Notes in Dynamics and Manuevering
ECEN 5008

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1 explaining some of the diff eqs from paper

$$\min \int_0^T l(x(\tau), u(\tau), \tau) d\tau + m(x(T))$$

such that $\dot{x} = f(x, u, t), x(0) = x_0$.
$\min(f(x))$ Thm: $x*$ is a local min $\implies \nabla f(x*) = 0$. First order necessary condition.
assume these functions are differentiable.
Pontryagin's maximum (or minimum) principle.
Baby PMP THM:
Suppose that $(\bar{x}(t), \bar{u}(t)), t \in [0, T]$ satisfies the dynnamics ($\dot{x} = f(x, u, t)$) and locally minimizes the cost ($\int_0^T l d\tau m$) (over trajectory).
THM: there is an absolutely continuous curve $\bar{p}(t), t \in [0, T]$ such that:

$$\dot{\bar{x}}(t) = H_p^T(\bar{x}(t), \bar{u}(t), \bar{p}(t), t),\ \bar{x}(0) = x_0$$
$$\dot{\bar{p}}(t) = -H_x^T(\bar{x}(t), \bar{u}(t), \bar{p}(t), t),\ \bar{p}(t) = m_x^T(x(T))$$
$$0 = H_u^T(\bar{x}(t), \bar{u}(t), \bar{p}(t), t)$$

where the control Hamiltonian is:

$$H(x, u, p, t) = l(x, u, t) + p^T f(x, u, t)$$

Written more directly,

$$\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t), t)\ \bar{x}(0) = x_0$$
$$\dot{\bar{p}}(t) = -A^T(\bar{x}(t), \bar{u}(t), t)\bar{p}(t) - l_x^T(\bar{x}(t), \bar{u}(t), t)\ \bar{p}(T) = m_x^T(x(T))$$
$$0 = B^T(\bar{x}(t), \bar{u}(t), t)\bar{p}(t) + l_u^T(\bar{x}(t), \bar{u}(t), t)$$

two point boundary value problem (TPBVP)
BVP 4c and BVP 5c solvers will do this, but need a good initial guess to use properl.
need x to be linear and cost to be quadratic.

2 Linear Dynamics, Quadratic Cost

(LQ opt ctrl LQR)

$$\min(\int_0^T \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u d\tau + \frac{1}{2}x(T)^T P_1 x(T))$$
$$Q = Q^T \geq 0$$

$$R = R^T > 0$$
$$P_1 = P_1^T \geq 0$$

take to be constant, but they don't have to be.
All of this such that:

$$\dot{x} = A(t)x + B(t)u, x(0) = x_0$$
$$H(x, u, p, t) = \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru + p^T(Ax + Bu)$$
$$\dot{x} = H_p^T = Ax + Bu$$
$$-\dot{p} = H_x^T = A^T p + Qx$$
$$0 = H_u^T = B^T p + Ru \implies u = -R^{-1}B^T p$$
$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}$$
$$x(0) = x_0$$
$$p(T) = P_1 x(T)$$
$$\begin{bmatrix} x(T) \\ p(T) \end{bmatrix} = \begin{bmatrix} I \\ P_1 \end{bmatrix} x(T)$$
$$p(T) = P_1 x(T)$$

maybe $p(t) = P(t)x(t)$(if we are lucky). Because of the sspecifications on definiteness of Q,R,P this is true here.
He likes to call this the Ricatti Transformation.

$$p = Px$$
$$\dot{p} = \dot{P}x + P\dot{x} = \dot{P}x + P(Ax - BR^{-1}B^T Px)$$
$$= -Qx - A^T Px$$
$$0 = (\dot{P} + A^T P + PA - PBR^{-1}B^T P + Q)x$$

this is a ricotti equation.
If $P(t), t \in [0, T]$ satisfies:

$$\dot{P} + A^T P + PA - PBR^{-1}B^T P + Q = 0 \quad P(T) = P_1$$

Then, $p(t) = P(t)x(t) \forall t \in [0, T]$.
and $u = -R^{-1}B^T Px = -K(t)x$.
so that the optimal trajectories one obtained using a linear feedback.

$$\dot{x} = (A - Bk)x, x(0) = x_0$$

3 Minimize this optimization

$$\min \int_0^T a^T(\tau)z(\tau) + b^T(\tau)v(\tau) + \frac{1}{2}\begin{bmatrix} z(\tau) \\ v(\tau) \end{bmatrix}^T \begin{bmatrix} Q(\tau) & S(\tau) \\ S^T(\tau) & R(\tau) \end{bmatrix} \begin{bmatrix} z(\tau) \\ v(\tau) \end{bmatrix} d\tau + \frac{1}{2}z(T)^T P_1 z(T) + r_1^T z(T)$$

$$\dot{z} = A(t)z + B(t)v, z(0) = 0$$

$$a(t) = l_x^T(t)$$

$$b(t) = L_u^T(t)$$

$$-\dot{r} == (A - BK_0)^T r + a - K_0^T b = A^T r + a - K_0^T(B^T r + b)$$

For the cheat sheet, we are at a trajectory, compute some minimums to figure out what is going downhill.

4 Newton Method (Pronto):

start with initial trajectory: $\epsilon \in \mathfrak{T}$
for $k = 0, 1, 2, \ldots.$
redisign $k(t)$.
descent direction:

$$\zeta_k = \arg\min_{\zeta \in T_{\epsilon_k}\mathfrak{T}} Dh(\epsilon_k)\zeta + \frac{1}{2}D^2 g(\epsilon_k) * (\zeta_1 \zeta)$$

or $\frac{1}{2}||\zeta||^2_{L2(\tilde{Q},\tilde{R}}$
or $\frac{1}{2}D^2 h(\epsilon_{12})(\zeta_1, \zeta)$.
line search (Aruijo backtracking)

$$\gamma_{12} = \arg\min_{\gamma \in [0,1]} h(\mathfrak{P}(\epsilon_k \gamma \zeta_k))$$

update:

$$\epsilon_{k+1} = \mathfrak{P}(\epsilon_k + \gamma_k \zeta_k)$$

$$\int_x (\tau)z(\tau) + l_u(\tau)v(t)d\tau + m_x(x(T)) * z(T)$$