Solutions 4

1. (a) In the World Series of baseball, two teams (call them A and B) play a sequence of games against each other, and the first team to win four games wins the series. Let p be the probability that A wins an individual game, and assume that the games are independent. What is the probability that team A wins the series?

   Solution: Let q = 1 - p. First let us do a direct calculation:
   P(A wins) = P(A winning in 4 games) + P(A winning in 5 games) +P(A wins in 6 games) + P(A winning in 7 games) = p^4 + $C_4^3 p^4 q$ + $C_5^3 p^4 q^2$+ $C_6^3 p^4 q^3$.
   To understand how these probabilities are calculated, note for example that P(A wins in 5) = P(A wins 3 out of first 4) · P(A wins 5th game|A wins 3 out of first 4) = $C_4^3 p^3 q$p
   (This value can also be found from the PMF of a distribution known as the Negative Binomial, which we will see later in the course.)
   An neater solution is to use the fact (explained in (b)) that we can assume that the teams play all 7 games no matter what. Then let X be the number of wins for team A, so that
   X ∼ Binomial(7, p). The probability that team A wins the series is

   P(X 4) = P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7)

   The PMF of the Bin(n, p) distribution is P(X = k) =$C_n^k p^k (1-p)^{n-k}$, and therefore
   P(X 4) = $C_7^4 p^4 q^3$ + $C_7^5 p^5 q^2$ + $C_7^6 p^6 q$+ p^7 ,
   which looks different from the above but is actually identical as a function of p (as can be verified by simplifying both expressions as polynomials in p).

   (b) Give a clear intuitive explanation of whether the answer to (a) depends on whether the teams always play 7 games (and whoever wins the majority wins the series), or the teams stop playing more games as soon as one team has won 4 games (as is actually the case in practice: once the match is decided, the two teams do not keep playing more games).

   Solution: The answer to (a) does not depend on whether the teams play all seven games no matter what. Imagine telling the players to continue playing the games even after the match has been decided, just for fun: the outcome of the match won't be affected by this, and this also means that the probability that A wins the match won't be affected by assuming that the teams always play 7 games!

2. A sequence of n independent experiments is performed. Each experiment is a success with probability p and a failure with probability q = 1 - p. Show that conditional on the number of successes, all possibilities for the list of outcomes of the experiment are equally likely (of course, we only consider lists of outcomes where the number of successes is consistent with the information being conditioned on).

   Solution: Let X_j be 1 if the jth experiment is a success and 0 otherwise, and let X = $X_1 + \cdots + X_n$ be the total number of successes. Then for any k and any $a_1,\ldots, a_n$ ∈ {0, 1} with $a_1 + \cdots + a_n$ = k,
   P($X_1 = a_1,\ldots, X_n = a_n$|X = k) = $\frac{P(X_1 = a_1,\ldots,X_n = a_n, X = k)}{\mathrm{P(X = k)}} = \frac{P(X_1 = a_1,\ldots,X_n = a_n)}{\mathrm{P(X = k)}} = \frac{p^k q^{n-k}}{C_n^k p^k q^{n-k}} = \frac{1}{C_n^k}$.
   This does not depend on $a_1 , \cdots , a_n$. Thus, for n independent Bernoulli trials, given that there are exactly k successes, the C_n^k possible sequences consisting of k successes and n - k failures are equally likely. Interestingly, the conditional probability above also does not depend on p .

3. (a) Consider the following 7-door version of the Monty Hall problem. There are 7 doors, behind one of which there is a car (which you want), and behind the rest of which there are goats (which you don't want). Initially, all possibilities are equally likely for where the car is. You choose a door. Monty Hall then opens 3 goat doors, and offers you the option of switching to any of the remaining 3 doors.
Assume that Monty Hall knows which door has the car, will always open 3 goat doors and offer the option of switching, and that Monty chooses with equal probabilities from all his choices of which goat doors to open. Should you switch? What is your probability of success if you switch to one of the remaining 3 doors?

Solution: Assume the doors are labeled such that you choose Door 1 (to simplify notation), and suppose first that you follow the "stick to your original choice" strategy. Let S be the event of success in getting the car, and let C_j be the event that the car is behind Door j. Conditioning on which door has the car, we have P(S) = P(S|C_1)P(C_1) + ⋯ + P(S|C_7)P(C_7) = P(C_1) = 1/7 .
Let M_{ijk} be the event that Monty opens Doors i, j, k. Then P(S) =$\sum_{i,j,k} P(S|M_{ijk})P(M_{ijk})$
By symmetry, this gives P(S|M_{ijk}) = P(S) = 1/7
for all i, j, k with $2 \leq i < j < k \leq 7$. Thus, the conditional probability that the car is behind 1 of the remaining 3 doors is 6/7, which gives 2/7 for each. So you should switch, thus making your probability of success 2/7 rather than 1/7.

(b) Generalize the above to a Monty Hall problem where there are n ≥ 3 doors, of which Monty opens m goat doors, with 1 ≤ m ≤ n - 2.

Solution: By the same reasoning, the probability of success for "stick to your original choice" is 1/n, both unconditionally and conditionally. Each of the n – m - 1 remaining doors has conditional probability $\frac{n-1}{(\mathrm{n-m-1})\mathrm{n}}$ (nm1)n of having the car. This value is greater than 1/n, so you should switch, thus obtaining probability $\frac{n-1}{(\mathrm{n-m-1})\mathrm{n}}$ of success (both conditionally and unconditionally).

4. Players A and B take turns in answering trivia questions, starting with player A answering the first question. Each time A answers a question, she has probability p_1 of getting it right. Each time B plays, he has probability p_2 of getting it right.

(a) If A answers m questions, what is the PMF of the number of questions she gets right?

Solution: The r.v. is Bin(m, p_1), so the PMF is $C_m^k p_1^k (1-p_1)^{m-k}$ for k ∈ {0, 1,...,m}.

(b) If A answers m times and B answers n times, what is the PMF of the total number of questions they get right (you can leave your answer as a sum)? Describe exactly when/whether this is a Binomial distribution.

Solution: Let T be the total number of questions they get right. To get a total of k questions right, it must be that A got 0 and B got k, or A got 1 and B got k - 1, etc. These are disjoint events so the PMF is P(T = k) = $\sum_{j=0}^{k} C_m^j\, p_1^j (1-p_1)^{m-j} C_n^{k-j} p_2^{k-j} (1-p_2)^{n-(k-j)}$ for k 2 {0, 1,...,m + n}, with the usual convention that C_n^k is 0 for k>n.

This is the Bin(m + n, p) distribution if p_1 = p_2 = p, as shown in class (using the story for the Binomial, or using Vandermonde's identity). For $p_1 \neq p_2$, it's not a Binomial distribution, since the trials have different probabilities of success; having some trials with one probability of success and other trials with another probability of success isn't equivalent to having trials with some "effective" probability of success.

(c) Suppose that the first player to answer correctly wins the game (with no predetermined maximum number of questions that can be asked). Find the probability that A wins the game.

Solution: Let r = P(A wins). Conditioning on the results of the first question for each player, we have r = p_1 + (1 - p_1) $p_2 \cdot 0$ + $(1-p_1)(1-p_2)$r, which gives r = $\frac{p_1}{1-(1-p_1)(1-p_2)} = \frac{p_1}{p_1+p_2-p_1p_2}$.

5. Calvin and Hobbes play a match consisting of a series of games, where Calvin has probability p of winning each game (independently). They play with a "win by two" rule: the first player to win two games more than his opponent wins the match. Find the probability that Calvin wins the match (in terms of p), in two different ways:

(a) by conditioning, using the law of total probability.

Solution: Let C be the event that Calvin wins the match, X $\sim$ Bin(2, p) be how many of the first 2 games he wins, and q = 1 - p. Then

P(C) = P(C|X = 0)q^2 + P(C|X = 1)(2pq) + P(C|X = 2)p^2 = 2pqP(C) + p^2 ,

so P(C) = $\frac{p^2}{1-2pq}$. This can also be written as $\frac{p^2}{p^2+q^2}$, since p + q = 1.

Miracle check: Note that this should (and does) reduce to 1 for p = 1, 0 for p = 0, and 1/2 for p = 1/2 . Also, it makes sense that the probability of Hobbes winning, which is 1 - P(C) =$\frac{p^2}{p^2+q^2}$, can also be obtained by swapping p and q.

(b) by interpreting the problem as a gambler's ruin problem.

Solution: The problem can be thought of as a gambler's ruin where each player starts out with \$2. So the probability that Calvin wins the match is

$$\frac{1-(\frac{q}{p})^2}{1-(\frac{q}{p})^4} = \frac{(p^2-q^2)/p^2}{(p^4-q^4)/p^4} = \frac{(p^2-q^2)/p^2}{(p^2-q^2)(p^2+q^2)/p^4} = \frac{p^2}{p^2+q^2}$$

which agrees with the above.