Attitude Determination Using a Single Sensor Observation: Analytic Quaternion Solutions and Property Discussion

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Abstract: This paper solves the attitude determination problem based on a single sensor observation. The rotation equation is transformed into a quadratic quaternion form and is then derived to a linear matrix equation with pseudoinverse matrices. The analytic solutions to the equation are computed via elementary row operations. The solutions show that the attitude determination from a single sensor observation has infinite solutions and the general one is governed by two limiting quaternions. Accordingly, the variance analysis is given in view of probabilistic characters. We explore the experimental results via the accelerometer attitude determination system. The properties of the two limiting quaternions are investigated in the experiment. The results show that the gravity-determination abilities of the two limiting quaternions are quite different. Using the rotation vector and eigenvalue decomposition of the attitude matrix, we prove that one limiting quaternion is better than another one geometrically. The singularity analysis is also performed revealing the nonexistence of singularities for limiting quaternions. The above findings are novel, which are quite different from the conclusions made in a previously published paper.

1. Introduction

Attitude determination is always served as an engineering problem. Representative methods mainly include the deterministic methods and stochastic methods [1]. Stochastic methods mainly consist of the Kalman filtering based methods [2, 3, 4, 5, 6] and the deterministic methods are always related to the so-called Wahba's problem [7].

In 1965, Wahba proposed the Wahba's problem via the least-squares approach which aims to find the optimal attitude matrix from various sensor observations [8]. Wahba's problem is from the following set of equations using rigid-body rotation:

$$\left\{ \begin{array}{c} \mathbf{b}_1 = \mathbf{C}\mathbf{r}_1 \\ \mathbf{b}_2 = \mathbf{C}\mathbf{r}_2 \\ \vdots \\ \mathbf{b}_n = \mathbf{C}\mathbf{r}_n \end{array} \right. \tag{1}$$

where $\mathbf{C}$ denotes the direction cosine matrix (DCM). $\mathbf{b}_i, \mathbf{r}_i$ denote the i-th normalized 3×1observation vector in the body frame and reference frame respectively. Wahba's problem is to minimize the target loss function which is given by

$$L(\mathbf{C}) = \frac{1}{2}\sum_{i=1}^{n} a_i |\mathbf{b}_i - \mathbf{C}\mathbf{r}_i|^2 \tag{2}$$

where a_i stands for the ith weight which indicates the variance or accuracy of the ith sensor. The sum of a_i satisfies

$$\sum_{i=1}^{n} a_i = 1 \tag{3}$$

and the DCM is an orthogonal matrix satisfying

$$\mathbf{C}\mathbf{C}^T = \mathbf{C}^T\mathbf{C} = \mathbf{I}_{3\times 3} \tag{4}$$

where $\mathbf{I}_{3\times 3}$ denotes the 3×3 identity matrix. A great deal of methods have been developed to solve the Wahba's problem, including QUaternion ESTimator [9], Fast Optimal Attitude Matrix Algorithm [10], Second Estimator of the Optimal Quaternion [11] and etc [12, 13, 14, 15].

Attitude determination using multi-sensor observations have been extensively discussed. The elementary problem from a single sensor observation is hardly be studied. In fact this would contribute to the method proposed by Shuster et.al in [16] using sequential quaternion multiplications.

Recently, Valenti et.al proposed the Algebraic Quaternion Algorithm (AQUA) for attitude determination based on the accelerometer and magnetometer [17]. The quaternion is obtained with a two-step sequential quaternion multiplication based on sensor observations from accelerometer and magnetometer respectively. However, some associated details of the general attitude determination using a single sensor observation have not been discussed yet.

Motivated by existing methods on multi-sensor observations, the main contributions of this paper are as follows:

1. By reformulating the DCM-based single sensor observation rotation model, two limit solutions to the attitude equation are obtained via elementary row operations. The general solution is equal to the linear combination of the two quaternions. Related variance analysis is given as well.

2. The accelerometer-based attitude determination system is studied. Using two limiting quaternions, their respective DCMs are calculated. By introducing the rotation vector, DCMs are investigated via the eigenvalue decomposition.

3. With experiment and Monte-Carlo simulation, the results show that distribution of the first set of eigenvalues are highly nonlinear rather than the second set. This proves that the second quaternion is better than the first one which is not the same with conclusions made in [17].

4. The singularity analysis is performed which shows the quaternions are nonsingular rather than the intuitive speculation in [17].

This paper is structured as follows: Section 2 presents the derivation of the optimal attitude quaternions for a single sensor observation. The variance analysis is also given. In Section 3, we also give the singularity analysis of the two limiting quaternion solutions. Section 4 introduces the static and dynamic accelerometer-based experiments involving some property verification and discussions. Section 5 contains the concluding remarks.

2. Equation for a Single Sensor Observation

2.1. Basis and Transformations

The observation vector of a single sensor in the body frame b and reference frame r (in this paper is NED) are given respectively by [18]

$$\begin{cases} \mathbf{D}^b = (D_x^b, D_y^b, D_z^b)^T \\ \mathbf{D}^r = (D_x^r, D_y^r, D_z^r)^T \end{cases} \tag{5}$$

The related rotation equation can be given by

$$\mathbf{D}^b = \mathbf{C}\mathbf{D}^r \tag{6}$$

The quaternion form of the DCM is defined by [19]

$$\mathbf{C} = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix} \tag{7}$$

which can be further derived using

$$\mathbf{C} = (\mathbf{J}_1\mathbf{q}, \mathbf{J}_2\mathbf{q}, \mathbf{J}_3\mathbf{q}) \tag{8}$$

where $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$ can be given by [20]

$$\begin{aligned} \mathbf{J}_1 &= \begin{pmatrix} q_0 & q_1 & -q_2 & -q_3 \\ q_3 & q_2 & q_1 & q_0 \\ -q_2 & q_3 & -q_0 & q_1 \end{pmatrix} \\ \mathbf{J}_2 &= \begin{pmatrix} -q_3 & q_2 & q_1 & -q_0 \\ q_0 & -q_1 & q_2 & -q_3 \\ q_1 & q_0 & q_3 & q_2 \end{pmatrix} \\ \mathbf{J}_3 &= \begin{pmatrix} q_2 & q_3 & q_0 & q_1 \\ -q_1 & -q_0 & q_3 & q_2 \\ q_0 & -q_1 & -q_2 & q_3 \end{pmatrix} \end{aligned} \tag{9}$$

Expanding (6), we obtain

$$D_x^r\mathbf{J}_1\mathbf{q} + D_y^r\mathbf{J}_2\mathbf{q} + D_z^r\mathbf{J}_3\mathbf{q} = \mathbf{D}^b \Leftrightarrow \mathbf{D}^b = (D_x^r\mathbf{J}_1 + D_y^r\mathbf{J}_2 + D_z^r\mathbf{J}_3)\mathbf{q} \tag{10}$$

Let $\mathbf{J}_D$ be

$$\mathbf{J}_D = (D_x^r\mathbf{J}_1 + D_y^r\mathbf{J}_2 + D_z^r\mathbf{J}_3) \tag{11}$$

(10) can be further given using the pseudoinverse of $\mathbf{J}_D$, such that

$$\mathbf{D}^b = \mathbf{J}_D\mathbf{q} \Leftrightarrow \mathbf{q} = \mathbf{J}_D^\dagger\mathbf{D}^b \tag{12}$$

(12) can be finally given with (13)

$$\mathbf{q} = \mathbf{J}_D^\dagger\mathbf{D}^b = \mathbf{H}\mathbf{q} \tag{13}$$

$$
\begin{pmatrix}
1 & \frac{a_3b_2-a_2b_3}{a_1b_1+a_2b_2+a_3b_3-1} & \frac{a_1b_3-a_3b_1}{a_1b_1+a_2b_2+a_3b_3-1} & \frac{a_2b_1-a_1b_2}{a_1b_1+a_2b_2+a_3b_3-1} \\
0 & \frac{a_1b_1(a_1b_1-2)-\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)+1}{a_1b_1+a_2b_2+a_3b_3-1} & a_2b_1+a_1b_2+\frac{(a_3b_1-a_1b_3)(a_3b_2-a_2b_3)}{a_1b_1+a_2b_2+a_3b_3-1} & a_3b_1+a_1b_3+\frac{(a_2b_1-a_1b_2)(a_2b_3-a_3b_2)}{a_1b_1+a_2b_2+a_3b_3-1} \\
0 & a_2b_1+a_1b_2+\frac{(a_3b_1-a_1b_3)(a_3b_2-a_2b_3)}{a_1b_1+a_2b_2+a_3b_3-1} & \frac{(a_2b_2-1)^2-\left(a_1^2+a_3^2\right)\left(b_1^2+b_3^2\right)}{a_1b_1+a_2b_2+a_3b_3-1} & a_3b_2+a_2b_3+\frac{(a_2b_1-a_1b_2)(a_3b_1-a_1b_3)}{a_1b_1+a_2b_2+a_3b_3-1} \\
0 & a_3b_1+a_1b_3+\frac{(a_2b_1-a_1b_2)(a_2b_3-a_3b_2)}{a_1b_1+a_2b_2+a_3b_3-1} & a_3b_2+a_2b_3+\frac{(a_2b_1-a_1b_2)(a_3b_1-a_1b_3)}{a_1b_1+a_2b_2+a_3b_3-1} & \frac{(a_3b_3-1)^2-\left(a_1^2+a_2^2\right)\left(b_1^2+b_2^2\right)}{a_1b_1+a_2b_2+a_3b_3-1}
\end{pmatrix} \tag{18}
$$

$$
\mathbf{N}=\begin{pmatrix}
1 & 0 & \frac{b_1b_3a_3^2-\left(a_1b_1^2-b_1+a_1\left(b_2^2+b_3^2\right)\right)a_3+\left(\left(a_1^2+a_2^2\right)b_1-a_1\right)b_3}{a_1b_1(a_1b_1-2)-\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)+1} & \frac{\left(a_1b_1^2-b_1+a_1\left(b_2^2+b_3^2\right)\right)a_2-a_2^2b_1b_2+\left(a_1-\left(a_1^2+a_3^2\right)b_1\right)b_2}{a_1b_1(a_1b_1-2)-\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)+1} \\
0 & 1 & \frac{b_1b_2a_2^2+\left(a_1b_1^2-b_1+a_1\left(b_2^2+b_3^2\right)\right)a_2+\left(\left(a_1^2+a_3^2\right)b_1-a_1\right)b_2}{a_1b_1(a_1b_1-2)-\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)+1} & \frac{a_1a_3b_3^2+\left(\left(a_1^2+a_2^2+a_3^2\right)b_1-a_1\right)b_3+a_3\left(a_1b_2^2+b_1(a_1b_1-1)\right)}{a_1b_1(a_1b_1-2)-\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)+1} \\
0 & 0 & -\frac{(a_1b_1+a_2b_2-a_3b_3-1)\left(\left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right)-1\right)}{a_1b_1(a_1b_1-2)-\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)+1} & \frac{(a_3b_2+a_2b_3)\left(\left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right)-1\right)}{a_1b_1(2-a_1b_1)+\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)-1} \\
0 & 0 & \frac{(a_3b_2+a_2b_3)\left(\left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right)-1\right)}{a_1b_1(2-a_1b_1)+\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)-1} & -\frac{(a_1b_1-a_2b_2+a_3b_3-1)\left(\left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right)-1\right)}{a_1b_1(a_1b_1-2)-\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)+1}
\end{pmatrix} \tag{19}
$$

where elements of $\mathbf{H}$ are listed in (14).

$$
\mathbf{H}=\begin{pmatrix}
D_x^bD_x^r+D_y^bD_y^r+D_z^bD_z^r & D_z^bD_y^r-D_y^bD_z^r & D_x^bD_z^r-D_z^bD_x^r & D_y^bD_x^r-D_x^bD_y^r \\
D_z^bD_y^r-D_y^bD_z^r & D_x^bD_x^r-D_y^bD_y^r-D_z^bD_z^r & D_x^bD_y^r+D_y^bD_x^r & D_x^bD_z^r+D_z^bD_x^r \\
D_x^bD_z^r-D_z^bD_x^r & D_x^bD_y^r+D_y^bD_x^r & -D_x^bD_x^r+D_y^bD_y^r-D_z^bD_z^r & D_y^bD_z^r+D_z^bD_y^r \\
D_y^bD_x^r-D_x^bD_y^r & D_x^bD_z^r+D_z^bD_x^r & D_y^bD_z^r+D_z^bD_y^r & -D_x^bD_x^r-D_y^bD_y^r+D_z^bD_z^r
\end{pmatrix} \tag{14}
$$

(13) would be further given by a homogeneous linear matrix equation, such that

$$(\mathbf{H}-\mathbf{I}_{4\times 4})\mathbf{q}=\mathbf{0} \tag{15}$$

Considering the rank of matrix $\mathbf{H}-\mathbf{I}_{4\times 4}$, on one hand, the determinant of the matrix $\mathbf{H}-\mathbf{I}_{4\times 4}$ can be given after simplification with

$$\det(\mathbf{H}-\mathbf{I}_{4\times 4})=\{-1+[(D_x^b)^2+(D_y^b)^2+(D_z^b)^2][(D_x^r)^2+(D_y^r)^2+(D_z^r)^2]\}^2=0 \tag{16}$$

where $\det(\cdot)$ denotes the determinant of a certain matrix. On the other hand, letting

$$\left\{\begin{array}{l} D_x^b=a_1 \\ D_y^b=a_2 \\ D_z^b=a_3 \end{array}\right., \left\{\begin{array}{l} D_x^r=b_1 \\ D_y^r=b_2 \\ D_z^r=b_3 \end{array}\right. \tag{17}$$

then $\mathbf{H}-\mathbf{I}_{4\times 4}$ can be transformed to reduced row echelon form using elementary row operations [21]. After processing the first column, the result is given by (18). Using the same methodology, we obtain the final result after the second column is processed (see (19)). Note that a common factor can be extracted from the 3rd and 4th rows of the matrix $\mathbf{N}$:

$$\left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right)-1=0 \tag{20}$$

Hence, the No. 3 and 4 rows of the matrix $\mathbf{N}$ are reduced to $(0,0,0,0)$ which indirectly makes $\det(\mathbf{H}-\mathbf{I}_{4\times 4})=0$.

2.2. Solutions to the Rotation Equation

The optimal quaternions can be solved by computing the normalized fundamental system of solutions. The public denominator of the elements $N_{13}, N_{14}, N_{23}, N_{24}$ is simplified using

$$\begin{aligned}&a_1b_1\left(a_1b_1-2\right)-\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)+1=a_1^2b_1^2-2a_1b_1+1-(1-a_1^2)(1-b_1^2)\\&=a_1^2b_1^2-2a_1b_1+1-(1+a_1^2b_1^2-a_1^2-b_1^2)=a_1^2-2a_1b_1+b_1^2\\&=(a_1-b_1)^2\end{aligned} \tag{21}$$

Numerators of $N_{13}, N_{14}, N_{23}, N_{24}$ can be simplified by

$$\begin{aligned}&a_3^2b_1b_3-a_3\left(a_1b_1^2+a_1\left(b_2^2+b_3^2\right)-b_1\right)+b_3\left(\left(a_1^2+a_2^2\right)b_1-a_1\right)\\&=a_3b_1-a_1b_3-a_1a_3(b_1^2+b_2^2+b_3^3)+b_1b_3(a_1^2+a_2^2+a_3^2)\\&=a_3b_1-a_1b_3-a_1a_3+b_1b_3\\&=(-a_3-b_3)(a_1-b_1).\\&a_1a_3b_3^2+b_3\left(\left(a_1^2+a_2^2+a_3^2\right)b_1-a_1\right)+a_3\left(a_1b_2^2+b_1\left(a_1b_1-1\right)\right)\\&=-a_2b_1+a_1b_2+a_1a_2(b_1^2+b_2^2+b_3^2)-b_1b_2(a_1^2+a_2^2+a_3^2)\\&=-a_2b_1+a_1b_2+a_1a_2-b_1b_2\\&=(a_2+b_2)(a_1-b_1).\\&a_2^2b_1b_2+a_2\left(a_1b_1^2+a_1\left(b_2^2+b_3^2\right)-b_1\right)+b_2\left(\left(a_1^2+a_3^2\right)b_1-a_1\right)\\&=-a_1b_2-a_2b_1+a_1a_2(b_1^2+b_2^2+b_3^2)+b_1b_2(a_1^2+a_2^2+a_3^2)\\&=-a_1b_2-a_2b_1+a_1a_2+b_1b_2\\&=(a_2-b_2)(a_1-b_1).\\&a_1a_3b_3^2+b_3\left(\left(a_1^2+a_2^2+a_3^2\right)b_1-a_1\right)+a_3\left(a_1b_2^2+b_1\left(a_1b_1-1\right)\right)\\&=-a_3b_1-a_1b_3+a_1a_3(b_1^2+b_2^2+b_3^2)+b_1b_3(a_1^2+a_2^2+a_3^2)\\&=-a_3b_1-a_1b_3+a_1a_3+b_1b_3\\&=(a_3-b_3)(a_1-b_1).\end{aligned} \tag{22}$$

respectively. Finally, $\mathbf{N}$ becomes

$$\mathbf{N}=\begin{pmatrix}1&0&\frac{-a_3-b_3}{a_1-b_1}&\frac{a_2+b_2}{a_1-b_1}\\0&1&\frac{a_2-b_2}{a_1-b_1}&\frac{a_3-b_3}{a_1-b_1}\\0&0&0&0\\0&0&0&0\end{pmatrix} \tag{23}$$

The fundamental solutions can be calculated by

$$\left\{\begin{array}{l}\tilde{\mathbf{q}}_I=(-a_2-b_2,-a_3+b_3,0,a_1-b_1)^T\\\tilde{\mathbf{q}}_{II}=(a_3+b_3,-a_2+b_2,a_1-b_1,0)^T\end{array}\right. \tag{24}$$

with norms of

$$\left\{\begin{array}{l}\|\tilde{\mathbf{q}}_I\|=\sqrt{(a_2+b_2)^2+(a_3-b_3)^2+(a_1-b_1)^2}=\sqrt{2+2(-a_1b_1+a_2b_2-a_3b_3)}.\\\|\tilde{\mathbf{q}}_{II}\|=\sqrt{(a_2-b_2)^2+(a_3+b_3)^2+(a_1-b_1)^2}=\sqrt{2+2(-a_1b_1-a_2b_2+a_3b_3)}.\end{array}\right. \tag{25}$$

Hence, the general quaternion solution is the linear combination of the two limit solutions

$$\mathbf{q}=\gamma\frac{\tilde{\mathbf{q}}_I}{\|\tilde{\mathbf{q}}_I\|}+(1-\gamma)\frac{\tilde{\mathbf{q}}_{II}}{\|\tilde{\mathbf{q}}_{II}\|} \tag{26}$$

where γ denotes the weight between two solutions.

2.3. Observability Discussion

Given the observation vector $\mathbf{D}^b$ and the reference vector $\mathbf{D}^r$, it is necessary to determine how many angles we can extract from the given sensor. For example, using the accelerometer we can only determine the roll and pitch angles while using the magnetometer we can just obtain the yaw angle. When we use the accelerometer to measure the yaw, the results would be definitely incorrect since it violates the observability of the sensor [22]. Likewise, when we use the magnetometer to obtain gravity estimation, it is impossible for us to achieve correct results. In real applications, we may use the consensus of the attitude solutions from two limiting quaternions to deduce how many angles can be observed by the specific sensor. In other words, we can model the quaternions using real-hardware experiments or Monte-Carlo simulations and then see how many angles from one quaternion are equal to another one's. Related experimental analysis is given in Section 4.

2.4. Variance Analysis

The proposed quaternionic solutions are perturbed by stochastic observation noises. Variance analysis is necessary in this case to provide probabilistic information of the calculated quaternions for further use e.g. they would be the input of a Kalman filter update. Although the quaternions are linear function of the input $\mathbf{D}^b$, its normalized form is a nonlinear function of $\mathbf{D}^b$. For this reason, the variance can be approximated by the Jacobian matrix of the quaternion

$$\Sigma_{\mathbf{q}} = \mathbf{J}_{\mathbf{q}} \Sigma_{\mathbf{D}^b} \mathbf{J}_{\mathbf{q}}^T \tag{27}$$

where $\Sigma_{\mathbf{q}}, \Sigma_{\mathbf{D}^b}$ denote the auto-covaraince matrix of $\mathbf{q}$ and $\mathbf{D}^b$ respectively. The Jacobian matrix of quaternion with respect to the input can be calculated by

$$\mathbf{J}_{\mathbf{q}} = \frac{\partial \mathbf{q}}{\partial \mathbf{D}^b} = \begin{pmatrix} \frac{\partial q_0}{\partial a_1} & \frac{\partial q_0}{\partial a_2} & \frac{\partial q_0}{\partial a_3} \\ \frac{\partial q_1}{\partial a_1} & \frac{\partial q_1}{\partial a_2} & \frac{\partial q_1}{\partial a_3} \\ \frac{\partial q_2}{\partial a_1} & \frac{\partial q_2}{\partial a_2} & \frac{\partial q_2}{\partial a_3} \\ \frac{\partial q_3}{\partial a_1} & \frac{\partial q_3}{\partial a_2} & \frac{\partial q_3}{\partial a_3} \end{pmatrix} \tag{28}$$

where $\Sigma_{\mathbf{D}^b}$ can be predefined by

$$\Sigma_{\mathbf{D}^b} = \begin{pmatrix} \sigma^2_{D^b_x} & 0 & 0 \\ 0 & \sigma^2_{D^b_y} & 0 \\ 0 & 0 & \sigma^2_{D^b_z} \end{pmatrix} \tag{29}$$

The Jacobian matrix of $\tilde{\mathbf{q}}_I, \tilde{\mathbf{q}}_{II}$ are calculated with (30).

$$\begin{aligned}
&\mathbf{J}_{\tilde{\mathbf{q}}_I} = \frac{1}{[2+2(-a_1b_1+a_2b_2-a_3b_3)]^{\frac{3}{2}}} \cdot \\
&\begin{pmatrix} -b_1(a_2+b_2) & -2+2a_1b_1-a_2b_2+b_2^2+2a_3b_3 & -b_3(a_2+b_2) \\ b_1(-a_3+b_3) & b_2(a_3-b_3) & -2+2a_1b_1-2a_2b_2+a_3b_3+b_3^2 \\ 0 & 0 & 0 \\ 2-a_1b_1-b_1^2+2a_2b_2-2a_3b_3 & b_2(-a_1+b_1) & b_3(a_1-b_1) \end{pmatrix} \\
&\mathbf{J}_{\tilde{\mathbf{q}}_{II}} = \frac{1}{[2+2(-a_1b_1-a_2b_2+a_3b_3)]^{\frac{3}{2}}} \cdot \\
&\begin{pmatrix} b_1(a_3+b_3) & b_2(a_3+b_3) & 2-2a_1b_1-2a_2b_2+a_3b_3-b_3^2 \\ b_1(-a_2+b_2) & -2+2a_1b_1+a_2b_2+b_2^2-2a_3b_3 & b_3(a_2-b_2) \\ 2-a_1b_1-b_1^2-2a_2b_2+2a_3b_3 & b_2(a_1-b_1) & b_3(-a_1+b_1) \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{30}$$

3. Singularity Analysis of The Quaternion Solutions

Above sections give the details of the solutions to the accelerometer based attitude determination system. As can be seen from (40) intuitively, the quaternions may become singular as the $a_z \to 1$ or $a_z \to -1$. The same conclusion is also made in [17]. However, this is inappropriate, because when $a_z \to 1$, $\mathbf{q}_I$ would approach $(0,0,0,0)^T$ as well. Mathematical derivations below show how we can find nonsingular quaternions for such limit conditions.

Let $\mathbf{a}_I = (a_1, a_2, a_3)^T$, $\mathbf{b}_I = (b_1, -b_2, b_3)^T$ and

$$\mathbf{c}_I = \mathbf{b}_I - \mathbf{a}_I = (-a_1 + b_1, -b_2 - a_2, b_3 - a_3)^T \tag{31}$$

$\mathbf{a}_I$ and $\mathbf{b}_I$ satisfies $\|\mathbf{a}_I\| = \|\mathbf{b}_I\| = 1$. $\mathbf{q}_I$ is then given by

$$\mathbf{q}_I = \frac{1}{\sqrt{2 - 2\mathbf{a}_I^T\mathbf{b}_I}}(c_2, c_3, 0, -c_1)^T \tag{32}$$

When $\mathbf{a}^T\mathbf{b} \to 1$, the angle θ between $\mathbf{a}$ and $\mathbf{b}$ approaches 0. Yet (32) has the same order with vector $\mathbf{c}$ thus the limit of quaternion exists if the following limit exists.

$$\mathbf{P}_I = \lim_{\mathbf{a}_I \to \mathbf{b}_I} \frac{\mathbf{c}_I}{\sqrt{2 - 2\mathbf{a}_I^T\mathbf{b}_I}} \tag{33}$$

Which can be further calculated via

$$\mathbf{P}_I = \lim_{\mathbf{a}_I \to \mathbf{b}_I} \frac{\mathbf{c}_I}{\sqrt{2-2\mathbf{a}_I^T\mathbf{b}_I}} = \lim_{\theta \to 0} \frac{\|\mathbf{c}_I\|\mathbf{v}_I}{\sqrt{2-2\left\|\mathbf{a}_I^T\right\|\|\mathbf{b}_I\|\cos\theta}} = \lim_{\theta \to 0} \frac{\|\mathbf{b}_I\|\theta\mathbf{v}_I}{\sqrt{2-2\cos\theta}} = \mathbf{v}_I \tag{34}$$

where $\mathbf{v}_I$ denotes the instant unitary vector that is perpendicular to the vector $\mathbf{b}_I$. This means $\mathbf{v}_I$'s distribution is on the normal plane of the vector $\mathbf{b}_I$. Likewise, define $\mathbf{q}_{II}, \mathbf{a}_{II}, \mathbf{b}_{II}, \mathbf{c}_{II}$ as

$$\begin{cases} \mathbf{a}_{II} = (a_1, a_2, a_3)^T \\ \mathbf{b}_{II} = (b_1, b_2, -b_3)^T \\ \mathbf{c}_{II} = \mathbf{b}_{II} - \mathbf{a}_{II} \end{cases} \tag{35}$$

In this case, $\mathbf{q}_{II}$ will be

$$\mathbf{q}_{II} = \frac{1}{\sqrt{2 - 2\mathbf{a}_{II}^T\mathbf{b}_{II}}}(-c_3, c_2, c_1, 0)^T \tag{36}$$

which has the limit of

$$\mathbf{P}_{II} = \lim_{\mathbf{a}_{II} \to \mathbf{b}_{II}} \frac{\mathbf{c}_{II}}{\sqrt{2 - 2\mathbf{a}_{II}^T\mathbf{b}_{II}}} = \mathbf{v}_{II} \tag{37}$$

Hence, $\mathbf{q}_{II}$ has the similar limit form with $\mathbf{q}_I$. This proves that the limit vector $\mathbf{P}_I, \mathbf{P}_{II}$ are nonsingular but they depend on the normal planes, which are decided by specific sensor configurations and reference vectors.

In terms of the accelerometer, the normal plane is parallel with the xy plane hence unitary vectors on it can be expressed by linear combinations of two orthonormal bases e.g. $\mathbf{i} = (1,0,0)^T$, $\mathbf{j} = (0,1,0)^T$. $\mathbf{v}$ can be given by $\mathbf{v} = (\cos\rho, \sin\rho, 0)^T$ which generates

$$\begin{aligned} \mathbf{q}_I &= (\sin\rho, 0, 0, -\cos\rho)^T \\ \mathbf{q}_{II} &= (0, \sin\rho, \cos\rho, 0)^T \end{aligned} \tag{38}$$

Corresponding Euler angles have zero components for roll and pitch which gives the verification of the above derivations.

4. Experimental Case: Accelerometer Based Attitude Determination

4.1. Static Test

The accelerometer-based attitude determination system is introduced hereinafter. Where the observation vector in the reference frame is defined as $(0, 0, 1)^T$. With (25), the quaternions from the accelerometer can be calculated by

$$\begin{aligned}\tilde{\mathbf{q}}_{a,I} &= (-a_y, -a_z + 1, 0, a_x)^T \\ \tilde{\mathbf{q}}_{a,II} &= (a_z + 1, -a_y, a_x, 0)^T\end{aligned} \tag{39}$$

where a_x, a_y, a_z denote the triaxial components of the acceleration. This can be normalized by means of

$$\begin{aligned}\mathbf{q}_{a,I} &= \frac{\tilde{\mathbf{q}}_{a,I}}{\left\|\tilde{\mathbf{q}}_{a,I}\right\|} = \frac{(-a_y,-a_z+1,0,a_x)^T}{\sqrt{a_x^2+a_y^2+(a_z-1)^2}} = \frac{1}{\sqrt{2-2a_z}}(-a_y, -a_z + 1, 0, a_x)^T \\ \mathbf{q}_{a,II} &= \frac{\tilde{\mathbf{q}}_{a,II}}{\left\|\tilde{\mathbf{q}}_{a,II}\right\|} = \frac{(a_z+1,-a_y,a_x,0)^T}{\sqrt{a_x^2+a_y^2+(a_z+1)^2}} = \frac{1}{\sqrt{2+2a_z}}(a_z + 1, -a_y, a_x, 0)^T\end{aligned} \tag{40}$$

which are the same with results given in [17] but the derivation is quite different.

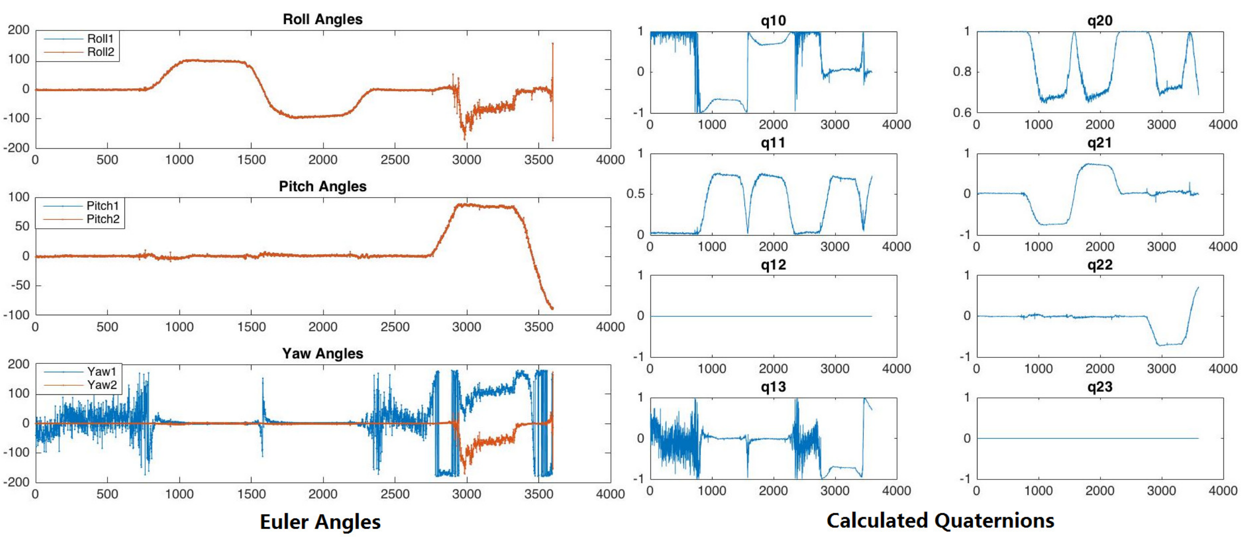

Fig. 1. *Left: Euler angles computed using (24) (in degree). x-axis stands for the sample number while y-axis represents the magnitudes of the Euler angles (in degree). Right: Calculated quaternions with two different types of quaternions.*

A test data set with 3500 samples is acquired with the Microstrain 3DM-GX3-25 Inertial Measurement Unit (IMU) at the sampling frequency of 50Hz. Figure 1 presents the calculated Euler angles from two limiting quaternions and the corresponding calculated quaternions. Figure 2 shows the differences between estimated gravity and real gravity. $\mathbf{q}_I, \mathbf{q}_{II}$ with a_x and a_y are plotted in the Figure 3.

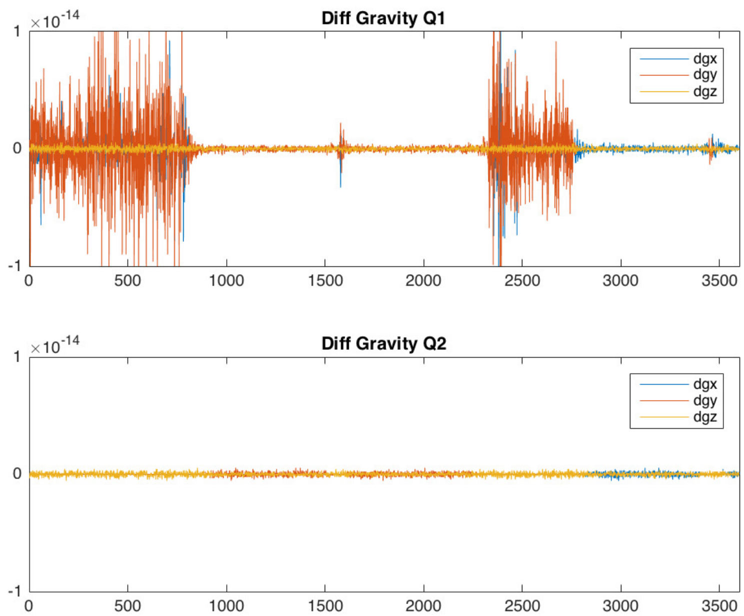

Fig. 2. *Differences between the estimated gravity and real gravity. The $x-$axis represents the sample index and the $y-$axis stands for the gravity's difference with the unit of G. As can be seen $\mathbf{q}_I$ is much worse than $\mathbf{q}_{II}$ in this studied case. dgx, dgy, dgz denote the three components of the differenced gravity vector.*

Figures presented provide us with the following findings:

1. $\mathbf{q}_I$ has different ability of estimating the pitch and roll angles with $\mathbf{q}_{II}$. However, the macroscopical performances of the roll and pitch angles are very similar.
2. $\mathbf{q}_{II}$ has a better determination accuracy.
3. $\mathbf{q}_{II}$'s corresponding yaw remains to be zero except for the case when the pitch approaches $\frac{\pi}{2}$ that forming the gimbal lock.

The first finding indicates that for accelerometer, it can only be used for determining the roll and pitch angles. This in fact corresponds to the observability of the accelerometer. As discussed in Section 2.3, the macroscopical results of the roll and pitch angles are basically the same while the yaw's performances are quite different. Hence, when we use the accelerometer to give attitude determination, the yaw solution is meaningless and should be neglected. The other two findings bring us a new problem: Which quaternion is better?

4.2. Discussion of The Two Limiting Quaternions

In fact, this can be solved via the Euler attitude dynamics. We know that the attitude of a rigid body can not only be represented by quaternions but can be given by rotation vectors as well [23, 24, 25]. In general, rotation vector is the eigenvector of the DCM with associated eigenvalue of 1 [26, 24], such that

$$\mathbf{C}\mathbf{\Phi} = \mathbf{\Phi} \tag{41}$$

where $\mathbf{\Phi}$ denotes the rotation vector that can be given by $\mathbf{\Phi} = (\Phi_x, \Phi_y, \Phi_z)^T$. Rotation vector provides a point of view of the attitude without the DCM because it reflects the character of rotation axis [25]. However, as the DCM is a 3×3 matrix, there should be three eigenvalues. The other two eigenvectors of the DCM $\mathbf{\Phi}_2^*, \mathbf{\Phi}_3^*$ are always useless in attitude determination but they decides the property of the DCM. Let the converted DCMs from $\mathbf{q}_I, \mathbf{q}_{II}$ be $\mathbf{C}_I, \mathbf{C}_{II}$. Then the two matrices can be decomposed with their eigenvalues and eigenvectors by

$$\begin{aligned} \mathbf{C}_I &= \mathbf{Q}_I \mathbf{\Sigma}_I \mathbf{Q}_I^{-1} \\ \mathbf{C}_{II} &= \mathbf{Q}_{II} \mathbf{\Sigma}_{II} \mathbf{Q}_{II}^{-1} \end{aligned} \tag{42}$$

where $\mathbf{Q}, \mathbf{\Sigma}$ stand for the matrix formed by eigenvectors and the diagonal matrix of eigenvalues respectively

$$\begin{aligned} \mathbf{Q} &= (\mathbf{\Phi}_1, \mathbf{\Phi}_2^*, \mathbf{\Phi}_3^*) \\ \Sigma &= diag(\lambda_1, \lambda_2, \lambda_3) \end{aligned} \tag{43}$$

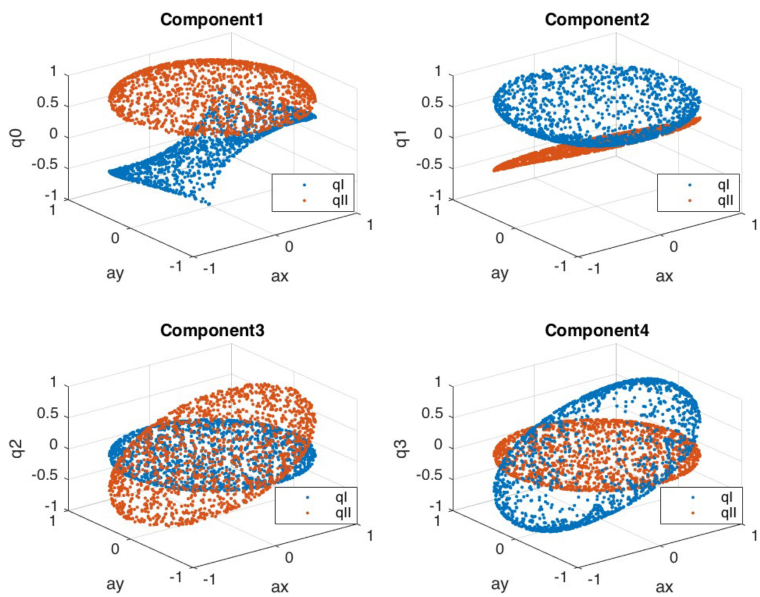

Fig. 3. *The distribution of the theoretical quaternions $\mathbf{q}_I, \mathbf{q}_{II}$ with the change of the acceleration. Due to the limitation of 3-d plot, a_z is not plotted.*

$$\begin{aligned} \mathbf{C}_{a,I} &= \frac{1}{a_x^2+a_y^2+(a_z-1)^2} \begin{pmatrix} -a_x^2+a_y^2+(a_z-1)^2 & 2a_xa_y & -2a_x(a_z-1) \\ -2a_xa_y & -a_x^2+a_y^2-(a_z-1)^2 & -2a_y(a_z-1) \\ -2a_x(a_z-1) & 2a_y(a_z-1) & a_x^2+a_y^2-(a_z-1)^2 \end{pmatrix} \\ \mathbf{C}_{a,II} &= \frac{1}{a_x^2+a_y^2+(a_z+1)^2} \begin{pmatrix} -a_x^2+a_y^2+(a_z+1)^2 & -2a_xa_y & 2a_x(a_z+1) \\ -2a_xa_y & a_x^2-a_y^2+(a_z+1)^2 & 2a_y(a_z+1) \\ -2a_x(a_z+1) & -2a_y(a_z+1) & -a_x^2-a_y^2+(a_z+1)^2 \end{pmatrix} \end{aligned} \tag{44}$$

For accelerometer, through (40), the corresponding DCMs can be calculated by (44). The characteristic polynomial of $\mathbf{C}_{a,I}$ is given by

$$\det(\lambda \mathbf{I}_{3\times3} - \mathbf{C}_{a,I}) = \frac{(\lambda-1)[\lambda^2+4(1-a_y^2-a_z)\lambda+1]}{2-2a_z} \tag{45}$$

then the corresponding eigenvalues are calculated by

$$\begin{cases} \lambda_{I,1} = 1 \\ \lambda_{I,2} = \frac{a_y^2 + a_z - 1 + \sqrt{a_y^2(a_y^2 + 2a_z - 2)}}{1 - a_z} \\ \lambda_{I,3} = \frac{a_y^2 + a_z - 1 - \sqrt{a_y^2(a_y^2 + 2a_z - 2)}}{1 - a_z} \end{cases} \tag{46}$$

where, $\lambda_{I,2}$ and $\lambda_{I,3}$ are conjugated with each other. Likewise, the eigenvalues of $\mathbf{C}_{a,II}$ can be computed by

$$\begin{cases} \lambda_{II,1} = 1 \\ \lambda_{II,2} = \frac{a_z^2 + a_z + \sqrt{(a_z^2 - 1)(a_z + 1)^2}}{1 + a_z} \\ \lambda_{II,3} = \frac{a_z^2 + a_z - \sqrt{(a_z^2 - 1)(a_z + 1)^2}}{1 + a_z} \end{cases} \tag{47}$$

Using Monte-Carlo simulation, the distributions of the eigenvalues are obtained, which is given in Figure (4).

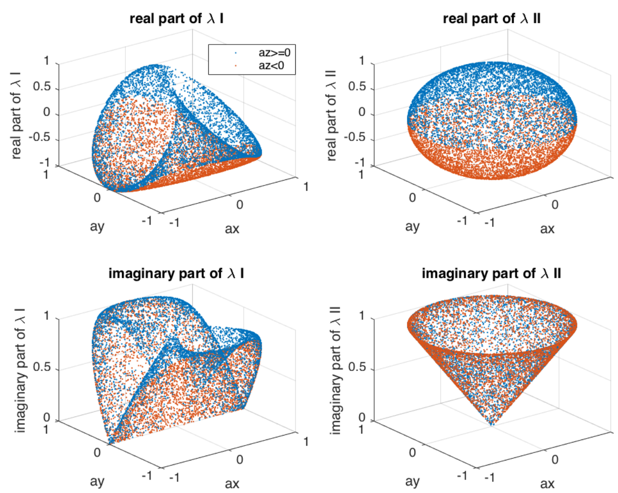

Fig. 4. *Real and imaginary parts of the eigenvalues λ_I, λ_{II}. Blue parts represent the conditions when $a_z >= 0$ while red parts represent conditions when $a_z > 0$. λ_{II} is constrained in the sphere and cone surface while the distribution of λ_I is irregular.*

The two eigenvalues have quite different distribution. The real and imaginary parts of λ_I are highly nonlinear while that of λ_{II} are constrained by a spheric and coning surface. In other words, λ_I is much more 'chaotic' than λ_{II}. In general, the eigenvalues reflect the most important characters of the DCM hence the internals of $\mathbf{C}_I$ and $\mathbf{C}_{II}$ are different. With the change of inputs, λ_{II} maintains more regular all the time than λ_I. The eigenvalues of two different quaternions using previously studied case are shown in Figure (5). This finding also shows λ_{II} is much more smooth.

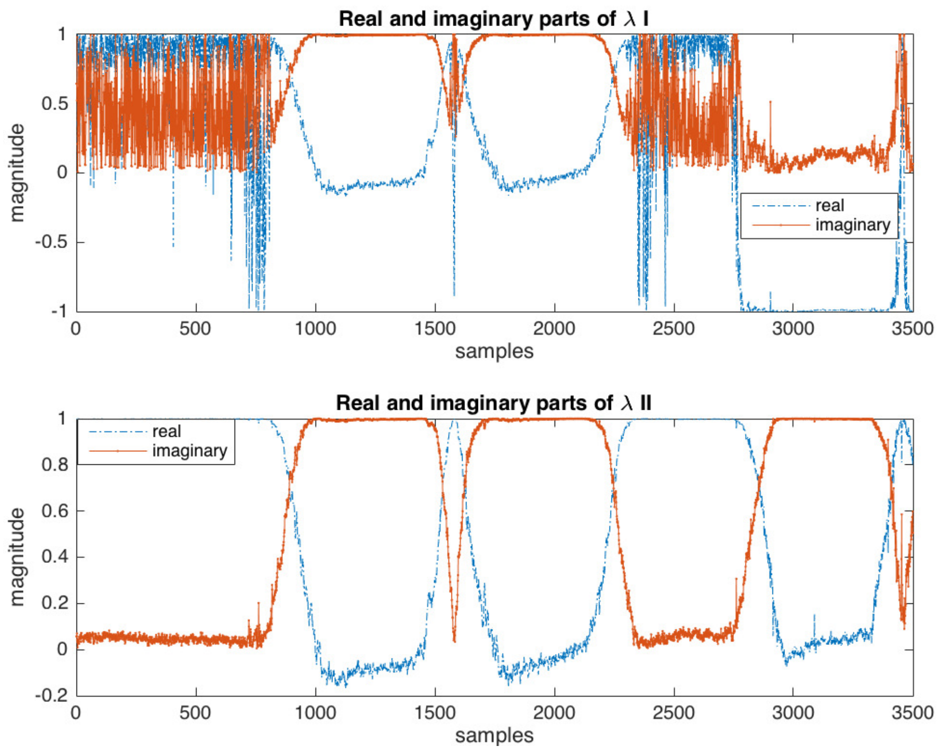

Fig. 5. *Different eigenvalues produced by accelerometer-based attitude determination system. As can be seen, λ_I is much more chaotic than λ_{II}.*

From another aspect, the rotation vectors of λ_I, λ_{II} can be respectively given by

$$\begin{array}{l}\mathbf{\Phi}_I = (\phi_{I,x}, \phi_{I,y}, \phi_{I,z})^T = \frac{\phi_I}{\sin\frac{\phi_I}{2}}(q_{I,1}, q_{I,2}, q_{I,3}) \\ = \frac{\phi_I}{\sqrt{2-2a_z}\sin\frac{\phi_I}{2}}(-a_z+1, 0, a_x)^T \\ \mathbf{\Phi}_{II} = (\phi_{II,x}, \phi_{II,y}, \phi_{II,z})^T = \frac{\phi_{II}}{\sin\frac{\phi_{II}}{2}}(q_{II,1}, q_{II,2}, q_{II,3}) \\ = \frac{\phi_{II}}{\sqrt{2+2a_z}\sin\frac{\phi_{II}}{2}}(-a_y, a_x, 0)^T\end{array} \tag{48}$$

where ϕ denotes the magnitude of the rotation angle about the rotation axis that can be given by [27]

$$\begin{array}{l}\phi_I = 2\cos^{-1} q_{I,0} \\ \phi_{II} = 2\cos^{-1} q_{II,0}\end{array} \tag{49}$$

Note that $\mathbf{\Phi}_{II}$ is perpendicular to the z rotation axis $\tilde{\mathbf{\Phi}}_z = (0,0,1)^T$ while $\mathbf{\Phi}_I$ is not. In fact, the observability of the accelerometer resolve that the yaw angle would not be influenced. Hence, if the rotation vector is not perpendicular to $\tilde{\mathbf{\Phi}}_z$, the yaw angle would definitely be changed. Hence in this case $\mathbf{q}_{II}$ is better than $\mathbf{q}_I$.

4.3. Dynamic Performance

According to the observability of consumer-level MEMS 3-axis accelerometer, it can only measure the pitch and roll angles of a certain object when it moves with low dynamic acceleration. However, in real applications external dynamic acceleration is everywhere. Hence, commonly, to prevent the signal from being covered by the external acceleration, we usually use the low-pass filter (LPF) to filter the accelerometer outputs [28]. Using the same hardware (Microstrain 3DM-GX3-25 IMU), we conducted another experiment containing dynamic motions. Figure 6 shows the related results.

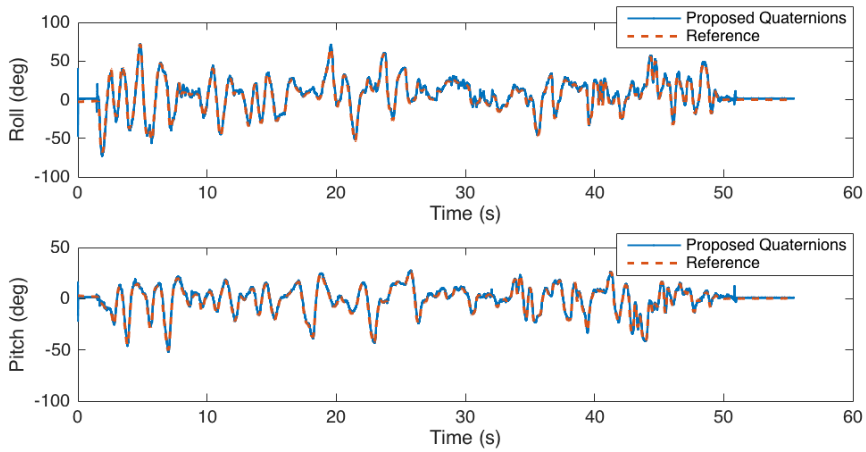

Fig. 6. *Dynamic performance of the proposed quaternion solution from the accelerometer. Corresponding performance of the reference angles is also plotted.*

In this experiment, the cutoff frequency of the LPF is set to 10Hz to filter the external acceleration, since we usually treat the acceleration with frequency of $> 10Hz$ as the external disturbance. As can be seen from the figure, the dynamic results of the attitude determination using accelerometer can well coincide with the reference angles. This reassures the effectiveness and correctness of the proposed method.

4.4. Epilogue

The above derivations and experiments reveal that although for a single sensor observation two different quaternions can be computed, their related DCMs have quite different internals. One solution has regular distribution of eigenvalues while the other seems 'chaotic'. Not only for accelerometer, we believe that similar phenomenon would occur in other attitude determination systems like magnetometer, sun sensor, nadir sensor and etc.

5. Discussion: Relationship with Two-Vector Attitude Determination

It should be noted that the proposed algorithm does not solves the Wahba's problem since in conventional Wahba's problem the amount of vector observation pairs is always more than 1. As far as the most simple condition i.e. the two-vector case is concerned, Markley proposed an efficient optimal algorithm for quaternion determination by means of Wahba's problem [29]. In his literature, the optimal attitude matrix is computed via

$$\mathbf{B}_3 = \mathbf{C}_{opt}\mathbf{R}_3 \tag{50}$$

where

$$\mathbf{B}_3 = \frac{\mathbf{D}_b^1 \times \mathbf{D}_b^2}{\|\mathbf{D}_b^1 \times \mathbf{D}_b^2\|}, \mathbf{R}_3 = \frac{\mathbf{D}_r^1 \times \mathbf{D}_r^2}{\|\mathbf{D}_r^1 \times \mathbf{D}_r^2\|} \tag{51}$$

in which $\mathbf{D}_b^i$ and $\mathbf{D}_r^i$ are i-th vector observations in body and reference frames respectively. It seems that this provides us with an approach to calculate the attitude quaternion by combining (24) and (50). However, in this way, the attitude determination can not be achieved. The reason is that there are zero components inside the limiting quaternions, which proves that the observability of such solution can only reach at most 2 angles. As the matter of fact, according to engineering practice, if the two reference vectors are not collinear, the attitude quaternion can be uniquely given. This means that for Wahba's problem in the presence of two vector measurements, we usually obtain quaternion solutions with observabilities of 3 angles, which does not coincide with the single-vector solution proposed by us.

For us, there may be several means by which we can obtain optimal attitude determination from multi-vector observations:

1. Although our proposed method seems to fail in solving two-vector attitude determination, combining (26) and (50) maybe helpful to obtain continuous and accurate quaternions. However, the work is quite complicated since we have to solve the nonlinear equation formed by $\mathbf{q}$ and γ. Hence, this may be achieved in a future publication.

2. According to [17], the solution to the two-vector model may undertake a series of proper extended quaternion transformations, making the solutions continuous, accurate and optimal. This provides us a motivation to calculate quaternions from single vectors respectively and then design a new way to obtain the general optimal quaternion satisfying the Wahba's problem by quaternion multiplications in the future.

6. Conclusion

This paper gives the closed form of the optimal quaternion for attitude determination using a single sensor observation. The attitude determination is transformed into an eigenvalue problem, which is similar with classical solution to Wahba's problem. Elementary row operations are performed to solve this problem. Then the form of the two limiting quaternions is obtained. With which the variance analysis is also discussed after approximation.

With respect to the accelerometer-based attitude determination system, the two quaternions are ultilized for experimental verification. To our surprise, although the two quaternions have almost the same ability of determining roll and pitch angles macroscopically, they have quite different gravity-estimation ability with a microscopic point of view. In order to know which quaternion is better, rotation vector is introduced. Eigen decomposition of the related DCM shows that distribution of the eigenvalues for the first solution is highly nonlinear. In contrast, distribution of the other limiting quaternion is quite regular geometrically. Besides, the rotation vector of the second quaternion is free of z component which would not influence the yaw angle. Consequently the second solution is better.

Different from concluding remarks made in [17], this paper also deals with the singularity analysis of the proposed two limiting quaternions. Results show that although through intuitive glimpse, singularities may occur when limit conditions are achieved, the quaternions are in fact nonsingular all the time. This again verifies the numerical robustness of the quaternion for attitude determination.

Although this paper mainly focuses on the accelerometer based attitude determination system, which is a specific condition for a single sensor observation vector, it reveals interesting and useful information. With the methodologies presented above, we believe for other sensors the same phenomena may also happen.

Acknowledgment

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8. Appendix A: Proofs for Section II

The proof of $\mathbf{J}_D^{\dagger}$ is conducted by MATLAB r2015b. Related source file can be obtained from the following link: `"http://www.zarathustr.com/wp-content/uploads/2016/10/solution_verify.txt"`.