Notes in ECEN 5448

Zahary Vogel

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1 Review of exam

I did good yay!!!
Starting systems at an eigen vector, the solutino will be in that direction times some exponential.

2 Input Output gain of a system

$L2$ gain of a system is the maximum gain over all frequencies.
Recap:

$$y = Hu$$

$$sup_{u\neq 0,||u||<\infty}\frac{||Hu||_2}{||u||_2}$$

Fact: For a linear system H:

$$L2\text{gain} = \sup_{\omega} ||H(j\omega)||_{\text{ind.}2}$$

$$= \sup_{\omega} \lambda_{\max}^{\frac{1}{2}} |H^H(j\omega)H(j\omega)|$$

This is in frequency domain, what about the time domain that we are used to working with.

3 Time Domain

For time-domain we want to go back to Lyapunov Analysis.

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Question is that $||H||_{\text{ind.}} \leq \delta$?
This is equivalent to saying that:

$$\frac{||H||_{\text{ind.}}}{\delta} \leq 1$$

Claim: Let P be a positive definite matrix:
For $V(x) = x^T P x$,

$$\dot{V}(x) \leq u^T(t)u(t) - y^T(t)y(t)$$

$$\implies ||H||_{\text{ind.}} \leq 1$$

So this $V(x)$ is almost a Lyapunov function.
Proof: Integrate both sides:

$$\int_0^T \dot{V}(x)dx \leq \int_0^T ||u(t)||^2 dt - \int_0^T ||y(t)||^2 dt$$
$$= V(x(T)) - V(x(0)) \leq ||u||^2_{\in,[0,T]} - ||y||^2_{\in[0,T]}$$

Since, $V(x(0))$ is 0 we get that:

$$\implies ||y||^2_{\in[0,T]} \leq ||u||^2_{\in[0,T]}$$

This holds for any capital T which means that:

$$\implies ||H||_{\text{ind}.2} \leq 1$$

You can drop the $V(x(T))$ because it is always positive based on its positive definiteness.

4 Looking at the meaning of this

Suppose $\dot{V}(x) \leq ||u||^2 - ||y||^2$ for $V(x) = x^T P x$.

$$x^T(PA + A^T P)x + u^T B^T P x + x^T P B u \leq u^T u - x^T C^T C x$$

This means that this is true if and only if you can take everything on the right hand side:

$$(xu) \begin{pmatrix} -(PA + A^T P) - C^T C & -PB \\ -B^T P & I \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \geq 0 \forall x, u$$

for any x and u we need this matrix to be positive semi-definite.
So $||H||_{\text{ind.},2} < 1$ iff

$$\begin{pmatrix} -(PA + A^T P + C^T C) & -PB \\ -B^T P & I \end{pmatrix} \geq 0$$

Find P such that this holds is a convex optimization problem.
So the converse holds as well, thus, if H induced is less than or equal to one you can always find a positive definite P.

5 Skipping the topic of small gain theorem

6 Moving on to Controllability

Will definetely be on the final.

$$\dot{x} = f(x, u)$$

Given how every many degrees of theorem that you have to effect, can you get a desired output?
Question:

$$\forall x_1, x_2 \in \mathbb{R}^n, \exists ?T, u : [0, T] \to \mathbb{R}^m$$

for the solution to $x(t)$ to the system above with $x(0) = x_1, x(T) = x_2$?
For linear systems, it is sufficient that you can steer a system from any point in the space to 0.
a system as above, $\forall x_1, x_2 \in \mathbb{R}^n \exists T, \hat{u} : [0, T] \to \mathbb{R}^m$ the solution $x(t)$ to the system with $x(0) = x_1$ and $u = \hat{u}$ $x(T) = x_2$, is called a controllable system.