Updated October 1, 2025

Homework problems for AMAT 540A (Topology I), Fall 2025. Over the course of the semester I'll add problems to this list, with each problem's due date specified. Each problem is worth 2 points.

Solutions will be gradually added (and may be hastily written without proofreading).

Problem 1 (due Weds 9/3): Let A be a set and let R be a relation on A that is symmetric and antisymmetric, but not reflexive. Prove that there exists $a \in A$ such that for all $b \in A$, aRb does not hold.

Solution: Since R is not reflexive, we can choose $a \in A$ such that aRa does not hold. Now for any $b \in A$, if aRb then by symmetry bRa, and by antisymmetry $a = b$, so aRa. We conclude aRb never holds. $\square$

Problem 2 (due Weds 9/3): Let $f\colon A \to B$ be a function. Suppose that for all $B_0 \subseteq B$ we have $f(f^{-1}(B_0)) = B_0$. Prove that f is surjective.

Solution: Let $b \in B$. We have $f(f^{-1}(\{b\})) = \{b\}$, so $f^{-1}(\{b\})$ is non-empty, i.e., there exists $a \in A$ with $f(a) = b$. Thus f is surjective. $\square$

Problem 3 (due Weds 9/3): Prove (using element arguments) that $(A \setminus B) \cap C = (A \cap C) \setminus B$ for sets A, B, and C.

Solution: ($\subseteq$): Let $x \in (A \setminus B) \cap C$, so $x \in A$ and $x \notin B$ and $x \in C$. Thus $x \in (A \cap C) \setminus B$.
($\supseteq$): Same thing really, this one was too easy. $\square$

Problem 4 (due Weds 9/10): Let X be a set and view the power set $\mathcal{P}(X)$ as a partially ordered set with the partial order $\subseteq$. Prove that every non-empty subset $\mathcal{A} \subseteq \mathcal{P}(X)$ has a least upper bound.

Solution: Let $U = \bigcup_{A \in \mathcal{A}} A$. Since $A \subseteq U$ for all $A \in \mathcal{A}$, U is an upper bound of $\mathcal{A}$. For any other upper bound V, we have $A \subseteq V$ for all $A \in \mathcal{A}$, and hence $\bigcup_{A \in \mathcal{A}} A \subseteq V$, i.e., $U \subseteq V$. Thus U is a least upper bound. $\square$

Problem 5 (due Weds 9/10): Let $X = \mathbb{N}$ with the partial order $|$ (this is "divides", i.e., "$a|b$" means "a divides b"). Prove that every non-empty subset $S \subseteq X$ has a greatest lower bound with respect to $|$. Does every such S have a least upper bound?

Solution: Let $\emptyset \neq S \subseteq X$. Let $D = \{d \in X \mid d|s \text{ for all } s \in S\}$, so D non-empty since $1 \in D$. Moreover, D is finite since each $s \in S$ has only finitely many divisors. Let $\ell \in X$ be the least common multiple of the elements of D. Since every element of S is a multiple of every element of D, ℓ is a lower bound (aka divisor) of every element of S, and since every element of D is a lower bound(aka divisor) of ℓ, it is a greatest lower bound of S.
As for least upper bounds, no, for example $S = X$ itself has no upper bound (no number is a multiple of every number), much less a least upper bound. $\square$

Problem 6 (due Weds 9/10): A total order $\leq$ on a set X is called a *well order* if for all $\emptyset \neq S \subseteq X$, S has a greatest lower bound in X, and moreover the greatest lower bound of S lies in S. Prove that the lexicographic order on $\mathbb{N} \times \mathbb{N}$ coming from the usual order on $\mathbb{N}$ is a well order.

Solution: Let $\emptyset \neq S \subseteq \mathbb{N} \times \mathbb{N}$. Let $x_0 = \min\{x \in \mathbb{N} \mid (x, y) \in S \text{ for some } y \in \mathbb{N}\}$. Since S is non-empty and $\mathbb{N}$ is well ordered, x_0 exists. Let $y_0 = \min\{y \in \mathbb{N} \mid (x_0, y) \in S\}$, which exists for the same reason. Now $(x_0, y_0) \in S$ and we claim it is a greatest lower bound for S. Indeed, for any $(x, y) \in S$ either $x = x_0$ and hence $y_0 \leq y$, so $(x_0, y_0) \leq (x, y)$, or else $x > x_0$ and hence $(x_0, y_0) < (x, y)$. This shows that it is a lower bound, and it is a greatest lower bound since it lies in S and so is an upper bound of every lower bound of S. $\square$

Problem 7 (due Weds 9/17): Let X be a set and $\spadesuit \in X$. Let $\mathcal{T} = \{U \subseteq X \mid \spadesuit \notin U\} \cup \{X\}$. Prove that $\mathcal{T}$ is a topology.

Solution: We have $X \in \mathcal{T}$ for free, and $\emptyset \in \mathcal{T}$ since $\spadesuit \notin \emptyset$. Now let $U_\alpha \in \mathcal{T}$ for some family $\{U_\alpha\}_{\alpha \in \Lambda}$, so for each α either $U_\alpha = X$ or $\spadesuit \notin U_\alpha$. Set $U := \bigcup_\alpha U_\alpha$. If $U \neq X$ then $U_\alpha \neq X$ for all α, so $\spadesuit \notin U_\alpha$ for all α, so $\spadesuit \notin U$. In this case $U \in \mathcal{T}$, and the other case is that $U = X$, so again $U \in \mathcal{T}$. Finally, let $U, V \in \mathcal{T}$, and we want to show $U \cap V \in \mathcal{T}$. If $U = V = X$ then $U \cap V = X \in \mathcal{T}$. If $U \neq X$ then $\spadesuit \notin U$, so $\spadesuit \notin U \cap V$, so $U \cap V \in \mathcal{T}$. Analogously if $V \neq X$ then $U \cap V \in \mathcal{T}$, so we are done. $\square$

Problem 8 (due Weds 9/17): Prove that $\{U \subseteq \mathbb{N} \mid |U| = \infty\} \cup \{\emptyset\}$ is not a topology on $\mathbb{N}$.

Solution: Let $U = \{1, 2, 4, 8, 16, \dots\}$ and $V = \{1, 3, 9, 27, 81, \dots\}$. Then U, V are in this set but $U \cap V = \{1\}$ is not. $\square$

Problem 9 (due Weds 9/17): Let X be a set and $F\colon X \to \mathcal{P}(X)$ a function, so for each $x \in X$ we have $F(x) \subseteq X$. Construct a subset $A \subseteq X$ that does not equal $F(x)$ for any $x \in X$, and conclude F is not surjective. (This shows that power sets always have strictly bigger cardinality.)

Solution: Set $A := \{x \in X \mid x \notin F(x)\}$. Now for any $x \in X$, if $x \in A$ then by the definition of A we have $x \notin F(x)$, so $A \cap F(x) = \emptyset$. This holds for all $x \in X$, so A cannot even intersect any of the $F(x)$, much less equal one of them, unless $A = F(x) = \emptyset$ for all $x \in X$.

But if $F(x) = \emptyset$ for all $x \in X$ then $x \notin F(x)$ holds for all $x \in X$, so in order to achieve $A = \emptyset$ we need $X = \emptyset$, and in this case F is the empty function from $\emptyset$ to $\{\emptyset\}$, which is not surjective. $\square$

Note: Remember there's an exam in class on Weds 9/24. It covers up through the product topology.

Problem 10 (due Weds 9/24): Let X and Y be topological spaces and let $\pi_1 \colon X \times Y \to X$ be the first projection function. Prove that for any open set W in $X \times Y$, the image $U := \pi_1(W)$ is open in X.

Solution: First note that if W is basic open, say $W = U \times V$ for U open in X and V open in Y, then $\pi_1(W) = U$ is indeed open in X. Now for arbitrary W, say $W = \bigcup_\alpha W_\alpha$ for W_α some basic open sets. Then $\pi_1(W) = \bigcup_\alpha \pi_1(W_\alpha)$ is a union of open sets, hence open. $\square$

Problem 11 (due Weds 9/24): Let X be a partially order set with partial order $\leq$. For each $x \in X$ let $B_x := \{y \in X \mid x \leq y\}$. Prove that $\mathcal{B} := \{B_x \mid x \in X\}$ is a basis for a topology on X.

Solution: Since $\leq$ is reflexive, for all $x \in X$ we have $x \in B_x$, and so the B_x cover X. Now suppose $z \in B_x \cap B_y$, so $x \leq z$ and $y \leq z$. Then for any $w \geq z$, by transitivity we have $x \leq w$ and $y \leq w$, so $w \in B_x \cap B_y$, which tells us $B_z \subseteq B_x \cap B_y$. Again by reflexivity we have $z \in B_z$, so all in all we have $z \in B_z \subseteq B_x \cap B_y$, so the B_x form a basis. $\square$

Problem 12 (due Weds 9/24): Let $X = \mathbb{R} \times \mathbb{R}$ with the lexicographic order. Let $\mathcal{T}_{lex}$ be the order topology and X and let $\mathcal{T}_{prod}$ be the standard (product) topology on X. Prove that $\mathcal{T}_{lex}$ is a refinement of $\mathcal{T}_{prod}$, but $\mathcal{T}_{prod}$ is not a refinement of $\mathcal{T}_{lex}$.

Solution: Let $U \times V$ be a basic open set in $\mathcal{T}_{prod}$ and let $(x, y) \in U \times V$. Choose $\varepsilon > 0$ such that $(y - \varepsilon, y + \varepsilon) \subseteq V$. Now the lex order interval $((x, y - \varepsilon), (x, y + \varepsilon))$ contains (x, y) and lies in $U \times V$, so we conclude that $\mathcal{T}_{lex}$ is a refinement of $\mathcal{T}_{prod}$. To see that the other direction is false, note that the lex order interval $((0, 0), (0, 1))$ is open in $\mathcal{T}_{lex}$, but not in $\mathcal{T}_{prod}$ since any open neighborhood of $(0, 0)$ in X contains points with first entry non-zero but all the points in $((0, 0), (0, 1))$ have first entry zero. $\square$

Problem 13 (due Weds 10/1): A topology on a set X is called *Alexandrov* if the collection of closed sets also forms a topology. Prove that if the finite complement topology on X is Alexandrov then it is discrete.

Problem 14 (due Weds 10/1): Let $P := \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2\}$ be the graph of $y = x^2$. Prove that P is closed in $\mathbb{R} \times \mathbb{R}$ (by proving its complement is open).

Problem 15 (due Weds 10/1): Let X be a topological space with topology $\mathcal{T}$, and $Y \subseteq X$ a subspace with the subspace topology $\mathcal{T}_Y$. Assume that $\{x\}$ is closed in X for all $x \in X$. Let $\phi\colon \mathcal{T} \to \mathcal{T}_Y$ be the function $\phi(U) := U \cap Y$. (So we know ϕ is surjective.) Prove that if ϕ is injective then $Y = X$.

Problem 16 (due Weds 10/8): Prove that if a topological space is Hausdorff and Alexandrov (see Problem 13), then it is discrete.

Problem 17 (due Weds 10/8): Let $f\colon X \to Y$ be a function. Suppose Y is a topological space. Let $\mathcal{T} := \{f^{-1}(U) \mid U \text{ is open in } Y\} \subseteq \mathcal{P}(X)$. Prove that $\mathcal{T}$ is a topology on X, and that any topology $\mathcal{T}'$ on X with respect to which f is continuous is a refinement of $\mathcal{T}$.

Problem 18 (due Weds 10/8): Let $X = \mathbb{N}$ with the finite complement topology. Let (x_n) be the sequence $x_n = n$. Prove that for all $y \in X$, the sequence (x_n) converges to y.
