Numerical Analysis Project 2
Flat Plate Flow

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Setting up the Problem

In this project, we will consider the flow of a fluid, initially at a temperature T_∞ with horizontal velocity U_∞, as it flows over a stationary flat plate that is uniformly held at temperature T_w. The goal of the project is to determine the temperature distribution $T(x,y)$, as well as the x and y components of the velocity, denoted $u(x,y)$ and $v(x,y)$.
Using Conservation of mass, x-momentum, and energy, we derive the following three coupled PDE's, known as the "boundary layer equations":

$$u_x + v_y = 0 \tag{1}$$

$$uu_x + vu_y = \nu u_{yy} \tag{2}$$

$$uT_x + vT_y = \alpha T_{yy} \tag{3}$$

where α is the thermal diffusivity of the fluid and ν is the kinematic viscosity. Based on the conditions at the edge of the plate, the horizontal velocity must be U_∞ and the temperature must be T_∞. This gives some conditions:

$$T(0,y) = T_\infty \quad u(0,y) = U_\infty \quad v(0,y) = 0 \tag{4}$$

We get more conditions based on the statements about the plate:

$$T(x,0) = T_w \quad u(x,0) = 0 \quad v(x,0) = 0 \tag{5}$$

Very far away from the plate, one can say that the fluid will reach its original temperature and velocity:

$$T(x,\infty) = T_\infty \quad u(x,\infty) = U_\infty \tag{6}$$

Far from the plate, there will be an induced vertical velocity, $v(x,y)$. Its value should be determined as part of the solution.
The equations above have never been solved analytically. To get a solution to the problem, we want to do a similarity transformation:

$$u = U_\infty F'(\eta) \tag{7}$$

$$v = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}(\eta F'(\eta) - F(\eta)) \tag{8}$$

$$G(\eta, Pr) = \frac{T - T_\infty}{T_w - T_\infty} \tag{9}$$

We define the similarity variable η as:

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}} \tag{10}$$

Pr is what's known as the Prandtl number and is defined as $Pr = \frac{\nu}{\alpha}$. With this similarity transformation, the conservation equations 1, 2, and 3 reduce to 2 ordinary differential equations given below:

$$F''' + \frac{1}{2}FF'' = 0 \tag{11}$$

$$G'' + \frac{Pr}{2}FG' = 0 \tag{12}$$

with the following conditions:

$$F(0) = 0 \tag{13}$$

$$F'(0) = 0 \tag{14}$$

$$F'(\infty) = 1 \tag{15}$$

$$G(0) = 1 \tag{16}$$

$$G(\infty) = 0 \tag{17}$$

These equations are convenient because F can be solved for independently of G. The other convenience of this problem is that it can be written as a set of first order equations using some basic substitutions. The problem comes from the conditions which don't work perfectly with an inital value problem solver.

Solving the Problem

To begin solving the problem, we wrote out the system as a set of first order ODEs:

$$u_1' = u2 = F'$$

$$u_2' = u_3 = F''$$

$$u_3' = F''' = -\tfrac{1}{2}u_1u_3$$

$$v_1' = v_2 = G'$$

$$v_2' = G'' = \frac{-Pr}{2}u_1v_2$$

The first 3 of these where indepedent of the last 2, so we solved them using RK4. The step size for this was $\Delta\eta = 0.1$, an initial η of 0, and the Prandtl Number is $Pr = 5$. We had to guess and check at the value of $F''(0)$ until equation 15 was satisfied. The value ended up being $F''(0) = 0.332055$. With this evaluated properly, we then solved the final two odes using RK4 (with the same conditions). Once again, we had to guess at the value of $G'(0)$ until equation 17 was satisfied. The value that satisfied our conditions was $G'(0) = -0.5607$. A table of all of these results can be found in Appendix 1. After that, a plot was made of the dimensionless velocities $\frac{v}{U_\infty}\sqrt{\frac{x}{L}}\sqrt{Re} = \frac{1}{2}(\eta F'(\eta) - F(\eta))$ and $\frac{u}{U_\infty} = F'(\eta)$ as functions of η where $Re = \frac{U_\infty L}{\nu}$ is the Reynold's number. This plot can be seen in Figure 1, with an indicated value of η corresponding to the thickness of the momentum boundary layer. Note that η_m corresponding to the thickness of the momentum boundary layer was around 5.

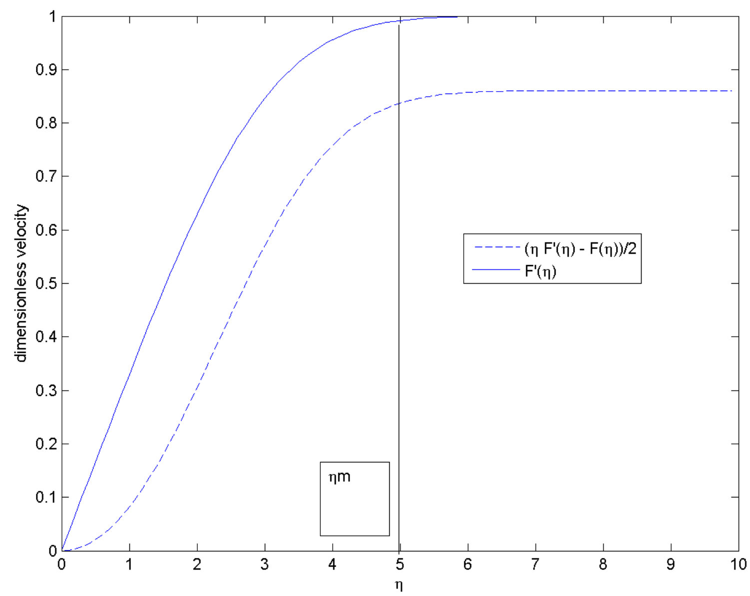

Figure 1: The dimensionless velocities as a function of η

After that, we made a plot of the dimensionless temperature $G(\eta)$ as a function of η (shown in figure 2). Note the value η_t which corresponds to the thickness of teh thermal boundary layer, as well as the various plots for $G(\eta)$ for different η values. Note that different $G'(0)$ values were used to get the correct solution and are listed in table 1.

Prandtl Number	$G'(0)$ estimate
0.2	-0.18366
2	-0.41362
5	-0.5607
10	-0.70285

Table 1: The various $G'(0)$ guesses used to meet the problem conditions

Prandtl Number	η_t
0.2	7.8588
2	3.3522
5	2.4449
10	1.9392

Table 2: The values of η_t determined for various Prandtl numbers

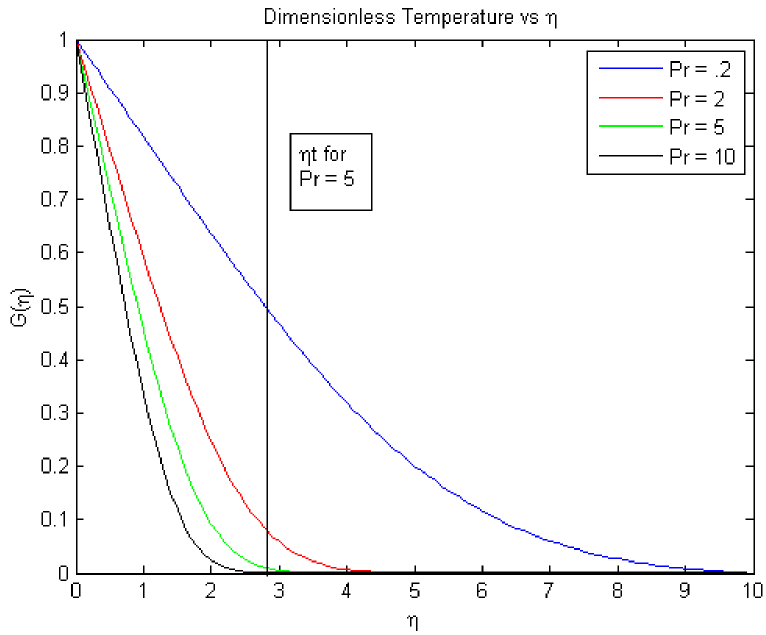

Figure 2: A plot of the dimensionless temperature $G(\eta)$ as a function of η for $\eta = 0.2, 2, 5, 10$

One of the last things we did was to use Lagrange Interpolation to find a more accurate value for η_m. The new value found from this interpolation was $\eta_m = 4.2628$. We did the same thing to find more accurate values of η_t, the results of which can be seen below in table 2: Then, we made a graph where we plotted η_t as a function of the Prandtl Number. This can be seen below in figure 3.

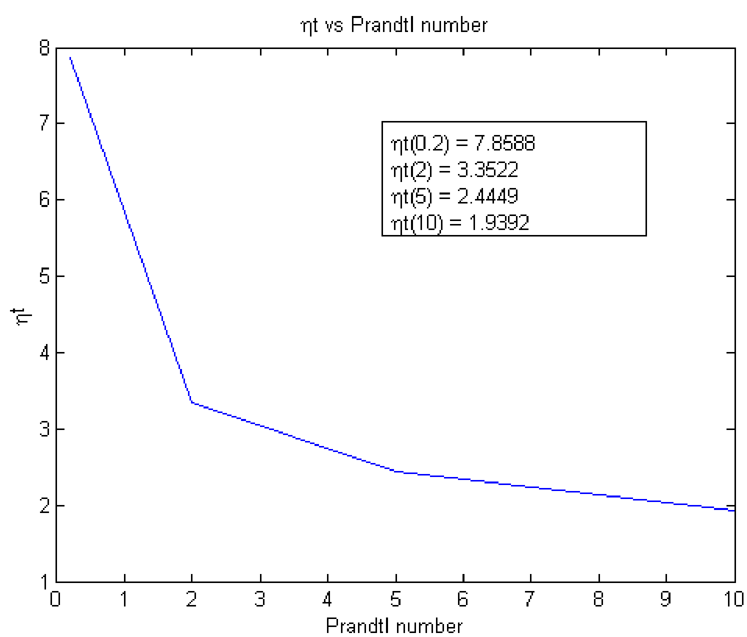

Figure 3: η_t as a function of Prandtl Number

Finally, we wanted to make a plot of δ_t and δ_m as functions of $\sqrt{\frac{x}{L}}$. The actual equations for these are:

$$\frac{\delta_m * \sqrt{Rey}}{L} = \eta_m \sqrt{\frac{x}{L}} \tag{18}$$

$$\frac{\delta_t * \sqrt{Rey}}{L} = \eta_t \sqrt{\frac{x}{L}} \tag{19}$$

The plots for these can be seen in figures 4 and 5.

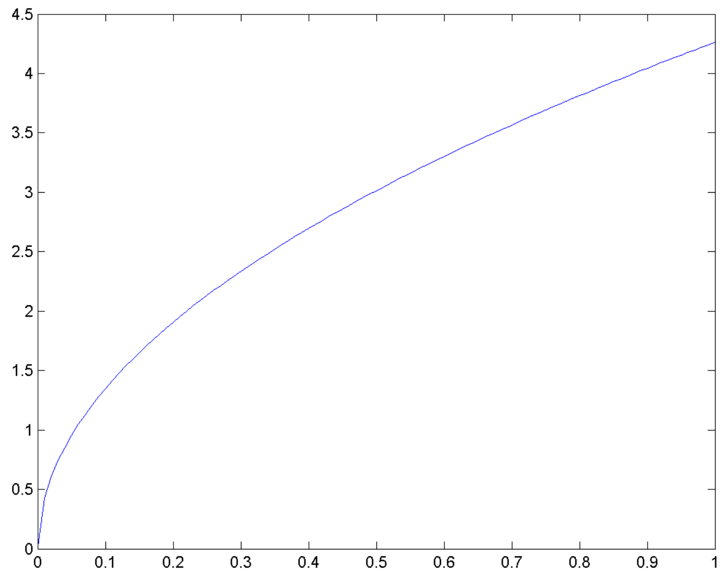

Figure 4: δ_m as a function of $\frac{x}{L}$

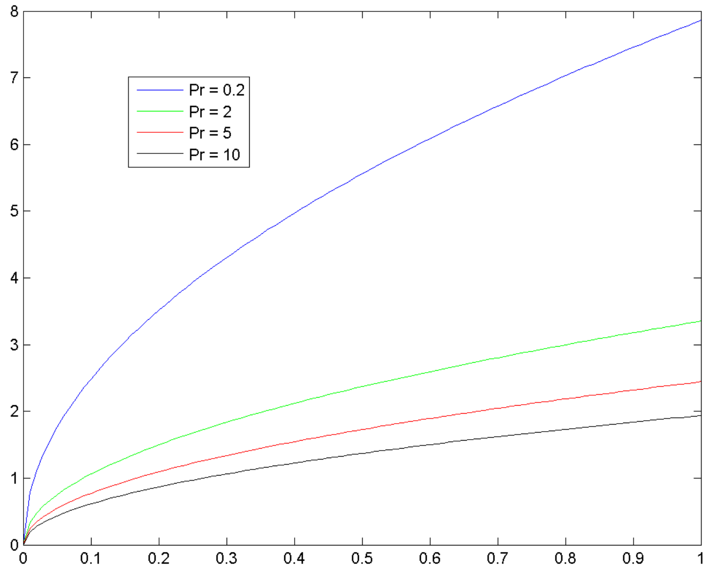

Figure 5: δ_t as a function of $\frac{x}{L}$ and various η_t (from various Prandtl Numbers)

Conclusion

Through this project, we have utilized various numerical techniques to determine characteristics of an extremely complex system of equations with no analytic solution. These solution methods included RK4 integration with the shooting method, as well as Lagrange Polynomial Interpolation. We examined how different Prandtl Numbers changed the η values which explains how physical characteristics effect dimensionless temperature and velocity through space. We also found η_m corresponding to teh momentum boundary layer. Using the various numerical methods to solve a complex real world problem has been illuminating in approaching real world problems with the techniques from this class.

Appendix 1: Data

η	F	F'	F''	G	G'
0	0	0	0.3321	1.0000	-0.5607
0.1000	0.0017	0.0332	0.3320	0.9439	-0.5607
0.2000	0.0066	0.0664	0.3320	0.8879	-0.5605
0.3000	0.0149	0.0996	0.3318	0.8319	-0.5595
0.4000	0.0266	0.1328	0.3315	0.7760	-0.5575
0.5000	0.0415	0.1659	0.3309	0.7205	-0.5538
0.6000	0.0597	0.1989	0.3301	0.6654	-0.5480
0.7000	0.0813	0.2319	0.3289	0.6110	-0.5399
0.8000	0.1061	0.2647	0.3274	0.5575	-0.5291
0.9000	0.1342	0.2973	0.3254	0.5053	-0.5152
1.0000	0.1656	0.3298	0.3230	0.4546	-0.4982
1.1000	0.2002	0.3619	0.3201	0.4058	-0.4780
1.2000	0.2379	0.3938	0.3166	0.3592	-0.4547
1.3000	0.2789	0.4252	0.3125	0.3151	-0.4284
1.4000	0.3230	0.4563	0.3079	0.2737	-0.3996
1.5000	0.3701	0.4868	0.3026	0.2353	-0.3686
1.6000	0.4203	0.5167	0.2967	0.2001	-0.3360
1.7000	0.4735	0.5461	0.2901	0.1682	-0.3025
1.8000	0.5295	0.5747	0.2829	0.1397	-0.2687
1.9000	0.5884	0.6027	0.2751	0.1145	-0.2354
2.0000	0.6500	0.6298	0.2667	0.0926	-0.2032
2.1000	0.7143	0.6560	0.2578	0.0739	-0.1727
2.2000	0.7812	0.6813	0.2483	0.0580	-0.1445
2.3000	0.8505	0.7056	0.2384	0.0449	-0.1188
2.4000	0.9223	0.7290	0.2281	0.0342	-0.0961
2.5000	0.9963	0.7512	0.2174	0.0256	-0.0763
2.6000	0.0725	0.7724	0.2065	0.0189	-0.0595
2.7000	0.1507	0.7925	0.1953	0.0137	-0.0455
2.8000	0.2310	0.8115	0.1840	0.0097	-0.0341
2.9000	0.3130	0.8293	0.1727	0.0068	-0.0251
3.0000	0.3968	0.8460	0.1614	0.0046	-0.0181
3.1000	0.4822	0.8616	0.1502	0.0031	-0.0127
3.2000	0.5691	0.8761	0.1391	0.0021	-0.0088
3.3000	0.6573	0.8894	0.1283	0.0013	-0.0059
3.4000	0.7469	0.9017	0.1179	0.0008	-0.0039
3.5000	0.8377	0.9130	0.1078	0.0005	-0.0025
3.6000	0.9295	0.9233	0.0981	0.0003	-0.0016
3.7000	0.0223	0.9327	0.0889	0.0002	-0.0010
3.8000	0.1160	0.9411	0.0801	0.0001	-0.0006
3.9000	0.2105	0.9487	0.0719	0.0001	-0.0004
4.0000	0.3057	0.9555	0.0642	0.0000	-0.0002
4.1000	0.4016	0.9616	0.0571	0.0000	-0.0001
4.2000	0.4980	0.9669	0.0505	0.0000	-0.0001
4.3000	0.5949	0.9717	0.0445	0.0000	-0.0000
4.4000	0.6923	0.9759	0.0390	0.0000	-0.0000
4.5000	0.7901	0.9795	0.0340	0.0000	-0.0000
4.6000	0.8882	0.9827	0.0295	0.0000	-0.0000
4.7000	0.9866	0.9854	0.0255	0.0000	-0.0000
4.8000	0.0853	0.9878	0.0219	0.0000	-0.0000

η	F	F'	F"	G	G'
4.9000	0.1841	0.9898	0.0187	0.0000	-0.0000
5.0000	0.2832	0.9915	0.0159	0.0000	-0.0000
5.1000	0.3824	0.9930	0.0135	0.0000	-0.0000
5.2000	0.4818	0.9942	0.0113	0.0000	-0.0000
5.3000	0.5813	0.9953	0.0095	0.0000	-0.0000
5.4000	0.6809	0.9961	0.0079	0.0000	-0.0000
5.5000	0.7805	0.9969	0.0066	0.0000	-0.0000
5.6000	0.8802	0.9975	0.0054	0.0000	-0.0000
5.7000	0.9800	0.9980	0.0045	0.0000	-0.0000
5.8000	0.0798	0.9984	0.0036	0.0000	-0.0000
5.9000	0.1797	0.9987	0.0030	0.0000	-0.0000
6.0000	0.2795	0.9990	0.0024	0.0000	-0.0000
6.1000	0.3795	0.9992	0.0019	0.0000	-0.0000
6.2000	0.4794	0.9993	0.0016	0.0000	-0.0000
6.3000	0.5793	0.9995	0.0012	0.0000	-0.0000
6.4000	0.6793	0.9996	0.0010	0.0000	-0.0000
6.5000	0.7792	0.9997	0.0008	0.0000	-0.0000
6.6000	0.8792	0.9998	0.0006	0.0000	-0.0000
6.7000	0.9792	0.9998	0.0005	0.0000	-0.0000
6.8000	0.0792	0.9998	0.0004	0.0000	-0.0000
6.9000	0.1792	0.9999	0.0003	0.0000	-0.0000
7.0000	0.2791	0.9999	0.0002	0.0000	-0.0000
7.1000	0.3791	0.9999	0.0002	0.0000	-0.0000
7.2000	0.4791	0.9999	0.0001	0.0000	-0.0000
7.3000	0.5791	1.0000	0.0001	0.0000	-0.0000
7.4000	0.6791	1.0000	0.0001	0.0000	-0.0000
7.5000	0.7791	1.0000	0.0001	0.0000	-0.0000
7.6000	0.8791	1.0000	0.0000	0.0000	-0.0000

Appendix 2: Code

All code was done in Matlab
First ODE

```matlab
function [ u1prime ] = F1( u1, u2, u3 )
%Differential equation for u1 (u1 is equal to F from original equations)
u1prime = u2;

end
```

Second ODE

```matlab
function [ u2prime ] = F2( u1, u2, u3 )
%Differential equation for u2 (u2 = F' from original problem)
u2prime = u3;

end
```

Third ODE

```matlab
function [ u3prime ] = F3( u1, u2, u3 )
%Differential equation for u3, (u3 = F'' from original problem)
u3prime = -.5 * u1 * u3;

```

```
end
```

Fourth ODE

```
function [ v1prime ] = G1(u1, v1, v2, Pr )
%Differential equation for v1 (v1 is equal to G from original problem)
v1prime = v2;


end
```

Fifth ODE

```
function [ v2prime ] = G2( u1, v1, v2, Pr )
%Differential equation for v2( v2 = G' from original problem)
v2prime = -Pr * u1 * v2 /2;

end
```

Runge-kutta 4

```
function [ uvnew ] = RK4( u1, u2, u3, v1, v2, h, Pr )
% uses the functions F1-F3, G1, G2 to perform one iteration of RK4 with
% step size h, returns a vector with updated values for u1, u2, u3, v1, v2
k11 = h * F1(u1, u2, u3);
k12 = h * F2(u1, u2, u3);
k13 = h * F3(u1, u2, u3);
k21 = h * F1(u1 + k11/2, u2 + k12/2, u3 + k13/2);
k22 = h * F2(u1 + k11/2, u2 + k12/2, u3 + k13/2);
k23 = h * F3(u1 + k11/2, u2 + k12/2, u3 + k13/2);
k31 = h * F1(u1 + k21/2, u2 + k22/2, u3 + k23/2);
k32 = h * F2(u1 + k21/2, u2 + k22/2, u3 + k23/2);
k33 = h * F3(u1 + k21/2, u2 + k22/2, u3 + k23/2);
k41 = h * F1(u1 + k31, u2 + k32, u3 + k33);
k42 = h * F2(u1 + k31, u2 + k32, u3 + k33);
k43 = h * F3(u1 + k31, u2 + k32, u3 + k33);
u1new = u1 + (k11 + 2*k21 + 2*k31 + k41)/6;
u2new = u2 + (k12 + 2*k22 + 2*k32 + k42)/6;
u3new = u3 + (k13 + 2*k23 + 2*k33 + k43)/6;
l11 = h * G1(u1, v1, v2, Pr);
l12 = h * G2(u1, v1, v2, Pr);
l21 = h * G1(u1, v1 + l11/2, v2 + l12/2, Pr);
l22 = h * G2(u1, v1 + l11/2, v2 + l12/2, Pr);
l31 = h * G1(u1, v1 + l21/2, v2 + l22/2, Pr);
l32 = h * G2(u1, v1 + l21/2, v2 + l22/2, Pr);
l41 = h * G1(u1, v1 + l31, v2 + l32, Pr);
l42 = h * G2(u1, v1 + l31, v2 + l32, Pr);
v1new = v1 + (l11 + 2*l21 + 2*l31 + l41)/6;
v2new = v2 + (l12 + 2*l22 + 2*l32 + l42)/6;
uvnew = [u1new, u2new, u3new, v1new, v2new];


end
```

Runke-Kutta 4 setup script

```
function [ m1 ] = RungeK( u1i, u2i, u3i,v1i, v2i, h, N, Pr )
```

```
%Run RK4 using Rk4 helper function for N iterations returns a matrix
% where the columns are eta, F, F', F'', G, G'. For each value of Pr, this
% function was called (with guesses for F''(0) and G'(0)) this matrix was
% then used to generate the plots and to interpolate the values of etam and
% etat
%
m1 = zeros (N, 6);
m1(1,1) = 0;
m1(1,2) = u1i;
m1(1,3) = u2i;
m1(1,4) = u3i;
m1(1,5) = v1i;
m1(1,6) = v2i;
eta = 0;
u1 = u1i;
u2 = u2i;
u3 = u3i;
v1 = v1i;
v2 = v2i;
for i = 2:N
    uv = RK4(u1, u2, u3, v1, v2, h, Pr);
    u1 = uv(1);
    u2 = uv(2);
    u3 = uv(3);
    v1 = uv(4);
    v2 = uv(5);
    eta = eta + h;
    m1(i, 1) = eta;
    m1(i, 2) = u1;
    m1(i, 3) = u2;
    m1(i, 4) = u3;
    m1(i, 5) = v1;
    m1(i, 6) = v2;




end
```

Lagrange Polynomial Interpolation

```
function [ eta ] = Interpolate( x0, x1, x2, f0, f1, f2 , xi)
%Takes in three points (x0, f0), (x1, f1), (x2, f2) and uses Lagrange
% polynomials to approximate the function value at the location xi
eta = f0 * (xi - x1) * (xi - x2)/((x0 - x1) * (x0 - x2)) + f1 * (xi - x0) * (
    xi - x2)/((x1 - x0) * (x1 - x2)) + f2 * (xi - x0) * (xi - x1)/((x2 - x0) *
    (x2 - x1));

end
```