More on Estimator Design and Duality

- **Observability matrix of observer canonical form**
  - Controllability matrix of control canonical form
- **Designing estimators for systems in observer canonical form**
  - Transformation method
- **Duality**

Observability Matrix Example

Example: Find $\mathcal{O}$ for the following 3^{rd}-order system in observer canonical form.

$$G(z) = \frac{b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}$$

$$\Phi_o = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix}, \quad \Gamma_o = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad H_o = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad J_o = 0$$

$$\mathcal{O}_o = \begin{bmatrix} H_o \\ H_o \Phi_o \\ H_o \Phi_o^2 \end{bmatrix}$$

General n^{th}-order observer canonical form system has $\mathcal{O}$:

$$\mathcal{O}_o = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \mathsf{X} & 1 & \ddots & \vdots \\ \mathsf{X} & \mathsf{X} & \ddots & 0 \\ \mathsf{X} & \mathsf{X} & \mathsf{X} & 1 \end{bmatrix}$$

What is the controllability matrix of a system in control canonical form?

Estimator Design for Systems in Observer Canonical Form

Closed-loop estimator system matrix:

$$\Phi_o - L_o H_o = \begin{bmatrix} -a_1 & 1 & 0 & 0 \\ -a_2 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} \ell_{1o} \\ \ell_{2o} \\ \vdots \\ \ell_{no} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a_1 - \ell_{1o} & 1 & & 0 \\ -a_2 - \ell_{2o} & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ -a_n - \ell_{no} & 0 & \cdots & 0 \end{bmatrix}$$

$$\Longrightarrow \det(z\mathbf{I} - \Phi_o + \mathbf{L}_o\mathbf{H}_o) = z^n + (a_1 + \ell_{1o})z^{n-1} + \cdots + (a_n + \ell_{no})$$

Match coefficients with desired characteristic equation:

$$\alpha_e(z) = z^n + \beta_1 z^{n-1} + \cdots + \beta_n$$

$$\Longrightarrow \begin{array}{ccccc} \ell_{1o} & = & \beta_1 & - & a_1 \\ \ell_{2o} & = & \beta_2 & - & a_2 \\ \vdots & \vdots & & \vdots & \\ \ell_{no} & = & \beta_n & - & a_n \end{array} \qquad \Longrightarrow \qquad \mathbf{L}_o = \boldsymbol{\beta} - \mathbf{a}$$

Transformation Method

So another way to design L (besides the Matching Coefficients Method) is to transform system to observer canonical form, compute L_o , and then transform back.

1. Find T : $\quad \{\Phi, \Gamma, \mathrm{H}, J\} \xrightarrow{\mathrm{x} = \mathrm{Tz}} \{\Phi_o, \Gamma_o, \mathrm{H}_o, J_o\}$

2. Estimator gain vector for observer canonical form:

$$\mathrm{L}_o = \beta - \mathrm{a}$$

3. Transform back.

What is the transform matrix T ?

Recall the transform matrix $\mathbf{T}$ to control canonical form:

Does similarity transformation affect observability?

$$\{\Phi_1, \Gamma_1, \mathbf{H}_1, J_1\} \xleftrightarrow{\mathbf{T}} \{\Phi_2, \Gamma_2, \mathbf{H}_2, J_2\}$$

$$\boxed{\begin{aligned} \Phi_2 &= \mathbf{T}^{-1}\Phi_1\mathbf{T} \\ \Gamma_2 &= \mathbf{T}^{-1}\Gamma_1 \\ \mathbf{H}_2 &= \mathbf{H}_1\mathbf{T} \end{aligned}}$$

$$\mathbf{O}_2 = \begin{bmatrix} \mathbf{H}_2 \\ \mathbf{H}_2\Phi_2 \\ \vdots \\ \mathbf{H}_2\Phi_2^{n-1} \end{bmatrix}$$

$$\boxed{\mathbf{O}_2 = \mathbf{O}_1\mathbf{T}}$$

Transformation Method Formula

Find $\mathbf{L}$ such that $\tilde{\mathbf{x}}(k+1) = (\Phi - \mathbf{L}\mathbf{H})\tilde{\mathbf{x}}(k)$

$$\mathbf{x} = \mathbf{T}\mathbf{z}$$

$$\hat{\mathbf{x}} = \mathbf{T}\hat{\mathbf{z}}$$

Then we have: $\tilde{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{T}^{-1}\tilde{\mathbf{x}}$

$$\tilde{\mathbf{z}}(k+1) = (\Phi_o - \mathbf{L}_o\mathbf{H}_o)\tilde{\mathbf{z}}(k)$$

$\boxed{\mathbf{L} = \mathbf{T}\mathbf{L}_o}$ $\Longrightarrow$ $\boxed{\mathbf{L} = \mathbf{O}^{-1}\mathbf{O}_o(\beta - \mathbf{a})}$

Example

$$G(s) = \frac{1}{s(s+1)} = \frac{Y(s)}{U(s)}$$

$$\xrightarrow{T = 2\sec} \quad G(z) = 1.135\frac{z+0.523}{(z-1)(z-0.135)} = \frac{1.135z+0.594}{z^2-1.135z+0.135}$$

$$\Phi = \begin{bmatrix} 1 & 1-e^{-T} \\ 0 & e^{-T} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} T+e^{-T}-1 \\ 1-e^{-T} \end{bmatrix}, \quad \mathrm{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad J = 0$$

Desired performance: $t_r < 2\sec$, $M_p < 40\%$

$$\Longrightarrow \quad \omega_n = 1 \text{ rad/sec}, \ \zeta = 0.3$$

$$\Longrightarrow \quad \alpha_c(z) = z^2 + 0.363z + 0.301$$

Designed K: $\mathrm{K} = \begin{bmatrix} 0.962 & 0.469 \end{bmatrix}$

If the states are not all available (measured), we need to design an estimator.

Estimator Design

Choose estimator poles to be faster:

$$\zeta = 0.5, \quad \omega_n = 1.5 \text{ rad/sec}$$

$$\Longrightarrow e^{sT}\Big|_{\substack{s=-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\ T=2}} = -0.1910 \pm j0.1154 = p_{1e}, \quad p_{2e}$$

$$\Longrightarrow \alpha_e(z) = (z - p_{1e})(z - p_{2e}) = z^2 + 0.382z + 0.0498$$

$$\mathbf{L} = \mathbf{O}^{-1}\mathbf{O}_o(\boldsymbol{\beta} - \mathbf{a})$$

$T = 2\,\text{sec}$

$$G(z) = \frac{1.135z + 0.594}{z^2 - 1.135z + 0.135}, \quad \Phi = \begin{bmatrix} 1 & 1 - e^{-T} \\ 0 & e^{-T} \end{bmatrix}, \quad \mathrm{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\boxed{\mathrm{L} = \begin{bmatrix} 1.5173 \\ 0.1385 \end{bmatrix}}$$

Duality

Given $G(z) = \dfrac{Y(z)}{U(z)} = \dfrac{b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n}$

Observer canonical form:

$$\Phi_o = \begin{bmatrix} -a_1 & 1 & 0 & 0 \\ -a_2 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix}, \quad \Gamma_o = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{H}_o = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \quad J_o = 0$$

Control canonical form:

$$\Phi_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Gamma_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{H}_c = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}, \quad J_c = 0$$

Canonical forms are dual: $\Phi_o = \Phi_c^T, \;\; \Gamma_o = \mathbf{H}_c^T, \;\; \mathbf{H}_o = \Gamma_c^T$

If system $\left\{\Phi, \Gamma, \mathrm{H}, J\right\}$ is controllable, its dual form $\left\{\Phi^T, \mathrm{H}^T, \Gamma^T, J\right\}$ is observable.

Control Problem

Find: $\mathbf{K} = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$

$\Longrightarrow$ **the eigenvalues of** $\Phi - \Gamma\mathbf{K}$ **are satisfactory.**

$\Longleftrightarrow$ $\det(z\mathbf{I} - \Phi + \Gamma\mathbf{K}) = 0$ $\Longrightarrow$ **satisfactory poles.**

Estimator Problem

Find: $\mathbf{L} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{bmatrix}$ $\Longrightarrow$ **the eigenvalues of** $\Phi - \mathbf{L}\mathbf{H}$ **are satisfactory.**

$\Longleftrightarrow$ $\det(z\mathbf{I} - \Phi + \mathbf{L}\mathbf{H}) = 0$ $\Longrightarrow$ **satisfactory poles.**

The algebra to solve the two problems is the same.

If we have an algorithm to solve the control problem:

$$\det\left(z\mathbf{I} - \Phi + \Gamma\mathbf{K}\right) = \alpha_c(z)$$

we can use the same algorithm to solve the estimator problem:

$$\det\left(z\mathbf{I} - \Phi + \mathbf{L}\mathbf{H}\right) = \alpha_e(z)$$

Solve for $\mathbf{K}$, let $\mathbf{L} = \mathbf{K}^T$.