Notes in ECEN 5448

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1 Detectability

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$\exists H_{n\times q}$ such that $(A - HC)$ is Hurwitz

$$\dot{\hat{x}} = (A\hat{x} + Bu) + H(y - \hat{y})$$
$$\hat{y} = C\hat{x}$$
$$\dot{e} = (x \dot{-} \hat{x}) = (A - HC)e$$

If H is such that $(A - HC)$ is Hurwitz then $e \to 0$.

1.1 Effect of Distrubance

$$\dot{x} = Ax + Bu + Ld$$
$$y = Cx + n$$

observer

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - \hat{y})$$
$$\hat{y} = C\hat{x}$$
$$\dot{e} = (x \dot{-} \hat{x}) = Ax + Bu + Ld - (A\hat{x} + Bu + H(C(x - \hat{x})) + n)$$
$$= (A - HC)e + Ld - Hn$$

No matter the disturbance you have, the effect of the noise on the error is finite. (Paraphrase) Since $(A-HC)$ is Hurwitz, the effect of the disturbances on the observer is finite (i.e. the Ln norm between the disturbance and state is finite).

1.2 Output stabilizable

Def: For $\dot{x} = Ax + Bu$, $y = Cx$ is output stabilizable if $\exists K$; if we let $u = Ky$, then the closed-loop system is stable.
So we have a feedback gain matrix K.

$$\dot{x} = Ax + BCKx = (A + BCK)x$$

Let, $u = \gamma y$ cause 1-D K.

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} x$$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ \gamma & 1 \end{pmatrix} x$$

$$C(\lambda) = \lambda^2 - \lambda - \gamma$$
$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+4\gamma}}{2}$$

which means one of the eigenvalues must be in the RHP, meaning it can't be Hurwitz.

To output stabilize the dynamics:

$$\begin{gathered}
\dot{x} = Ax + Bu \\
y = Cx \\
\dot{\hat{x}} = A\hat{x} + Bu + H(y - \hat{y}) \\
\hat{y} = C\hat{x} \\
u = -K\hat{x}
\end{gathered}$$

So that $(A - BK)$ is Hurwitz. Then, we get:

$$\dot{x} = Ax - BK\hat{x}$$

$$\dot{\hat{x}} = (A - BK - HC)\hat{x} + HCx$$

$$\dot{\begin{pmatrix} x \\ \hat{x} \end{pmatrix}} = \begin{pmatrix} A & -BK \\ HC & A - BK - HC \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$

rename part of the state:

$$\begin{gathered}
e = x - \hat{x} \\
\begin{pmatrix} x \\ \hat{x} \end{pmatrix} \to \begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \\
\dot{x} = Ax - BK(x - e) = (A - BK)x + BKe
\end{gathered}$$

$$\dot{e} = \dot{(x - \hat{x})} = Ae + HCe = (A + HC)e$$

$$\dot{\begin{pmatrix} x \\ e \end{pmatrix}} = \begin{pmatrix} A - BK & BK \\ 0 & A + HC \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}$$

Then, call the block matrix $\tilde{A}$ and the eigenvalues of $\tilde{A}$ are the union of the eigenvalues of $A - BK$ and $A + HC$ (why?).

1.3 Realization

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Time-dommain transfer function: $H(\tau) = Ce^{A\tau}B$.
Frequency-domain transfer function: $H(s) = C(sI - A)^{-1}B$.
A realization of a transfer function $H(\tau)$ is a linear system.

$$\begin{gathered}
\dot{x} = Ax + Bu \\
y = Cx
\end{gathered}$$

such that $H(\tau) = Ce^{A\tau}B$. Note that realization is not unique.
Minimal REalization: We say that the realization (A, B, C) of $H(\tau)$ is minimal if $\forall$ realization $(\bar{A}, \bar{B}, \bar{C})$ of $H(\tau)$,

$$\dim(A) \leq \dim(\bar{A})$$

Fact: A realization (A, B, C) is minimal iff (A, B) is controllable and (A, C) is observable.
PROOF: (A, B, C) minimal $\implies$ (A, B) controllable, (A, C) observable.
If (A, B) is not controllable, we can transform them into:

$$\begin{pmatrix} A_c & A_{cu} \\ 0 & A_u \end{pmatrix}, \begin{pmatrix} B_c \\ 0 \end{pmatrix}, (C_c C_{cu})$$

IN this case, you can show that:

$$Ce^{A\tau}B = C_c e^{A_c\tau} B_c$$

$\implies$ (A_c, B_c, C_c) is a realization of smaller dimension.
same thing for observability.

Now, for the $\leftarrow$. So we have observability and controllability, let us show that the realization is minimal.

$$Ce^{A(t-\tau)}B = \tilde{C}e^{\tilde{A}(t-\tau)}\tilde{B}$$

multiply from the left by $e^{A^T t}C^T$ will give:

$$\int_0^T e^{A^T t}C^T C e^{At} dt e^{-A\tau} B = \int_0^T e^{A^T t}C^T \tilde{C} e^{\tilde{A}t} dt e^{-\tilde{A}\tau}\tilde{B}$$

The integral term is called M.

$$= W_O(0, T)e^{-A\tau}B = Me^{-\tilde{A}\tau}\tilde{B}$$

now multiply from the right by $B^T e^{-A^T\tau}$ gives:

$$W_O(0, T)\int_{-T}^{0} e^{-A\tau}BB^T e^{-A^T\tau} d\tau$$

$$= M\int_{-T}^{0} e^{-\tilde{A}\tau}\tilde{B}B^T e^{-A^T\tau} d\tau$$

call the integral N to get:

$$\implies W_O(0, T)W_C(0, T) = WN$$

these two matrices must be invertible, so LHS is invertible, so MN should be invertible. Note that if n is the dimension of A and $\tilde{n}$ is the dimension of $\tilde{A}$.
then $M_{n\times\tilde{n}}$ and $N_{\tilde{n}\times n}$, so $\tilde{n} \geq n$ because of matrix dimesnsions.

2 next time

talk a little bit about linear quadratic regulators, touching on optimal control.

3 Notes

the K function mapping y to u is not linear in general, your notes will say otherwise.