Homework 3

1. For each statement below, either show that it is true or give a counterexample. Throughout, X, Y, Z are discrete random variables.
   (a) If X and Y are independent and Y and Z are independent, then X and Z are independent.
   (b) If X and Y are independent, then they are conditionally independent given Z.
   (c) If X and Y are conditionally independent given Z, then they are independent.
   (d) If X and Y have the same distribution given Z, i.e., for all a and z, we have $P(X = a|Z = z) = P(Y = a|Z = z)$, then X and Y have the same distribution.

2. Consider the following conversation from an episode of The Simpsons:

   Lisa: Dad, I think he's an ivory dealer! His boots are ivory, his hat is ivory, and I'm pretty sure that check is ivory.
   Homer: Lisa, a guy who's got lots of ivory is less likely to hurt Stampy than a guy whose ivory supplies are low.

   Here Homer and Lisa are debating the question of whether or not the man (named Blackheart) is likely to hurt Stampy the Elephant if they sell Stampy to him. They clearly disagree about how to use their observations about Blackheart to learn about the probability (conditional on the evidence) that Blackheart will hurt Stampy.

   (a) Define clear notation for the various events of interest here.

   (b) Express Lisa's and Homer's arguments (Lisa's is partly implicit) as conditional probability statements in terms of your notation from (a).

   (c) Assume it is true that someone who has a lot of a commodity will have less desire to acquire more of the commodity. Explain what is wrong with Homer's reasoning that the evidence about Blackheart makes it less likely that he will harm Stampy.

3. A gambler repeatedly plays a game where in each round, he wins a dollar with probability 1/3 and loses a dollar with probability 2/3. His strategy is "quit when he is ahead by \$2," though some suspect he is a gambling addict anyway. Suppose that he starts with a million dollars. Show that the probability that he'll ever be ahead by \$2 is less than 1/4.

4. A sequence of n independent experiments is performed. Each experiment is a success with probability p and a failure with probability $q = 1 - p$. Show that conditional on the number of successes, all possibilities for the list of outcomes of the experiment are equally likely (of course, we only consider lists of outcomes where the number of successes is consistent with the information being conditioned on).

5. (a) Consider the following 7-door version of the Monty Hall problem. There are 7 doors, behind one of which there is a car (which you want), and behind the rest of which there are goats

(which you don't want). Initially, all possibilities are equally likely for where the car is. You choose a door. Monty Hall then opens 3 goat doors, and offers you the option of switching to any of the remaining 3 doors.

Assume that Monty Hall knows which door has the car, will always open 3 goat doors and offer the option of switching, and that Monty chooses with equal probabilities from all his choices of which goat doors to open. Should you switch? What is your probability of success if you switch to one of the remaining 3 doors?

(b) Generalize the above to a Monty Hall problem where there are n 3 doors, of which Monty opens m goat doors, with $1 \leq m \leq n - 2$.

6. Calvin and Hobbes play a match consisting of a series of games, where Calvin has probability p of winning each game (independently). They play with a "win by two" rule: the first player to win two games more than his opponent wins the match. Find the probability that Calvin wins the match (in terms of p), in two different ways:

   (a) by conditioning, using the law of total probability.

   (b) by interpreting the problem as a gambler's ruin problem.