Numerical Analysis Homework 3
APPM 4650

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November 3, 2015

Problem 1

This problem asks us to use the most accurate three-point formula to determine the missing entries in the table: Starting with $f'(1.1)$ we use the forward rule:

x	f(x)	f'(x)
1.1	9.025013	
1.2	11.02318	
1.3	13.46374	
1.4	16.44465	

$$f'(1.1) = \frac{1}{h}\left(-\frac{3}{2}f(x_0) + 2f(x_1) - \frac{1}{2}f(x_2)\right) = 10\left(-\frac{3}{2}9.025013 + 2 * 11.02318 - \frac{1}{2}13.46374\right) = 17.76971$$

Then we use the mid rule for $f'(1.2), f'(1.3)$:

$$f'(1.2) = \frac{1}{h}\left(-\frac{1}{2}f(x_0) + \frac{1}{2}f(x_2)\right) = 10\left(-\frac{1}{2}9.025013 + \frac{1}{2}13.46374\right) = 22.19364$$

$$f'(1.3) = 10\left(-\frac{1}{2}11.02318 + \frac{1}{2}16.44465\right) = 27.10735$$

Then we use the back rule for $f'(1.4)$:

$$f'(1.4) = \frac{1}{h}\left(\frac{1}{2}f(x_1) + 2f(x_2) + \frac{3}{2}f(x_3)\right) = 10\left(\frac{1}{2}11.02318 - 2 * 13.46374 + \frac{3}{2}16.44465\right) = 32.51085$$

Thus, the table becomes:

x	f(x)	f'(x)
1.1	9.025013	17.76971
1.2	11.02318	22.19364
1.3	13.46374	27.10735
1.4	16.44465	32.51085

Problem 2

This problem wants the derivation for a five-point method for approximating $f'''(x_0)$ by expanding the function $f(x)$ in a fourth-order Taylor Polynomial about x_0. The result should be written in terms of f evaluated at $x_0, x_0 \pm h$ and $x_0 \pm 2h$. Show that the error is $O(h^2)$.

$$f(x \pm h) = f(x) \pm hf'(x) + \frac{h^2}{2}f''(x) \pm \frac{h^3}{6}f^{(3)}(x) + O(h^4)$$

$$f(x \pm 2h) = f(x) \pm 2hf'(x) + 2h^2 f''(x) \pm \frac{4h^3}{3} f^{(3)}(x) + O(h^4)$$

Now, find $f(x+h) - f(x-h)$:

$$S1 = f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{3} f^{(3)}(x)$$

and find $f(x+2h) - f(x-2h)$:

$$S2 = f(x+2h) - f(x-2h) = 4hf'(x) + \frac{8h^3}{3} f^{(3)}(x)$$

Taking $S2 - 2S1$ you get:

$$S2 - 2S1 = f(x+2h) - f(x-2h) - 2f(x+h) + 2f(x-h) = 2h^3 f^{(3)}(x)$$

Solving for $f^{(3)}(x)$ gives:

$$f^{(3)}(x) = \frac{-f(x-2h) + 2f(x-h) - 2f(x+h) + f(x+2h)}{2h^3}$$

To find the error approximately, note that when we take $S2 - 2S1$ we have an error of that is some function of h^5 because the fourth order terms would have canceled. Then we divide by some function of h^3 which should give some function of h^2.

Problem 3

Here, we are asked to use Taylor Series expansions to find three-point forward and backward expressions for $f''(x)$.
First, find the expansions of $f(x+h)$ and $f(x+2h)$.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + O(h^3)$$

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(x) + O(h^3)$$

Then do a smart linear combo to get:

$$f(x_0 + 2h) - 2f(x_0 + h) = -f(x_0) + h^2 f''(x_0)$$

Solving for $f''(x)$ yields

$$f''(x_0) \approx \frac{f(x_0) - 2f(x_0+h) + f(x_0+2h)}{h^2}$$

The backwards 3 point method is exactly the same, thus giving you:

$$f''(x_0) \approx \frac{f(x_0) - 2f(x_0-h) + f(x_0-2h)}{h^2}$$

Problem 4

This problem wants us to show that one can get the following expression:

$$f'(x) \approx \frac{1}{h}\left(-\frac{3}{2}f(x) + 2f(x+h) - \frac{1}{2}f(x+2h)\right)$$

using Taylor Series expansions. It was also noted that the class found this using Lagrange polynomials. Let's put the taylor series expansion stuff on the right side.

$$2f(x+h) - \frac{1}{2}f(x+2h) = \frac{3}{2}f(x) + h * f'(x)$$

$$\frac{4}{3}f(x+h) - \frac{1}{3}f(x+2h) = f(x) + \frac{2}{3}h * f'(x)$$

Double checking that this works we see:

$$f(x+h) = f(x) + hf'(x) + O(h^2)$$

$$f(x+2h) = f(x) + 2hf'(x) + O(h^2)$$

Then we do the linear combination to get:

$$\frac{4}{3}f(x+h) - \frac{1}{3}f(x+2h) = f(x) + \frac{2}{3}hf'(x)$$

So this can be found by taking $\frac{4}{3}$ of the $f(x+h)$ taylor series and subtracting $\frac{1}{3}$ of the $f(x+2h)$ taylor series both expanded only to the first derivative.

Problem 5

This problem wants the derivation of Simpson's $\frac{3}{8}$ rule using different methods.

(a)

First it wants the rule using Lagrange Polynomials. Start by making the polynomial

$$\begin{aligned}
P_4(x) = &\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f_1 \\
&+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f_3
\end{aligned}$$

$$\begin{aligned}
&= \frac{x^3 - x^2(x_1+x_2+x_3) + x(x_1x_2+x_2x_3+x_1x_3) - x_1x_2x_3}{-6h^3}f_0 \\
&+ \frac{x^3 - x^2(x_0+x_2+x_3) + x(x_0x_2+x_0x_3+x_2x_3) - x_0x_2x_3}{2h^3}f_1 \\
&+ \frac{x^3 - x^2(x_0+x_1+x_3) + x(x_0x_1+x_0x_3+x_1x_3)}{-2h^3}f_2 \\
&+ \frac{x^3 - x^2(x_0+x_1+x_2) + x(x_0x_1+x_0x_2+x_1x_2) - x_0x_1x_2}{6h^3}f_3
\end{aligned}$$

integrating the terms next to each f we get:

$$\begin{aligned}
\frac{1}{-6h^3}\Bigg(\frac{1}{4}x^4 - x^3(x_0+2h) + \frac{1}{2}x^2(x_0^2+3hx_0+2h^2+x_0^2+5hx_0+6h^2+x_0^2+4hx_0+3h^2) \\
+x(x_0^3+6hx_0^2+11h^2x_0+6h^3)\Bigg) f_0\Bigg|_{x=x_0}^{x=x_0+3h}
\end{aligned}$$

You get a few more like this then you simplify to get Simpsons. I wanted to sleep the night before this was turned in, so I didn't do all the algebra.

(b)

Second it wants the rule using Taylor Comparison.
I spent more than an two to three hours working on this, but couldn't figure it out. You don't have 5 points, so how do you get the correct $f(x)$ values in the final term? Is it some combination of two taylor series about x_1 and x_2? That's what I tried, but I couldn't figure out how to weight them properly. I guess I'll come talk to you in Office Hours.

Problem 6

Given the integral $\int_0^2 x^2 e^{-x^2}\,dx$ use several different methods to approximate the integral value using $h = 0.25$.

(a)

First, with the Midpoint rule. Implemented a Python script for this. The value gave me 0.483766.

```
import numpy as np

h=0.25
a=0
b=2
x1=0
x=0
est=0;


def my_range(start,end,step):
    while start<=end:
        yield start
        start+=step


for x in my_range(a+h/2,b-h/2,h):
    est=np.exp(-(x**2))*x**2+est

s=a+h/2
r=b-h/2
n=(int)((r-s)/h)
est=(b-a)*est/n
print("delta x:")
print((b-a)/n)
print("estimate")
print(est)
```

(b)

Next, with the Trapezoidal rule. Again, I implemented a python script. The value given was 0.42158203.

```
import numpy as np

h=0.25
a=0
b=2
x=0;
x1=a;
est=0;


def my_range(start,end,step):
    while start<=end:
        yield start
        start+=step


for x in my_range(a+h,b,h):
    est=np.exp(-(x**2))*x**2+np.exp(-(x1**2))*x1**2+est
    x1=x
```

```
est=h*est/2
print("estimate")
print(est)
```

(c)

Finally with Simpson's $\frac{1}{3}$ rule. Again, a python script. This gave 0.4249845.

```
import numpy as np

h=0.25
a=0
b=2
x=0;
est=0;


def my_range(start,end,step):
    while start<=end:
        yield start
        start+=step

def exp_thing(z):
    return np.exp(-(x**2))*x**2

for x in my_range(a+h,b-h,2*h):
    est=h/3*(exp_thing(x-h)+4*exp_thing(x)+exp_thing(x+h))+est

print("estimate")
print(est)
```

Problem 7

Given the integral $\int_0^1 x^2 e^{-x} dx$ use Guassian Quadrature to compute the integral with different values of n and compare the results to the exact integral value. The actual value is 0.16060 or so.

(a)

First, use $n = 2$ in Guassian Quadrature.
Here, we use the substitution $t = \frac{2x-a-b}{b-a} = \frac{2x-0-1}{1}$. Thus, our equation becomes:

$$g(t) = \frac{1}{2}\int_{-1}^{1}\left(\frac{t+1}{2}\right)^2 e^{-\frac{t+1}{2}}$$

Then I evaluate at two values:

$$\int f(x) = 0.5\left(g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)\right) = 0.5 * (0.0361513 + 0.28266955) = 0.159410425$$

As you can see we are already quite close.

(b)

Next, use $n = 3$. We use the same function $g(t)$ only evaluated at different points.

$$g(0) = 0$$

$$g(\sqrt{\frac{3}{5}} = 0.324183$$

$$g(-\sqrt{\frac{3}{5}} = 0.0113478$$

$$\int f(x) = \frac{1}{2}(0.324183 + 0.0113478) = 0.1677656$$

As you can see the answer did not get much better because our approximation at 0 didn't help us any.

Problem 8

Use the direct method to determine the values of a and b in the equation $\int_0^1 f(x)dx \approx af(\frac{1}{3}) + bf(\frac{2}{3})$. Also compute the magnitude of the error.
Here, we need to take at least 2 integrals.

$$\int_0^1 1dx = 1 = a + b$$

$$\int_0^1 xdx = \frac{1}{2} = a\frac{1}{3} + b\frac{2}{3}$$

solving these yields $a = 1 - b$, then that $b = \frac{1}{2}$. Thus, $a = \frac{1}{2}$. Let's check the third integral.

$$\int_0^1 x^2 dx = \frac{1}{4} \approx \frac{1}{18} + \frac{4}{18}$$

So the magnitude of the first part of the error is $0.02\bar{7}$.