Notes in ECEN 5448

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1 Optimal Control Problem

This class doesn't actually cover interesting things, but it provides the basis for interesting things. Here we cover a basic optimal control lecture to illustrate the power of this class.
general non-linear dynamics:

$$\dot{x} = f(x, u, t), \quad x(t_0) \in \mathbb{R}^n$$

This lecture is about one of the ways that u is often chosen.
One question is how to optimally take $x(t_0)$ to x_T at a given time.
minimize star below:

$$\int_{t_0}^{t_1} l(x(\tau), u(\tau), \tau)d\tau + m(x(t_1)) = V(u)$$

subject to:

$$\dot{x} = f(x, u, t)$$

with a given $x(t_0)$. Here, $l : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ and $m : \mathbb{R}^n \to \mathbb{R}$ m is cost.

How to solve this? Through dynamic programming:
Let $V^O(x_0, t_0)$ be $\min_{u(t_0,t_1)} V(u)$ subject to $x(t_0) = x_0$ in star.
v^O is also called the value function of star.

$$V^O(x_0, t_0) = \min_{u[t_0,t_1]} V(u) = \min_{u(t_0,t_1)} \int_{t_0}^{t_1} l(x(\tau), u(\tau), \tau)d\tau + m(x(t_1))$$

break up the integral between t_0 and t_1 with an intermediate value t_m where $t_0 \leq t_m \leq t_1$:

$$= \min_{u(t_0,t_1)} \int_{t_0}^{t_m} l(x(\tau), u(\tau), \tau)d\tau + \int_{t_m}^{t_1} l(x(\tau), u(\tau), \tau)d\tau + m(x(t_1))$$

$$= \min_{u[t_0,t_m]} \left(\int_{t_0}^{t_m} l(x(\tau), u(\tau), \tau)d\tau + min_{t_m,t_1} \int_{t_m}^{t_1} l(x(\tau), u(\tau), \tau)d\tau + m(x(t_1)) \right)$$

the second integral is $V^O(x(t_m), t_m)$.

$$= \min_{u[t_0,t_m]} \int_{t_0}^{t_m} l(x(\tau), u(\tau), \tau)d\tau + V^O(x(t_m), t_m)$$

Let $t = t_0$, let $x = x_0$ and let $t_m = t + \delta t$. Then:

$$V^O(x, t) \approx \min_{u \in \mathbb{R}^m} (l(x, u, t) * \delta t + V^O(x, t) + \frac{d}{dt} V(x, t)\delta t + \frac{\partial V(x, t)}{\partial x} \delta x)$$

so the last two terms with derivatives are $\approx V^O(x(t, \delta t), \delta t)$ and $\delta x = x(t + \delta t) - x(t)$.

$$\implies \frac{d}{dt} V^O(x, t) \approx \min_{u \in \mathbb{R}^n} (l(x, u, t) + \frac{\partial V^O(x, t)}{\partial x} f(x, u, t))$$

dynamic programming equation.
The equation $-\frac{dV^O(x,t)}{dt} = \min_{u\in\mathbb{R}^m}(l(x,u,t) + \nabla_x V^O(x,t)f(x,u,t))$ note that $v^O(x,t_1) = m(x)$ with the terminal condition $V^O(x,t) = m(x)$ is called Hamilton-Jacobi-Bellman equation.
Theorem: Suppose that $V^O(x,t)$ is a function that has continous partial derivatives and also, $u^O(t)$ is such that:

$$\min_{u\in\mathbb{R}^n}(l(x,u,t) + \nabla_x V^O(x,t)f(x,u,t)) = l(x,u^O,t) + \nabla_x v^O(x,t)f(x,u^O,t).$$

Then $u^O(t)$ is the optimal control and V^O is the value function iff V^O satisfies HJB equation.

Define the hamiltonian:

$$H(x,u,p,t) = l(x,u,t) + p^T * f(x,u,t)$$

HJB eqution can be written:

$$-frac{-}ddtV^O(x,t) = \min_{u\in\mathbb{R}^m} H(x,u,\nabla_x V^O,t)$$

with $v^O(x,t_1) = m(x)$

so here, u is a state feedback.

2 Example

Suppose we have an integrator with everything being 1.

$$\dot{x} = u \quad x_0\exists$$

$$\min \int_0^{t_1}(u^2 + x^4)d\tau$$

$$l(x,u,t) = u^2 + x^4$$

$$m(x) = 0$$

$$-\frac{d}{dt}V^O(x,t) = \min_u(u^2 + x^4 + \frac{\partial v^O}{\partial x}u)$$

$$u^O(x,t) = -\frac{1}{2}\frac{dV^O}{dx}(x,u)$$

$$\dot{x}^O = +u^O = -\frac{1}{2}\frac{dV^O}{dx}(x,t)$$

closed loop dynamics above, now find $V^O(t,x)$ can be obtained by solving:

$$-\frac{d}{dt}V^O(x,t) = x^4 - \frac{1}{4}(\frac{dV^O}{dx}(x,t))^2$$

with $V^O(x,t) = 0$

3 LQR

An important subset of optimal control problems is called Linear Quadratic Regulator problems. Here, we talk about the Linear time varying case.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) \in \mathbb{R}^n$$
$$\min \int_{t_0}^{t_1} (x^T(\tau)Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau))d\tau + x^T(t_1)Mx(t_1)$$

with p.d. TV $Q(t), R(t)$ and p.d. M. This is the standard setting of these problems.

Let us use the machinery discussed before for this problem. What is $u(t)$, it is the minimizer of the HJB. (below is still functions of t)

$$u^O(t) = \arg\min_u (x^T Qx + u^T Ru + \nabla_x V^O(x,t)(Ax + Bu))$$

from first order condition

$$\implies 2Ru^O + B^T \nabla_x V^O(x,t) = 0$$
$$\implies u^O = -\frac{1}{2}R^{-1}B^T \nabla_x V^O(x,t)$$

HJB:

$$-\frac{d}{dt}V^O(x,t) = x^T Qx + \frac{(\nabla_x V^O)^T BR^{-1}RR^{-1}B^T(\nabla_x V^O)}{4} + \nabla_x V^O(x,t)(Ax - \frac{BR^{-1}}{2}B^T \nabla_x V^O)$$
$$V(x,t_1) = x^T Mx$$

This suggest that the value function itself, $V^O(x,t) = x(t)^T P(t)x(t)$ should be a quadratic function.
Let's examine this to see if it works.

$$-x^T \dot{P}x = x^T Qx + x^T P(t)BR^{-1}B^T P(t)x + 2x^T PAx - 2x^T P(t)BR^{-1}B^T Px$$
$$= x^T Qx + x^T(PA + A^T P)x - x^T PBR^{-1}BPx$$

This should hold for all x in the space therefore:

$$\dot{P} = Q + PA + A^T P - PBR^{-1}B^T P$$
$$P(t_1) = M$$

This ↑ ode is called the Riccati differential equation.
As long as P, Q, A, and R are nice enough, teh solution is definite and solves the LQR problem.
Theorem: Suppose that $A(t), B(t), Q(t), R(t)$ are piecewise continuous matrix functions of $t \in [t_0, t_1]$. Suppose that $R(t)$ is positive definite, then the solution to the Ricatte Differential Equation (RDE) exists, and is unique. Furthermore, $P(t)$ is positive definite and teh optimal control to the LQR problem is:

$$u(t) = -K(t)x(t)$$

with $K(t) = R^{-1}B^T P$.

P is always time-varying even when the other stuff isn't. $x^T Px$ is a lyapunov function.
Pontryagin max principle.