Notes in ECEN 5448

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1 Stabilizability

Lyapunov Test:
(A, B) is stabilizable iff $\exists$ p.d. W s.t.:

$$AW + WA^T - BB^T = -Q$$

for some p.d. Q (that equation is *)

$\dot{x} = Ax + Bu$ $u = -Kx$, is stable.
How to get feedback Kx?
Multiply * from left and right by $P = W^{-1}$, we get:

$$PA + A^T P - PBB^T P = -PQP$$

$$B^T P = 2K$$

$$\implies P(A - BK) + (A^T - K^T B^T)P = -PQP$$

Therefore, $(A - BK)$ is Hurwitz (stable).

Fact: Suppose:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_c & A_{cu} \\ 0 & A_u \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \tilde{B}_c \\ 0 \end{pmatrix} u$$

is the controllability transformation of $\dot{x} = Ax + Bu$. Then the system is stabilizable iff A_u is Hurwitz.

PBH Test for Stabilizability:
(A, B) is stabilizable iff rank$(A - \lambda I, B) = n$ for $\forall \lambda$ with $Re(\lambda) > 0$

2 Observability

Dual to Controllability, related to output.
We say that $\dot{x} = Ax + Bu$, $y = Cx + Du$ is observable if $\forall x(0) \in \mathbb{R}^n$, $(\forall u)$, $\exists T$ such that by observing $y(t)$ in $[0, T]$, $x(0) \in \mathbb{R}^n$ ccan be determined uniquely.
Note that for arbitrary $u(t)$, $y(t) = Ce^{At}x(0) + \int_0^T Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$ where the integral term is $\tilde{y}(t)$.
Then, $(y - \tilde{y}(t))$ is the output for the unforced system $(u = 0)$.
Therefore, observability is a property of (A, C) pair like controllability was a property of (A, B) pair.

$$\begin{pmatrix} y(0) \\ y(h) \\ \vdots \\ y(Nh) \end{pmatrix} = \begin{pmatrix} Cx(0) \\ Ce^{Ah}x(0) \\ \vdots \\ Ce^{NAh}x(0) \end{pmatrix}$$

So this basically requires that all those Ce^{Anh} are LI.
Uniquely $x(0) \implies \notin \tilde{x}(0) = x(0)$:

$$\tilde{y} = \begin{pmatrix} C \\ Ce^{Ah} \\ \vdots \\ Ce^{NAh} \end{pmatrix} \tilde{x}$$

$$\implies 0 = \begin{pmatrix} C \\ Ce^{Ah} \\ \vdots \\ Ce^{Nah} \end{pmatrix} x$$

should have a unique solution $x = 0$. Which holds iff the columns of that matrix are L.I.

A system is observable iff the columns of Ce^{At} are linearly independent over the interval $t \in [0, T]$.
columns of Ce^{At} are L.I. over $[0, T]$ iff:

$$W_0(0,T) = \int_0^T e^{A^T t} C^T C e^{At} dt$$

is a positive definite matrix.
Reminder: $x^T W_0(0,T) x = \int_0^T x^T e^{A^T t} C^T C e^{At} x dt = \int_0^T ||Ce^{At}x||^2 dt \geq 0$ iff the columns of Ce^{At} are L.I.

The matrix $W_0(0,T)$ is called the observability Gramian of $\dot{x} = Ax$ and $y = Cx$. Note that:

$$W_c(0,T) = \int_0^T eAtBB^T e^{A^T t} dt$$

Therefore, a system is observable iff (A^T, C^T) is controllable.

So (A, C) is observable iff (A^T, C^T) is controllable:
or if the observability matrix:

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

is full rank.
or if rank$\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = n \forall \lambda \in \mathbb{C}$.
or the columns of Ce^{At} are linearly independent for $t \in [0, T]$
or if $W_O(0,T)$ is p.d. for all $T > 0$.

What is $x(0)$?

$$y(t) = Ce^{At} x(0)$$
$$\implies x(0) = W_O(0,T)^{-1} \int_0^T c^T e^{A^T t} y(t) dt$$

Observability form:
Similar to controllability, $\exists T; z = Tx$ results in:

$$z = \begin{pmatrix} A_O & 0 \\ A_u & A_{uO} \end{pmatrix} z$$

$$y = \begin{pmatrix} c_1 & 0 \end{pmatrix} z$$

where A_O, C_1 is an observable pair.
Note that any initial condition $z_0 = \begin{pmatrix} 0 \\ u \end{pmatrix}$ where $u \in \mathbb{C}$ the output $y(t) = 0 \; \forall t \in \mathbb{R}^+$.

Duality in controls defaults to mean the duality between controllability and observability. The dual to stabalizability is Detectability.

3 Detectability

The pair (A, C) is called detectable if $A \pm HC$ is Hurwitz for some $n \times q$ matrix H.
Model-Based Observer aka Luenberger Observer (deterministic version of Kallman filter).
Original setting:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Observers reconstruct x from whatever you know.
A mapping that maps (u, y) to something is called an observer for linear system if:

$$||x(t) - \hat{x}(t)|| \to 0$$

as $t \to \infty$.
Model-Based Observer because you know the model:

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - \hat{y})$$
$$\hat{y} = C\hat{x}$$

for some H.

Suppose that $A - HC$ is Hurwitz.
Let $e = x - \hat{x}$. Then,

$$\begin{aligned} \dot{e} &= Ax + Bu - A\hat{x} - Bu - H(y - \hat{y}) \\ &= Ax - A\hat{x} - HCe \\ \dot{e} &= (A - Hc)e \end{aligned}$$

Therefore, $\hat{x}(t) \to x(t)$ exponentially fast

4 outside of topic

PBH test for controllability: (A, B) controllable, iff rank$(A - \lambda IB) = n$ for $\forall \lambda$.
Rank$(A - \lambda_1 IB) < n$ iff $\exists c \; c^T(A - \lambda_1 IB) = 0 \implies c^T A = -\lambda c^T$, $c^T B = 0$
$c^T(BAB \ldots A^{n-1}B) = 0$