Homework 9: Advanced Linear Systems (ECEN 5448)

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Problem 1

Given $a < b$. Show that the set of polynomials on any interval (a, b) is an independent set in the vector space of functions.
In this problem we are effectively trying to show that:

$$\sum_{n=0}^{\infty} c_n * t^n \neq 0$$

if and only if $c_n = 0$. Starting with:

$$c_1 * 1 + c_2 * t = 0$$

If we assume this is true for $c_1, c_2 \neq 0$, we can construct a contradiction by taking the first derivative:

$$c_2 * t = 0$$

which implies $c_2 = 0$, and $c_1 = 0$, giving us a contradiction. Next, we move on to the case with 3 elements.

$$c_1 * 1 + c_2 * t + c_3 * t^2 = 0$$

By making the same assumption that $c_n \neq 0$ and taking the second derivative we get:

$$2 * c_3 = 0 \implies c_3 = 0 \implies c_2 = 0 \implies c_1 = 0$$

Now, we show that if the c_{n+1} term is zero for the $n + 1$ element case, the rest of the c elements are zero by taking the nth derivative. So, we have:

$$c_1 1 + c_2 t + c_3 t^2 + \cdots + c_n t^n + c_{n+1} t^{n+1} = 0$$

It should be clear that if $c_{n+1} = 0$ all other c's are zero. Taking the $n + 1$th derivative we get:

$$c_{n+1}(n + 1)! = 0$$

which implies c_{n+1} is zero, implying all c's are zero.

Problem 2

Perform the controllability transformation on the system $\dot{x} = Ax + Bu$ where:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad , \; B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

First, I need A^2.

$$A^2 = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

Then the controllability Gramian is:

$$C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Note the rank of this matrix is 2. Finding the orthonormal basis of C, we get two vectors:

$$v_1 = \begin{pmatrix} 0.48723 \\ 0.72472 \\ 0.48723 \end{pmatrix} \quad v_2 = \begin{pmatrix} -0.51246 \\ 0.68904 \\ -0.51246 \end{pmatrix}$$

Now we need another vector v_3 orthonormal to our first two to form the T matrix.

$$v_3 = \begin{pmatrix} -0.70711 \\ 0 \\ 0.70711 \end{pmatrix}$$

$$T = \begin{pmatrix} 0.48723 & 0.72472 & 0.48723 \\ -0.51256 & 0.68904 & -0.51256 \\ -0.70711 & 0 & 0.70711 \end{pmatrix}$$

Now we can do the controllability transformation:

$$\dot{z} = TAT^{-1}z + TBu = \begin{pmatrix} 2.4124 & -0.071338 & 0 \\ -0.071338 & -0.41241 & 0 \\ 0 & 0 & 1 \end{pmatrix} z + \begin{pmatrix} 0.72472 \\ 0.68904 \\ 0 \end{pmatrix} u$$

Problem 3

Is the system from Problem 2 stabalizable?
Stabalizability is proven by showing the matrix:

$$(A - \lambda * I \;\; B)$$

has full row rank for all $Re(\lambda) \geq 0$. The eigenvalues of the original matrix are -0.41421, 1, and 2.41421. Therefore,

$$S1 = (A - I \;\; B) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

must be full rank, but it only has rank 2. Therefore, the system is not stabalizable.

Problem 4

For the linear system:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

show that linear state transformation $z = Rx$, preserves the transfer function where R is an invertible matrix.
The transfer function of the original system is:

$$Y(s) = C(s * I - A)^{-1}B + D$$

The new system for the transform is:

$$\dot{z} = TAT^{-1}z + TBu$$
$$y = C * T^{-1}z + Du$$

Then, the transfer function becomes

$$\frac{Y(s)}{U(s)} = CT^{-1}(sI - TAT^{-1})^{-1}TB + D$$

$$= CT^{-1}(T^{-1}(s * I - TAT^{-1}))^{-1}B + D$$
$$= C(T^{-1}(sI - TAT^{-1})T)^{-1}B + D$$
$$= C(T^{-1}sIT - IAI)^{-1}B + D$$
$$= C(sI - A)^{-1}B + D$$

Completing the proof.

Problem 5

We know that for dynamics:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

there exists a matrix R, from the controllability transformation, such that the linear state transformation $z = Rx$ results in dynamics:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_c & A_{cu} \\ 0 & A_u \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_c \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} C_c & C_u \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + Du$$

where A_c, B_c are a controllable pair. Show that the transfer function of the original system and the transfer function of the reduced system:

$$\dot{z}_1 = A_c z_1 + B_c u$$
$$y = C_c z_1 + Du$$

are the same. As a result, if a system is not controllable, one can reduce the dimension of the internal state.

$$\frac{Y(s)}{U(s)} = (C_c \; C_u) \left(sI - \begin{pmatrix} A_c & A_{cu} \\ 0 & A_u \end{pmatrix} \right)^{-1} \begin{pmatrix} B_c \\ 0 \end{pmatrix} + D$$

Evaluating $(sI - A)^{-1}$:

$$(sI - A)^{-1} = \begin{pmatrix} sI - A_c & -A_{cu} \\ 0 & sI - A_u \end{pmatrix}^{-1} = \begin{pmatrix} (sI - A_c)^{-1} & -(sI - A_c)^{-1} A_{cu} A_u^{-1} \\ 0 & A_u^{-1} \end{pmatrix}$$

Then, plugging into the transfer function we get:

$$\frac{Y(s)}{U(s)} = C_c(sI - A_c)^{-1}B_c + D$$

Then, due to the last problem, we know this is equivalent to the transfer function for the original system.