Colored Cube

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Let A be the event that the face pointing down is painted. Let B be the event that the five visible faces are not painted. We need to find $P(A \mid B)$ - the probability that the face pointing down is painted given that the five faces that are visible are not painted.

According to **Bayes' theorem**, we have

Definition 0.1 – Bayes' Theorem

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)},$$

where $P(A)$ is the prior probability that the face pointing down is painted, $P(B \mid A)$ is the likelihood, i.e. the probability that the five faces that are visible are not painted given that the face pointing down is painted, and $P(B)$ is simply the probability that the five visible faces are not colored.

First, we compute $P(A)$. We have

$$\begin{aligned} P(A) &= \frac{\text{\# of painted faces}}{\text{\# of total faces}} \\ &= \frac{1 \times 6 + 2 \times 12 + 3 \times 8}{27 \times 6} \\ &= \frac{48}{162} \\ &= \frac{1}{3}. \end{aligned}$$

We now compute $P(B \mid A)$. For a cube with

1. one painted face, there is a 1 in 6 chance that the painted face is pointind down;
2. two painted faces, there is a 2 in 6 chance;
3. three painted faces, there is a 3 in 6 chance.

We need to consider that when a painted face points down, that the other five faces are visible; this can only occur in the case of one painted face. Thus,

$$\begin{aligned} P(B \mid A) &= \frac{(1 \times 6) \times 1 + (2 \times 12) \times 0 + (3 \times 8) \times 0}{1 \times 6 + 2 \times 12 + 3 \times 8} \\ &= \frac{6}{54} \qquad = \frac{1}{9}. \end{aligned}$$

Next, we compute $P(B)$, the probability that the visible faces are not painted. We consider four types of cubes:

1. 1 cube with no painted faces: when this cube is rolled, the probability that visible faces are not painted is 1.
2. cube with 1 painted face: there is a $\frac{1}{6}$ chance that one of these cubes has its face pointing down.
3. the remaining two types of cubes have probability 0 because at least one of the non-downward pointing faces would be visible.

$$P(B) = \frac{1}{27} \cdot 1 + \frac{6}{27} \cdot \frac{1}{6} + \frac{12}{27} \cdot 0 + \frac{8}{27} \cdot 0 = \frac{2}{27}.$$

Therefore, our final result is

$$\begin{aligned} P(A \mid B) &= \frac{P(B \mid A) \cdot P(A)}{P(B)} \\ &= \frac{\frac{1}{9} \cdot \frac{1}{3}}{\frac{2}{27}} \\ &= \frac{1}{2}. \end{aligned}$$