Homework 5

1. Let X be a random day of the week, coded so that Monday is 1, Tuesday is 2, etc. (so X takes values 1, 2,..., 7, with equal probabilities). Let Y be the next day after X (again represented as an integer between 1 and 7). Do X and Y have the same distribution? What is P(X$< Y$)?

2. Are there discrete random variables X and Y such that E(X) > 100E(Y) but Y is greater than X with probability at least 0.99?

3. A group of 50 people are comparing their birthdays (as usual, assume their birthdays are independent, are not February 29, etc.). Find the expected number of pairs of people with the same birthday, and the expected number of days in the year on which at least two of these people were born.

4. A total of 20 bags of Haribo gummi bears are randomly distributed to the 20 students in a certain Stat 110 section. Each bag is obtained by a random student, and the outcomes of who gets which bag are independent. Find the average number of bags of gummi bears that the first three students get in total, and find the average number of students who get at least one bag.

5. There are 100 shoelaces in a box. At each stage, you pick two random ends and tie them together. Either this results in a longer shoelace (if the two ends came from different pieces), or it results in a loop (if the two ends came from the same piece). What are the expected number of steps until everything is in loops, and the expected number of loops after everything is in loops? (This is a famous interview problem; leave the latter answer as a sum.)

   Hint: for each step, create an indicator r.v. for whether a loop was created then, and note that the number of free ends goes down by 2 after each step.

6. Randomly, k distinguishable balls are placed into n distinguishable boxes, with all possibilities equally likely. Find the expected number of empty boxes.

7. Athletes compete one at a time at the high jump. Let X_j be how high the jth jumper jumped, with X_1, X_2,... i.i.d. with a continuous distribution. We say that the jth jumper set a record if X_j is greater than all of X_{j-1},..., X_1.

   (a) Is the event "the 110th jumper sets a record" independent of the event "the 111th jumper sets a record"? Justify your answer by finding the relevant probabilities in the definition of independence and with an intuitive explanation.

   (b) Find the mean number of records among the first n jumpers (as a sum). What happens to the mean as n $\rightarrow \infty$?