Solutions 2

1. Alice attends a small college in which each class meets only once a week. She is deciding between 30 non-overlapping classes. There are 6 classes to choose from for each day of the week, Monday through Friday. Trusting in the benevolence of randomness, Alice decides to register for 7 randomly selected classes out of the 30, with all choices equally likely. What is the probability that she will have classes every day, Monday through Friday? (This problem can be done either directly using the naive definition of probability, or using inclusion-exclusion.)

   Solution:

   Direct Method: There are two general ways that Alice can have class every day: either she has 2 days with 2 classes and 3 days with 1 class, or she has 1 day with 3 classes, and has 1 class on each of the other 4 days. The number of possibilities for the former is $C_5^2 {C_6^2}^2 6^3$ (choose the 2 days when she has 2 classes, and then select 2 classes on those days and 1 class for the other days). The number of possibilities for the latter is $C_5^1 C_6^3 6^4$. So the probability is

   $$\frac{C_5^2 {C_6^2}^2 6^3 + C_5^1 C_6^3 6^4}{C_{30}^7} = \frac{114}{377} \approx 0.302$$

   Inclusion-Exclusion Method: we will use inclusion-exclusion to find the probability of the complement, which is the event that she has at least one day with no classes. Let $B_i = A_i^c$.

   Then $P(B_1 \cup B_2 \cup \dots \cup B_5) = \sum_i P(B_i) - \sum_{i<j} P(B_i \cap B_j) + \sum_{i<j<k} P(B_i \cap B_j \cap B_k)$

   (terms with the intersection of 4 or more B_i's are not needed since Alice must have classes on at least 2 days). We have $P(B_1)=\frac{C_{24}^7}{C_{30}^7}$ $\quad P(B_1 \cap B_2) = \frac{C_{18}^7}{C_{30}^7}$ $\quad P(B_1 \cap B_2 \cap B_k)=\frac{C_{12}^7}{C_{30}^7}$

   and similarly for the other intersections. So

   $P(B_1 \cup B_2 \cup \dots \cup B_5)=5\frac{C_{24}^7}{C_{30}^7} - C_5^2 \frac{C_{18}^7}{C_{30}^7} + C_5^3 \frac{C_{12}^7}{C_{30}^7} = \frac{263}{377}$

   Therefore, $P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)=\frac{114}{377} \approx 0.302$

2. Is it possible that an event is independent of itself? If so, when?

   Solution: Let A be an event. If A is independent of itself, then P(A) = P(A∩A) = $P(A)^2$, so P(A) is 0 or 1. So this is only possible in the extreme cases that the event has probability 0 or 1.

3. Give an example of 3 events A, B, C which are pairwise independent but not independent. Hint: find an example where whether C occurs is completely determined if we know whether A occurred and whether B occurred, but completely undetermined if we know only one of these things.

   Solution: Consider two fair, independent coin tosses, and let A be the event that the first toss is Heads, B be the event that the second toss is Heads, and C be the event that the two tosses have the same result. Then A, B, C are dependent since P(A∩B∩C) = P(A∩B) = P(A)P(B)= $\frac{1}{4} \neq \frac{1}{8} =$

P(A)P(B)P(C), but they are pairwise independent: A and B are independent by definition; A and C are independent since P(A ∩ C) = P(A ∩ B)= $\frac{1}{4}$ = P(A)P(C), and similarly B and C are independent.

4. A bag contains one marble which is either green or blue, with equal probabilities. A green marble is put in the bag (so there are 2 marbles now), and then a random marble is taken out. The marble taken out is green. What is the probability that the remaining marble is also green?

   Solution: Let A be the event that the initial marble is green, B be the event that the removed marble is green, and C be the event that the remaining marble is green. We need to find P(C|B). There are several ways to find this; one natural way is to condition on whether the initial marble is green: P(C|B) = P(C|B,A)P(A|B) + P(C|B,A^c)P(A^c |B)=1P(A|B)+0P(A^c |B).

   P(A|B) = $\frac{\text{P(B|A)P(A)}}{\text{P(B)}} = \frac{1/2}{\text{P(B|A)P(A) + P(B|}A^c\text{P(}A^c\text{)}} = \frac{1/2}{1/2+1/4} = \frac{2}{3}$

   So P(C|B)=2/3.

5. Let G be the event that a certain individual is guilty of a certain robbery. In gathering evidence, it is learned that an event E_1 occurred, and a little later it is also learned that another event E_2 also occurred.

   (a) Is it possible that individually, these pieces of evidence increase the chance of guilt (so P(G|E_1) > P(G) and P(G|E_2) > P(G)), but together they decrease the chance of guilt (so P(G|E_1, E_2) < P(G))?

   Solution: Yes, this is possible. In fact, it is possible to have two events which separately provide evidence in favor of G, yet which together preclude G. For example, suppose that the crime was committed between 1 pm and 3 pm on a certain day. Let E_1 be the event that the suspect was at a nearby coffeeshop from 1 pm to 2 pm that day, and let E_2 be the event that the suspect was at the nearby coffeeshop from 2 pm to 3 pm that day. Then P(G|E_1) > P(G), P(G|E_2) > P(G), yet P(G|E_1 ∩ E_2) < P(G).

   (b) Show that the probability of guilt given the evidence is the same regardless of whether we update our probabilities all at once, or in two steps (after getting the first piece of evidence, and again after getting the second piece of evidence). That is, we can either update all at once (computing P(G|E_1, E_2) in one step), or we can first update based on E_1, so that our new probability function is P_{new}(A) = P(A|E_1), and then update based on E_2 by computing P_{new} (G|E_2).

   Solution: This follows from the definition of conditional probability:
   P_{new} (G|E_2)=$\frac{P_{new}\text{ (G|}E_2\text{)}}{P_{new}\text{ (}E_2\text{)}} = \frac{P\text{ (G,}E_2|E_1)}{P\text{ (}E_2|E_1)} = \frac{P\text{ (G,}E_2,E_1)/\text{P(}E_1)}{P\text{ (}E_2,E_1)/\text{P(}E_1)} = P(G|E_1, E_2)$

6. You are going to play 2 games of chess with an opponent whom you have never played against before (for the sake of this problem). Your opponent is equally likely to be a beginner, intermediate, or a master. Depending on which, your chances of winning an individual game are 90%, 50%, or 30%, respectively.

   (a) What is your probability of winning the first game?

Solution: By the law of total probability, P(W_1) = (0.9+0.5+0.3)/3 = 17/30.

(b) Congratulations: you won the first game! Given this information, what is the probability that you will also win the second game (assume that, given the skill level of your opponent, the outcomes of the games are independent)?

Solution: Let W_i be the event of winning the i-th game. We have P(W2|W1) = P(W2, W1)/P(W1). The denominator is known from (a), while the numerator can be found by conditioning on the skill level of the opponent: P(W1, W2) = $\frac{1}{3}$ P(W1, W2|beginner)+1 3 P(W1, W2|intermediate)+1 3 P(W1, W2|expert). Since W1 and W2 are conditionally independent given the skill level of the opponent, this becomes P(W1, W2) = (0.9^2 + 5^2 + 0.3^2)/3 = 23/60. So P(W2|W1) = $\frac{23/60}{17/30}$= 23/34.

7. A card player is dealt a 13 card hand from a well-shuffled, standard deck of cards. What is the probability that the hand is void in at least one suit ("void in a suit" means having no cards of that suit)?

   Solution: Let S, H, D, C be the events of being void in Spades, Hearts, Diamonds, Clubs, respectively. We want to find P(S∪D∪H∪C). By inclusion-exclusion and symmetry, P(S ∪ D ∪ H ∪ C)=4P(S) - 6P(S ∩ H)+4P(S ∩ H ∩ D) - P(S ∩ H ∩ D ∩ C). The probability of being void in a specific suit is $\frac{C_{39}^{13}}{C_{52}^{13}}$. The probability of being void in 2 specific suits is $\frac{C_{26}^{13}}{C_{52}^{13}}$. The probability of being void in 3 specific suits is $\frac{1}{C_{52}^{13}}$. And the last term is 0 since it's impossible to be void in everything. So the probability is $4\frac{C_{39}^{13}}{C_{52}^{13}} - 6\frac{C_{26}^{13}}{C_{52}^{13}} + 4\frac{1}{C_{52}^{13}} \approx 0.051$.

8. A family has two children. Assume that birth month is independent of gender, with boys and girls equally likely and all months equally likely, and assume that the elder child's characteristics are independent of the younger child's characteristics).

   (a) Find the probability that both are girls, given that the elder child is a girl who was born in March.

   Solution: Let G_j be the event that the jth born child is a girl and M_j be the event that the jth born child was born in March, for j ∈ {1, 2}. Then P($G_1 \cap G_2 | G_1 \cap M_j$) = P($G_2 | G_1 \cap M_1$), since if we know that G_1 occurs, then $G_1 \cap G_2$ occurring is the same thing as G2 occurring. By independence of the characteristics of the children, P($G_2 | G_1 \cap M_1$) = P(G_2)=1/2.

   (b) Find the probability that both are girls, given that at least one is a girl who was born in March.

   Solution: P(both girls|at least one March-born girl) = $\frac{\text{P(both girls,at least one born in March)}}{\text{P(at least one March−born girl)}}$ = $\frac{(1/4)(1-(11/12)^2)}{1-(23/24)^2}$ = 23/ 47 ≈ 0.489. In contrast, P(both girls|at least one girl) = 1/3. So the seemingly irrelevant "born in March" information actually matters! By symmetry, the answer would stay the same if we replaced "born in March" by, say, "born in July"; so it's not the awesomeness of March that matters, but rather the fact that the information brings "at least one" closer to being "a specific one." The more detailed the information being conditioned on, the closer this becomes to specifying one of the children, and thus the closer the answer gets to 1/2.

9. A fair coin is flipped 3 times. The toss results are recorded on separate slips of paper (writing "H" if Heads and "T" if Tails), and the 3 slips of paper are thrown into a hat.

   (a) Find the probability that all 3 tosses landed Heads, given that at least 2 were Heads.

Solution: Let's define the events:A: All 3 tosses landed Heads, B: At least 2 tosses landed Heads. We want to find P(A|B), the probability that A occurs given that B occurs. The probability of getting at least 2 Heads can be calculated as follows: P(B) = P(2 Heads) + P(3 Heads)

P(2 Heads) = $\frac{C_3^2}{2^3}$= $\frac{3}{8}$ P(3 Heads) =$\frac{1}{2^3}$ = $\frac{1}{8}$ P(B) = $\frac{3}{8}$ + $\frac{1}{8}$ = $\frac{1}{2}$.
P(A ∩ B) represents the probability that both A and B occur, which means getting 3 Heads and at least 2 Heads simultaneously. Since all 3 tosses being Heads is a subset of at least 2 Heads, P(A ∩ B) is simply the probability of getting 3 Heads: P(A ∩ B) = P(3 Heads) = 1/8
P(A|B) = $\frac{1/8}{1/2} = \frac{1}{4}$. So, the probability that all 3 tosses landed Heads, given that at least 2 were Heads, is 1/4.

(b) Two of the slips of paper are randomly drawn from the hat, and both show the letter H. Given this information, what is the probability that all 3 tosses landed Heads?

Solution: Let A_i be the event that i tosses landed Heads, and B be the event that two slips paper drawn both show H. From low of total probability follows that P(A_3|B)=$\frac{P(B|A_3)P(A_3)}{\sum_{i=0}^{3} P(B|A_i)P(A_i)}$.

10. The Jack of Spades (with cider), Jack of Hearts (with tarts), Queen of Spades (with a wink), and Queen of Hearts (without tarts) are taken from a deck of cards. These four cards are shuffled, and then two are dealt.
    Note: Literary references to cider, tarts, and winks do not need to be considered when solving this problem.

    (a) Find the probability that both of these two cards are queens, given that the first card dealt is a queen.

    Solution: Let A be the event that the first card dealt is a Queen, and B be the event that the second card dealt is a Queen.P(A) = 2/4 =1/2 P(B) = 1/3 P(A∩B) = Probability of drawing both cards as Queens = 1/4 P(B|A)= P(A∩B) / P(A)=1/2

    (b) Find the probability that both are queens, given that at least one is a queen.

    Solution: To find the probability that both cards are Queens given that at least one is a Queen, we need to calculate P(A∩B|A∪B). P(A∩B) =1/4 P(A∪B) = Probability of drawing at least one Queen among two cards . To calculate P(A∪B), we'll consider two cases: Drawing a Queen and then a non-Queen: P(A) * P(B^c |A) = (1/2) * (2/3) = 1/3 or drawing a non-Queen and then a Queen: P(A^c) * P(B|A^c) = (1/2) * (1/3) = 1/6. So P(A∪B) =5/12. P(A∩B|A∪B)=$\frac{1/4}{5/12}$=3/5.

(c) Find the probability that both are queens, given that one is the Queen of Hearts

Solution: Let C be the event that one of the cards is the queen of Hearts. We need to calculate P(A∩B|C). P(A ∩ B ∩ C) = Probability of drawing both cards as Queens, given that one is the Queen of Hearts. Since we know that one card is the Queen of Hearts, there is only one Queen left to draw (the Queen of Spades), and we have a total of 3 cards left in the deck. P(A ∩ B ∩ C) = 1/3 * 1/2 = 1/6, P(C) = 1/2 . P(A∩B|C)=$\frac{1/6}{1/2}$=1/3.