Notes in ECEN 5448

Zahary Vogel

December 9, 2015

Fundamental Theorem of Lin Alg Proof

Very non-trivial.
row rank of $A_{m\times n}$=column rank A.
Let $v_1, \dots, v_k$ be the independent rows of A that span the rowspace of A.

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}$$

row rank(A)=k
Let $u_i \in \mathbb{R}^n = Av_i^T$ for $i = 1, \dots, k$.
We show that $u_1, \dots, u_k$ are independent.
Suppose that this isn't true, and prove by contradiction. $\implies \exists \alpha_1, \dots, \alpha_k$:

$$\alpha_1 u_1 + \cdots + \alpha_k u_k = 0$$
$$\leftrightarrow \alpha_1 A v_1^T + \cdots + \alpha_k A v_k^T = 0$$

this implies that:

$$A(\alpha_1 v_1^T + \cdots + \alpha_k v_k^T) = 0$$
$$z = \alpha_1 v_1^T + \cdots + \alpha_k v_k^T$$

this means that z is orthogonal to A_i including $v_1, \dots, v_k$. IN particular, z is orthogonal to z, which implies $||z||^2 = 0 \implies z = 0$. Since, $v_1, \dots, v_k$ are linearly independent $\alpha_1 = \cdots = \alpha_k = 0$ which means u_is are linearly indepedent.
So we have found k indepedent vectors that are in the column space that are independent. Therefore, the column rank is greater than or equal to the row rank.
Repeat this argument for $u_i * A = v_i$, or use the same argument for A transpose. Then you get the other inequality.

Controllability Comment

want to move from one point in space to another.
$\exists u(t)$:

$$-x(0) = \int_0^T e^{-A\tau} B u(\tau) d\tau$$

$$y = \int_0^T e^{-A\tau} B u(\tau) d\tau \text{for some control} u(\tau)$$

is equal to the range of the controllability matrix. So just find a vector that isn't in the range of the controllability matrix and then find a vector that isn't in that.

Homework 10

Problem 1

$$\dot{x} = Ax + Bu$$
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(a)

Use the rank test to conclude that (A, B) is not controllable.

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

It's rank is clearly 1.

(b)

Use the PBH test to conclude that (A, B) is stabilizable.

$$\text{rank}(A - \lambda I B) = n$$

for all λ with $Re(\lambda) \geq 0$
$\lambda_1 = 1$, $\lambda_2 = -1$.

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

Note that:

$$det(\begin{pmatrix} -\lambda & 1 \\ 1 & 1 \end{pmatrix} = -\lambda - 1 \neq 0$$

for λ with $\text{Re}(\lambda) \geq 0$

(c)

Find a T such that the transformation $z = Tx$ takes the system to triangular Controller Form.
Want T such that $TAT^{-1} = \begin{pmatrix} A_c & A_{cu} \\ 0 & A_u \end{pmatrix}$ and $TB = \begin{pmatrix} B_c \\ 0 \end{pmatrix}$.

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad T^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$TAT^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$TB = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

(d)

Find a stabilizing feedback in the transformed coordinates $u = -Gz$ for $z = Tx$.

$$\dot{z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z + \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} u(t)$$
$$u(t) = -\begin{pmatrix} -\sqrt{2} & 0 \end{pmatrix} z$$

Thus, the system is:

$$\dot{z} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} z$$

(e)

Now express this in the original coordinates $u = -Kx$.

$$\begin{aligned} u &= (-\sqrt{2} \quad 0)z = (-\sqrt{2} \quad 0)Tx \\ &= (-sqrt2 \quad 0)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} x \\ &= (-1 \quad -1)x \end{aligned}$$

checking this:

$$\begin{aligned} Ax + Bu &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} (-1 \quad -1)x \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x \end{aligned}$$

which is Hurwitz.

Problem 2

Show that for any A, there exists a $\mu > 0$ such that:

$$-\mu I - A$$

is a stability matrix.

$$C_{A\mu}(\lambda) = \det(A_\mu - \lambda I) = \det(-A - \mu I - \lambda I) = \det(-A - (\mu + \lambda)I) = c_{-A}(\mu + \lambda)$$

So if λ_0 is an eigenvalue of $-A - \mu I$ then, $C_{-A}(\mu + \lambda_0) = 0$ i.e. $\mu + \lambda_0$ is an eigenvalue for $-A$. Thus, for large enough μ, $\lambda_0 < 0$.
Let $(\lambda_1, \ldots, \lambda_n)$ be eigenvalues of $-A$.
If $\mu > |Re(\lambda_1)|, \ldots, |Re(\lambda_n)|$ then $\mu + \lambda_0 = \lambda_i$ for some $\lambda_0 = \lambda_i - \mu$

$$\implies Re(\lambda_0) = Re(\lambda_i) - \mu < 0$$

because μ is bigger than the real part of λ_1.

Problem 3

Use the PBH test to show that if (A, B) is a controllable pair, then $(-\mu I - A, B)$ is a controllable pair.
(A, B) controllable $\leftrightarrow$ $(-A - \mu I, B)$ is controllable.

$$\text{rank}(A - \lambda I \quad B) < n$$

iff $c^T(A - \lambda I) = 0 \leftrightarrow c^T A = \lambda c^T \; c^T B = 0$
Left eigenvectors of $A =$ left eigenvectors of $(-A - \mu I)$

$$c^T(-A - \mu I) = -c^T A - \mu c^T = -\lambda c^T - \mu c^T = (-\lambda - \mu)c^T$$

Problem 4

Consider the inverted pendulum model:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} x_2 \\ \sin(x_1) - (x_2 + x_2|x_2|) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

where $u(t)$ is the input torque. Linearize th dynamics around the equilibrium point $(0,0)$. Let $\dot{x} = Ax + bu$ be that linearization. For the linearized model, derive the equations for the minimum energy controller that derives the dynamics from $(0.2, 0)$ to close neighborhood of $(0,0)$ in 1 second. In other words, derive the Riccati equation and the coresponding controller for the following optimal control problem:

$$\min_{u_{[0,1]}} \left(\int_0^1 u^2(\tau) d\tau + 10||x(1) - (0,0)||^2 \right)$$

subject to: $\dot{x} = Ax + bu$.

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

$$Q(t) = 0, \quad R(t) = 1$$

$$M = 10I$$

$$\min \int_0^! u^T R u d\tau + x^T(1) M x(1)$$

$$\dot{P} = PA + A^T P - PBR^{-1}B^T D, \quad P(1) = M$$

$$= P \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} P + P \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P$$

we get:

$$V^O(x,t) = x^T P(t) x$$

The optimizer would be the state feedback:

$$u^0(t) = -K(t)x(t) = -R^{-1}B^T P(t)x(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} P(t)x(t)$$