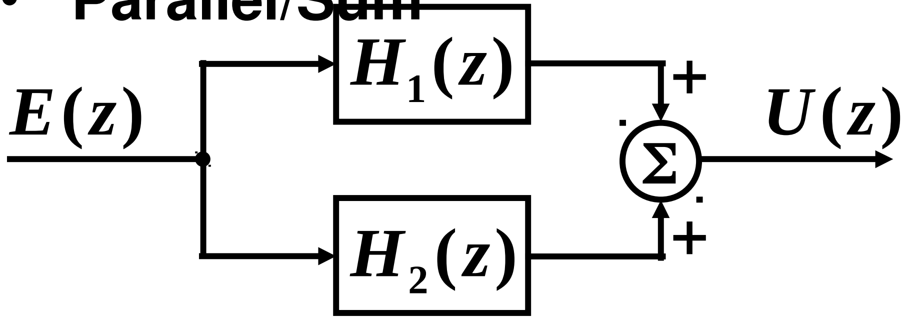
Block Diagram Manipulation

Four rules of block diagram manipulation, analysis, and simplification:

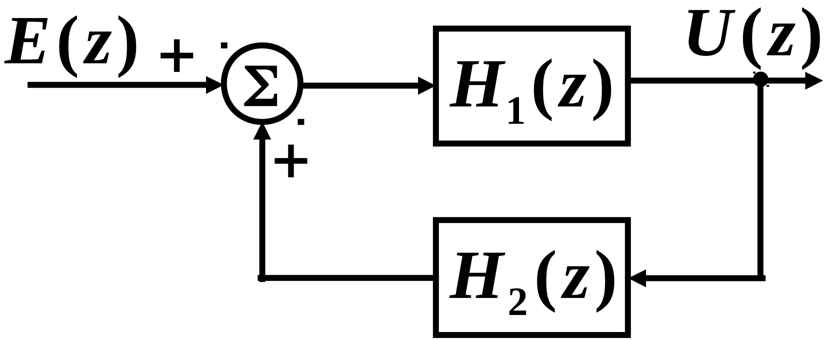
- **Series/Product**



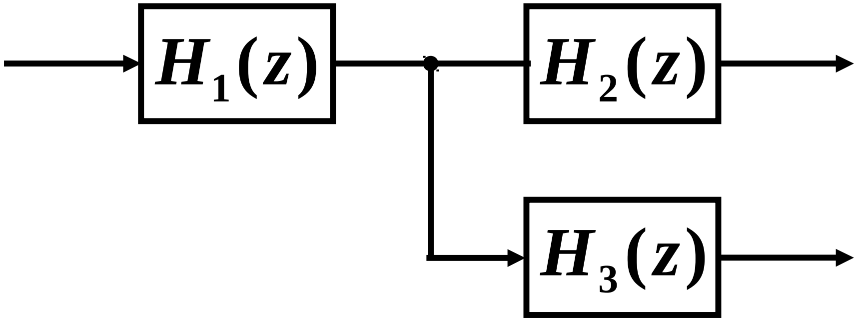
- **Parallel/Sum**



- **Single loops**



- **Moving nodes across blocks (or blocks across nodes)**



Multi-path, multi-loop block diagrams can be simplified using

- **Above techniques**
- **Mason's rule**

Relation of Transfer Function to Pulse Response

- **In continuous-time systems:**

$$\xrightarrow{E(s)} \boxed{H(s)} \xrightarrow{U(s)}$$

  - **The transfer function of a system is equal to the Laplace Transform of the impulse response.**

- **We find a similar result in discrete-time systems:**

$$\xrightarrow{E(z)} \boxed{H(z)} \xrightarrow{U(z)}$$

BIBO Stability

- **Internal vs. External stability**
  - **Internal stability is concerned with all the internal variables (states) of the system.**

$$E(z) \longrightarrow \boxed{H(z)} \longrightarrow U(z)$$

  - **External stability is concerned only with the output.**
    - **BIBO stability is the most common type of external stability used:**
      - **If for every Bounded Input, we have a Bounded Output, we say the system is BIBO stable.**

A Test for BIBO Stability

- can be given in terms of the unit-pulse response.
  - First, consider a sufficient condition.
    - Suppose the input e_k is bounded.
    - The output can be written as $U(z) = H(z)E(z)$
    - Can we find a bound on the magnitude of the output?

      - This condition is also necessary . . .

- Consider the input

$$e_j = \mathrm{sgn}(h_{-j}) = \mathrm{sgn}(h(-j)) = \begin{cases} -1, & h(-j) < 0 \\ 0, & h(-j) = 0 \\ 1, & h(-j) > 0 \end{cases}$$

a bounded input . . .

- Above provides requirement on unit-pulse response h_k .

- **What are the requirements on $H(z)$ for BIBO stability?**
  - **That is, what are the requirements in the frequency domain?**

$$H(z) = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n} = \frac{b(z)}{a(z)}$$

$$= c_0 + \frac{c_1 z}{z - p_1} + \frac{c_2 z}{z - p_2} + \cdots + \frac{c_n z}{z - p_n} \qquad \text{for} \quad m \le n$$

Example

Suppose in our general difference equation

$$u_k = -a_1 u_{k-1} - a_2 u_{k-2} - \cdots - a_n u_{k-n} + b_0 e_k + b_1 e_{k-1} + \cdots + b_m e_{k-m}$$

that all the coefficients are zero except a_1 and b_0.

- What is the unit-pulse response?
- Is the system BIBO stable? Or, under what specific conditions will the system be BIBO stable?

Jury Test

- Provides a quick way to determine by hand whether the roots of a polynomial are all inside the unit circle.

- Analogous to Routh Test for continuous-time systems

- Useful for determining stability of a class of systems where certain parameters may not have fixed quantities.