Notes in Dynamics and Manuevering
ECEN 5008

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1 Cheat Sheet

To get a descent direction, try to minimize a quadratic model function.

$$f : \mathbb{R}^n \to \mathbb{R}$$
$$-\nabla f(x) = arg \min_z Df(x)z + \frac{1}{2}||z||^2$$

linear plus quadratic minimization. Can do this many ways.
when n is large, structure is important which is why it becomes better to solve the quadratic linear function instead of just taking the gradient.

$$-H(x)^{-1}\nabla f(x) arg \min_z Df(x) * z + \frac{1}{2}D^2 f(x)(z, z)$$

where H is the hessian.

$$=< \nabla f(x), z > + \frac{1}{2} < z, H(x)z >$$

Top of cheat sheet we have:

$$\zeta_k = arg \min_{\zeta \in T_{\xi_k}\mathfrak{T}} Dh(\xi_k) * \zeta + \frac{1}{2}D^2 g(\xi_k) * (\zeta, \zeta)$$

$$\min \int_0^T a^T(\tau)z(\tau) + b^T(\tau)v(\tau) + \frac{1}{2}\begin{pmatrix} z(\tau) \\ v(\tau) \end{pmatrix}^T \begin{bmatrix} Q_0(\tau) & S_0(\tau) \\ S_0^T(\tau) & R_0(\tau) \end{bmatrix} \begin{pmatrix} z(\tau) \\ v(\tau) \end{pmatrix} d\tau + \frac{1}{2}z^T(T)P_1 z(T) + r^T z(T)$$

s.t. $\dot{z} = A(t)z + B(t)v, z(0) = 0$.
need the hessian to be positive definite for this to work. otherwise you might not get a single minimizer.
we still need:

$$R(t) = R^T(t) \geq r_0 I > 0$$

because R has to be invertible, but:
$Q(t) \geq 0$
is way too strict for optimization!
Call $\begin{bmatrix} Q(\tau) & S(\tau) \\ S^T(\tau) & R(\tau) \end{bmatrix} = W(\tau)$. Then let:

$$g * (\zeta, \zeta) = \int_0^\pi \zeta^T(\tau)W(\tau)\zeta(\tau)d\tau + z^T(T)P_1 z(T), \zeta(\tau) = \begin{pmatrix} z(\tau) \\ v(\tau) \end{pmatrix}$$

and let:

$$\mathcal{L} = \{\zeta = (z(\cdot), v(\cdot)) : \dot{z} = A(t)z + B(t)v, z(0) = 0\}$$

THM: Suppose that $R(t) = R^T(t) \geq r_0 I > 0, \forall t \in [0,T]$. Then, q is p.d. on $\mathcal{L} \leftrightarrow$ for the following Riccati equation has a bounded solution on $[0,T]$:

$$\dot{P} + \tilde{A}(t)P + P\tilde{A}(t) - PB(t)R^{-1}(t)B^T(t)P + \tilde{Q}(t) = 0, P(T) = P_1$$

for:

$$\tilde{A} = A - BR^{-1}S^T, \quad \tilde{Q} = Q - SR^{-1}S^T$$

L_2 trajectory exploration:

$$\min \int_0^T \frac{1}{2}||x(\tau) - x_d(\tau)||_Q^2 + \frac{1}{2}||u(\tau) - u_d(\tau)||_R^2 d\tau + \frac{1}{2}||x(T) - x_d(T)||_{P_1}^2$$

typical: Q & R are diagnol:
$R > 0$ and $Q \geq 0$ but usually $Q > 0$.
FACT: in the LQ minimization problem,
if $R(t) = R^T(t) \geq r_0 I > 0$ and $Q(t) = Q^T(t) \geq 0$, and $P_1 = P_1^T \geq 0$, then there is a unique solution to the problem.

Sketch of proof:
consider $\min \frac{1}{2} \int_{t_0}^T \begin{pmatrix} z \\ v \end{pmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{pmatrix} z \\ v \end{pmatrix} d\tau + \frac{1}{2} z^T(T) P_1 z(T)$
such that $\dot{z} = Az + Bv, z(t) = z_0$
the minimum value is then:

$$V(z_0, t_0) = \frac{1}{2} z_0^T P(t_0) z_0 \geq 0$$

2 break

$$h(\xi) \quad g(\xi) = h(\mathcal{P}(\xi))$$

$$Dg(\xi)\zeta = Dh(\mathcal{P}(\xi)) * D\mathcal{P}(\xi)\zeta \Big|_{\xi \in \mathfrak{T}, \zeta \in T_\xi \mathfrak{T}} = Dh(\xi)\zeta$$

you don't want all the stuff in the middle of that equation you just want the end.

$$D^2 g(\xi) \cdot (\zeta, \zeta) = D^2 h(\mathcal{P}(\xi)) \cdot (D\mathcal{P}(\xi)\zeta, D\mathcal{P}(\xi)\zeta) + Dh(\mathcal{P}(\xi)) D^2\mathcal{P}(\xi)(\zeta, \zeta)$$

restriccted to $\xi \in \mathfrak{T}$ and $\zeta \in \mathfrak{T}T_\xi$ trajectory and tanget trajectory.

$$= D^2 h(\xi)(\zeta, \zeta) + Dh(\xi) D^2\mathcal{P}(\xi)(\zeta, \zeta)$$

first part is associated with $\begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}$.
Second part will be associated with $\begin{bmatrix} q^T f_{xx} & q^T f_{xu} \\ q^T f_{ux} & q^T f_{uu} \end{bmatrix}$.
q^T means summation over the ks.

$$(x(), u()) = \mathcal{P}(\xi) := \dot{x} = f(x, u) \quad u = u(t) + K(t)(\alpha(t) - x) x(0) = x_0 \quad \xi = (\alpha(\cdot), \mu(\cdot))$$
$$(z(), v()) = D\mathcal{P}(\xi)\zeta := \dot{z} = A(t)z + B(t)v v = \nu(t) + K(t)(B(t) - z) \quad z(0) = 0 \quad \zeta = (\beta(\cdot), \nu(\cdot))$$
$$(y(), w()) = D^2\mathcal{P}(\xi)(\zeta, \zeta) := \dot{y} = A(t)y + B(t)w + D^2 f(x(t), u(t))((z(t), v(t)), (z(t), v(t))) \quad y(0) = 0$$
$$w = -K(t)y$$

3 Q optimal

$$(Q_0)_{ij} = Q_{ij} + \sum_k g_k \frac{\partial f_k}{\partial x_i \partial x_j}$$

4 aside

might want:

$$\min b^T z + \frac{1}{2} z^T Q z$$

s.t. $A^T z = 0$.
Let the columns of Z span the null space of A^T. then look for $z = Zw$.
problem becomes:

$$\min b^T Z w + \frac{1}{2} w^T Z^T Q Z w$$

need $Z^T Q Z > 0$. You end up looking at matrices like:

$$\begin{pmatrix} Q & A \\ A^T & 0 \end{pmatrix}$$

will need to analyze this.

5 book to torrent

anderson and moore on linear quadratic optimal control book.
Linear quadratic regulator problems.
the book title is "Optimal Control: Linear Quadratic Methods".
Typical assumption:
A,B, constant (or A(t), B(t))
$R = R^T > 0$
$Q = Q^T \geq 0$
can't guarantee these things.

6 checking real derivatives

$f(x)$ built $Df(x)$
define $g(\epsilon) = f(x + \epsilon z)$ for $-1 \leq \epsilon \leq 1$.
Plot the components of $g(\epsilon)$

$$Dg(\epsilon) = Df(x + \epsilon z) * z$$
$$g'(\epsilon) = \frac{g(\epsilon + \delta) - g(\epsilon)}{\delta}$$

if teh forward difference method and your D are different, you did it wrong.

7 consider other things

consider haveing sliding car model at the end of the car, or a thrust vector at the end of the car instead of the hand of god control over ω.