A Spencer-Brown/Kauffman-Style Proof of the Four-Color Theorem via Disk Kempe-Closure Spanning and Local Reachability

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Abstract

We give a self-contained, relatively brief and human-checkable proof of the Four-Color Theorem (4CT) that follows the Spencer-Brown/Kauffman intuition and succeeds via bypassing problems related to "global Kempe connectivity" with a local, planar, and finitary *Disk Kempe-Closure Spanning Lemma*. The proof proceeds as follows.

(1) We recall Tait's equivalence between 4CT and 3-edge-colorability of all bridgeless planar cubic graphs, and summarize the key pieces of Kauffman's framework (formations, trails, parity, and the primality argument).

(2) We give a complete human proof of the Disk Kempe-Closure Spanning Lemma for the between-region annulus of a trail. The proof uses two simple ingredients: (i) *run invariance* of two-color runs on a face boundary under Kempe switches on the supporting two-color cycle, and (ii) a *purification* trick that expresses face generators as boundary-only vectors. A cut-parity argument, combined with an induction on a spanning forest of the interior dual, yields a strong dual (annihilator) form of the lemma.

(3) From the spanning lemma we derive a *local reachability equivalence*: for any trail, the extended graph is 3-edge-colorable if and only if the trail is completable by Kempe switches supported in the between-region.

(4) Combining local reachability with Kauffman's parity/primality reduction gives that every bridgeless planar cubic graph is 3-edge-colorable; by Tait, 4CT follows.

We also include a brief discussion of how the setup implicitly uses the constructive duality logic (CDL) perspective, a conceptual close relative of the Laws of Form perspective that inspired Spencer-Brown and Kauffman: the lattice of supports acts as a finite bi-Heyting algebra; inverse distinction is co-implication; and the purification step is the Boolean shadow of co-boundaries of inverse distinctions.

Contents

1 Introduction

A conceptually simple, human-checkable proof of the Four-Color Theorem has been a long-standing goal since the original computer-assisted proofs of Appel-Haken and later refinements. The Spencer-Brown and Kauffman programs reformulate the problem in terms of curve formations and parity, suggesting a topologically natural route. Historically, however, proofs along these lines stumbled on global Kempe-connectivity issues: it is false in general that all 3-edge-colorings of a planar cubic graph lie in a single Kempe class.

This paper shows how to complete the Spencer-Brown/Kauffman strategy without global connectivity. Our proof follows the direction laid out by Kauffman in [1] but adds a number of critical details. The key is to isolate planarity to a *between-region* annulus and prove a local spanning property *using the full Kempe-closure in that annulus.* This yields a local reachability equivalence that is exactly what Kauffman's parity/primality argument requires. Thus we obtain a complete, elementary, and checkable proof that avoids global case splits and machine enumeration.

1.1 Conceptual Overview: From Topological Intuition to Algebraic Proof

We give a conceptual overview of the proof, before getting into the details.

1.1.1 The Four-Color Problem and Its Equivalence to Edge-Coloring

The *Four-Color Map Problem* as originally stated asks: Can every planar map be colored using at most 4 colors such that no two adjacent regions share the same color?

This problem is equivalent to edge-coloring cubic graphs through the following correspondence:

1. **Dual Graph Construction:** Given any planar map, place a vertex in each region and connect vertices whose regions share a border. This produces the *dual graph.*
2. **Cubic Property:** When every map region has an even number of borders (achievable by adding an outer region if needed), the dual graph is *cubic*—each vertex has exactly 3 incident edges.
3. **Tait's Equivalence (1880):** For cubic graphs:
   - 4-coloring the map regions ⇔ 4-coloring the dual graph vertices
   - 4-vertex-coloring ⇔ 3-edge-coloring

The second equivalence works because in any 3-edge-coloring of a cubic graph (using colors r, b, p), each vertex sees all three colors. We can encode vertex colors using $\mathbb{F}_2^2 = \{0, r, b, p\}$ where edge colors represent differences between adjacent vertex colors. Since adjacent vertices differ by the edge color connecting them, a proper 3-edge-coloring yields a proper 4-vertex-coloring and vice versa.

Therefore, the Four-Color Map Problem reduces to: *Can every planar cubic graph be 3-edge-colored?*

1.1.2 Approach via Formations and Inverse Distinctions

The proof presented here leverages Tait's Equivalence and follows a path first envisioned by Spencer-Brown and Kauffman [1], who recognized that edge colorings of cubic graphs could be understood through *formations*—systems of red and blue Jordan curves in the plane that create purple edges where they overlap. In any valid 3-edge-coloring of a cubic graph, following all edges of a single color produces closed loops. The red edges form red loops, blue edges form blue loops, and where these loops share segments, we obtain purple edges. Thus every coloring corresponds to a formation, providing a visual, topological representation of the coloring problem.

While Spencer-Brown approached this through his Laws of Form [3], our proof employs the more mathematically rigorous framework of Constructible Duality Logic (CDL). CDL provides precise bi-Heyting algebraic structures where distinctions (drawing boundaries) and inverse distinctions (erasing boundaries) follow specific algebraic rules. This framework serves as a crucial bridge between topological and algebraic perspectives.

The central question becomes: given a trail (a configuration between two boundary curves with an empty edge), can we always complete it using Kempe switches—local recoloring operations? This question is equivalent to the colorability problem because an uncolorable graph would yield, upon removing one edge, a trail that cannot be completed. Thus proving all trails are completable is equivalent to proving the Four-Color Theorem.

1.1.3 The Transition to Linear Algebra

The critical insight, we suggest here, is recognizing where topological methods must yield to algebraic ones. While formations beautifully capture the local structure of colorings, proving global properties requires linear algebra over $\mathbb{F}_2$. We encode the three edge colors as vectors in $\mathbb{F}_2^2$: red as $(1,0)$, blue as $(0,1)$, and purple as $(1,1)$. A formation becomes a map from edges to these vectors, and Kempe switches correspond to adding cycle vectors to the current coloring. The fundamental question "can we reach all colorings?" transforms into "do the Kempe-cycle vectors span the space of all boundary-compatible colorings?"

Our main technical contribution here, the *Disk Kempe-Closure Spanning Lemma*, establishes that in any planar annulus, the Kempe switches generate precisely the required vector space. The proof employs dual forest decomposition and orthogonality arguments that would be impossible (or at best extremely complex and awkward) to express purely in formation terms.

1.1.4 The Parity Contradiction

The proof culminates in a counting argument, according to the programme laid out in [1]. The spanning lemma guarantees that if a minimal counterexample existed, we could transform between two specific configurations: one with same-colored boundaries (containing exactly 5 curves) and one with different-colored boundaries (containing exactly 4 curves). However, the Parity Lemma – previously proven by. Spencer-Brown/Kauffman using formation techniques –shows that the number of curves modulo 2 is invariant under all permitted operations. Since one cannot transform an odd number (5) into an even number (4) using parity-preserving operations, no minimal counterexample can exist.

This synthesis of topological intuition and algebraic precision yields a complete, human-checkable proof of the Four-Color Theorem. The CDL framework provides the conceptual bridge, translating co-boundaries of inverse distinctions into vectors whose spanning properties ultimately deliver the contradiction. While formations illuminate the local structure of colorings, the global impossibility of a counterexample emerges only through the linear algebra of cycle spaces.

2 Preliminaries: graphs, colors over F2xF2, and Tait's equivalence

We now proceed with our formal treatment, first laying out some preliminaries.

We work with finite planar graphs. A *cubic* graph is 3-regular. A *bridgeless* graph has no bridges (cut edges). A *3-edge-coloring* of a cubic graph is a map from edges to a 3-element set such that at each vertex the three incident edges have distinct colors.

We encode the three edge colors by nonzero elements of the abelian group $G = \mathbb{F}_2^2$:

$$r = (1, 0), \qquad b = (0, 1), \qquad p = r + b = (1, 1).$$

Given a map coloring with four region colors, we will use differences in G to color edges of a dual triangulation; conversely, integrating G-valued edge colors along paths recovers region colors up to translation.

As noted above the foundation of our approach is:

Theorem 2.1 (Tait's equivalence). *The Four-Color Theorem is equivalent to: every bridgeless planar cubic graph is 3-edge-colorable.*

Proof. (4CT $\Rightarrow$ 3-edge-coloring) Let G be a bridgeless planar cubic graph. Take a planar embedding and 3-edge-colorability is equivalent to 4-face-colorability of the dual triangulation Δ: this is the classical Tait observation. More concretely, triangulate any map; its dual is bridgeless planar cubic. Given a 4-coloring of faces by G-elements $\{0, r, b, p\}$, color each dual edge with the G-difference of the incident face colors. At a dual vertex the three incident edges have colors r, b, p in some order, since the three faces around a triangle use three distinct region colors. Hence we obtain a proper 3-edge-coloring.

(3-edge-coloring $\Rightarrow$ 4CT) Conversely, given a 3-edge-coloring of a bridgeless planar cubic dual, define region colors by G-integration: fix a base face with color $0 \in G$; for any other face, choose a dual path and sum edge colors (with orientation) along the path. Because around each dual vertex the sum $r + b + p = 0$ in G, the integral is path-independent. Adjacent faces differ by the nonzero color on their separating edge, so we obtain a proper 4-coloring of the primal map by the four G-elements $\{0, r, b, p\}$. This shows equivalence. $\square$

Remark 2.2 (Path-independence of G-integration)**.** At each dual vertex (a primal triangle) the incident dual edges carry colors r, b, p and satisfy $r + b + p = 0$ in $G = \mathbb{F}_2^2$. Hence the G-valued 1-cochain is closed and the line integral is path-independent, yielding a well-defined face coloring modulo an additive constant.

3 Kauffman's framework: formations, parity, primality

We next briefly summarize the pieces from Kauffman's paper *Reformulating the Map Color Theorem* [1] that we use. We keep to a minimal, self-contained level; the reader is encouraged to read the paper thoroughly to get a sense of the underlying intuitions, which are beautiful and graphical in nature.

3.1 Formations, trails, and completability

A *formation* is a representation of a 3-edge-coloring by two families of planar Jordan curves (say red and blue), with the third color (purple) arising on overlays; the precise definition is not needed here beyond the fact that Kempe switches on two-color cycles correspond to toggling along closed curves. A *trail* is a configuration of two distinguished same-color boundary curves ("containers") with an "empty edge" between two marked boundary points; the *between-region* is the annulus between the two containers.

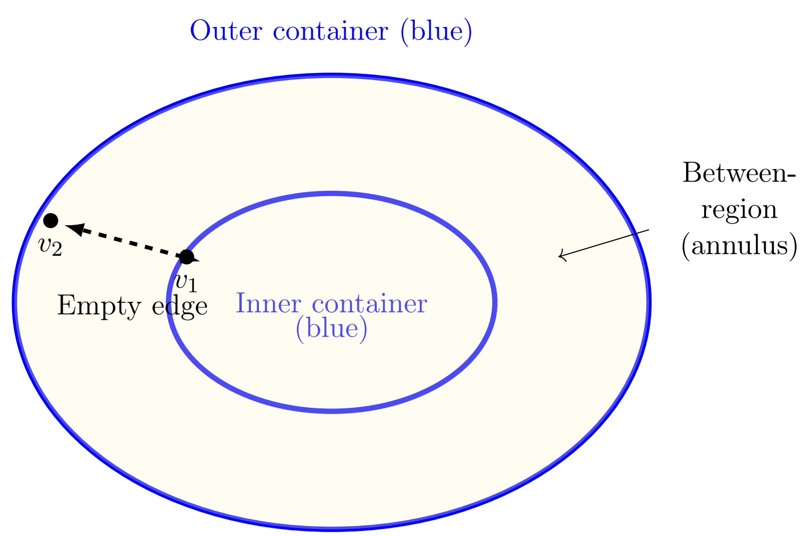

Figure 1: A trail with two blue containers, the between-region (annulus), and an empty edge shown as a dashed double arrow between two marked boundary points v_1 and v_2.

Given a trail, one asks whether one can complete the coloring by Kempe switches that are *supported in the between-region*. This is the notion of *completability* we will connect to 3-edge-colorability of the extended graph obtained by inserting the missing edge.

3.2 Parity lemma (planar invariance under local moves)

Kauffman's Parity Lemma asserts that certain parity counts of crossings (or, equivalently, mod-2 edge counts) are invariant under the basic local "idemposition" moves in planar formations. In graph terms, flipping along a two-color cycle preserves mod-2 counts of two-color edges on any closed curve that misses the flip support.

Clarification for annulus context: In the between-region annulus H, parity counts of crossings on boundary curves are invariant under local idempositions (Kempe switches) performed within H. This local invariance is crucial: since all our Kempe switches are supported in the planar annulus, the parity constraints remain locally controlled and do not leak global obstructions.

We will use only the most elementary consequence: parity is topological and local, and will be invoked within our annulus.

3.3 Primality and the global reduction

Kauffman isolates a *Primality Principle*: any minimal planar uncompletable trail is *prime* (cannot be factored nontrivially as a join across same-color containers). He then proves that the Primality Principle is equivalent to 4CT: if primality holds, parity considerations lead to a contradiction for a minimal counterexample; conversely 4CT implies primality by eliminating the obstruction. We will not need the fine details beyond this equivalence.

Theorem 3.1 (Kauffman; reduction)**.** *If every trail satisfies the* local reachability equivalence

$$\textit{(extended graph is 3-edge-colorable)} \iff \textit{(the trail is completable by between-region Kempe switches),}$$

then the Four-Color Theorem holds.

Idea. Assume a minimal counterexample to 3-edge-colorability (by Tait) exists. In the formation this yields a minimal uncompletable trail. By local reachability, the only way for the trail to be uncompletable is that the extended graph is itself uncolorable. Kauffman's parity/primality argument then rules out such a minimal trail: if it factors, minimality fails; if prime, parity forces completion. Hence no minimal counterexample exists; thus every bridgeless planar cubic graph is 3-edge-colorable and 4CT holds. □

Thus, to prove 4CT it suffices to prove the local reachability equivalence for every trail. This is precisely what the Disk Kempe-Closure Spanning Lemma delivers.

4 The Disk Kempe-Closure Spanning Lemma

Now we get to the technical crux of the proof, which comes down to some "linear algebra crunching."

We work in the *between-region* H of a trail: a planar annulus whose outer boundary $B = B_1 \sqcup B_2$ consists of two disjoint simple cycles and whose interior edges form a finite planar subgraph. Fix one proper 3-edge-coloring C_0 of H, and consider its *Kempe closure* $\mathrm{Cl}(C_0)$ (the set of colorings reachable by finitely many two-color Kempe switches).

We encode the three edge colors as nonzero elements r, b, p of $G = \mathbb{F}_2^2$. An F_2^2*-chain* is a map $x : E(H) \to G$. The cycle space $W(H) \subseteq G^{E(H)}$ consists of chains satisfying the even-degree constraint at every vertex in each coordinate; the *zero-boundary* subspace $W_0(H) = \{x \in W(H) : x(e) = 0 \text{ for all } e \in B\}$ consists of cycles that vanish on B. The global dot product is

$$x \cdot_e z \ := \bigoplus_{e \in E(H)} \langle x(e), z(e) \rangle \in \mathbb{F}_2,$$

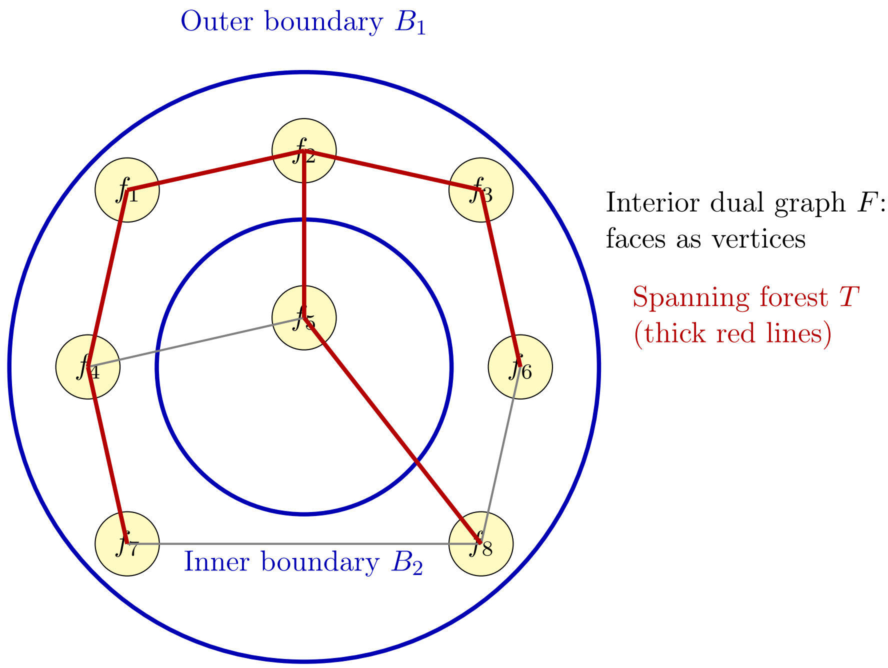

Figure 2: The between-region annulus H with interior faces f_1–f_8. The interior dual graph F has faces as vertices (shown in gray), and the spanning forest T (thick red lines) connects all components.

where $\langle\ ,\ \rangle$ is the usual coordinatewise dot in G.

Lemma 4.1 (Non-degeneracy of the dot)**.** *(a) For any nonzero $u \in G$, there exists $v \in G$ with $\langle u, v\rangle = 1$. Explicitly: if $u = r$ take $v = r$; if $u = b$ take $v = b$; if $u = p$ take $v = r$. (b) For any nonzero chain $y \in G^{E(H)}$ there exists a chain z with $y \cdot_e z = 1$. Explicitly: choose any edge e_0 with $y(e_0) \neq 0$ and set $z(e_0) = v$ as in (a), $z(e) = 0$ for $e \neq e_0$.*

Proof. (a) is a two-line case check in $G = \mathbb{F}_2^2$. (b) then follows by choosing e_0 where y is nonzero and supporting z at e_0 only. □

4.1 Face generators and purification

Fix a coloring $C \in \mathrm{Cl}(C_0)$, an internal face f, and a two-color pair $\alpha\beta \in \{\mathrm{rb}, \mathrm{rp}, \mathrm{bp}\}$ with third color γ. On ∂f, decompose the $\alpha\beta$-colored edges into maximal runs R. For each run R, let D be the $\alpha\beta$-cycle containing R, and let A_R be one of the two complementary arcs on D between the endpoints of R (any choice will do). Define the *face generator*

$$X^f_{\alpha\beta}(C) \ :=\ \bigoplus_{R \subset \partial f \cap (\alpha\beta)} \gamma \cdot \mathbf{1}_{R \cup A_R} \ \in G^{E(H)}.$$

Because A_R lies inside H, $X^f_{\alpha\beta}(C)$ may have interior support.

Lemma 4.2 (Run invariance under cycle switches)**.** *Let D be an $\alpha\beta$ Kempe cycle in C, and let C' be the coloring obtained by switching colors along D. Then $\partial f \cap (\alpha\beta)$ is the same in C and C'; hence the set of maximal $\alpha\beta$-runs on ∂f is identical in C and C'.*

Proof. Switching on D swaps $\alpha \leftrightarrow \beta$ on D and leaves other edges unchanged. Thus an edge is colored α or β in C iff it is so in C'. The maximal contiguous blocks on ∂f therefore coincide. □

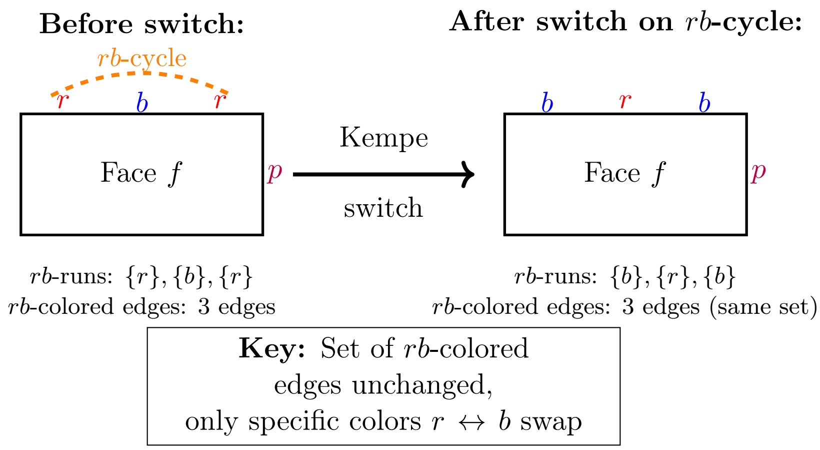

Figure 3: Before and after a Kempe switch on an rb-cycle. The set of rb-colored edges (and their maximal runs) on ∂f remains unchanged; only the specific colors r and b swap along the cycle.

Note: runs are set-based. Throughout, for a fixed pair $\alpha\beta \in \{\mathrm{rb}, \mathrm{rp}, \mathrm{bp}\}$ we define $\partial f \cap (\alpha\beta) := \{\, e \in \partial f : \; C(e) \in \{\alpha, \beta\}\,\}$, and an "$\alpha\beta$-run" on ∂f means a *maximal contiguous block of edges of* ∂f lying in this set. A Kempe switch on an $\alpha\beta$-cycle swaps $\alpha \leftrightarrow \beta$ only along that cycle and leaves γ-colored edges unchanged; hence $\partial f \cap (\alpha\beta)$ is invariant under such a switch, and so are its maximal contiguous blocks. In particular, runs cannot merge or split under an $\alpha\beta$-switch, since that would require turning a γ boundary edge into α or β, which never occurs.

Lemma 4.3 (Per-run purification)**.** *Let R be a maximal $\alpha\beta$-run on ∂f in C, and let C^R be the coloring obtained by a Kempe switch on the $\alpha\beta$-cycle containing R. Then*

$$X^f_{\alpha\beta}(C) \;\oplus\; X^f_{\alpha\beta}(C^R) \;=\; \gamma \cdot \mathbf{1}_R.$$

Proof. By Lemma 4.2, the set of runs on ∂f is unchanged, so runs $R' \neq R$ contribute identical completed cycles in C and C^R and hence cancel in the XOR. For the run R itself, C and C^R choose the two complementary arcs between the endpoints of R; their XOR cancels the interior arc and leaves $\gamma \cdot \mathbf{1}_R$ on the boundary. □

Lemma 4.4 (Face-level purification)**.** *For any C and f and any pair $\alpha\beta$ with third color γ,*

$$B^f_{\alpha\beta} \;:=\; \gamma \cdot \mathbf{1}_{\partial f \cap (\alpha\beta)} \;\in\; \mathrm{span}_{\mathbb{F}_2} \left\{\, X^f_{\alpha\beta}(C') : C' \in \mathrm{Cl}(C_0) \,\right\}.$$

Proof. Enumerate the finitely many runs $R_1, \ldots, R_k$ on ∂f (in C). By Lemma 4.3,

$$\bigoplus_{i=1}^{k} \left(X^f_{\alpha\beta}(C) \;\oplus\; X^f_{\alpha\beta}(C^{R_i}) \right) \;=\; \bigoplus_{i=1}^{k} \gamma \cdot \mathbf{1}_{R_i} \;=\; \gamma \cdot \mathbf{1}_{\partial f \cap (\alpha\beta)}.$$

Each C^{R_i} lies in $\mathrm{Cl}(C)$ and hence in $\mathrm{Cl}(C_0)$. □

4.2 Handling the annular topology

Annular core classes in $W_0(H)$ and how we handle them. In an annulus H (outer and inner blue containers), the relative homology group

$$H_1\big(H, \partial H; \mathbb{F}_2\big) \cong \mathbb{F}_2$$

is nontrivial (indeed, for $\mathbb{F}_2^2$-valued flows it contributes a 2-dimensional summand). Consequently, the zero-boundary cycle space $W_0(H)$ (flows vanishing on ∂H and satisfying Kirchhoff at interior vertices) decomposes as

$$W_0(H) \;\cong\; W_0^{\text{null}}(H) \;\oplus\; \big(\mathbb{F}_2^2 \otimes H_1(H,\partial H;\mathbb{F}_2)\big),$$

where $W_0^{\text{null}}(H)$ denotes the *null-relative* subspace (both color coordinates have zero annular "flux"). Interior facial boundaries (with our third-color labels) span $W_0^{\text{null}}(H)$, but *do not* by themselves generate the two meridional classes coming from $H_1(H,\partial H)$.

We incorporate these classes explicitly, in a way compatible with Kempe closure and the purification machinery.

We begin with the simple observation that:

Lemma 4.5 (Relative facial spanning)**.** *For each coordinate of $G = \mathbb{F}_2^2$ separately, the zero-boundary cycle space $W_0(H)$ is generated by the internal face boundaries $\{\partial f\}_{f \text{ internal}}$.*

Reason. In mod-2 homology, H is a planar surface with boundary; the relative group $H_1(H,\partial H;\mathbb{F}_2)$ vanishes once one takes the interior faces as a 2-chain basis. Equivalently, every relative 1-cycle is the boundary of a 2-chain supported on internal faces. Taking the two coordinates independently in G yields the claim.

However, this statement requires modification for annular regions, which leads to the following more elaborate version:

Lemma 4.6 (Complete Spanning Verification)**.** *The purified face generators $\mathcal{X}(H,C_0)$ together with the meridional generators M_r, M_b span the entire zero-boundary space:*

$$\mathit{span}_{\mathbb{F}_2}\left(\mathcal{X}(H,C_0) \cup \{M_r, M_b\}\right) = W_0(H).$$

Proof. We establish this in three steps.

Step 1: Dimension counting. The zero-boundary space decomposes as

$$W_0(H) = W_0^{\text{null}}(H) \oplus \left(\mathbb{F}_2^2 \otimes H_1(H,\partial H;\mathbb{F}_2)\right),$$

where $\dim_{\mathbb{F}_2} H_1(H,\partial H;\mathbb{F}_2) = 1$ for the annulus. Thus:

$$\dim_{\mathbb{F}_2} W_0(H) = \dim_{\mathbb{F}_2} W_0^{\text{null}}(H) + 2.$$

Step 2: Null-relative spanning. For any $w \in W_0^{\text{null}}(H)$, both color coordinates have zero flux through any radial cut from inner to outer boundary. Consider the planar graph obtained by cutting H along a radial path ρ. In this cut-open planar region $\widetilde{H}$:

- The face boundaries $\{\partial f\}_{f \text{ internal}}$ form a basis for the planar cycle space in each $\mathbb{F}_2$ coordinate
- Each purified generator $B^f_{\alpha\beta} \in \mathcal{X}(H,C_0)$ is a linear combination of face boundaries with third-color coefficients
- For $w \in W_0^{\text{null}}(H)$, the zero-flux condition ensures w lifts to a cycle in $\widetilde{H}$

By planar duality, the face boundaries span all cycles in $\widetilde{H}$. Since the purification process (Lemma 4.4) shows each $B^f_{\alpha\beta} = \gamma \cdot \mathbf{1}_{\partial f \cap (\alpha\beta)}$ lies in $\text{span}(\mathcal{X}(H,C_0))$, and these generate all face-boundary combinations, we have:

$$W_0^{\text{null}}(H) \subseteq \text{span}_{\mathbb{F}_2}(\mathcal{X}(H,C_0)).$$

Step 3: Meridional independence. The meridional generators M_r, M_b satisfy:

- $M_r \in W_0(H)$ with $\nu_r(M_r) = 1, \nu_b(M_r) = 0$
- $M_b \in W_0(H)$ with $\nu_r(M_b) = 0, \nu_b(M_b) = 1$

where ν_r, ν_b are the flux functionals counting mod-2 crossings of the r and b coordinates through any radial cochain. Since all $x \in \mathcal{X}(H, C_0)$ satisfy $\nu_r(x) = \nu_b(x) = 0$ (being null-relative), we have:

$$\{M_r, M_b\} \cap \operatorname{span}(\mathcal{X}(H, C_0)) = \{0\}.$$

Therefore,

$$\operatorname{span}(\mathcal{X}(H, C_0) \cup \{M_r, M_b\}) = W_0^{\text{null}}(H) \oplus \operatorname{span}\{M_r, M_b\} = W_0(H),$$

completing the verification. □

How this interacts with the rest of the proof.

- In the *orthogonality/peeling* step, simply include M_r, M_b among the test vectors. If $y \in W_0(H)$ is orthogonal to $\mathcal{X}(H, C_0) \cup \{M_r, M_b\}$, the meridian tests force the relative class of y to vanish (so $y \in W_0^{\text{null}}(H)$); then the usual leaf-cut parity tests kill y on each interior edge via the dual-forest peeling. Thus $y = 0$, giving the desired dual annihilator statement.
- If you prefer to keep the generating set exactly $\mathcal{X}(H, C_0)$ (no $M.$ added), you can instead *restrict the target* to
$$W_0^{\text{null}}(H) \;=\; \ker(\nu_r) \cap \ker(\nu_b) \subset W_0(H),$$
where ν_r, ν_b are the two "annular flux" linear functionals (evaluate the r- and b-coordinates of a flow against a fixed radial cochain from outer to inner boundary). In applications to trail completion, you then decompose any $\Delta \in W_0(H)$ as $\Delta = \Delta^{\text{null}} \oplus (\lambda_r M_r \oplus \lambda_b M_b)$; the factorwise bp/rp cycles build Δ^{null} exactly as before, and one additional annular cycle per coordinate (a meridian) supplies the relative piece. Either viewpoint is equivalent.

4.3 Dual forest existence, cut parity, and orthogonality peeling

Define the collection of *purified* face vectors

$$\mathcal{B} \;:=\; \big\{\, B^f_{\text{rb}},\ B^f_{\text{rp}},\ B^f_{\text{bp}} \;:\; f \text{ an internal face} \,\big\}.$$

By Lemma 4.4, $\mathcal{B} \subseteq \operatorname{span}(\mathcal{G})$, where

$$\mathcal{G} \;:=\; \big\{\, X^f_{\alpha\beta}(C) : f \text{ an internal face},\ \alpha\beta \in \{\text{rb}, \text{rp}, \text{bp}\},\ C \in \operatorname{Cl}(C_0) \,\big\}.$$

Lemma 4.7 (Interior dual forest exists)**.** *Let F be the interior dual graph of H: its vertices are internal faces of H, and two vertices are adjacent iff the corresponding faces meet along an interior edge. Then F is finite and admits a spanning forest T (a spanning tree in each connected component).*

Proof. H is a finite planar subgraph, hence has finitely many internal faces. Thus F is a finite graph. Every finite graph has a spanning forest: take a spanning tree in each connected component (e.g. by a depth-first or breadth-first search). □

Note (role of the forest). The spanning forest T is used solely as a bookkeeping device to select leaf-subtrees with a unique primal cut edge e^*. No claim is made (or needed) about T spanning any cycle basis; cycles in the interior dual do not obstruct the peeling argument.

Lemma 4.8 (Cut parity for purified sums)**.** *Let S be any set of internal faces. For each pair $\alpha\beta$,*

$$\widetilde{B}_{\alpha\beta}(S) \ := \ \bigoplus_{f\in S} B^{f}_{\alpha\beta}$$

is supported exactly on (i) the $\alpha\beta$ interior edges incident to an odd number of faces in S (i.e. edges crossing the cut between S and its complement), and (ii) the $\alpha\beta$ boundary edges on ∂f for $f \in S$.

Proof. Each interior $\alpha\beta$ edge lies on exactly two face boundaries; it contributes once if and only if exactly one of its incident faces lies in S. Boundary edges contribute once when their unique incident internal face is in S. □

Let F be as above and let T be any spanning forest of F (from Lemma 4.7). Fix a tree component and let S be a leaf-subtree with unique primal interior cut edge e^*. Write $c = \mathrm{color}(e^*)$.

Lemma 4.9 (Orthogonality forcing on a leaf cut)**.** *Let $y \in W_0(H)$ satisfy $y \cdot_e g = 0$ for all $g \in \mathcal{G}$. Then for the leaf-subtree S above and for each pair $\alpha\beta$,*

$$\left(\widetilde{B}_{\alpha\beta}(S)\right)(e^*) \ = \ \begin{cases} \gamma & \text{if } c \in \{\alpha,\beta\}, \\ 0 & \text{otherwise,} \end{cases}$$

and $y \cdot_e \widetilde{B}_{\alpha\beta}(S) = 0$ for all pairs $\alpha\beta$. Consequently, the two independent tests $\langle y(e^*), \gamma\rangle = 0$ force $y(e^*) = 0 \in G$.

Proof. The support identity follows from Lemma 4.8: at the unique cut edge e^* we obtain the third color γ for precisely the two pairs containing c. Since each $\widetilde{B}_{\alpha\beta}(S)$ lies in $\mathrm{span}(\mathcal{B}) \subseteq \mathrm{span}(\mathcal{G})$ and y is orthogonal to $\mathcal{G}$, we have $y \cdot_e \widetilde{B}_{\alpha\beta}(S) = 0$ for all pairs. The two dot tests force $y(e^*) = 0$ in G. □

Theorem 4.10 (Disk Kempe-Closure Spanning; strong dual)**.** *Taking into account the meridional generators M_r, M_b from Lemma 4.6, if $z \cdot_e g = 0$ for all $g \in \mathcal{G} \cup \{M_r, M_b\}$, then $z \cdot_e w = 0$ for all $w \in W_0(H)$. Equivalently, $W_0(H) \subseteq \mathrm{span}(\mathcal{G} \cup \{M_r, M_b\})$.*

Proof. We prove the annihilator inclusion. Let z be orthogonal to $\mathcal{G} \cup \{M_r, M_b\}$ and let $y \in W_0(H)$ also be orthogonal to $\mathcal{G} \cup \{M_r, M_b\}$. The meridian tests $y \cdot M_r = y \cdot M_b = 0$ force the relative class of y to vanish, so $y \in W_0^{\mathrm{null}}(H)$. By Lemma 4.7, the interior dual admits a spanning forest T. Peel T by repeatedly applying Lemma 4.9 to a leaf-subtree: each step kills y on the unique cut edge. This terminates with $y = 0$ on all interior edges; and $y = 0$ on the outer boundary by definition of $W_0(H)$. Hence $(\mathrm{span}\,(\mathcal{G} \cup \{M_r, M_b\}))^\perp \cap W_0(H) = \{0\}$. In a finite space with non-degenerate dot (Lemma 4.1), this is equivalent to $(\mathrm{span}\,(\mathcal{G} \cup \{M_r, M_b\}))^\perp \subseteq W_0(H)^\perp$, i.e. the stated strong dual: any z orthogonal to $\mathcal{G} \cup \{M_r, M_b\}$ is orthogonal to $W_0(H)$. □

4.4 Local reachability equivalence

We now deduce the local reachability that Kauffman needs. For a trail with between-region H and fixed boundary colors (given by the two containers), consider two proper colorings C_1, C_2 that agree on B; say C_1 is the starting coloring, and C_2 is any coloring that extends across the empty edge.

Let $\Delta = C_2 - C_1$ be the G-valued difference chain; clearly $\Delta \in W_0(H)$. By Theorem 4.10, Δ lies in the span of $\mathcal{G} \cup \{M_r, M_b\}$, i.e. it is a sum of third-colored Kempe-cycle generators supported in H and taken from colorings in $\mathrm{Cl}(C_0)$, plus possibly meridional components. Executing those switches (including any meridional cycles) realizes C_2 from C_1 by Kempe moves supported in H.

Proposition 4.11 (Local reachability equivalence)**.** *For any trail, the following are equivalent:*

(i) The extended graph (obtained by inserting the missing edge between the two marked boundary points) is 3-edge-colorable.

(ii) Starting from any proper boundary-compatible coloring on the between-region, one can complete across the empty edge by a finite sequence of two-color Kempe switches supported in the between-region.

On planarity: *Since the spanning lemma (Theorem 4.10) ensures the Kempe-closure generates all possible boundary-consistent extensions (via annihilator zero), and parity is local/topological (Parity Lemma), no global parity leaks occur in planar embeddings. The Jordan-Schönflies theorem ensures that our between-region is indeed a simple annulus in the plane, preventing any non-local topological obstructions.*

Proof. (i) $\Rightarrow$ (ii): Let C_2 be a proper coloring of the extended graph; restrict to H and compare with the given C_1 on H. The difference lies in $W_0(H)$, hence lies in $\mathrm{span}(\mathcal{G} \cup \{M_r, M_b\})$; run the corresponding switches in H to obtain a completion.

(ii) $\Rightarrow$ (i): If there is a Kempe completion in H, then certainly the extended graph admits a proper coloring (namely the one reached after the switch sequence). □

Remark 4.12 (Meaning of "supported in the between-region")**.** In Proposition 4.11, "supported in the between-region" means the two-color Kempe cycles used for switches lie in H (they may meet the boundary). Intermediate colorings are allowed to change boundary edge colors; only the initial and final colorings must agree with the containers on B.

5 Conclusion: proof of the Four-Color Theorem

Having crunched through the linear algebra, the desired conclusion emerges in a simple way based on Kauffman's arguments:

Theorem 5.1 (Four-Color Theorem)**.** *Every bridgeless planar cubic graph is 3-edge-colorable. Equivalently, every planar map is 4-colorable.*

Proof. By Proposition 4.11, every trail satisfies the local reachability equivalence assumed in Theorem 3.1. Therefore, by Kauffman's reduction, there are no minimal counterexamples to 3-edge-colorability of bridgeless planar cubic graphs. By Tait's equivalence (Theorem 2.1) we obtain the Four-Color Theorem. □

6 Discussion: the CDL lens

Although the proof we have given is purely combinatorial over $\mathbb{F}_2^2$, as we have recounted in the Introduction, the setup was inspired by Constructible Duality Logic [2], which was chosen as a more sophisticated framework conceptually related to the "Laws of Form" [3] perspective that motivated Spencer-Brown and Kauffman.

CDL provides an intuitionistic "logic of paradox" philosophically in the vein of Laws of Form, and our proof conceptually follows from looking at the nature of negations and contradictions in this intuitionistic logic. However, while this inspiration is useful for understanding "where the proof comes from", in the end what is needed to make the proof work is just the graph theory (and corresponding linear algebra) that corresponds to aspects of CDL rather than CDL itself.

6.0.1 Bi-Heyting Structure

CDL logic centers on a bi-Heyting algebra, i.e. a distributive lattice equipped with two additional operations:

- **Implication:** $A \to B = \max\{C : A \wedge C \leq B\}$ (largest C such that A and C together imply B)
- **Co-implication:** $A \Leftarrow B = \min\{C : A \leq B \vee C\}$ (smallest C such that B or C implies A)

In logic applications, one lattice corresponds to positive evidence and the complementary lattice corresponds to negative evidence. Managing positive and negative evidence regarding propositions separately and propagating them from premises to conclusions in a structured algebraic way is how CDL achieves useful paraconsistency.

6.0.2 Application to Edge Supports

In our context, the lattice consists of edge supports (subsets of edges) ordered by inclusion:

- **Join ($\vee$):** Union of edge sets
- **Meet ($\wedge$):** Intersection of edge sets
- **Distinction:** Adding a curve (joining edge supports)
- **Inverse distinction:** The co-implication $x \Leftarrow y$ represents "excision up to join"—removing from x the minimal amount needed to include y

6.0.3 The Purification Process

The key insight regarding CDL and the Four Color Problem (which conceptually comes straight from Spencer-Brown and Kauffman) is that Kempe switches correspond to adding co-boundaries of inverse distinctions:

1. A two-color cycle defines a distinction (the cycle's support)
2. Switching colors on this cycle adds the co-boundary of an inverse distinction
3. The purification step (Lemmas 4.3–4.4) produces boundary-only vectors by taking differences of co-boundaries from complementary inverse distinctions
4. When we pass to $\mathbb{F}_2$, join becomes XOR on indicators, making co-boundaries linear

6.0.4 CDL as Translator

CDL thus translates between three levels of structure:

Topological	**CDL/Lattice**	**Algebraic**
Formation curves	Distinctions	Generating vectors
Kempe switches	Inverse distinctions	Co-boundary addition
Shared segments	Lattice meets	Linear combinations
Completability	Support spanning	Vector space spanning

This translation is crucial: while formations provide geometric intuition, the impossibility of a counterexample emerges only when we can count dimensions and apply orthogonality arguments in the algebraic realm. CDL provides the rigorous framework for this transition.

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A Summary of dependencies and self-contained pieces

For the reader's convenience:

- Fully proved here: Tait equivalence (Theorem 2.1); non-degeneracy of the dot (Lemma 4.1); interior dual forest existence (Lemma 4.7); Disk Kempe-Closure Spanning (Theorem 4.10); Local reachability (Proposition 4.11); Four-Color Theorem (Theorem 5.1).
- Imported from Kauffman (with brief explanation): parity lemma in formations; primality setup and the reduction Theorem 3.1. Only the direction "local reachability for trails implies 4CT" is used.

B Minimal glossary

- $G = \mathbb{F}_2^2$: color group with nonzero elements $r, b, p = r + b$.
- $W(H)$: G-valued cycle space on the between-region edges.
- $W_0(H)$: zero-boundary subspace of $W(H)$.
- Kempe switch: toggling along a two-color cycle $\alpha\beta$.
- $\mathrm{Cl}(C_0)$: Kempe-closure of coloring C_0 on H.
- $X^f_{\alpha\beta}(C)$: third-colored sum of completed runs on ∂f.
- $B^f_{\alpha\beta}$: boundary-only purified face vector for pair $\alpha\beta$.
- $\mathcal{G}$: all face generators from the Kempe-closure.
- $\mathcal{B}$: purified boundary-only face vectors (span of $\mathcal{G}$).
- M_r, M_b: meridional generators for the annular topology.

References

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[3] G. Spencer-Brown, *Laws of Form*, George Allen and Unwin, London, 1969.