Notes in ECEN 5448

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1 A test that will be useful

Fact: Let A be Horwitz. Then (A,B) is controllable iff $\exists$ p.d. W such that:
(1)

$$WA^T + AW = BB^T$$

Proof: Let (A,B) be controllable. Then:

$$W = \int_0^\infty e^{A\tau}BB^Te^{A^T\tau}d\tau$$

will solve the equation and W is p.d.
Basically the controllability gramian from 0 to ∞.

$$\frac{dM}{dt} = M(t)A^T + AM(t) \quad M(0) = BB^T$$

The solution to the above ode is:

$$M = e^{At}BB^Te^{A^Tt}$$

On the other hand, by integration of both sides:

$$\int_0^T \frac{dM}{dt}dt = \int_0^T (e^{At}BB^Te^{A^Tt})dtA^T + A\int_0^T (e^{At}BB^Te^{A^Tt})dt$$

$$\implies -BB^T = WA^T + AW$$

$\leftarrow$:So let (λ, v): $v^TA = \lambda v^T$. We need to show that $v^TB \neq 0$.

Note that if (1) has a solution, then:

$$||B^Tv||^2 = v^TBB^Tv = v^T(WA^T + AW)v$$
$$= \lambda v^TWv + \lambda v^TWv < 0$$
$$\implies ||B^Tx||^2 \neq 0$$

assuming $\lambda \neq 0$.

Why is this nice?
Fact: Let (A,B) be controllable.
Then, $\exists$ p.d. W;

$$WA^T + AW + BB^T = -Q$$

for some p.d. Q.

Proof: The key B to look at $(-A-\mu I)$. The eigenvectors of A and $(-A-\mu I)$ are the same. For large enough μ, $(-A-\mu I)$ is Horwitz (?).
(A, B) controllable $\implies$ $(-A-\mu I, B)$ is controllable for $\mu > 0$.

$$\implies \exists W; W(-A-\mu I)^T + (-A-\mu I)W = -BB^T$$
$$\implies WA^T + AW - BB^T = -2\mu W$$

Definition: For the system:

$$\dot{x} = Ax + Bu$$

we say it is feedback stabilizable if $\exists K$; if

$$u = Kx \implies \dot{x} = Ax + BKx = (A+BK)x$$

is stable, i.e. $(A+BK)$ is Horwitz for some K.

Fact: If (A, B) is controllable $\implies$ the system is feedback stabilizable.
Proof: By previous result $\exists W - Q > 0$

$$WA^T + AW - BB^T = -Q$$

Let $P = W^{-1}$

$$PWA^TP + PAWP - PBB^TP = -PQP = -\tilde{Q}$$
$$\implies A^TP + PA - PBB^TP = -\tilde{Q}$$

Let $K = \frac{1}{2}B^TP \implies$

$$(A^TP - K^TB^TP) + (PA - PBK) = -\tilde{Q}$$
$$(A-BK)^TP + P(A-BK) = -\tilde{Q}$$

$\implies$ $(A-BK)$ is horwitz because P,Q are positive definite. So a K exists that makes the system stable.

Theorem: (A, B) is controllable iff $\forall \lambda_1, \ldots, \lambda_n \in \mathbb{C} \exists K \in \mathbb{C}^{n\times n}; (A+BK)$ has eigenvalues $\lambda_1, \ldots, \lambda_n$

2 State Transfomrations

$$\dot{z} = T\dot{x} = TAx + TBu = TAT^{-1}z + TBu$$
$$y = CT^{-1}z + Du$$

Many properties of the system (are naturally) invariant under under state transformation.

1. Unforced system stability 2. Controllabilityi

$$(BAB \ldots A^{n-1}B)(TBTABTA^2B \ldots TA^{n-1}B)$$

Controllabilliyt of original system and transfomred system. $\implies$ rank$(BAB \ldots A^{n-1}B)$ =rank $(TBTAB \ldots TA^{n-1}B)$ because T is invertible.

3. Transfer Function.

Recall that for an unctrollable system:

$$\dot{x} = Ax + Bu$$

$\exists T$ s.t. $T^{-1} = T^T$; $\tilde{x} = Tx$

$$\implies \dot{\tilde{x}} = (A = \begin{pmatrix} A_c & A_{cu} \\ 0 & A_u \end{pmatrix})\tilde{x} + \begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix} u$$

and $(A_c, \tilde{B})$ is controllable.

For this form if $v^T \tilde{A} = \lambda v^T$ and $v^T \begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix} = 0$.

, then $v = \begin{pmatrix} 0 \\ \tilde{v} \end{pmatrix}$ s.t. $\tilde{v}^T A_u = \lambda \tilde{v}$.

Because if $v = \begin{pmatrix} v_1 \\ \tilde{v} \end{pmatrix} \implies v_1^T Ac = \lambda v_1^T$

and $v_1^T \tilde{B} = 0 \implies v_1 = 0$ by controllability of $A_c, \tilde{B}$.

Definition: We say that a:

$$\dot{x} = Ax + Bu$$

is stabilizable if $\forall x(0) \in \mathbb{R}^n, \exists u : [0, \infty) \to \mathbb{R}^m$.
if $x(t)$ is a solution, then $\lim_{t\to\infty} x(t) = 0$
similar to controllability, but with infinite time.

Fact: $\dot{x} = Ax + Bu$ is stabilizable iff $\forall v : v^T A = \lambda v^T$ and $\mathrm{Re}(\lambda) \geq 0$, $v^T B \neq 0$
Proof: Stabilizable $\implies$ this.
Suppose this doesn't hold $\implies \exists v; v^T A = \lambda v^T$, $\mathrm{Re}(\lambda) \geq 0$ and $v^T B = 0$.
For $x(0) = v; \exists u; x(t) \to 0$ as $t \to \infty$.

$$v^T \left(x(t) = e^{At} v + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \right)$$

$$\implies v^T x(t) = e^{\lambda t} ||v|| + \int_0^t e^{\lambda(t-\tau)} v^T Bu(\tau) d\tau$$

but $v^T B = 0$ so it breaks the assumption that x goes to zero.
$\implies v^T x(t) \to \infty$ as $t \to \infty$.

3 Lyapunov test for stabilizability

(A, B) are stabilizable iff $\exists$ p.d. W,Q such that:

$$AW + A^T W - BB^T = -Q$$