We note that Bessel functions of half-integer order are expressible in closed form in terms of trigonometric functions, as illustrated in the following example.

▶ *Find the general solution of*

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$$

This is Bessel's equation with v=1/2, so from (18.80) the general solution is simply

$$y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$$

However, Bessel functions of half-integral order can be expressed in terms of trigonometric functions. To show this, we note from (18.79) that

$$J_{\pm 1/2}(x) = x^{\pm 1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n\pm 1/2} n! \Gamma(1 + n \pm \frac{1}{2})}.$$

Using the fact that $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we find that, for $v = 1/2$

$$\begin{aligned}
J_{1/2}(x) &= \frac{(\frac{1}{2}x)^{1/2}}{\Gamma(\frac{3}{2})} - \frac{(\frac{1}{2}x)^{5/2}}{1!\Gamma(\frac{5}{2})} + \frac{(\frac{1}{2}x)^{9/2}}{2!\Gamma(\frac{7}{2})} - ... \\
&= \frac{(\frac{1}{2}x)^{1/2}}{(\frac{1}{2})\sqrt{\pi}} - \frac{(\frac{1}{2}x)^{5/2}}{1!(\frac{3}{2})(\frac{1}{2})\sqrt{\pi}} + \frac{(\frac{1}{2}x)^{9/2}}{2!(\frac{5}{2})(\frac{3}{2})(\frac{1}{2})\sqrt{\pi}} - ... \\
&= \frac{(\frac{1}{2}x)^{1/2}}{(\frac{1}{2})\sqrt{\pi}} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - ...\right) = \frac{(\frac{1}{2}x)^{1/2}}{(\frac{1}{2})\sqrt{\pi}} \frac{\sin x}{x} = \sqrt{\frac{2}{\pi x}} \sin x,
\end{aligned}$$

whereas for $v = -1/2$ we obtain

$$\begin{aligned}
J_{-1/2}(x) &= \frac{(\frac{1}{2}x)^{-1/2}}{\Gamma(\frac{1}{2})} - \frac{(\frac{1}{2}x)^{3/2}}{1!\Gamma(\frac{3}{2})} + \frac{(\frac{1}{2}x)^{7/2}}{2!\Gamma(\frac{5}{2})} - ... \\
&= \frac{(\frac{1}{2}x)^{-1/2}}{\sqrt{\pi}} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...\right) = \sqrt{\frac{2}{\pi x}} \cos x.
\end{aligned}$$

Therefore the general solution we require is

$$y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x) = c_1 \sqrt{\frac{2}{\pi x}} \sin x + c_2 \sqrt{\frac{2}{\pi x}} \cos x. \blacktriangleleft$$

18.5.2 Bessel functions for integer v

The definition of the Bessel function $J_v(x)$ given in (18.79) is, of course, valid for all values of v, but, as we shall see, in the case of integer v the general solution of Bessel's equation cannot be written in the form (18.80). Firstly, let us consider the case $v = 0$, so that the two solutions to the indicial equation are equal, and we clearly obtain only one solution in the form of a Frobenius series. From (18.79), a4paper,total=180mm,270mm,left=20mm,top=25mm,right=25mm

```
\documentclass[12pt,letterpaper]{article}
\usepackage{nopageno}
\usepackage{geometry}
\geometry{a4paper,total={180mm,270mm},left=20mm,top=25mm,right=25mm}
\usepackage{tcolorbox}
\usepackage{amsmath,amsthm,amsfonts,amssymb,amscd}
\usepackage{fancyhdr}
\pagestyle{fancyplain}
\chead{\textbf{ 18.5 BESSEL FUNCTIONS}}


\setlength{\parindent}{0pt}%
\begin{document}
We note that Bessel functions of half-integer order are expressible in closed form in te

\begin{tcolorbox}
\item\blacktriangleright\textit{Find the general solution of}

$$x^2y''+xy'+(x^2-\frac{1}{4})y=0$$
\end{tcolorbox}


This is Bessel's equation with v=1/2, so from (18.80) the general solution is simply
$$y(x)=c_1J_{1/2}(x)+c_2J_{-1/2}(x)$$

However, Bessel functions of half-integral order can be expressed in terms of trigonomet

$$J_{\pm1/2}(x)=x^{\pm1/2}\sum_{n=0}^\infty\frac{(-1)^nx^{2n}}{2^{2n\pm1/2}n!\Gamma(1+n\

Using the fact that $ \Gamma(x+1)=x\Gamma(x)$ and  $ \Gamma(\frac{1}{2})=\sqrt{\pi}$, we

\begin{alignat*}{2} J_{1/2}(x) &=\frac{(\frac{1}{2}x)^{1/2}}{\Gamma(\frac{3}{2})}-\frac{
\end{alignat*}

whereas for $v=-1/2$ we obtain

\begin{alignat*}{2} J_{-1/2}(x) &=\frac{(\frac{1}{2}x)^{-1/2}}{\Gamma(\frac{1}{2})}-\fra
\end{alignat*}

Therefore the general solution we require is

$$y(x)=c_1J_{1/2}(x)+c_2J_{-1/2}(x)=c_1\sqrt{\frac{2}{\pi x}}\sin{x}+c_2\sqrt{\frac{2}{\



$$\textbf{\textit{18.5.2 Bessel functions for integer v}}$$
The definition of the Bessel function $J_v(x)$ given in (18.79) is, of course, valid for

$$605$$
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