Grad Project Equations in MCEN 5115: Rotating Inverted Pendulum

Zachary Vogel Maurice Woods III

Derek Reamon Mechatronics

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Normal Inverted Pendulum

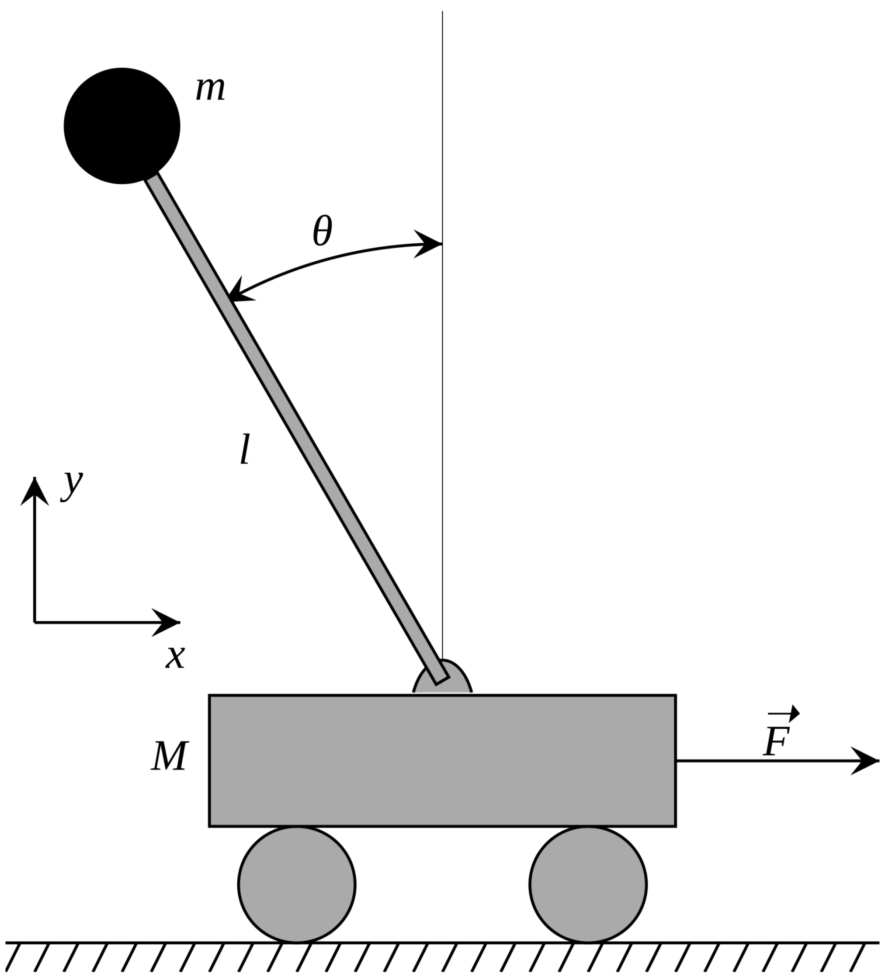

Figure 1: sup sup

Here we have the normal inverted pendulum equations for the image above:

$$(M+m)\ddot{x} - ml\ddot{\theta} + ml(\dot{\theta})^2\sin(\theta) = F$$
$$l\ddot{\theta} - g\sin(\theta) = \ddot{x}\cos(\theta)$$

The x position for us is actually ω because we are spinning. Also, the input force isn't F. What we have is a torque $\tau = ||r||||F||\sin(\theta)$. Given that the θ here is 90 degrees or 270 degrees, force can be equated to $F = \frac{\tau}{r}$.
Torque is some function of the motor current and the motor properties that we can put in later. We also note that $x = \omega * r$ and thus $\ddot{x} = \ddot{\omega} r$.
Thus our equations become:

$$\frac{\tau}{r} = ml(\dot{\theta})^2 \sin(\theta) - ml\ddot{\theta} + (M+m)\ddot{\omega} r$$
$$\ddot{\omega} r \cos(\theta) = l\ddot{\theta} - g\sin(\theta)$$

rewriting with $\ddot{\omega} r = \frac{l\ddot{\theta} - g\sin(\theta)}{\cos(\theta)}$

$$\frac{\tau}{r} = ml(\dot{\theta})^2 \sin(\theta) - ml\ddot{\theta} + \frac{(M+m)(l\ddot{\theta} - g\sin(\theta))}{\cos(\theta)}$$
$$= ml(\dot{\theta})^2 \sin(\theta) + \ddot{\theta}\frac{((M+m)l - ml\cos(\theta))}{\cos(\theta)} - \frac{(M+m)g\sin(\theta)}{\cos(\theta)}$$
$$\ddot{\theta} = \frac{\cos(\theta)}{(M+m)l - ml\cos(\theta)}\left(\frac{\tau}{r} - ml(\dot{\theta})^2\sin(\theta) - \frac{(M+m)g\sin(\theta)}{\cos(\theta)}\right)$$

hmmm, those equations suck to take partial derivatives.

$$\ddot{\theta} = \dot{\theta}^2 \sin(\theta) - \frac{\tau}{rml} + \frac{(M+m)\ddot{\omega} r}{ml}$$
$$\ddot{\omega} = \frac{l\ddot{\theta} - g\sin(\theta)}{r\cos(\theta)}$$

Now let's pick some variables. $x_1 = \dot{\theta}$, $x_2 = \theta$, $x_3 = \dot{\omega}$ and $x_4 = \omega$. We need to linearize around $\omega = 0$ and $\dot{\omega} = 0$.
State matrix will look like:

$$\dot{x} = Ax = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ 1 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Here we have $a_0 = 0$, $a_1 = \frac{(M+m)g}{(M+m)l - ml}$, $a_2 = 0$, and $a_3 = 0$. Now we write out the equations in the other form.

$$l\ddot{\theta} = \frac{-\tau}{rm} + l(\dot{\theta})^2 \sin(\theta) + \frac{(M+m)\ddot{\omega} r}{m}$$

this gives:

$$\ddot{\omega} r\cos(\theta) = -\frac{\tau}{rm} + l(\dot{\theta})^2\sin(\theta) + \frac{(M+m)\ddot{\omega} r}{m} - g\sin(\theta)$$

Simplifying we get:

$$\ddot{\omega}\left(r\cos(\theta) - \frac{(M+m)r}{m}\right) = \frac{-\tau}{rm} + l(\dot{\theta})^2\sin(\theta) - g\sin(\theta)$$

etc:

$$\ddot{\omega} = \frac{-\tau}{r^2(m\cos(\theta) - (M+m))} + \frac{ml(\dot{\theta})^2\sin(\theta)}{r(\cos(\theta) - (M+m))} - \frac{mg\sin(\theta)}{r(\cos(\theta) - (M+m))}$$

small angle

This is not wrong, but we will be doing the small angle approximation then the jacobian. I don't need to do jacobian I need to do small angle approximation. For small θ $\cos(\theta) \approx 1$ and $\sin(\theta) \approx \theta$ Thus we get:

$$\ddot{\theta} = \frac{1}{Ml}\left(\frac{\tau}{r} - ml(\dot{\theta})^2\theta - (M+m)g\theta\right)$$

and:

$$\ddot{\omega} = \frac{-\tau}{-r^2M} + \frac{ml(\dot{\theta})^2\theta}{r(1-(M+m))} - \frac{mg\theta}{r(1-(M+m))}$$

Our state equations are thus:

$$\begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \\ \ddot{\omega} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & -(M+m)g & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \dfrac{-mg}{r(1-(M+m))} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \theta \\ \dot{\omega} \\ \omega \end{bmatrix} + \begin{bmatrix} \dfrac{1}{Mlr} \\ 0 \\ \dfrac{1}{r^2M} \\ 0 \end{bmatrix} \tau$$

The output is then:

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \theta \\ \dot{\omega} \\ \omega \end{bmatrix}$$

Note here that a_0 and b_1 are zero. If this doesn't turn out to be good enough we can take the third part of the taylor series for the $\dot{\theta}^2\theta$ term which will be non-zero unlike the first 2.