Notes in ECEN 5448

Zahary Vogel

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1 Some matrix Algebra

$A_{n\times m}$ Range$(A) = Im(A) := \{y : y = Ax \text{ for some } x \in \mathbb{R}^m\}$

$$A = \left[a_1 \middle| a_2 \middle| \dots \middle| a_m\right]$$

$a_i \in \mathbb{R}^n$ and $Ax = x_1a_1 + x_2a_2 + x_3a_3 + \cdots + x_ma_m$.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{pmatrix}$$

$$A_{\text{reduced}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{pmatrix}$$

$$Im(A) = IM(A_{\text{reduced}})$$

Reminder about a subspace, V is a subspace of $\mathbb{R}^n$ if $\forall v_1, v_2, \dots \in V$ and any $\alpha \in \mathbb{R}, v_1 + \alpha v_2 \in \mathbb{V}$.
For a given subspace V find a matrix such that the image of A is V.
For arbitrar subspace $V \subseteq \mathbb{R}^n$ we know there exists a basis $\{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$ for V. If we let $A = [v_1v_2\dots v_k]$, then $IM(A) = V$.
$rank(A) :=$ minimum number of columns of A that span$Im(A) = dim(Im(A))$.

Basis: We say that $\{b_1, \dots, b_k\}$ is a basis for a subspace S if the bs are linearl independent and $\forall s \in S, \exists c \in \mathbb{R}^d, s = Bc$ where B is the matrix made up of the b vectors (linear combinations of b form every point in S).

Fact: Suppose that columns of $B_{n\times n}$ are a basis for S. $\forall$ invertible matrices $T_{n\times m}$; the columns of $\tilde{B} := BT$, are a basis for S.
Proof: columns of $\tilde{B}$ are independent iff $\tilde{B}x = 0$ implies that $x = 0$.
if: $0 = \tilde{B}x = B(Tx) \implies Tx = 0 \implies x = 0$ because T is invertible.

Second part: any vector in S can be constructed with $\tilde{B}$ just like B.
Let $s \in S, \exists c \in \mathbb{R}^m; s = Bc$. If we let $\tilde{c} = T^{-1}c$. Then, $\tilde{B}\tilde{c} = BT\tilde{c} = BTT^{-1}c = Bc = s$.

The columns of B are an orthonormal basis for S if in addition to being a basis for S, $B^TB = I$.
Note that $B^TB = I$ iff $b_i^Tb_j = 0, i \neq j$ and $b_i^Tb_j = 1, i = j$.

IMportant fact: For a symmetric (or Hermetian) matrix A, there always exists M and diagnol matrix Γ;

$$A = M\Gamma M^T, MM^T = I$$

If $Av = \lambda_1 v$, $Aw = \lambda w$, then $w^T Av = \lambda_1 w^T v = \lambda_2 w^T v$ because column eigenvectors are also row eigenvectors for symmetric matrices.

FACT: For symmetric positive definite (p.d.) matrix P, $\exists$ a symmetric matrix R; $P = R^2$.
Proof: for a p.d. P, $P = M\Gamma M^T = M\sqrt{\Gamma}M^T M\sqrt{\Gamma}M^T = R * R$.
$R = M\sqrt{\Gamma}M^T$.
For a basis matrix B, $B^T B$ is p.d. $\implies$

$$B^T B = R * R$$

for some invertible R. Now define $\tilde{B} = BR^{-1}$. Then $\tilde{B}$ is an orthonormal basis for S.

$$\tilde{B}^T \tilde{B} = R^{-T} B^T B R^{-1} = R^{-T} R R R^{-1} = I$$

so there is always an orthonormal basis for any basis.
Let S be a subspace of $\mathbb{R}^n$. Then we define the orthogonal compliment of S by:

$$S^{\perp} := \{x : x^T s = 0 \forall s \in S\}$$

Suppose the columns of B form an orthonormal basis for S. $\forall x \in \mathbb{R}^n$,

$$x = x_s + x_{s\perp}$$

where $x_s \in S$ and $x_{s\perp} \in S^{\perp}$.
Proof: Let $x_s = BB^T x$ and $x_{s\perp} = (I - B^T B)x$.
can show that this projects x onto S and $S^{\perp}$.
x_s has to be in S because it is constructed by B times something which is spanned by the image of B.
Then, $x_s \in S$. show that $x_{s\perp} \in S^{\perp}$, we need to show that $\forall c$ $(Bc \perp x_s)$.

$$c^T B^T (I - BB^T)x = c^T B^T x - c^T B^T x = 0$$

Corollary 1: $S^{\perp} = IM(I - B^T B)$
Corollary 2: $n = \dim(S) + \dim(S^{\perp})$.

2 BACK TO CONTROLLABILITY

$\dot{x} = Ax + Bu$ is controllable if $\leftrightarrow$ of $e^{-A\tau}$ are independent.
$\leftrightarrow \Omega(0,T) = \int_0^T e^{-A\tau} BB^T e^{-A^T \tau} d\tau$ is non-singular
$\leftrightarrow rank(BABA^2B \ldots A^{n-1}B) = n$.
Proof: Use Calley Hamilton,

$$e^{-A\tau} = \alpha_0(\tau)I + \alpha_1(\tau)A + \cdots + \alpha_{n-1}(\tau)A^{n-1}$$
$$\implies e^{-A\tau} = \alpha_0(\tau)B + \alpha_1(\tau)AB + \cdots + \alpha_{n-1}(\tau)A^{n-1}B$$
$$\implies \exists x, x(BAB \ldots A^{n-1}B) = 0,$$
$$\implies x^T e^{-A\tau} B = \alpha_0(\tau)x^T B + \alpha_1(\tau)x^T AB + \cdots + \alpha_{n-1}(\tau)x^T A^{n-1}B = 0$$

Question: Show that $rank(A) < n$, $\exists x \neq 0, x^T A = 0$.
Controllability $\implies$ rank$(BAB \ldots A^{n-1}B) = n$.

Suppose that, $\dot{x} = Ax + Bu$ is not controllable:

$$\implies \exists y; y^T e^{-A\tau} B = 0 \implies y^T B = 0$$
$$\implies \frac{d}{d\tau}(y^T e^{-A\tau} B = -y^T e^{-A\tau} AB\Big|_{\tau=0} = -y^T AB$$

$$\implies \frac{d^{n-1}}{d\tau^{n-1}} y^T e^{-A\tau} \implies (\pm) y^T A^{n-1} B = 0$$