Updated November 12, 2025

Homework problems for AMAT 540A (Topology I), Fall 2025. Over the course of the semester I'll add problems to this list, with each problem's due date specified. Each problem is worth 2 points.

Solutions will be gradually added (and may be hastily written without proofreading).

Problem 1 (due Weds 9/3): Let A be a set and let R be a relation on A that is symmetric and antisymmetric, but not reflexive. Prove that there exists $a \in A$ such that for all $b \in A$, aRb does not hold.

Solution: Since R is not reflexive, we can choose $a \in A$ such that aRa does not hold. Now for any $b \in A$, if aRb then by symmetry bRa, and by antisymmetry $a = b$, so aRa. We conclude aRb never holds. $\square$

Problem 2 (due Weds 9/3): Let $f \colon A \to B$ be a function. Suppose that for all $B_0 \subseteq B$ we have $f(f^{-1}(B_0)) = B_0$. Prove that f is surjective.

Solution: Let $b \in B$. We have $f(f^{-1}(\{b\})) = \{b\}$, so $f^{-1}(\{b\})$ is non-empty, i.e., there exists $a \in A$ with $f(a) = b$. Thus f is surjective. $\square$

Problem 3 (due Weds 9/3): Prove (using element arguments) that $(A \setminus B) \cap C = (A \cap C) \setminus B$ for sets A, B, and C.

Solution: ($\subseteq$): Let $x \in (A \setminus B) \cap C$, so $x \in A$ and $x \notin B$ and $x \in C$. Thus $x \in (A \cap C) \setminus B$.
($\supseteq$): Same thing really, this one was too easy. $\square$

Problem 4 (due Weds 9/10): Let X be a set and view the power set $\mathcal{P}(X)$ as a partially ordered set with the partial order $\subseteq$. Prove that every non-empty subset $\mathcal{A} \subseteq \mathcal{P}(X)$ has a least upper bound.

Solution: Let $U = \bigcup_{A \in \mathcal{A}} A$. Since $A \subseteq U$ for all $A \in \mathcal{A}$, U is an upper bound of $\mathcal{A}$. For any other upper bound V, we have $A \subseteq V$ for all $A \in \mathcal{A}$, and hence $\bigcup_{A \in \mathcal{A}} A \subseteq V$, i.e., $U \subseteq V$. Thus U is a least upper bound. $\square$

Problem 5 (due Weds 9/10): Let $X = \mathbb{N}$ with the partial order $|$ (this is "divides", i.e., "$a|b$" means "a divides b"). Prove that every non-empty subset $S \subseteq X$ has a greatest lower bound with respect to $|$. Does every such S have a least upper bound?

Solution: Let $\emptyset \neq S \subseteq X$. Let $D = \{d \in X \mid d|s \text{ for all } s \in S\}$, so D non-empty since $1 \in D$. Moreover, D is finite since each $s \in S$ has only finitely many divisors. Let $\ell \in X$ be the least common multiple of the elements of D. Since every element of S is a multiple of every element of D, ℓ is a lower bound (aka divisor) of every element of S, and since every element of D is a lower bound(aka divisor) of ℓ, it is a greatest lower bound of S.
As for least upper bounds, no, for example $S = X$ itself has no upper bound (no number is a multiple of every number), much less a least upper bound. $\square$

Problem 6 (due Weds 9/10): A total order $\leq$ on a set X is called a *well order* if for all $\emptyset \neq S \subseteq X$, S has a greatest lower bound in X, and moreover the greatest lower bound of S lies in S. Prove that the lexicographic order on $\mathbb{N} \times \mathbb{N}$ coming from the usual order on $\mathbb{N}$ is a well order.

Solution: Let $\emptyset \neq S \subseteq \mathbb{N} \times \mathbb{N}$. Let $x_0 = \min\{x \in \mathbb{N} \mid (x, y) \in S \text{ for some } y \in \mathbb{N}\}$. Since S is non-empty and $\mathbb{N}$ is well ordered, x_0 exists. Let $y_0 = \min\{y \in \mathbb{N} \mid (x_0, y) \in S\}$, which exists for the same reason. Now $(x_0, y_0) \in S$ and we claim it is a greatest lower bound for S. Indeed, for any $(x, y) \in S$ either $x = x_0$ and hence $y_0 \leq y$, so $(x_0, y_0) \leq (x, y)$, or else $x > x_0$ and hence $(x_0, y_0) < (x, y)$. This shows that it is a lower bound, and it is a greatest lower bound since it lies in S and so is an upper bound of every lower bound of S. $\square$

Problem 7 (due Weds 9/17): Let X be a set and $\spadesuit \in X$. Let $\mathcal{T} = \{U \subseteq X \mid \spadesuit \notin U\} \cup \{X\}$. Prove that $\mathcal{T}$ is a topology.

Solution: We have $X \in \mathcal{T}$ for free, and $\emptyset \in \mathcal{T}$ since $\spadesuit \notin \emptyset$. Now let $U_\alpha \in \mathcal{T}$ for some family $\{U_\alpha\}_{\alpha \in \Lambda}$, so for each α either $U_\alpha = X$ or $\spadesuit \notin U_\alpha$. Set $U := \bigcup_\alpha U_\alpha$. If $U \neq X$ then $U_\alpha \neq X$ for all α, so $\spadesuit \notin U_\alpha$ for all α, so $\spadesuit \notin U$. In this case $U \in \mathcal{T}$, and the other case is that $U = X$, so again $U \in \mathcal{T}$. Finally, let $U, V \in \mathcal{T}$, and we want to show $U \cap V \in \mathcal{T}$. If $U = V = X$ then $U \cap V = X \in \mathcal{T}$. If $U \neq X$ then $\spadesuit \notin U$, so $\spadesuit \notin U \cap V$, so $U \cap V \in \mathcal{T}$. Analogously if $V \neq X$ then $U \cap V \in \mathcal{T}$, so we are done. $\square$

Problem 8 (due Weds 9/17): Prove that $\{U \subseteq \mathbb{N} \mid |U| = \infty\} \cup \{\emptyset\}$ is not a topology on $\mathbb{N}$.

Solution: Let $U = \{1, 2, 4, 8, 16, \dots\}$ and $V = \{1, 3, 9, 27, 81, \dots\}$. Then U, V are in this set but $U \cap V = \{1\}$ is not. $\square$

Problem 9 (due Weds 9/17): Let X be a set and $F\colon X \to \mathcal{P}(X)$ a function, so for each $x \in X$ we have $F(x) \subseteq X$. Construct a subset $A \subseteq X$ that does not equal $F(x)$ for any $x \in X$, and conclude F is not surjective. (This shows that power sets always have strictly bigger cardinality.)

Solution: Set $A := \{x \in X \mid x \notin F(x)\}$. Now for any $x \in X$, if $x \in A$ then by the definition of A we have $x \notin F(x)$, so $A \cap F(x) = \emptyset$. This holds for all $x \in X$, so A cannot even intersect any of the $F(x)$, much less equal one of them, unless $A = F(x) = \emptyset$ for all $x \in X$.

But if $F(x) = \emptyset$ for all $x \in X$ then $x \notin F(x)$ holds for all $x \in X$, so in order to achieve $A = \emptyset$ we need $X = \emptyset$, and in this case F is the empty function from $\emptyset$ to $\{\emptyset\}$, which is not surjective. $\square$

Note: Remember there's an exam in class on Weds 9/24. It covers up through the product topology.

Problem 10 (due Weds 9/24): Let X and Y be topological spaces and let $\pi_1 \colon X \times Y \to X$ be the first projection function. Prove that for any open set W in $X \times Y$, the image $U := \pi_1(W)$ is open in X.

Solution: First note that if W is basic open, say $W = U \times V$ for U open in X and V open in Y, then $\pi_1(W) = U$ is indeed open in X. Now for arbitrary W, say $W = \bigcup_\alpha W_\alpha$ for W_α some basic open sets. Then $\pi_1(W) = \bigcup_\alpha \pi_1(W_\alpha)$ is a union of open sets, hence open. $\square$

Problem 11 (due Weds 9/24): Let X be a partially order set with partial order $\leq$. For each $x \in X$ let $B_x := \{y \in X \mid x \leq y\}$. Prove that $\mathcal{B} := \{B_x \mid x \in X\}$ is a basis for a topology on X.

Solution: Since $\leq$ is reflexive, for all $x \in X$ we have $x \in B_x$, and so the B_x cover X. Now suppose $z \in B_x \cap B_y$, so $x \leq z$ and $y \leq z$. Then for any $w \geq z$, by transitivity we have $x \leq w$ and $y \leq w$, so $w \in B_x \cap B_y$, which tells us $B_z \subseteq B_x \cap B_y$. Again by reflexivity we have $z \in B_z$, so all in all we have $z \in B_z \subseteq B_x \cap B_y$, so the B_x form a basis. $\square$

Problem 12 (due Weds 9/24): Let $X = \mathbb{R} \times \mathbb{R}$ with the lexicographic order. Let $\mathcal{T}_{lex}$ be the order topology and X and let $\mathcal{T}_{prod}$ be the standard (product) topology on X. Prove that $\mathcal{T}_{lex}$ is a refinement of $\mathcal{T}_{prod}$, but $\mathcal{T}_{prod}$ is not a refinement of $\mathcal{T}_{lex}$.

Solution: Let $U \times V$ be a basic open set in $\mathcal{T}_{prod}$ and let $(x, y) \in U \times V$. Choose $\varepsilon > 0$ such that $(y - \varepsilon, y + \varepsilon) \subseteq V$. Now the lex order interval $((x, y - \varepsilon), (x, y + \varepsilon))$ contains (x, y) and lies in $U \times V$, so we conclude that $\mathcal{T}_{lex}$ is a refinement of $\mathcal{T}_{prod}$. To see that the other direction is false, note that the lex order interval $((0, 0), (0, 1))$ is open in $\mathcal{T}_{lex}$, but not in $\mathcal{T}_{prod}$ since any open neighborhood of $(0, 0)$ in X contains points with first entry non-zero but all the points in $((0, 0), (0, 1))$ have first entry zero. $\square$

Problem 13 (due Weds 10/1): A topology on a set X is called *Alexandrov* if the collection of closed sets also forms a topology. Prove that if the finite complement topology on X is Alexandrov then it is discrete.

Solution: Let $x \in X$, so $\{x\}$ is closed in the finite complement topology. Since this topology is Alexandrov, arbitrary unions of closed sets are closed. Every set is the union of its singleton subsets, so every subset is closed, so the topology is discrete. $\square$

Problem 14 (due Weds 10/1): Let $P := \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2\}$ be the graph of $y = x^2$. Prove that P is closed in $\mathbb{R} \times \mathbb{R}$ (by proving its complement is open).

Solution: Let $(x_0, y_0) \in (\mathbb{R}\times\mathbb{R}) \setminus P$, so $x_0^2 \neq y_0$. The function $d(x, y) := \sqrt{(x - x_0)^2 + (y - y_0)^2}$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ restricted to P has outputs of the form $\sqrt{(x - x_0)^2 + (x^2 - y_0)^2}$, which is bounded below by some $\varepsilon > 0$ since $x_0^2 \neq y_0$. Now the open ball of radius ε centered at (x_0, y_0) lies outside P, and we conclude P is closed. $\square$

Problem 15 (due Weds 10/1): Let X be a topological space with topology $\mathcal{T}$, and $Y \subseteq X$ a subspace with the subspace topology $\mathcal{T}_Y$. Assume that $\{x\}$ is closed in X for all $x \in X$. Let $\phi \colon \mathcal{T} \to \mathcal{T}_Y$ be the function $\phi(U) := U \cap Y$. (So we know ϕ is surjective.) Prove that if ϕ is injective then $Y = X$.

Solution: Suppose ϕ is injective. Let $x \in X$. We know $X \setminus \{x\}$ is open, and does not equal the open set X, so by injectivity $\phi(X \setminus \{x\}) \neq \phi(X)$, i.e., $(X \setminus \{x\}) \cap Y \neq Y$. We conclude $x \in Y$, and so $X = Y$. $\square$

Problem 16 (due Weds 10/8): Prove that if a topological space is Hausdorff and Alexandrov (see Problem 13), then it is discrete.

Solution: Let A be a subset of such a space X. Since X is Hausdorff, $\{a\}$ is closed in X for all $a \in A$. Since X is Alexandrov, the union $A = \bigcup_{a \in A}\{a\}$ of closed sets is closed. Thus every subset of X is closed, i.e., X is discrete. $\square$

Problem 17 (due Weds 10/8): Let $f \colon X \to Y$ be a function. Suppose Y is a topological space. Let $\mathcal{T} := \{f^{-1}(U) \mid U \text{ is open in } Y\} \subseteq \mathcal{P}(X)$. Prove that $\mathcal{T}$ is a topology on X, and that any topology $\mathcal{T}'$ on X with respect to which f is continuous is a refinement of $\mathcal{T}$.

Solution: Since $\emptyset = f^{-1}(\emptyset)$ and $X = f^{-1}(Y)$ we have $\emptyset, X \in \mathcal{T}$. Now let $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{T}$, so for each α we have $U_\alpha = f^{-1}(V_\alpha)$ for some open $V_\alpha \subseteq Y$. Then $\bigcup_{\alpha \in \Lambda} U_\alpha = \bigcup_{\alpha \in \Lambda} f^{-1}(V_\alpha) = f^{-1}\left(\bigcup_{\alpha \in \Lambda} V_\alpha\right)$ is of the form $f^{-1}(V)$ for V open in Y, hence this union is in $\mathcal{T}$. Finally, an analogous thing works for intersections. Now if $\mathcal{T}'$ is any topology on X such that f is continuous, then $f^{-1}(V) \in \mathcal{T}'$ for all open V in Y, and so every set in $\mathcal{T}$ lies in $\mathcal{T}'$. $\square$

Problem 18 (due Weds 10/8): Let $X = \mathbb{N}$ with the finite complement topology. Let (x_n) be the sequence $x_n = n$. Prove that for all $y \in X$, the sequence (x_n) converges to y.

Solution: Let U be an open neighborhood of y, so $X \setminus U$ is finite. Let $N = \max(X \setminus U)$. Then $x_n \in U$ for all $n > N$, so in particular $x_n \in U$ for all but finitely many n, i.e., (x_n) converges to y. $\square$

Nothing due Weds 10/15, since Monday and Tuesday are October break.

Problem 19 (due Weds 10/22): Let X and Y be topological spaces with the finite complement topology. Prove that X and Y are homeomorphic if and only if they have the same cardinality.

Solution: It suffices to prove that every bijection in this situation is continuous. Indeed, the preimage of any closed set is closed, since bijections preserve finiteness. $\square$

Problem 20 (due Weds 10/22): Let $\{X_\alpha\}_{\alpha\in\Lambda}$ and $\{Y_\alpha\}_{\alpha\in\Lambda}$ be two families of topological spaces (with the same indexing set Λ). Suppose that $X_\alpha \cong Y_\alpha$ for each $\alpha \in \Lambda$ (here "$\cong$" means homeomorphic). Prove that $\prod\limits_{\alpha\in\Lambda} X_\alpha \cong \prod\limits_{\alpha\in\Lambda} Y_\alpha$ (using the product topology for both).

Solution: For each $\alpha \in \Lambda$ let $\phi_\alpha \colon X_\alpha \to Y_\alpha$ be a homeomorphism. Define a function Φ from $\prod\limits_{\alpha\in\Lambda} X_\alpha$ to $\prod\limits_{\alpha\in\Lambda} Y_\alpha$ as follows. Given $f \in \prod\limits_{\alpha\in\Lambda} X_\alpha$, let $\Phi(f)$ be the element of $\prod\limits_{\alpha\in\Lambda} Y_\alpha$ defined via $\Phi(f)(\alpha) := \phi_\alpha(f(\alpha))$. Since $f(\alpha) \in X_\alpha$, this is well defined. Writing π_α^X and π_α^Y for the respective projection maps, we have $\pi_\alpha^Y \circ \Phi = \phi_\alpha \circ \pi_\alpha^X$ for all α, and each $\phi_\alpha \circ \pi_\alpha^X$ is continuous, so by a theorem from class Φ is continuous. By the same argument, $\Psi(g)(\alpha) := \phi_\alpha^{-1}(g(a\alpha))$ gives a well defined continuous map Ψ from $\prod\limits_{\alpha\in\Lambda} Y_\alpha$ to $\prod\limits_{\alpha\in\Lambda} X_\alpha$, and $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the respective identity maps, so Φ is invertible with inverse Ψ, hence a homeomorphism. $\square$

Problem 21 (due Weds 10/22): Let $\phi \colon \prod\limits_{n\in\mathbb{N}} [0,1] \to [0,1]$ be given by $\phi(a_1, a_2, a_3, \dots) := \sum\limits_{n\in\mathbb{N}} \frac{a_n}{2^n}$. (This series converges and lands in $[0,1]$ by the comparison test, so we know ϕ is well defined.) Let $A \subseteq \prod\limits_{n\in\mathbb{N}} [0,1]$ be the subspace of all $(a_1, a_2, \dots)$ such that for some $N \in \mathbb{N}$, $a_n = 0$ for all $n > N$ (we're using the product topology). Prove that A is not open, but its image $\phi(A)$ is open.

Solution: To see that A is not open, note that any basic open set $\prod\limits_{n\in\mathbb{N}} U_n$ must have $U_n = [0,1]$ for all large enough n, and so this basic open set is not contained in A. To see that $\phi(A)$ is open, we claim that it equals $[0,1)$, which is open in $[0,1]$. Note that for any $(a_1, a_2, \dots) \in A$, the sequence $\sum\limits_{n=k}^{\infty} \frac{a_n}{2^n}$ is eventually constant 0 as k goes to ∞, and so $\phi(a_1, a_2, \dots)$ is bounded above by a finite sum of the form $1/2 + 1/4 + \cdots 1/2^m$, which is strictly less than 1. This shows that $\phi(A) \subseteq [0,1)$. For the reverse inclusion we note that the sequence of partial sums

$\sum_{n=1}^{m} \frac{1}{2^n}$ converges to 1, and these are all of the form $\phi(1,1,\dots,1,0,0,\dots) \in \phi(A)$, so 1 is a limit point of $\phi(A)$. $\square$

Note: Remember there's an exam in class on Weds 10/29. It covers up through connected spaces.

Problem 22 (due Weds 10/29): Let (X,d) be a metric space, with the metric topology. Suppose that for all $x,y,z \in X$ we have $d(x,z) \leq \max\{d(x,y), d(y,z)\}$. Prove that any intersection of basic open sets is a basic open set.

Solution: Let $B_r(x)$ and $B_s(y)$ be basic open sets. If their intersection is empty then this is technically a basic open set, being a ball of radius zero. Now suppose their intersection is non-empty, say $z \in B_r(x) \cap B_s(y)$, so $d(z,x) < r$ and $d(z,y) < s$. In particular $d(x,y) < \max\{r,s\}$, say without loss of generality it's r. Now we claim that $B_s(y) \subseteq B_r(x)$. Let $w \in B_s(y)$. Then $d(w,x) \leq \max\{d(w,y), d(y,x)\} < r$, so $w \in B_r(x)$ and we are done. $\square$

Problem 23 (due Weds 10/29): Let X and Y be topological spaces, both with the finite complement topology. Prove that every continuous surjection $X \to Y$ is a quotient map. [This feels familiar, but I don't think (?) I already proved it in class...if I did, free points.]

Solution: Let $f\colon X \to Y$ be a continuous surjection. Let $N \subseteq Y$ be a non-open subset, so $Y \setminus N$ is infinite and not all of Y. Since f is surjective, $f^{-1}(Y \setminus N)$ is infinite and not all of X, hence not open. $\square$

Problem 24 (due Weds 10/29): Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and let $\Gamma(f) := \{(x,y) \in \mathbb{R}^2 \mid y = f(x)\}$ be the *graph* of f (with the subspace topology in $\mathbb{R}^2$). Prove that if f is continuous then $\Gamma(f)$ is connected. [Don't use "path connectivity", we haven't talked about that yet. I suggest thinking about maps onto S^0.]

Solution: Suppose f is continuous. The identity map on $\mathbb{R}$ is continuous, so the map $\mathrm{id} \times f\colon \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ sending x to $(x, f(x))$ is continuous. The image of $\mathrm{id} \times f$ is precisely $\Gamma(f)$, so $\Gamma(f)$ is the continuous image of the connected space $\mathbb{R}$, hence is connected. $\square$

Nothing due Weds 11/5.

Problem 25 (due Weds 11/12): Prove that every connected component of a topological space is connected.

Solution: Let C be a connected component of a space X (with the subspace topology). Fix some $c_0 \in C$. For each $c \in C$ let D_c be a connected subspace of X containing both c_0 and c (this exists by the definition of connected component). Since D_c is connected and intersects C it must be contained in C. Now C equals the union of all the D_c, and this is connected since the D_c are connected and all share a point, namely c_0. $\square$

Problem 26 (due Weds 11/12): A topological space is *totally disconnected* if every connected component is a singleton. Prove that if X is totally disconnected and locally connected, then it is discrete.

Solution: Let $x \in X$, and we need to prove $\{x\}$ is open. Since X is locally connected, x is contained in a connected open set U. Since every connected component is a singleton, U must be contained in a singleton, which must be $\{x\}$ since $x \in U$. We conclude $\{x\} = U$ is open. $\square$

Problem 27 (due Weds 11/12): Prove that $\mathbb{R}$ with the finite complement topology is locally path connected.

Solution: We claim that every open subset of the space is path connected, from which local path connectivity will follow. The empty set is vacuously path connected, so consider an arbitrary non-empty open set U, i.e., $\mathbb{R} \setminus U$ is finite. Let $p\colon [a,b] \to \mathbb{R} \setminus U$ be an arbitrary injective function. A subset of $\mathbb{R} \setminus U$ is closed in $\mathbb{R} \setminus U$ if and only if it is finite or all of $\mathbb{R} \setminus U$. Thus by injectivity, the preimage under p of any closed set is either finite or all of $[a,b]$. In particular it is closed, since $[a,b]$ is Hausdorff. We conclude p is continuous. Having shown that every injective function from $[a,b]$ to $\mathbb{R} \setminus U$ is a path, we conclude that $\mathbb{R} \setminus U$ is path connected, since we can choose $p(a)$ and $p(b)$ to be whatever points we want. $\square$

Problem 28 (due Weds 11/19): Let X be a topological space with subspaces A and B. Prove that if A and B are compact then $A \cup B$ is compact. Give a counterexample to show that $A \cap B$ is not necessarily compact.

Problem 29 (due Weds 11/19): Let $f\colon X \to Y$ be a surjective open map. Prove that if X is second countable then so is Y.

Problem 30 (due Weds 11/19): Let X be a topological space. Suppose X admits two bases $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{B}_1$ is countable and every member of $\mathcal{B}_2$ is closed. Prove that X is metrizable.
