SOME OPEN PROBLEMS

MATTHEW C. B. ZAREMSKY

Here are some open problems that I specifically like. In general these are just stated, with minimal explanation or background. Some are famous, some are not; some might be easy in the end, most probably not. Any new problems I've added over the years include the date added; the original list was made on 6/15/23. The numbering of the problems is fixed forever.

1. Finiteness properties, BNSR-invariants, and Dehn functions

1. Does every group of type F_n embed in a group of type F_{n+1}, for $n \geq 2$? What about in a group of type F_∞?
2. Does every group of type F_3 have solvable word problem? [The answer is no – Xiaolei Wu pointed out a result of Collins and Miller producing a group of type F with unsolvable word problem.]
3. If a finitely generated group is quasi-isometric to an $(n-1)$-connected proper metric space, is the group of type F_n? (The converse is true.)
4. Are type FP_n and type FH_n equivalent? That is, does every group of type FP_n admit a proper, cocompact action on an $(n-1)$-acyclic complex?
5. Are all Artin groups of type F_∞?
6. Compute the BNS-invariant $\Sigma^1(A)$ for an arbitrary Artin group.
7. Conjecture of Juan-Pineda–Leary: If a group admits a finite classifying space for the family of virtually cyclic subgroups, then the group is virtually cyclic. Related Conjecture of von Puttkamer–Wu: If a finitely presented group has finitely many virtually cyclic subgroups into which all virtually cyclic subgroups are conjugate, then the group is virtually cyclic.
8. Compute the BNSR-invariants of graph braid groups.
9. Compute the BNSR-invariants of pure braid groups.
10. Find a finitely presented (or F_∞), infinite, simple, amenable group.
11. Find a finitely presented (or F_∞), infinite, torsion group.
12. Find an amenable, non-elementary amenable group of type F_∞.
13. Every $\Sigma^n(G)$ is an open subset of the character sphere $\Sigma(G)$, but what about $\Sigma^\infty(G)$? Must that be open?
14. Compute the BNSR-invariants of Röver–Nekrashevych groups.
15. Find finitely presented groups G_n ($n \geq 2$) with quadratic Dehn function such that G_n is of type F_n but not F_{n+1}. (E.g., do Bieri–Stallings groups all have quadratic

Date: September 24, 2025.

Dehn function?) [Actually this isn't open, it was proved by Carter and Forester using Bieri–Stallings groups.]

16. Find finitely presented simple groups with arbitrarily large (recursive) Dehn function.
17. (Vague:) If a finitely presented group acts cocompactly on a highly connected complex with finitely presented cell stabilizers, prove that the Dehn function is determined by the filling function of the complex, the Dehn functions of the stabilizers, and the distortion of the stabilizers.
18. Does there exist a group G of type F_∞ such that the BNSR-invariants $\Sigma^1(G) \supsetneq \Sigma^2(G) \supsetneq \cdots$ are all properly nested?
19. Conjecture: If a right-angled Artin group has a finite index subgroup that maps onto $\mathbb{Z}$ with kernel of type F_n, then so does the right-angled Artin group itself.
20. (Added 6/12/25) Are there uncountably many simple groups of type FP_2? Of type FP?
21. (Added 7/24/25) For a group G, if $G \times \mathbb{Z}$ is of type F then is G of type F? This turns out to be equivalent to: If G is of type F_∞ and has finite cohomological dimension then is G of type F?

2. Thompson groups

1. Is Thompson's group F amenable?
2. Is F automatic?
3. Is every subgroup of F either elementary amenable or else contains a copy of F? (Brin–Sapir Conjecture)
4. (Bridson) Is F quasi-isometric to $F \times \mathbb{Z}$? (Related:) Is F quasi-isometric to $F \times F$? Is F quasi-isometric to T? (Surely not, but why not?) "Easier": Does F quasi-retract onto $F \times \mathbb{Z}$ or $F \times F$? Does T quasi-retract onto F?
5. Prove T and V have quadratic Dehn function. Compute the higher-rank Dehn functions of F, T, and V (are they all euclidean?) [Update: Matteo Migliorini has proved that T has quadratic Dehn function: `https://arxiv.org/abs/2410.23088`.]
6. Prove the Stein group $F_{2,3}$ does not embed into V.
7. Do the Brin–Thompson groups nV ($n \geq 2$) have the Haagerup property? Or Property (T)?
8. Find an infinite dimensional torsion-free group of type F_∞ that does not contain F.
9. Is braided V acyclic?
10. Is braided V Hopfian? Is every quotient of braided V finitely presented? (This would imply Hopfian.)
11. Is braided V inner amenable? (Probably not.)
12. Prove "loop braided V" is of type F_∞.
13. Compute the BNSR-invariants of the pure braided Brown–Higman–Thompson groups $bF_{n,r}$.

14. Boone–Higman Conjecture: A finitely generated group has solvable word problem iff it embeds in a finitely presented simple group. (I put this in the "Thompson groups" section since a resolution to this conjecture would presumably come from the world of Thompson-like groups.)
15. (Added 2/10/24) Does every torsion-free subgroup of V with no non-abelian free subgroups embed into F?
16. (Added 2/10/24) Lehnert's conjecture: Every group with context-free co-word problem embeds in Thompson's group V. So, question, given a cloning system on a family of finite groups G_n, is the Thompson-like group $\mathcal{T}(G_*)$ coCF and/or does it embed into V?
17. (Added 2/10/24) Does T contain a copy of $T \times T$? (Surely yes, but how?) [In fact no! Jim Belk points out that every finite subgroup of T is cyclic, so T cannot contain $T \times T$.]
18. (Added 6/19/24) Does Thompson's group F admit a cobounded action by isometries on the hyperbolic plane?
19. (Added 10/2/24) Various questions about which groups embed in Thompson's group V and/or the Brin–Thompson groups $2V$, $3V$, etc. Does every hyperbolic group embed in one of these? Is there a hyperbolic group that does not embed in V? Does every $\mathrm{GL}_n(\mathbb{Z})$ embed in some mV? Maybe even in $2V$? Does $\mathbb{Q}$ embed in $2V$?

3. Matrix groups, braid groups, etc

1. Is the Torelli group of $\mathrm{Out}(F_n)$ (or $MCG(S_g)$, or $\mathrm{Aut}(F_n)$) finitely presented for large enough n or g?
2. Do braid groups embed in $\mathrm{GL}_n(\mathbb{Z})$?
3. Is $\mathrm{SL}_2(\mathbb{Z}[t, t^{-1}])$ finitely generated?
4. Is $\mathrm{SL}_3(\mathbb{Z})$ coherent? (Meaning every finitely generated subgroup is finitely presented.) (Famous problem of Serre.)
5. Is the 4-strand Burau representation faithful?
6. Are braid groups CAT(0)? Even without requiring cocompactness this is unknown.
7. Do braid groups have the Haagerup property?
8. Given an Artin group A, we can embed it in the finitely generated simple twisted Brin–Thompson group AV_A. Does this group embed in a finitely presented group? If so, then this would prove that Artin groups have solvable word problem.
9. (Added 5/18/24) Are braid groups self-similar? More generally, are there any examples of (say, finitely presented) groups that are residually finite but don't embed in a (say, finitely presented) self-similar group?
10. (Added 3/20/25) Does every $\mathrm{Out}(F_m)$ embed in some $\mathrm{Aut}(F_n)$? Does every $\mathrm{MCG}(S_g)$ embed in some $\mathrm{Out}(F_n)$ and/or some $\mathrm{Aut}(F_n)$?

4. Vietoris–Rips complexes

1. If a group is amenable and automatic, does it have a contractible Rips complex? (This would show that F cannot be both amenable and automatic.)
2. Is every (connected) Rips complex of every (finite?) subset of the plane homotopy equivalent to a wedge of spheres?
3. Is the Rips complex of $\mathbb{Z}^n$ with the standard word metric contractible for large enough Rips parameter? [The answer is yes, recently proved by Ziga Virk in arXiv:2405.09134.]
4. If a group is of type F_n does it admit an $(n-1)$-connected Rips complex? For $n = 2$ the answer is yes, but for $n > 2$ we don't know.
5. (Added 10/2/24): Is the Rips complex of any RAAG with the standard word metric contractible for large enough Rips parameter?
6. (Added 10/2/24): If a group is of type F_∞ and has finite cohomological dimension, then does it have a contractible Rips complex? (This would imply it's of type F, so this would solve Problem 21 in Section 1 above.)
7. (Added 7/24/25): Relatedly, does every group of type F have a contractible Rips complex?

Department of Mathematics and Statistics, University at Albany (SUNY), Albany, NY
Email address: mzaremsky@albany.edu