Notes in ECEN 5448

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1 input output stability

Input Output Characteristics
norm of a linear system is:

$$||h|| = \sup_{u\neq 0, ||u||<\infty} \frac{||Hu||}{||u||}$$

for a given functional norm.
Definition: For $u : \mathbb{R}^+ \to \mathbb{R}^n$:

$$||u||_\infty := \sup_t ||u(t)||_\infty$$

vector $||\cdot||_\infty$
Let $\mathscr{H} y(t) = \int_0^\infty H(t-\tau)u(\tau)d\tau$ for an integrable $H : \mathbb{R} \to \mathbb{R}$.
The goal is to study input-output norm of this system for functional norm being the infinity norm defined above.
Claim: For this system $\gamma = ||\mathscr{H}||_{\text{ind.}} = \int_0^\infty |H(t)| d\tau$.
IN particular if $\mathscr{H}$ is a BIBO system if that integral is bounded.

Proof: For a given t, and u(t) with $||u(t)||_\infty < \infty$

$$|y(t)| = |\int_0^\infty H(t-\tau)u(\tau)d\tau| \leq \int_0^\infty |H(t-\tau)||u(\tau)|\tau \leq ||u||_\infty \int_0^\infty |H(t-\tau)|d\tau = ||u|| \int_0^\infty |H(\tau)|d\tau$$

$$\implies ||y|| \leq \gamma ||u||_\infty \implies \frac{||y||_\infty}{||u||_\infty} \leq \gamma$$

$$\implies \sup_{||u||\leq\infty, u\neq 0} \frac{||Hu||_\infty}{||u||_\infty} \leq \gamma$$

To show that $\sup_{\text{same}} \frac{\text{same}}{\text{same}} \leq \gamma$,
Let:

$$u(\tau)_T = \{ \begin{array}{ll} \text{sign}(H(T-\tau)) & \text{for} \tau \in [0,\tau] \\ 0 & \text{else} \end{array} .$$

For u_T defined this way:

$$y(T) = \int_0^{\infty} H(T-\tau)u(\tau)d\tau = \int_0^{T} H(T-\tau)sgn(T-\tau)d\tau$$
$$= \int_0^{T} |H(\tau)|d\tau$$

For these u_T,

$$||y||_{\infty} \geq |y(T)| = \int_0^{T} |H(\tau)|d\tau ||u_T||_{\infty}$$
$$\implies ||\mathscr{H}||_{\text{ind.}} \geq \int_0^{T} |H(\tau)|d\tau$$

for any T. Thus,

$$\implies ||\mathscr{H}||_{\text{ind.}} \geq \gamma$$

Implication: The linear system $\mathscr{H}$ is BIBO stable iff $\int_0^{\infty} |H(\tau)|d\tau < \infty$.

2 MIMO BIBO STAB

LEt's look at the same thing for multi input output systems.

$$u : \mathbb{R}^+ \to \mathbb{R}^m, H : \mathbb{R} \to \mathbb{R}^{p\times m}$$

then we get that:

$$||\mathscr{H}||_{\text{ind.}} = ||\mathscr{M}||$$

$$\mathscr{M} = \begin{pmatrix} ||H_{11}||_{\text{ind.}} & ||H_{12}||_{\text{ind.}} \dots ||H_{1m}||_{\text{ind.}} & & \\ \dots & \dots & \dots & \dots \\ ||H_{p1}||_{\text{ind.}} & \dots & \dots & ||H_{pm}||_{\text{ind.}} \end{pmatrix}$$

Note that $H_{ij}(t)$ is the transfer function from jth input to the i'th output and $||H_{ij}||_{\text{ind.}}$ is the $||\cdot||_{\text{ind.},\infty}$ of this SISO system.

3 Relationship between state space Stability and INput Output STability(BIBO)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$x(0) = 0$$

$$y(t) = \int_0^t C * e^{A(t-\tau)} Bu(\tau) d\tau + Du(t)$$

D doesn't have any real impact on io stab so we just assume it is zero.
For convenience let $D = 0$.
Claim: If the system is exponentially stable, then it is BIBO Stable regardless of which norm you use.
PROOF:

$$||y(t)||_\infty = || \int_0^t Ce^{A(t-\tau)} Bu(\tau) d\tau ||_\infty$$

$$\leq \int_0^t ||Ce^{A(t-\tau)} Bu(\tau)|| d\tau \leq \int_0^t ||Ce^{A(t-\tau)} B||_{\text{ind.}} ||u(\tau)||_\infty d\tau$$

$$\leq \int_0^t ||Ce^{A(t-\tau)} B||_{\text{ind.}} d\tau ||u||_{\text{ind.}\infty}$$

$$\leq ||C||_{\text{ind.}} ||B||_{\text{ind.}} \int_0^t ||e^{A(t-\tau)}|| d\tau \leq ||u||_{\text{ind.}} ||C||_{\text{ind.}} ||B||_{\text{ind.}} m \int_0^t e^{-\lambda_0 (t-\tau)} d\tau$$

where $\lambda_0 \geq 0 \implies$ the system is BIBO.

4 Euclidian norm of functions

Let $||u||_2 := (\int ||u(t)||^2 dt)^{\frac{1}{2}}$.
This is also known as the L2 norm of $u(t)$ This is a generalization of Euclidian norm to functions.
Goal:Characterize the L2 gain of a linear system: $y(t) = \int_0^\infty H(t-\tau) u(\tau) d\tau$.
Claim: $||\mathscr{H}||_{\text{ind.},2} = \sup_{\omega \in \mathbb{R}} ||H(j\omega)||_{\text{ind.},2} = \gamma$.
where $H(j\omega) = \mathfrak{L}(H(t)) \Big|_{s=j\omega}$.

$$u(t) = \sin(\alpha t^2) = \sin((\alpha t = f)t)$$

to test in real world.

Proof: Parsevals:

$$||y||_2^2 = \int_0^\infty y^T(t) y(t) dt = \frac{1}{2\pi} \int_{-\infty}^\infty Y(j\omega)^H Y(j\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty U^H(j\omega) H^H(j\omega) H(j\omega) U(j\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^\infty ||H(j\omega) U(j\omega)||_2^2 d\omega \leq \frac{1}{2\pi} \int_{-\infty}^\infty ||H(j\omega)||_{\text{ind.},2}^2 ||U(j\omega)||^2 d\omega$$

$$\leq \frac{1}{2\pi} \gamma^2 \int_{-\infty}^\infty ||U(j\omega)||_2^2 d\omega$$

$$= \gamma^2 ||u||_2^2$$

design a u that exites the system. Some chain of exponentially decaying sinusoids to get H.