Notes in ECEN 5448

Zahary Vogel

November 16, 2015

1 Controllability

$$\dot{x} = Ax + Bu$$

System is controllable:

1. iff $e^{-A\tau}B$ is linearly independent
2. iff $W(t,0) = (\int_0^T e^{-A\tau}BB^Te^{-A^T\tau}d\tau)$ is full rank.
3. iff $\text{rank}(BABA^2B\dots A^{n-1}B) = n$

to drive the system from $x(0) \to 0 \ \forall x(0)$.

We say that a state $x \in \mathbb{R}^n$ is reachable if $\exists T, \exists u$ such that:

$$x(T) = \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau = \int_0^T e^{As} * B * (u(T-s) = g(s))dx$$

System is reachable if for any state in $\mathbb{R}^n$ we can drive the system from 0 to that point.
A system is reachable iff rows of: $e^{A\tau}B$ are linearly independent iff:

$$W_R(0,T) = -\int_0^T e^{A\tau}BB^Te^{A^T\tau}d\tau$$

is invertible, iff $\text{rank}(BABA^2B\dots A^{n-1}B) = n$
AKA a system is reachable iff it is controllable. All states that are reachable are controllable.

2 Controllability of LTV Systems

Supposedly, all this analysis except the rank condition will work for time varying systems.
Baby version of time varying systems:
Consider the system:

$$\dot{x} = B(t)u(t)$$

For a given initial condition, $x(t_0) \in \mathbb{R}^n$, what states $x(t_1)$ are reachable at t_1?
Suppose that you can make $x(t_1)$ with u

$$x(t_1) = \int_{t_0}^{t_1} B(t)u(t)dt + x(t_0)$$

$$\implies x(t_1) - x(t_0) \in \text{range}\left(\int_{t_0}^{t_1} B(t)dt\right) = \mathcal{R}(B(t))$$

The claim is that $\mathcal{R}(B(t)) = \text{range}\left(\int_{t_0}^{t_1} B(t)B^T(t)dt\right)$
How to show this?
Suppose that

$$y \in \text{range}\left(\int_{t_0}^{t_1} B(t)B^T(t)dt\right) \implies \exists \tilde{u} \in \mathbb{R}^m$$

$$y = \int_{t_0}^{t_1} B(t)B^T(t)dt\tilde{u}$$

Note that $y \in \mathcal{R}(B(t))$ as $u(t) = B^T(t)\tilde{u}$ would give $\int_{t_0}^{t_1} B(t)B^T(t)\tilde{u}dt = y$.

Suppose that $y \in \mathcal{R}(B(t))$ and $y \not\in \text{range}\left(\int_{t_0}^{t_1} B(t)B^T(t)dt\right)$.

Let $V \in \mathbb{R}^n$ be a linear subspace and $y \not\in V$. Then, if $y \neq 0$ there exists a $u \in \mathbb{R}^n$, s.t. $u^Tv = 0 \forall v \in V$ and $u^Ty \neq 0$.
Proof: Let B be an orthonormal basis for V and C be an orthonormal basis for $\mathbb{R}^n$(?). Furthermore, $y^Tc_i \neq 0$ for some $c_i \in \mathbb{C}$(?).
Note that $c_i \perp v, \forall v \in V$.
$V^\perp = \{u \Big| u^Tv = 0 \forall u \in V\}$.

$$\implies \exists c \in \mathbb{R}^n, c^Ty \neq 0$$

but $c^T \int_{t_0}^{t_1} B(t)B^T(t)dt \neq 0$.

$$0 = c^T \int_{t_0}^{t_1} B(t)B^T(t)dt$$

(note: B must be continous) iff

$$c^t \int_{t_0}^{t_1} B(t)B^T(t)dtc = 0 = \int_{t_0}^{t_1} ||B^T(t)c||^2 dt$$

so this holds iff

$$B^T(t)c = 0$$

But $y \in \text{range}(B(t))$ iff $\exists u(t)$

$$y = \int_{t_0}^{t_1} B(t)u(t)dt \implies 0 \neq c^Ty = \int_{t_0}^{t_1} c^TB(t)u(t)dt = 0$$

thus we have a contradiction.
Also note that the matrix forming the range of $B(t)$ is extremely dependent on t_0, t_1.

Now for the general system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$x(t_1) \in \mathbb{R}^n$ at time t_1 is reachable from $x(t_0)$ at t_0 if $\exists u(t)$ such that:

$$x(t_1) = \Phi(t_1, t_0)x(t_0) + \int_{t_0}^{t_1} \Phi(t_1, t))Bu(t)dt$$

Theorem: $x(t_1)$ at t_1 is reachable from $x(t_0)$ at t_0 iff:

$$\Phi(t_0, t_1)x(t_1) - x(t_0) \in \text{range}\left(\int_{t_0}^{t_1} \Phi(t_0, t)B(t)B^T(t)\Phi^T(t_0, t)dt\right)$$

Reachability Gramian:

$$W_R(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,t_1)B(t)B^T(t)\Phi^T(t_0,t)dt$$

3 Controllability Decomp

Fact: Suppose that $\text{rank}(BAB\dots A^{n-1}B) = R < n$. Then, $\exists$ invertible T such that if we let $z = Tx$, then:

$$\dot{z} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{pmatrix} z + \begin{pmatrix} \bar{B} \\ 0 \end{pmatrix} u$$

Proof: Let

$$T = \begin{pmatrix} v_1^H \\ v_2^H \\ \dots \\ v_r^H \\ w_1^H \\ \dots \\ w_{n-1}^H \end{pmatrix}$$

such that $(v_1,\dots,v_r)$ is an orthonormal basis for $\text{range}(B\dots A^{n-1}B)$ and $(w_1,\dots,w_{n-1})$ is an orthonormal basis for $R^\perp$.
This will result in the controllability decomp.
Showing this in homework is bonus.

4 Fourth check of Controllability

PBH-eigenvector test: Popov-Belevitch-Hautus
any P in controls is Popov, any K is kalman.
The pair (A,B) is controllable iff $\text{rank}(A-\lambda I; B) = n\ \forall \lambda \in \mathbb{R}$
Proof Sketch: Case that A is diagnolizable. Suppose that $wA = \Lambda w$ where:

$$w = \begin{pmatrix} w_1^H \\ \dots \\ w_n^H \end{pmatrix}$$

and

$$\Lambda = \text{diag}(\lambda_1,\lambda_2,\dots,\lambda_n$$

If we let $z = wx$ then:

$$\dot{z} = w\dot{x} = wAw^Hz + wBu$$

Note that $\text{rank}(A-\lambda I, B) < n$ iff there exists w such that $w^T(A-\lambda I) = 0$ and $w^TB = 0$ which would mean W^T is an eigenvector for A^T.
So a left eigenvector for A must be orthogonal to B.
$\implies$ if $\text{rank}() < n \implies \exists w_1,\dots,w_n \perp B$ for some $r \geq 1 \implies$

$$\dot{z} = \Lambda z + \begin{pmatrix} \bar{B} \\ 0 \end{pmatrix} u$$

and this system is clearly not controllable.

It can be shown (?) that linear transformation $z = Tx$ using an invertible T does not effect the controllability.

What are the various controllability tests useful for?
Suppose that we have $\dot{x} = Ax + Bu$. A is not stable.
Question: $\exists$ $m \times n$ matrix K; $u = Kx$ such that the system is stable.
i.e. $\dot{x} = Ax + BKx = (A + BK)x$ is stable?

This is called Stabilizability.
The difference between this and controllability is that controllability is finite time horizon.