Principle of Inclusion and Exclusion (PIE)

The **principle of inclusion and exclusion (PIE)** is a counting technique that computes the number of elements that satisfy at least one of several properties while guaranteeing that elements satisfying more than one property are not counted twice.

An underlying idea behind PIE is that summing the number of elements that satisfy at least one of two categories and subtracting the overlap prevents double counting. For instance, the number of people that have at least one cat or at least one dog can be found by taking the number of people who own a cat, adding the number of people that have a dog, then subtracting the number of people who have both.

PIE is particularly useful in combinatorics and probability problem solving when it is necessary to devise a counting method that ensures an object is not counted twice.

Contents

Two Sets

In the case of objects being separated into two (possibly disjoint) sets, the principle of inclusion and exclusion states

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

where $|S|$ denotes the cardinality, or number of elements, of set S in set notation.

PROOF

To prove this statement, we will show that every element which belongs in one of these sets is counted exactly once, and every element that is not in these sets is counted exactly zero times.

- Case 1. Element is not in A, and not in B.

  It is obvious it is counted zero times in the LHS. It is obvious it is counted zero times in the RHS.

- Case 2. Element is in A, and not in B.

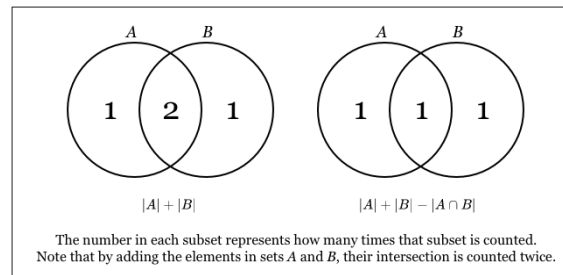 It will be counted once on the LHS. On the RHS, it is counted once in $|A|$.

- Case 3. Element is not in A, and is in B.

  It will be counted once on the LHS. On the RHS, it is counted once in $|B|$.

- Case 4. Element is in A, and in B.

  It will be counted once on the LHS. On the RHS, it is counted $+1$ in $|A|$, $+1$ in $|B|$ and -1 in $|A \cap B|$. Hence, it is counted exactly once.

As a Venn diagram, PIE for two sets can be depicted easily:

The number in each subset represents how many times that subset is counted.
Note that by adding the elements in sets A and B, their intersection is counted twice.

How many integers from 1 to 100 are multiples of 2 or 3?

Let A be the set of integers from 1 to 100 that are multiples of 2, then $|A| = 50$.

Let B be the set of integers from 1 to 100 that are multiples of 3, then $|B| = 33$.

Now, $A \cap B$ is the set of integers from 1 to 100 that are multiples of both 2 and 3, and hence are multiples of 6, implying $|A \cap B| = 16$.

Hence, by PIE,

$$|A \cup B| = |A| + |B| - |A \cap B| = 50 + 33 - 16 = 67. \ \square$$

In a higher secondary examination, 80% of the examinees have passed in English and 85% in mathematics, while 75% passed in both English and mathematics. If 45 candidates failed in both the subjects, find the total number of candidates.

Submit your answer

A local grocery store in the outback newly opened. They were offering 1 free bottle Marmite to every 11th customer and 1 free pound of kangaroo meat for every 13th customer. If there were 1000 customers that visited them on opening day, how many customers walked away with free goodies?
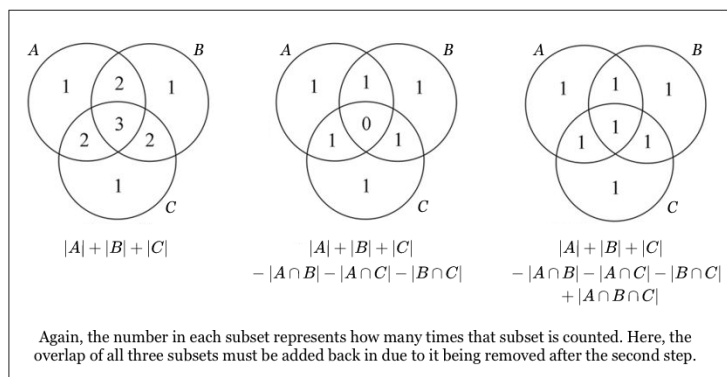
○ 160

○ 166

○ 172

○ 1000

Three Sets and More

We've already looked at the case of 2 sets. Before we dive into the general case, let's consider having 3 sets.

If there are three sets, the principle of inclusion and exclusion states

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

We can verify these statements for ourselves by considering the Venn diagram of events:

Again, the number in each subset represents how many times that subset is counted. Here, the overlap of all three subsets must be added back in due to it being removed after the second step.

EXAMPLE

There are exactly three types of students in a school: the geeks, the wannabees, and the athletes. Each student is classified into at least one of these categories. And the total number of students in the school is 1000. Suppose that the following is given:

- The total number of students who are geeks is 310.
- The total number of students who are wannabees is 650.
- The total number of students who are athletes is 440.

- The total number of students who are both geeks and wannabees is 170.

- The total number of students who are both geeks and athletes is 150.

- The total number of students who are both wannabees and athletes is 180.

What is the total number of students who fit into all 3 categories?

Let G, W, A denote the set for geeks, wannabees, and athletes, respectively. Then by the principle of inclusion and exclusion, we have

$$n(G \cup W \cup A) = n(G) + n(W) + n(A) - n(W \cap G) - n(G \cap A) - n(W \cap A) + n(G \cap W \cap A),$$

which gives us $1000 = 310 + 650 + 440 - 170 - 150 - 180 + n(G \cap W \cap A) = 900 + n(G \cap W \cap A)$.

So the total number of students who fit into all 3 categories is 100. $\square$

TRY IT YOURSELF

How many positive integers less than or equal to 60 are divisible by 3, 4, or 5?

Incorrect. 👇
The answer is 36.
View the solution to learn why!

Continue

View solutions

We defer the proof to the general case. If you are interested, you can duplicate the proof above and check that every element in $A \cup B \cup C$ is counted exactly once on the RHS.

More generally, if $A_1, A_2, \ldots, A_n$ are finite sets, then the principle of inclusion and exclusion states

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|.$$

TRY IT YOURSELF

At the Winter Sochi Olympics Press Conference, there are 200 foreign journalists. Out of them,

- 175 people can speak German,
- 150 people can speak French,
- 180 people can speak English, and
- 160 people can speak Japanese.

What is the minimum number of foreigners that can speak all the four languages?

Submit your answer

Derangements

Main article: Derangements

EXAMPLE

There are eight guests at a secret Santa party. Each guest brings a gift and each receives another gift in return. No one is allowed to receive the gift they brought. How many ways are there to distribute the gifts?

Let A denote the set of ways to distribute gifts such that everyone receives a gift, possibly their own. Let A_i denote the set of ways to distribute gifts such that person i receives his or her own gift. Then we would like to find

$$|A| - |A_1 \cup A_2 \cup \cdots \cup A_8|.$$

Since A_i is the set of ways for person i to receive his/her own gift, there are 7 choices of gifts for the next person, 6 choices of gifts for the following person, and so on. By the rule of product,

$$|A_i| = 7 \times 6 \times \cdots \times 2 \times 1 = 7!.$$

Since $A_i \cap A_j$ is the set of ways person i and person j both receive their own gifts, there are 6 choices of gifts for the next person, 5 choices of gifts for the following person, and so on. Again by the rule of product,

$$|A_i \cap A_j| = 6 \times 5 \times \cdots \times 2 \times 1 = 6!.$$

By continuing this argument, if k people receive their own gifts, then there are $(8 - k)!$ possible ways. Applying PIE, we obtain

$$|A| - |A_1 \cup A_2 \cup \cdots \cup A_8| = 8! - \binom{8}{1} \times 7! + \binom{8}{2} \times 6! - \cdots + \binom{8}{8} \times 1! = 14833. \ \square$$

Note: A **derangement** of n objects is a permutation of the objects such that none of them stays in the same place. The number of ways this can be done is denoted D_n, and this calculation shows $D_8 = 14833$.

Problem Solving

What is the sum of all integers from 1 to 100 that are multiples of 2 or 3?

While PIE is often used to count the elements of a set, if we remove the $|\cdot|$ symbols, the statement is still true. For example, in two variables, $A \cup B = A + B - A \cap B$. The same proof using Venn diagrams works to show that each element is included once. As such, the sum of elements in $A \cup B$ is equal to the sum of elements in A plus the sum of elements in B minus the sum of elements in $A \cap B$. Let A be the set of multiples of 2 and B be the set of multiples of 3, then $A \cap B$ is the set of multiples of 6, and hence the sum of $A \cup B$ is

$$\frac{(2 + 100) \times 50}{2} + \frac{(3 + 99) \times 33}{2} - \frac{(6 + 96) \times 16}{2} = 3417. \; \square$$

EXAMPLE

We have 7 balls each of different colors (red, orange, yellow, green, blue, indigo, violet) and 3 boxes each of different shapes (tetrahedron, cube, dodecahedron). How many ways are there to place these 7 balls into the 3 boxes such that each box contains at least 1 ball?

Let X be the total number of ways we can distribute the balls if there are no restrictions. Each ball can be placed into any one of the 3 boxes, so $|X| = 3^7$. Let T be the set of ways such that the tetrahedron box has no balls, C the set of ways such that the cube box has no balls, and D the set of ways such that the dodecahedron box has no balls. We would like to find

$$|X| - |T \cup C \cup D|.$$

We have $|T| = |C| = |D| = 2^7$, since the balls can be placed into one of the two other boxes, and $|T \cap C| = |C \cap D| = |D \cap T| = 1^7$, since all the balls must be placed in the remaining box, and $|T \cap C \cap D| = 0$. Hence, PIE implies

$$|X| - |T \cup C \cup D| = 3^7 - 3 \times 2^7 + 3 \times 1^7 - 0 = 1806. \; \square$$

Katie is teaching a class of thirty pupils, of which fourteen are girls. She knows that there are twenty-two pupils who are right-handed.

What is the minimum number of girls who are right-handed in this class?

Submit your answer

At a construction site, Jorge is in charge of hiring skilled workers for the project. Out of 80 candidates that he interviewed, he found that

- 45 were painters,
- 50 were electricians,
- 50 were plumbers,
- 15 had skills in all three areas, and
- all of them had skills in at least one of these areas.

If he hired everyone who was skilled in exactly 2 areas, how many candidates were hired?

○ 35

○ 45

○ 55

○ 65

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