Analyzing the Dynamic Response of Linear Constant Discrete-Time Systems

- Basic signals
  - Unit pulse
  - Unit step
  - Exponential
  - Sinusoid

- Correspondence of z-plane pole locations with s-plane pole locations

- Step response

Analyzing the Dynamic Response of Linear Constant Discrete-Time Systems

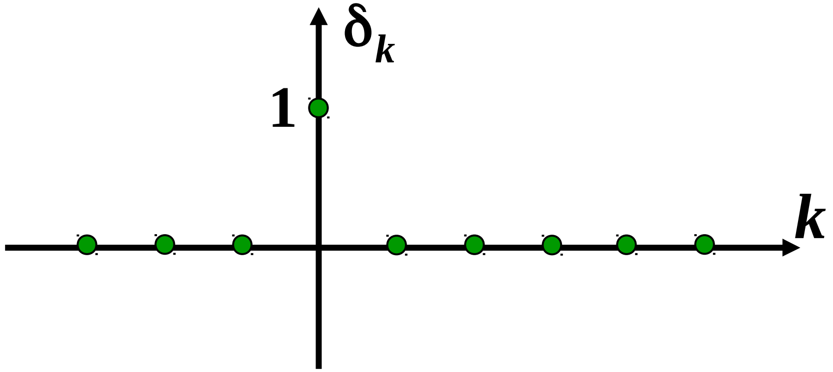
- Given a difference equation relating an input signal e_k and an output signal u_k, and given e_k, find u_k.

- We can solve for u_k by either
  - Solving difference equation directly
  - Going to z-domain and back:
    - Find $H(z)$ (transfer function)
    - Compute $E(z)$
    - Compute $U(z) = H(z)\,E(z)$
    - Find inverse z-transform of $U(z)$: u_k

Basic Signals

- **Unit Pulse**

$$e_1(k) = \delta_k = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

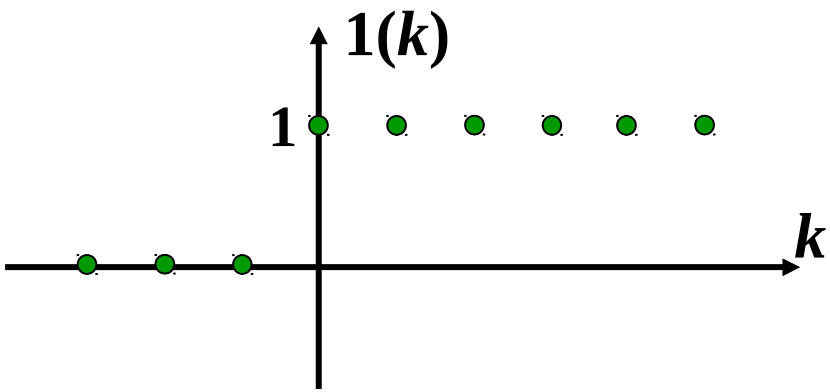


$$E_1(z) = \sum_{k=-\infty}^{\infty} \delta_k z^{-k} = z^0 = 1$$

- No finite poles or zeros
- Similar to continuous-time unit impulse function (whose Laplace transform is 1).
- One way to experimentally determine $H(z)$ is to input δ_k into the system, then the Z-transform of the output is $U(z) = H(z)\, E(z) = H(z)$.

- ## Unit Step

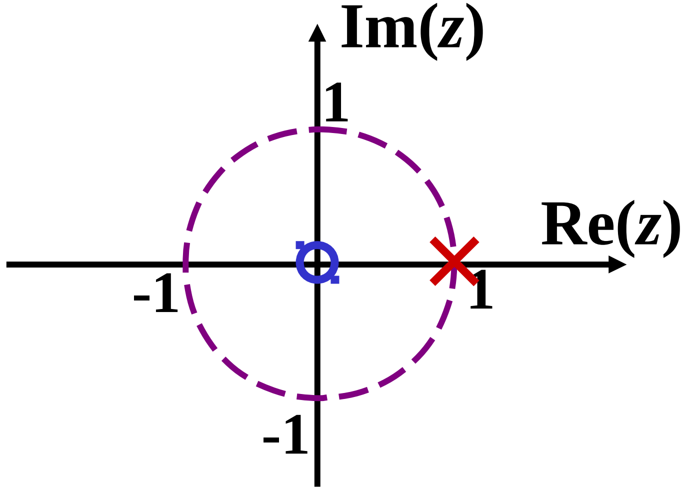
$$e_2(k) = 1(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$



$$E_2(z) = \sum_{k=-\infty}^{\infty} 1(k) z^{-k} = \sum_{k=0}^{\infty} z^{-k}$$

$$= \frac{1}{1 - z^{-1}}, \qquad |z^{-1}| < 1$$

$$= \frac{z}{z - 1}, \qquad |z| > 1$$

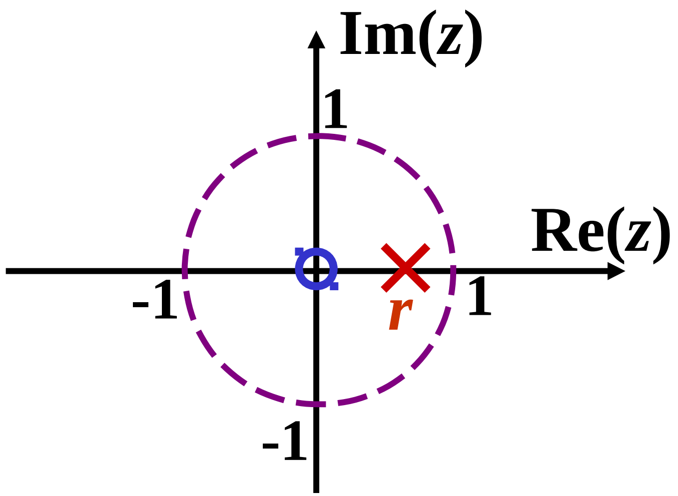


Zero: 0

Pole: 1

• Exponential

$$e_3(k) = r^k 1(k) = \begin{cases} r^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$



$$E_3(z) = \frac{z}{z - r}, \qquad |z| > |r|$$

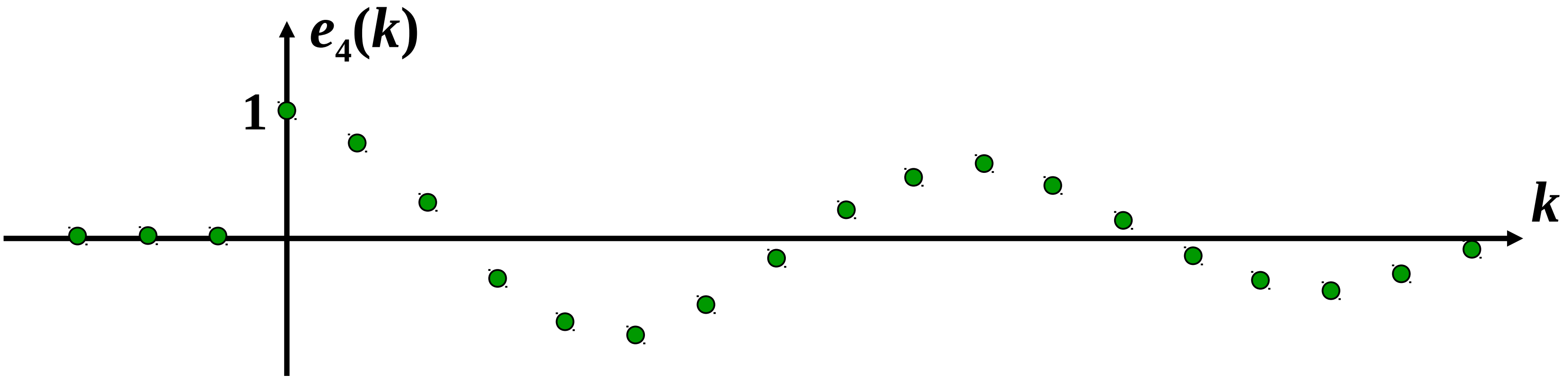
Zero: 0

Pole: r

- $|r| < 1$ ➔ **stable, decaying signal, pole inside unit circle**
- $|r| = 1$ ➔ **unit step, marginally stable, pole on unit circle**
- $|r| > 1$ ➔ **unstable signal, pole outside unit circle**

- **Sinusoid**

$$e_4(k) = r^k \cos(k\theta)1(k), \quad r > 0, \quad \theta \text{ real}$$



$$E_4(z) = \sum_{k=0}^{\infty} r^k \cos(k\theta) z^{-k}$$

$$E_4(z) = \frac{1}{2}\left[\frac{z}{z - re^{j\theta}} + \frac{z}{z - re^{-j\theta}}\right]$$

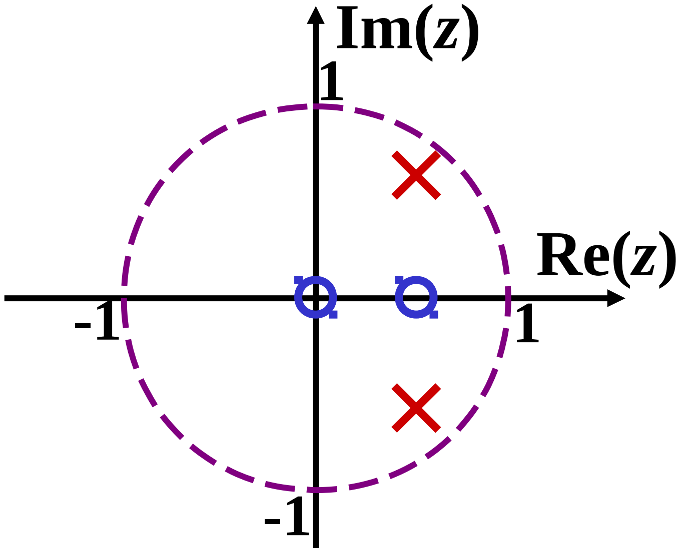
$$= \frac{1}{2}\,\frac{2z^2 - r\left(e^{-j\theta} + e^{j\theta}\right)z}{z^2 - r\left(e^{j\theta} + e^{-j\theta}\right)z + r^2}$$

$$E_4(z) = \frac{z(z - r\cos\theta)}{z^2 - 2r(\cos\theta)z + r^2}, \qquad |z| > r$$

$$E_4(z) = \frac{z(z - r\cos\theta)}{z^2 - 2r(\cos\theta)z + r^2}, \qquad |z| > |r|$$

Zeros:

Poles:



- $r < 1$ ➔ decaying sinusoid, stable, poles inside unit circle
  - The closer the poles are to the origin, the faster the decay.

- $r = 1$ ➔ constant amplitude sinusoid, marginally stable, poles on unit circle

- $r > 1$ ➔ growing sinusoid, unstable, poles outside unit circle

- Note that the number of samples per period (or oscillation) is determined by θ .

$$e_4(k) = r^k \cos(k\theta) 1(k)$$

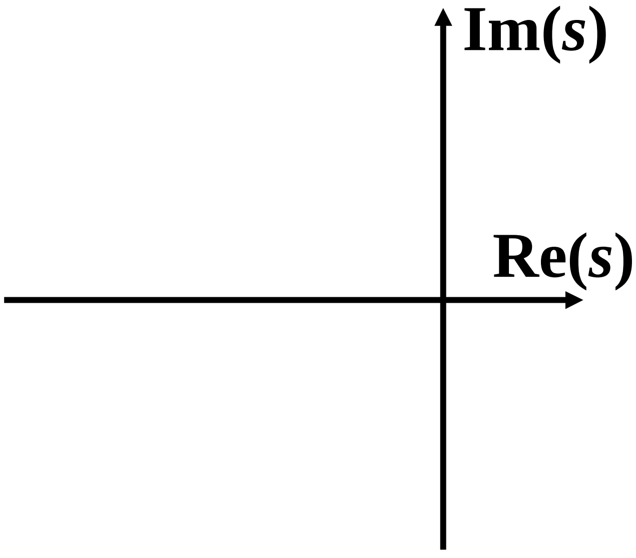
See Figure 4.24 of text (also last page of Lab 1 handout).

Correspondence of z-Plane Pole Locations with s-Plane Pole Locations

- **Recall for 2nd-order Laplace transform:**
  - **Stable if poles in LHP**

poles: $s_1, \; s_2 = -a \pm jb$

$$= -\zeta\omega \pm j\omega\sqrt{1-\zeta^2}$$



- **Sampled signal:**

$$Y(z) = \frac{z(z - r\cos\theta)}{z^2 - 2r\cos\theta\, z + r^2}$$

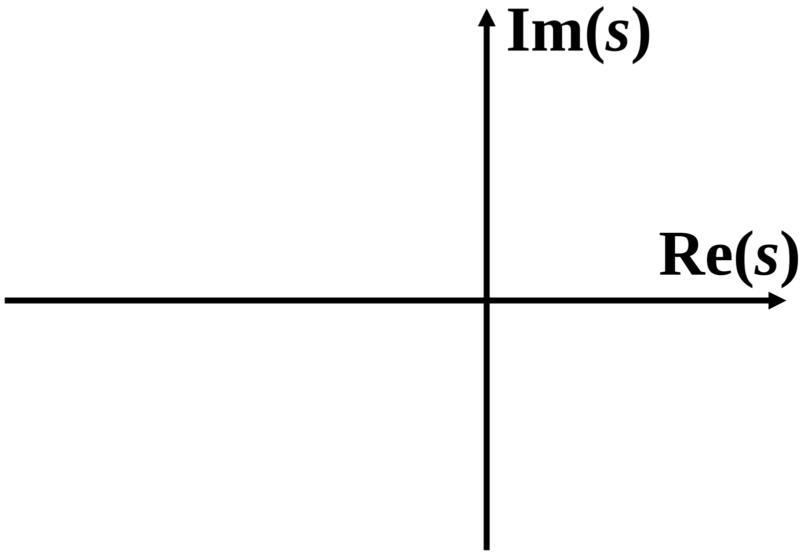
- **Poles:**

- **In general, if a continuous signal $y(t)$ has Laplace transform $Y(s)$ with poles $s_1, s_2, \ldots$, then the sampled (discrete) signal $y(kT)$ has z-transform $Y(z)$ with poles $z_1, z_2, \ldots$, where**

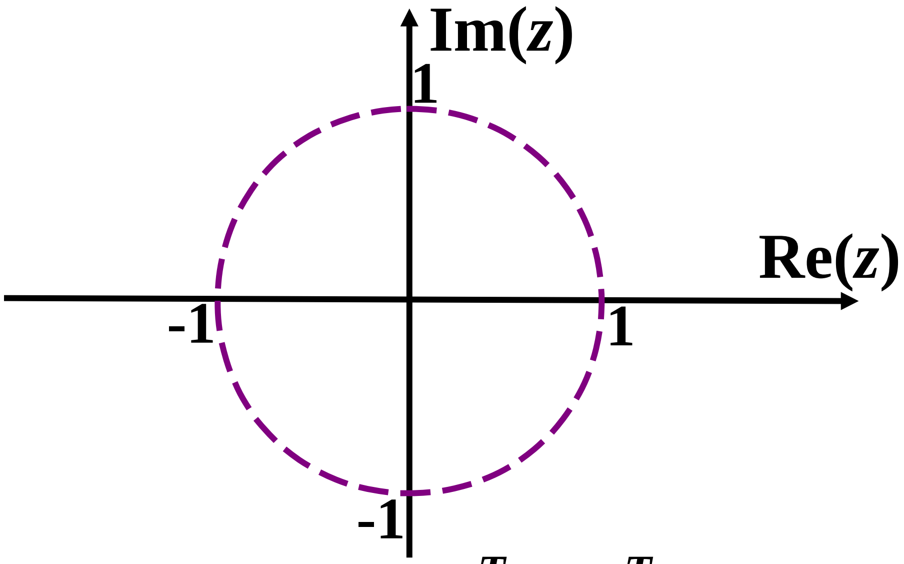
$$z_i = e^{s_i T}$$

Example Correspondences

s-plane	*z*-plane

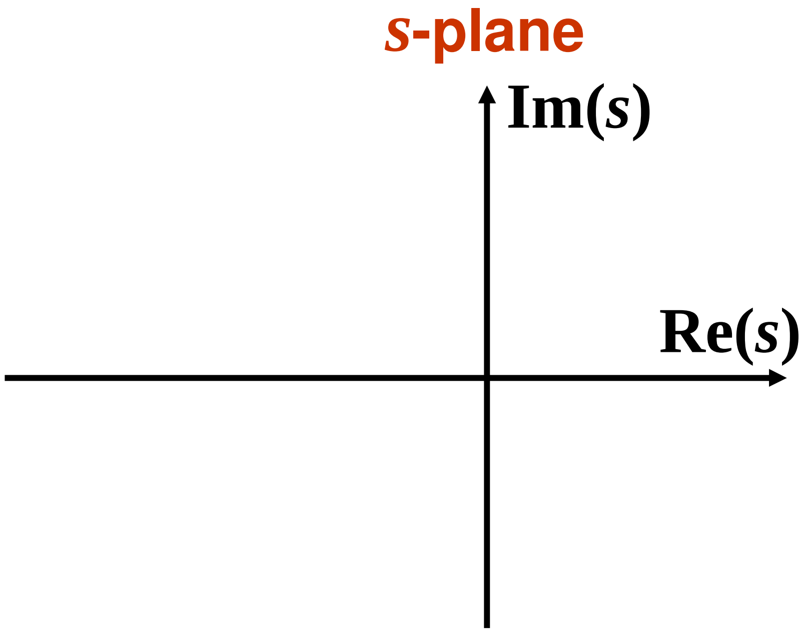


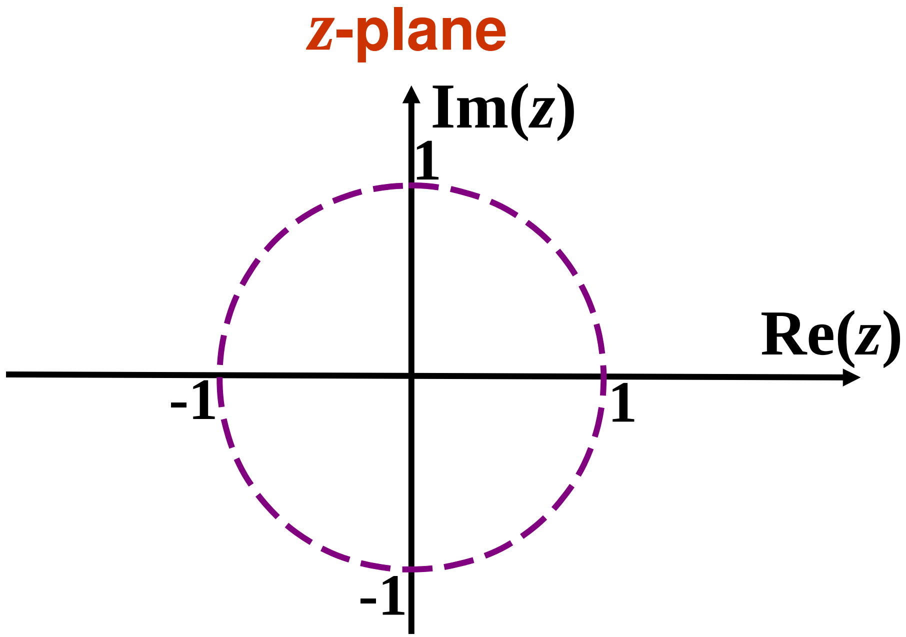
$$s_1, s_2 = -a \pm jb$$



$$z_1, z_2 = e^{s_1 T}, e^{s_2 T}$$

- Stability boundary





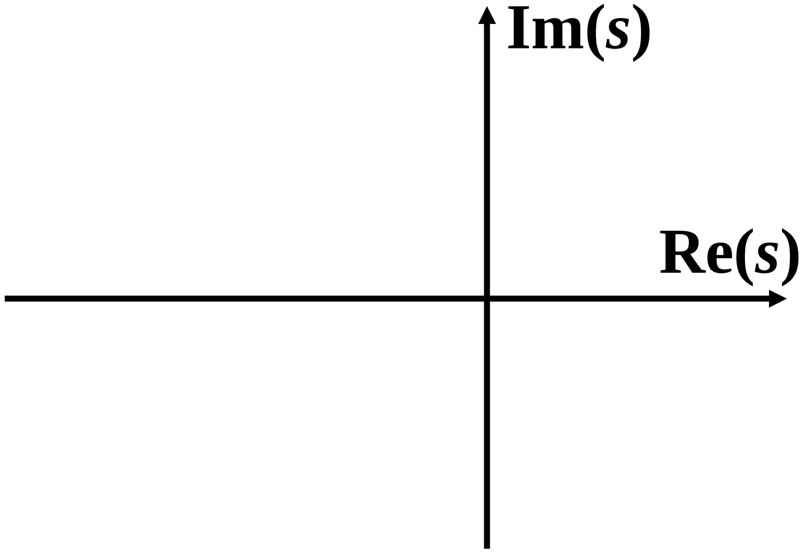
$$s_1, s_2 = -\zeta\omega \pm j\omega\sqrt{1-\zeta^2}$$

$$z_1, z_2 = e^{-\zeta\omega T} e^{\pm j\omega\sqrt{1-\zeta^2}\,T}$$

- **Constant damping**

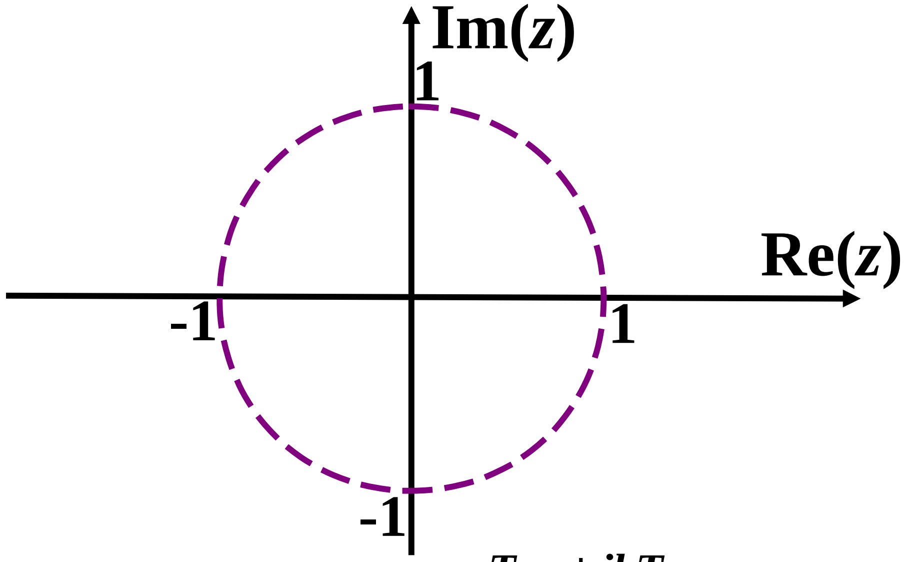
See Figure 4.24 of text.

***s*-plane**



$$s_1, s_2 = -a \pm jb$$

***z*-plane**



$$z_1, z_2 = e^{-aT} e^{\pm jbT}$$

- **Constant *b***

- **Mapping from *s*-plane to *z*-plane is many-to-one:**

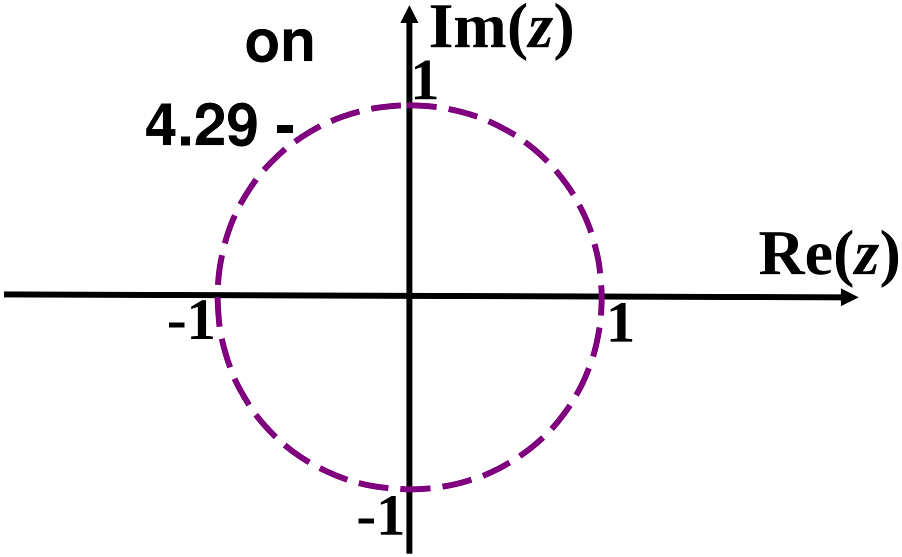
Step Response

- What are step-response characteristics given the poles and zeros of the system?
- Consider $$H(z) = \frac{(z - z_1)(z - z_2)}{(z - p_1)(z^2 - a_1 z + a_2)}$$
- First, let $z_1 = p_1$ so that the system is of 2nd order.
- Z-transform of unit pulse response is $z^{-1} E_4(z)$.

$$H(z) = \frac{(z - z_2)}{(z^2 - a_1 z + a_2)}$$

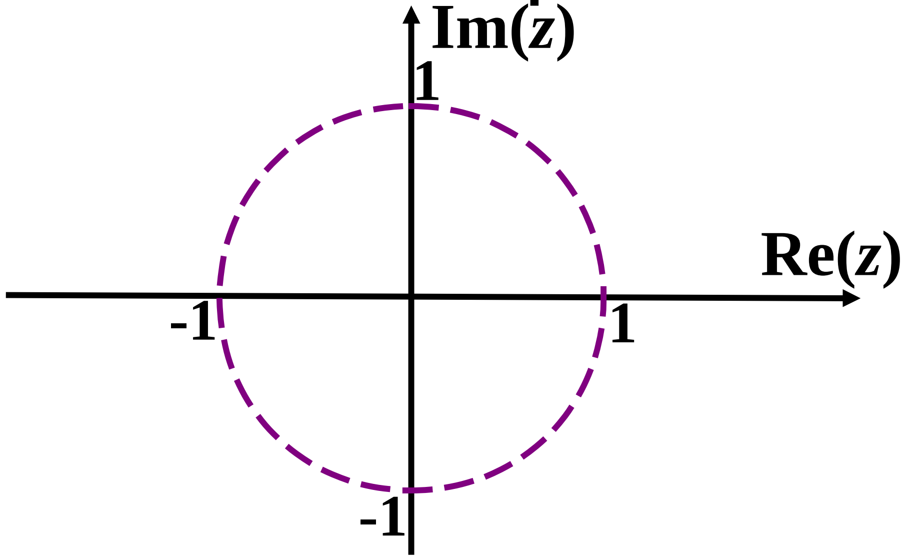
- Effect of zero location on step response

- Major effect of zero z_2 is on overshoot. See Figures 4.29 - 4.31 of text.



$$H(z) = \frac{(z - z_1)(z - z_2)}{(z - p_1)(z^2 - a_1 z + a_2)}$$

- Effect of 3rd pole p_1
- Let $z_1 = z_2 = -1$ (so zeros have little influence on step response)
- Main effect of p_1 is on rise time.
  See Figure 4.32 of text.



- In general, extra zeros and (stable) poles have small effect if on the negative real axis.
- As extra zeros approach +1, overshoot increases.
- As extra poles approach +1, rise time increases.