Notes in ECEN 5448

Zahary Vogel

November 9, 2015

1 Controllability

Controllability: $\forall x_0, x \in \mathbb{R}^n$.
$\exists \{T, u : [0,T] \to \mathbb{R}^m$, for:

$$\dot{x} = f(x,u), t \in [0,T]$$
$$x(0) = x_0, x(T) = x$$

For LTI systems, it suffice to verify controllability condition for any initial condition $x_0 \in \mathbb{R}^n$ and $x_1 = 0$.(why?)

Question: Under what condition for any $x(0) \in \mathbb{R}^n$, $\exists T, u : [0,T] \to \mathbb{R}^m$,

$$0 = e^{AT}x(0) + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau$$

What causes non-controllable systems?

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

no way to change the first coordinate.

Let us determine the necessary condition for controllability.
Suppose $x(0) \in \mathbb{R}^n$ is such that:

$$0 = e^{AT}x(0) + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau$$

for some $T > 0$ and $u : [0,T] \to \mathbb{R}$ (B is nx1)
This implies that:

$$-x(0) = \int_0^T e^{-A\tau}Bu(\tau)d\tau = \int_0^T \begin{pmatrix} f_1(\tau) \\ f_2(\tau) \\ \vdots \\ f_n(\tau) \end{pmatrix} u(\tau)d\tau$$

where $f_1(\tau) = [e^{-A\tau}B]$
$\approx$ by discretization of $[0,T]$ to $h, 2h, 3h, \dots, Nh$. This integral would bee close to:

$$\approx h \begin{pmatrix} f_1(h)u(h) + f_1(2h)u(2h) + \dots f_1(nh)u(nh) \\ \vdots \\ f_n(h)u(h) + f_n(2h)u(2h) + \cdots + f_n(Nh)u(Nh) \end{pmatrix}$$

So controllability is related to solving $y = Mu$. for a given vector y and matrix $M_{n\times N}$ N¿¿n.

Fact: $y = Mu$ has a solution for all $y \in \mathbb{R}^n$ if and only if the rows of M are linearly independent.
Proof: Suppose the row vectors, $M_1, M_2, \ldots, M_n$ are linearly dependent, and non-negative $c_1, c_2, \ldots, c_n \in \mathbb{R} \neq 0$ such that $c_1M_1 + c_2M_2 + \cdots + c_nM_n = 0 = c^TM = 0$. Let $y = c$ and suppose that $\exists u : c = Mu$. Multiply everything by c^T, then:

$$c^Tc = ||c||^2 = c^TMu = 0$$

that's a contradiction because c has to have some positive element.
Suppose that $M_1, M_2, \ldots, M_n$ are linearly independent.
IN this case, the matrix $P = MM^T$ is a p.d. matrix because for any $x \in \mathbb{R}^n$, $x^TPx = x^TMM^Tx = ||M^Tx||^2 > 0$ for $x \neq 0$
For solving $y = Au$, let:

$$u = A^T(AA^T)^{-1}y$$
$$Au = (Aa^T)(AA^T)^{-1}y = y$$

Definition: We say that $g_1, \ldots, g_n : [0, T] \to \mathbb{R}$ are linearly independent if:

$$c_1g_1(t) + c_2g_2(t) + \cdots + c_ng_n(t) = O(t)$$

$\forall t \in [0, T]$ implies that $c_1 = c_2 = c_n = 0$.

Example: Let $g_1 = 1, g_2 = t, g_3 = t^2$.
Suppose $c_1 + c_2t + c_3t^2 = 0(t)\forall t \in [0, T]$.

We want to solve:

$$-x(0) = \int_0^T \begin{pmatrix} f_1(z) \\ \ldots \\ f_n(z) \end{pmatrix} u(\tau)d\tau$$

This has a solution $u(\tau)\forall x(0) \in \mathbb{R}^n$ iff $f_1(\tau), \ldots, f_n(\tau)$ are linearly independent.
Proof:Suppose $f_1, \ldots, f_n$ are not independent $\implies \exists c_1, c_2, \ldots, c_n : c_1f_1(t) + \cdots + c_nf_n(t) = 0\forall t \in [0, T]$. Let $x(0) = c$ and suppose $\exists u : [0, T]$

$$c = \int_0^T \begin{pmatrix} f_1(\tau) \\ \vdots \\ f_n(\tau) \end{pmatrix} u(\tau)d\tau$$

multiply by c^T:

$$||c||^2 = -\int_0^T c^T \begin{pmatrix} f_1(\tau) \\ \vdots \\ f_n(\tau) \end{pmatrix} u(\tau)d\tau = 0$$

$\implies c = 0$.

Suppose that $f_1, \ldots, f_n$ are independent. Let:

$$M = \int_0^T F(\tau)F^T(\tau)d\tau$$

where $F(\tau) = \begin{pmatrix} f_1(\tau) & \ldots & f_n(\tau) \end{pmatrix}^T$. Again W is a pd matrix when $f_1's$ are continuous.

$$x^TWx = \int_0^T x^TF(\tau)F^T(\tau)xd\tau = \int_0^T ||F(\tau)x||^2d\tau > 0$$

Let $u(\tau) = -F^T(\tau)W^{-1}(0,T)x(0)$.

$$\implies \int_0^T f(\tau)u(\tau)d\tau = -\int_0^T F^T(\tau)F(\tau)W^{-1}[0,T]x(0)d\tau$$
$$= -\int_0^T F^T(\tau)F(\tau)d\tau W^{-1}[0,T]x(0) = -x(0)$$

So the system:

$$\dot{x} = Ax + Bu$$

is controllable iff:

$$W[0,T] = \int_0^T e^{-A\tau}BB^Te^{-A^T\tau}d\tau$$

is invertible for some $T > 0$.
corollary that this does not depend on what T you pick for this.

$W[0,T]$ is the controllability Gramian.
An implication of the previous result, is that the time interval is not dependent on x_0. Thus, the T is global, but also the T doesn't matter.

2 Controllability rank test

$$\dot{x} = Ax + Bu$$

is controllabe iff $\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$ is full rank. This also again tells us that the time T is irrelevant.

We will continue to this test and observability next time.