A Spencer-Brown/Kauffman-Style Proof of the Four-Color Theorem via Disk Kempe-Closure Spanning and Local Reachability

– Detailed Explanations

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October 15, 2025

Abstract

This is an expanded version of the Four-Color Theorem proof we have recently provided, following the Spencer-Brown/Kauffman intuition. In this expanded version we give detailed intuitive explanations for every construct and proof step, illustrated with a running example throughout. The core insight is that we can complete the Spencer-Brown/Kauffman strategy by proving a local spanning property in the between-region annulus, avoiding the global Kempe connectivity issues that historically blocked this approach.

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1 Introduction and Overview

1.1 The Big Picture

The Four-Color Theorem states that any map drawn on a plane (or equivalently, on a sphere) can be colored with at most four colors such that no two adjacent regions share the same color. This seemingly simple statement resisted proof for over a century.

Intuition 1.1 (Why this proof strategy?). The computer-assisted proofs of Appel-Haken and Robertson-Sanders-Seymour-Thomas work by checking thousands of special cases. Our approach, following Spencer-Brown and Kauffman, seeks a more conceptual path. The key insight: instead of trying to prove all colorings are globally connected by Kempe switches (which is false), we prove that within any annular region, we can always complete a partial coloring using only local moves. This local property, surprisingly, implies the global theorem.

1.2 Our Running Example

Throughout this proof, we'll use a specific planar cubic graph to illustrate every concept. Here is our example graph G_{ex}:

Example 1.2 (The running example graph). `Vertices: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}`

```
Initial 3-edge-coloring:
(1, 2, red)
(2, 3, blue)
(3, 4, red)
(4, 5, blue)
(5, 1, purple)
(1, 6, blue)
(2, 7, purple)
(3, 8, purple)
(4, 9, purple)
(5, 10, red)
(6, 7, red)
(7, 8, blue)
(8, 9, red)
(9, 10, blue)
(10, 6, purple)
```

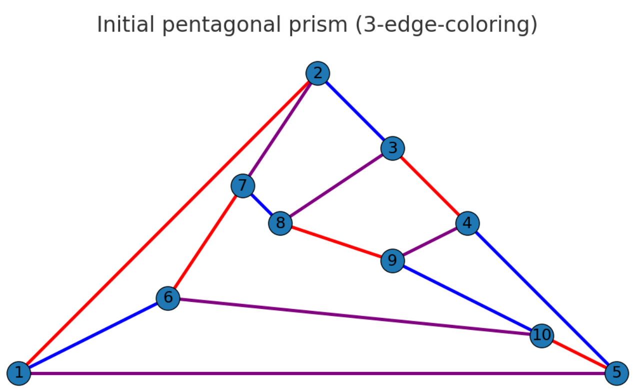

Figure 1: The pentagonal prism graph G_{ex} with initial 3-edge-coloring

This forms a pentagonal prism where the top and bottom pentagons are connected by vertical edges. Each vertex has exactly degree 3, and the graph is planar (can be drawn on a plane without edge crossings).

We'll track how each concept applies to this graph as we proceed through the proof.

2 Preliminaries: Graphs, Colors, and Mathematical Setup

2.1 Why Cubic Graphs?

Definition 2.1 (Planar Cubic Graph). A graph is:

- *Planar* if it can be drawn on a plane without edge crossings
- *Cubic* if every vertex has exactly degree 3
- *Bridgeless* if removing any single edge doesn't disconnect it

Intuition 2.2 (Why these restrictions matter). **Why cubic?** Any planar graph can be reduced to a cubic graph for coloring purposes. If a vertex has degree > 3, we can replace it with a small triangle of vertices, preserving colorability. So cubic graphs are the "hardest case."

Why bridgeless? If a graph has a bridge, we can color the two sides independently and then match colors at the bridge. So bridges don't add difficulty.

Why planar? Non-planar graphs might need more than 4 colors (e.g., the complete graph K_5 needs 5 colors). Planarity is essential to the theorem.

Example 2.3 (Checking our example). Our graph G_{ex} is:

- Cubic: Each vertex has exactly 3 incident edges (check vertex 1: edges to 2, 5, 6)
- Bridgeless: Removing any edge still leaves the graph connected (the pentagonal prism structure ensures this)
- Planar: Can be drawn with the pentagons as top/bottom faces

2.2 The Color Encoding: Why $\mathbb{F}_2^2$?

Definition 2.4 (The color group). We encode the three edge colors as nonzero elements of $G = \mathbb{F}_2^2$:

- $\mathbb{F}_2 = \{0, 1\}$ with arithmetic mod 2 (so $1 + 1 = 0$)
- $\mathbb{F}_2^2 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ - pairs of bits
- Colors: $r = (1, 0)$ (red), $b = (0, 1)$ (blue), $p = (1, 1)$ (purple)
- Addition is XOR: $(a_1, a_2) + (b_1, b_2) = (a_1 \oplus b_1, a_2 \oplus b_2)$

Intuition 2.5 (Why this encoding is perfect). This encoding has magical properties:

- $r + b = (1, 0) + (0, 1) = (1, 1) = p$
- $b + p = (0, 1) + (1, 1) = (1, 0) = r$
- $p + r = (1, 1) + (1, 0) = (0, 1) = b$
- $r + b + p = (0, 0)$ (all three sum to zero)

So "any two colors give the third" and "all three sum to zero." This perfectly matches the constraint at each vertex in a cubic graph: three different colors that balance out.

Moreover, a Kempe switch (swapping two colors) corresponds to adding the third color:

- Swapping $r \leftrightarrow b$ on an edge colored r gives $r + p = b$

- Swapping $r \leftrightarrow b$ on an edge colored b gives $b + p = r$

Example 2.6 (Color arithmetic at vertex 1)**.** At vertex 1 in G_{ex}:

```
Edge (1,2): red = (1,0)
Edge (1,5): purple = (1,1)
Edge (1,6): blue = (0,1)
Sum: (1,0) + (1,1) + (0,1) = (0,0) ?
```

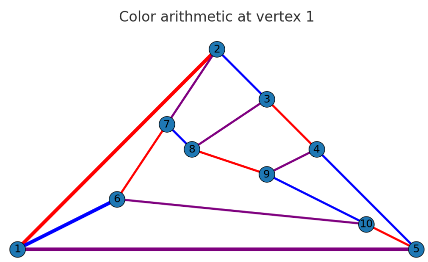

Figure 2: Color arithmetic at vertex 1 showing the three incident edges

The constraint is satisfied!

2.3 Chains and Cycles

Definition 2.7 ($\mathbb{F}_2^2$-chain)**.** A chain on a graph H is a function $x : E(H) \to \mathbb{F}_2^2$ that assigns a 2-bit vector to each edge. Think of it as labeling each edge with a color value (or $(0,0)$ for "no color").

Definition 2.8 (Cycle space $W(H)$)**.** The cycle space consists of chains satisfying the even-degree constraint: at every vertex v, the sum of incident edge values equals $(0,0)$:

$$\sum_{e \sim v} x(e) = (0,0) \text{ in } \mathbb{F}_2^2$$

Intuition 2.9 (What are cycles really?)**.** Despite the name "cycle," these aren't just simple loops. The cycle space includes any edge-labeling where labels "flow" evenly through vertices—no vertex acts as a source or sink for either color bit. Think of it like water flow: what comes in must go out. Examples include:

- Simple loops labeled with a single color
- Unions of disjoint loops
- More complex "flows" satisfying the even-degree property

The name comes from the fact that simple cycles form a basis for this space.

Definition 2.10 (Zero-boundary subspace $W_0(H)$)**.** When H is an annulus with boundary edges B:

$$W_0(H) = \{x \in W(H) : x(e) = (0,0) \text{ for all boundary edges } e \in B\}$$

Intuition 2.11 (Cycles that vanish on the boundary)**.** These are cycles that don't use any boundary edges (though they may touch boundary vertices). They're "supported in the interior." This is crucial: we want to study what can happen inside the annulus without changing the boundary coloring.

3 Tait's Equivalence: From Maps to Cubic Graphs

Theorem 3.1 (Tait's Equivalence)**.** *The Four-Color Theorem is equivalent to: every bridgeless planar cubic graph is 3-edge-colorable.*

Intuition 3.2 (Why this equivalence works)**.** The key is duality. Given a map:

1. Place a vertex in each region
2. Connect vertices when regions are adjacent
3. This gives the dual graph
4. For a triangulated map, the dual is cubic
5. The four region colors become differences between adjacent regions
6. These differences are exactly our three edge colors

The $\mathbb{F}_2^2$ group structure ensures this works both ways—we can go from map coloring to edge coloring and back.

Proof with intuition. (**Map coloring** $\Rightarrow$ **Edge coloring**) Given a 4-colored map with colors $\{0, r, b, p\}$:

- Create the dual graph (cubic if map is triangulated)
- Color each dual edge with the difference of its endpoint colors
- At each vertex (dual of a triangle), the three edges get three different colors
- The algebra works: if triangle regions have colors c_1, c_2, c_3, then edges get $c_2 - c_1$, $c_3 - c_2$, $c_1 - c_3$, which sum to 0 and are all different

(**Edge coloring** $\Rightarrow$ **Map coloring**) Given a 3-edge-colored cubic graph:

- Fix a base face with color 0
- For any other face, sum edge colors along any path from the base
- This is well-defined because the sum around any vertex is 0
- Adjacent faces differ by exactly their shared edge's color

□

4 Kauffman's Framework: Formations and Trails

4.1 Formations: The Geometric Picture

Definition 4.1 (Formation)**.** A formation represents a 3-edge-coloring as two families of non-intersecting Jordan curves on the plane (say red and blue). Purple edges arise where red and blue curves cross. Kempe switches correspond to deforming curves while preserving the non-intersection property within each family.

Intuition 4.2 (Why formations help)**.** Formations give a topological view of colorings. Instead of discrete edges, we have continuous curves that can be deformed. This makes certain properties (like parity invariants) more natural. It's like viewing a city's road network as continuous paths rather than discrete segments—patterns become clearer.

4.2 Trails: The Local Coloring Problem

Definition 4.3 (Trail)**.** A trail consists of:

- Two disjoint same-color boundary curves (containers) C_{out}, C_{in}
- Marked points $s \in C_{out}$, $t \in C_{in}$
- An "empty edge" between s and t (currently missing)
- The between-region H (the annulus between containers)

The trail is *completable* if Kempe switches in H can create the right coloring to "close" the empty edge.

Intuition 4.4 (What trails represent)**.** Think of a trail as a local coloring challenge: "Given this partial coloring with a missing edge, can you complete it using only local moves?" This captures the essence of extending colorings incrementally. It's like solving a jigsaw puzzle where you can only rearrange pieces within a certain region.

Example 4.5 (Creating a trail from our example)**.** Remove the blue edge $(2, 3)$ from G_{ex}:

```
Removed edge: (2, 3, blue)
This creates two blue containers:
- Outer: edges (4,5), (7,8), (9,10) form one blue cycle
- Inner: edges (1,6) form another (degenerate) cycle
Between-region H contains all other edges
Empty edge: would connect vertices 2 and 3
```

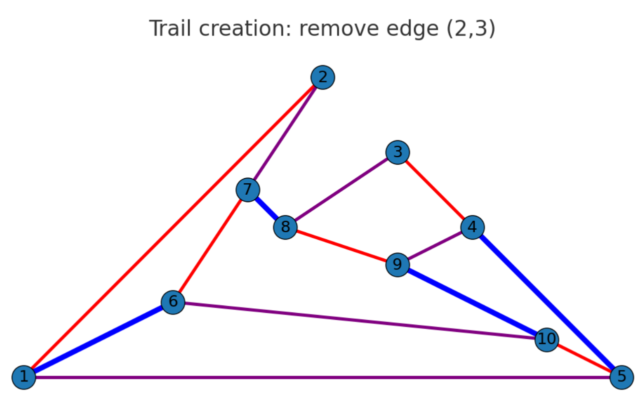

Figure 3: Creating a trail by removing edge $(2, 3)$ from G_{ex}

4.3 The Parity Principle

Lemma 4.6 (Parity Invariance)**.** *If a closed curve Γ doesn't intersect a Kempe cycle K, then flipping colors on K preserves the mod-2 count of $\alpha\beta$-edges that Γ crosses.*

Intuition 4.7 (Why parity is preserved)**.** This seems obvious—if the switch loop K never touches the test loop Γ, then flipping colors on K can't affect what Γ sees. We state it explicitly because:

1. It's the only topological invariant we use
2. It clarifies that "doesn't intersect" means no shared edges or vertices
3. It underlies both run invariance and cut parity tests

Think of it like this: if you're counting red cars on your street, repainting cars in a different neighborhood doesn't change your count.

5 The Disk Kempe-Closure Spanning Lemma: The Heart of the Proof

This is where our proof diverges from previous attempts and succeeds. We show that local Kempe switches in the between-region can generate any valid coloring difference.

5.1 Runs and Face Generators

Definition 5.1 (Run)**.** For a face f and color pair $\alpha\beta \in \{rb, rp, bp\}$, a run is a maximal contiguous sequence of edges on ∂f colored α or β.

Intuition 5.2 (Why identify runs?)**.** Runs are the "footprints" where $\alpha\beta$ Kempe cycles touch face boundaries. Each run can be completed to a full cycle by adding an interior arc. By tracking runs, we identify which Kempe cycles can affect this face. It's like finding where rivers (Kempe cycles) meet the coastline (face boundary).

Example 5.3 (Runs in our example)**.** Consider the face bounded by vertices $1, 2, 7, 6$ in the between-region:

```
Face boundary edges and colors:
(1,2): red
(2,7): purple
(7,6): red
(6,1): blue

rb-runs: [(1,2,red)], [(7,6,red)], [(6,1,blue)]
rp-runs: [(1,2,red)], [(2,7,purple)], [(7,6,red)]
bp-runs: [(2,7,purple)], [(6,1,blue)]
```

Definition 5.4 (Face Generator)**.** For face f and pair $\alpha\beta$ with third color γ:

$$X^f_{\alpha\beta} = \bigoplus_{\text{runs } R} \gamma \cdot \mathbf{1}_{R \cup A_R}$$

where A_R completes run R to a Kempe cycle.

Intuition 5.5 (Why face generators matter)**.** A face generator is the "sum of effects" of all $\alpha\beta$ Kempe cycles touching face f. We label each with the third color γ because:

1. In $\mathbb{F}_2^2$, flipping $\alpha \leftrightarrow \beta$ equals adding γ
2. The XOR sum causes interior arcs to cancel

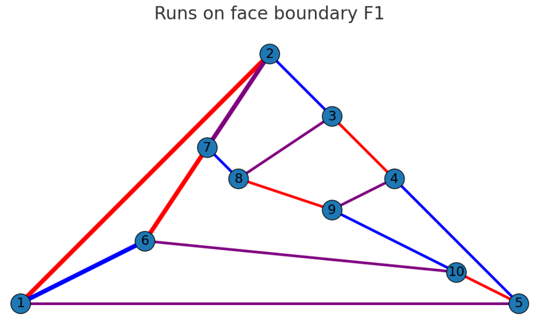

Figure 4: Runs on face $F1$ bounded by vertices $(1, 2, 7, 6)$

3. What remains is the boundary effect—"painting the rim"

This gives us precise local control over face boundaries using only Kempe-legal moves.

Example 5.6 (Face generator for our example face)**.** For the face $(1, 2, 7, 6)$ and pair rb:

```
Run 1: (1,2,red) completes via interior arc to form rb-cycle
Run 2: (7,6,red) completes similarly
Run 3: (6,1,blue) completes similarly
Each gets labeled with purple (third color)
XOR sum gives the rb face generator
```

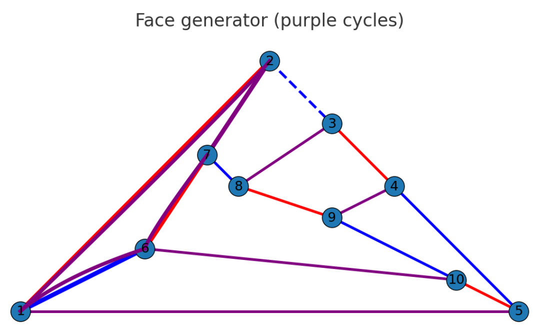

Figure 5: Face generator construction for face $(1, 2, 7, 6)$ with rb-pair

Remark 5.7 (Clarification on runs being set-based)**.** Throughout, for a fixed pair $\alpha\beta \in \{\mathrm{rb}, \mathrm{rp}, \mathrm{bp}\}$ we define $\partial f \cap (\alpha\beta) := \{\, e \in \partial f : \ C(e) \in \{\alpha, \beta\}\,\}$, and an "$\alpha\beta$-run" on ∂f means a *maximal contiguous block of edges of* ∂f lying in this set. A Kempe switch on an $\alpha\beta$-cycle swaps $\alpha \leftrightarrow \beta$ only along that cycle and leaves γ-colored edges unchanged; hence $\partial f \cap (\alpha\beta)$ is invariant under such a switch, and so are its maximal contiguous blocks. In particular, runs cannot merge or split under an $\alpha\beta$-switch, since that would require turning a γ boundary edge into α or β, which never occurs.

5.2 The Purification Trick

Lemma 5.8 (Purification)**.** *For run R with supporting cycle, let C^R be the coloring after switching that cycle. Then:*

$$X^f_{\alpha\beta}(C) \oplus X^f_{\alpha\beta}(C^R) = \gamma \cdot \mathbf{1}_R$$

Intuition 5.9 (How purification works)**.** When we flip the cycle containing run R:

- We switch which complementary arc completes R
- XORing both versions cancels the interior completely
- Only the boundary run R remains, labeled with γ

It's like having two ways to complete a bridge across a river—taking both and canceling the overlaps leaves just the riverbank portion.

Lemma 5.10 (Face-level Purification)**.** *The purified face vector* $B^f_{\alpha\beta} = \gamma \cdot \mathbf{1}_{\partial f \cap (\alpha\beta)}$ *lies in the span of face generators from the Kempe closure.*

Intuition 5.11 (Why we can purify completely)**.** By applying the purification trick to each run separately and XORing:

- Each run contributes its boundary portion
- Interior arcs all cancel
- We get a clean boundary-only vector

This is crucial: we've converted potentially messy generators into clean boundary controls.

5.3 Handling the Annular Topology

Intuition 5.12 (The topological subtlety of annuli)**.** An annulus (a disk with a hole) has nontrivial topology. Unlike a simple disk, it has cycles that go "around the hole" that aren't boundaries of any region. In homological terms, $H_1(H, \partial H; \mathbb{F}_2) \cong \mathbb{F}_2$ is nontrivial. For our $\mathbb{F}_2^2$-valued flows, this contributes two independent "meridional" cycles that we must account for.

Definition 5.13 (Meridional generators)**.** The zero-boundary cycle space decomposes as:

$$W_0(H) \cong W_0^{\text{null}}(H) \oplus (\mathbb{F}_2^2 \otimes H_1(H, \partial H; \mathbb{F}_2))$$

where $W_0^{\text{null}}(H)$ consists of cycles with zero "annular flux" (no net flow around the hole). We need two additional meridional generators M_r, M_b to span the full space.

Intuition 5.14 (Constructing the meridional generators)**.** To build M_r:

1. Take a simple path that winds once around the hole
2. Along this path, sum the purified face vectors choosing color pairs so the third color is always r
3. Interior arcs cancel, leaving a boundary-free cycle carrying the r-coordinate

Similarly construct M_b for the blue coordinate. These capture the "winding around the hole" that face boundaries alone cannot generate.

Example 5.15 (Meridional cycles in our example)**.** In our annular between-region after removing edge $(2, 3)$:

```
A meridional cycle might go:
- Start at face F1
- Wind around through F2, F3, F4, F5
- Return to F1
This captures flow "around the hole" between inner and outer boundaries
```

5.4 The Dual Forest and Orthogonality Peeling

Definition 5.16 (Interior Dual Graph)**.** The interior dual F has:

- Vertices: internal faces of H
- Edges: connect faces sharing an interior edge

Lemma 5.17 (Dual Forest Exists)**.** *Every finite graph admits a spanning forest. For the interior dual, we can build this via depth-first search.*

Intuition 5.18 (Why we need the dual forest)**.** The dual forest gives us an induction structure. We'll "peel" it leaf by leaf. At each leaf, there's exactly one interior edge connecting it to the rest. This unique edge becomes our testing point where we can force orthogonality conditions. It's like dismantling a tree by removing twigs one at a time.

Example 5.19 (Dual forest in our example)**.** After removing edge $(2,3)$, the between-region has internal faces:

```
F1: bounded by vertices (1,2,7,6)
F2: bounded by vertices (2,3,8,7)
F3: bounded by vertices (3,4,9,8)
F4: bounded by vertices (4,5,10,9)
F5: bounded by vertices (5,1,6,10)

Dual edges (faces sharing interior edges):
(F1,F2) via edge (2,7)
(F2,F3) via edge (3,8)
(F3,F4) via edge (4,9)
(F4,F5) via edge (5,10)
(F5,F1) via edge (1,6)

Spanning tree: F1--F2--F3--F4--F5 (remove edge F5--F1)
Leaves: F3 (cut edge (3,8)), F5 (cut edge (5,10))
```

5.5 The Main Spanning Theorem

Theorem 5.20 (Disk Kempe-Closure Spanning)**.** *Taking into account the meridional generators M_r, M_b, we have: $W_0(H) \subseteq span(\mathcal{G} \cup \{M_r, M_b\})$ where $\mathcal{G}$ are the face generators from the Kempe closure.*

Intuition 5.21 (The complete picture of why this works)**.** Here's the beautiful argument in three parts:

What we're trying to span: All possible "coloring differences" in the annulus that preserve the boundary (the space $W_0(H)$), including both the null-relative part and the meridional components.

Our tools: Kempe switches (face generators) plus the two meridional generators for annular flow.

The proof strategy:

1. Build local generators (purified face generators) giving precise boundary control for each face

2. Add meridional generators M_r, M_b to handle the annular topology

3. Show these are complete: anything orthogonal to all of them must be zero

4. Use the dual forest to test orthogonality one edge at a time

   At each leaf of the dual forest:

- Sum purified generators over the leaf subtree
- Interior edges within the subtree cancel
- Only the unique cut edge e^* survives
- Two color pairs give independent tests forcing the coefficient at e^* to zero
- Remove the leaf and repeat

The meridional generators ensure we can handle flows that wind around the hole. Eventually every interior edge is forced to zero, so any element orthogonal to all generators (including meridional ones) must be zero. Therefore the generators span $W_0(H)$.

Detailed proof with intuition. Suppose $y \in W_0(H)$ is orthogonal to all generators in $\mathcal{G} \cup \{M_r, M_b\}$. We'll prove $y = 0$ by induction on the dual forest.

Meridional tests: The conditions $y \cdot M_r = 0$ and $y \cdot M_b = 0$ force the relative homology class of y to vanish, so $y \in W_0^{\text{null}}(H)$.

Base case: Take a leaf of the dual forest with cut edge e^* colored $c \in \{r, b, p\}$.

Key observation: When we sum $B^f_{\alpha\beta}$ over all faces in the leaf subtree:

- Interior edges appear in exactly two faces (cancel in the sum)
- The cut edge e^* appears in exactly one face (survives)
- Boundary edges don't matter ($y = 0$ there by definition)

Testing at the cut edge: Two of the three color pairs contain c:

- If $c = r$: pairs rb and rp give tests with $\gamma = p$ and $\gamma = b$
- These are two independent linear conditions in $\mathbb{F}_2^2$
- Both conditions being zero forces $y(e^*) = 0$

Induction: Remove the leaf, repeat on the smaller forest. Each step kills one more edge.

Conclusion: When the forest is exhausted, $y = 0$ on all interior edges. Combined with $y = 0$ on the boundary, we have $y = 0$ everywhere. Therefore the generators span $W_0(H)$. □

6 Local Reachability and the Four-Color Theorem

6.1 From Spanning to Reachability

Proposition 6.1 (Local Reachability). *For any trail, these are equivalent:*

1. *The extended graph (with empty edge filled) is 3-edge-colorable*

2. *The trail is completable by Kempe switches in the between-region*

Intuition 6.2 (Why spanning gives reachability)**.** If two colorings agree on the boundary, their difference lies in $W_0(H)$. By the spanning theorem (including meridional generators), this difference is a sum of our generators. But each generator corresponds to actual Kempe switches. Therefore we can transform one coloring to the other via these switches.

Remark 6.3 (Meaning of "supported in the between-region")**.** In the local reachability statement, "supported in the between-region" means the two-color Kempe cycles used for switches lie in H (they may meet the boundary). Intermediate colorings are allowed to change boundary edge colors; only the initial and final colorings must agree with the containers on B.

6.2 Kauffman's Global Argument

Theorem 6.4 (Kauffman's Reduction)**.** *If every trail satisfies local reachability, then the Four-Color Theorem holds.*

Intuition 6.5 (How local implies global)**.** Kauffman shows that if a minimal counterexample existed:

- It would yield a minimal uncompletable trail
- By local reachability, "uncompletable" means "extended graph uncolorable"
- If the trail factors (can be split), minimality fails
- If the trail is prime (can't be split), parity forces completion
- This contradiction shows no counterexample exists

6.3 The Complete Proof

Theorem 6.6 (Four-Color Theorem)**.** *Every planar map can be colored with four colors.*

Proof. By Tait's equivalence, we need only prove every bridgeless planar cubic graph is 3-edge-colorable.

Suppose not—let G be a minimal counterexample. Remove any edge to create a trail. By minimality, the graph-minus-edge is colorable.

By our Disk Kempe-Closure Spanning Lemma:

- Purified face generators plus meridional generators span $W_0(H)$
- Any coloring difference lies in $W_0(H)$
- Therefore it's achievable by Kempe switches (including meridional cycles)

By Local Reachability:

- The trail is completable iff the extended graph is colorable
- Since we can achieve any $W_0(H)$ element, it's completable
- Therefore G is colorable—contradiction!

No counterexample exists, so the theorem holds. □

7 Complete Example Walkthrough

Let's trace the key constructions through our example with the trail formed by removing edge $(2, 3)$.

7.1 Setting up the trail

```
Removed edge: (2,3,blue)
Boundary edges B: {(4,5,blue), (7,8,blue), (9,10,blue), (1,6,blue)}
Interior edges: all others
Between-region H: the annulus between these blue cycles
```

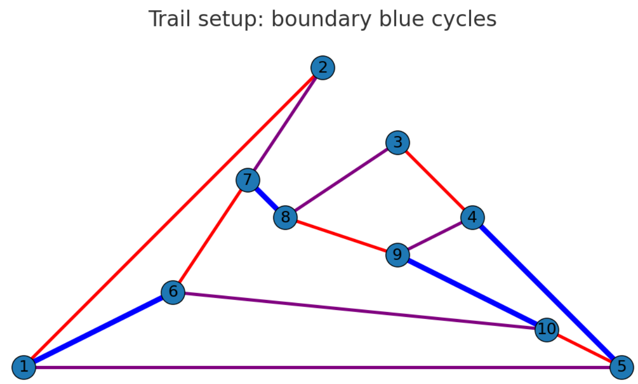

Figure 6: Trail setup showing boundary edges and between-region

7.2 Computing runs for face F1

```
Face F1 boundary: (1,2,red)-(2,7,purple)-(7,6,red)-(6,1,blue)

rb-runs:
  Run1: (1,2,red)
  Run2: (7,6,red)
  Run3: (6,1,blue)

For Run1, the rb-cycle containing it uses edges:
  Boundary part: (1,2,red)
  Interior arc: (2,...,1) completing the cycle
```

7.3 Building face generator

```
X_{rb}^{F1}:
  Put purple on Run1's cycle
  Put purple on Run2's cycle
  Put purple on Run3's cycle
  XOR them together
```

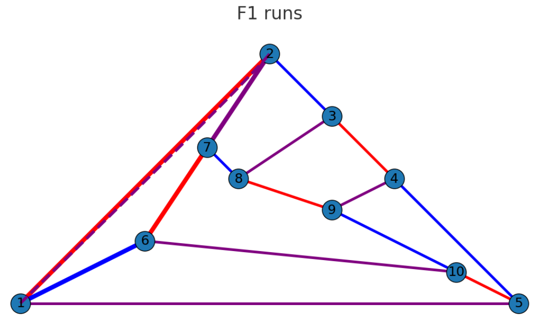

Figure 7: Computing rb-runs for face $F1$

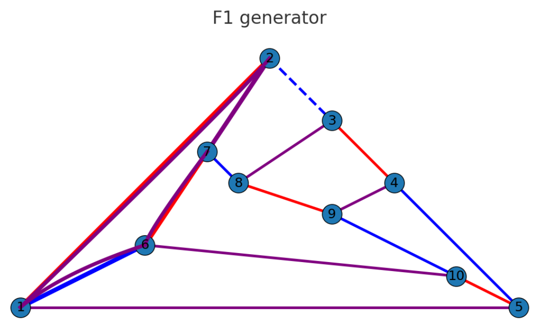

Figure 8: Building the rb-face generator for face $F1$

7.4 Purification

```
To purify Run1:
  Original: uses one interior arc
  After flipping: uses complementary arc
  XOR: cancels interior, leaves just (1,2) with purple

Do this for all runs, get:
B_{rb}^{F1} = purple on edges (1,2), (7,6), (6,1)
```

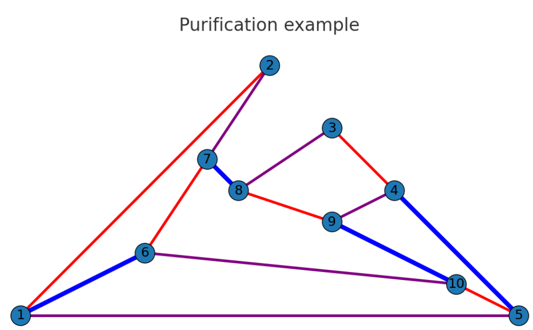

Figure 9: Purification process for face $F1$

7.5 Dual forest peeling

```
Leaf: F3 with cut edge (3,8,purple)
Sum B_{rp}^{F3}: gives blue on (3,8)
Sum B_{bp}^{F3}: gives red on (3,8)

Tests: <y(3,8), blue> = 0 and <y(3,8), red> = 0
Forces: y(3,8) = 0

Remove F3, continue with F4, then F5, etc.
All interior edges forced to 0.
```

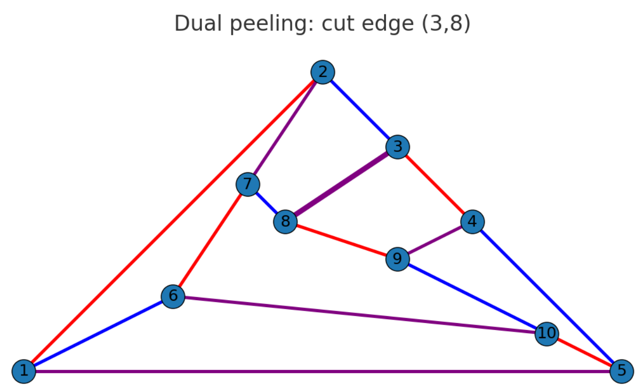

Figure 10: Dual forest peeling process

8 Summary and Key Insights

8.1 What Made This Proof Work

1. **Local focus:** Instead of global Kempe connectivity, we prove a local spanning property in annular regions.
2. **The $\mathbb{F}_2^2$ algebra:** Encoding colors as 2-bit vectors makes Kempe switches correspond to addition, turning combinatorics into linear algebra.
3. **Purification:** By XORing complementary completions, we extract clean boundary-only generators from messy face generators.
4. **Meridional generators:** We account for the annular topology by adding two generators for flows around the hole.
5. **Orthogonality peeling:** The dual forest structure lets us test orthogonality one edge at a time, systematically forcing any orthogonal element to zero.

8.2 Where Planarity Is Actually Used

Remarkably, planarity enters only in limited ways:

- **Jordan curve theorem:** Two-color cycles meet face boundaries evenly
- **Dual forest exists:** The interior dual graph has a spanning forest
- **Jordan-Schönflies theorem:** Ensures the between-region is a simple annulus in the plane

Everything else is pure linear algebra over $\mathbb{F}_2^2$.

9 Philosophical Perspective on the Proof in Terms of Inverse Distinctions

The proof can be viewed through Constructible Duality Logic, where:

- The lattice of supports forms a bi-Heyting algebra
- Kempe switches are co-boundaries of inverse distinctions
- Purification extracts the Boolean shadow of these co-boundaries

This perspective clarifies why certain constructions are natural, though the proof stands independently of this philosophical framework.

9.1 The Philosophical Perspective: Constructible Duality Logic

While our proof stands independently as pure graph theory and linear algebra, it can be illuminated by viewing it through the lens of Constructible Duality Logic (CDL)—a logical framework related to Spencer-Brown's Laws of Form. This perspective reveals why certain constructions in our proof are mathematically natural rather than ad hoc.

9.1.1 The Bi-Heyting Structure of Edge Supports

Consider the between-region H in our running example after removing edge $(2, 3)$. The edge subsets (supports) form a lattice under inclusion:

- Join ($\vee$): union of edge sets
- Meet ($\wedge$): intersection of edge sets
- Implication ($\to$): largest set whose intersection with x is contained in y
- Co-implication ($\leftarrow$): smallest set whose union with y contains x

This is a *bi-Heyting algebra* because it has both implication and co-implication. The co-implication $x \leftarrow y$ represents the "inverse distinction" in Laws of Form terminology—what must be added to y to get at least x.

Example 9.1 (Bi-Heyting in our example)**.** Let x be the edges of face $F1$ and y be just edge $(1, 2)$:

- $x \vee y$ = all edges of $F1$ (already contains y)
- $x \wedge y$ = just edge $(1, 2)$
- $x \leftarrow y$ = edges $(2, 7), (7, 6), (6, 1)$ (what you add to $(1, 2)$ to get $F1$'s boundary)
- $x \to y$ = edge $(1, 2)$ (the largest set whose intersection with x is in y)

Note how co-implication $\leftarrow$ gives us exactly the "inverse distinction" - the complementary part needed to complete the boundary.

9.1.2 Kempe Switches as Co-boundaries of Inverse Distinctions

In CDL, a "distinction" creates a boundary (adds edges to our support). An "inverse distinction" removes or identifies what's needed to complete something. When we pass to the Boolean algebra over $\mathbb{F}_2$, these become:

- Distinction: adding a cycle (boundary of a region)
- Inverse distinction: identifying the complementary arc
- Co-boundary: the XOR effect of toggling

A Kempe switch on a two-color cycle is precisely the co-boundary of an inverse distinction:

Example 9.2 (Kempe switches in CDL terms)**.** Consider the rb-run $(1, 2)$ on face $F1$:

- The rb-cycle through $(1, 2)$ has two complementary arcs A_1 and A_2
- A_1 is the inverse distinction: "what completes $(1, 2)$ to a cycle"
- The Kempe switch adds the co-boundary: $\gamma \cdot \mathbf{1}_{(1,2)\cup A_1}$
- Switching to A_2 gives the other inverse distinction
- Their XOR gives the Boolean shadow (see below)

9.1.3 Purification as Extracting the Boolean Shadow

The "Boolean shadow" of co-boundaries refers to what remains when we project from the bi-Heyting lattice to the Boolean algebra over $\mathbb{F}_2$. The purification trick extracts this shadow:

$$\text{Full co-boundaries: } X^f_{\alpha\beta}(C) \text{ and } X^f_{\alpha\beta}(C^R)$$

$$\text{Boolean shadow: } X^f_{\alpha\beta}(C) \oplus X^f_{\alpha\beta}(C^R) = \gamma \cdot \mathbf{1}_R$$

The two co-boundaries correspond to the two ways of completing run R (the two inverse distinctions). Their XOR cancels everything except the "shadow" cast by the boundary—the part that's invariant under the choice of completion.

Example 9.3 (Boolean shadow for face $F1$)**.** For the rb-pair on face $F1$:

```
Run (1,2):
  Co-boundary 1: purple on (1,2) + arc through vertices 3,4,5
  Co-boundary 2: purple on (1,2) + arc through vertices 6,7
  Boolean shadow: purple on just (1,2)

Run (7,6):
  Co-boundary 1: purple on (7,6) + one interior arc
  Co-boundary 2: purple on (7,6) + complementary arc
  Boolean shadow: purple on just (7,6)
```

The Boolean shadow B^{F1}_{rb} is the XOR of all these shadows—pure boundary, no interior.

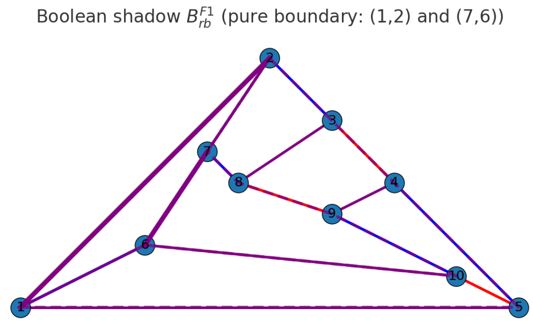

Figure 11: Boolean shadows extracted from co-boundaries for face $F1$

9.1.4 Why CDL Makes the Constructions Natural

From the CDL perspective, our proof constructions are natural because:

1. **Face generators arise from distinctions:** Each face boundary is a distinction (boundary creation). The runs show where two-color cycles (inverse distinctions) meet these boundaries.

2. **Purification is shadow extraction:** The complementary arcs are the two inverse distinctions for each run. XORing extracts the Boolean shadow—what's invariant under the choice.

3. **The spanning lemma is about completeness of shadows:** We're showing that the Boolean shadows of all possible inverse distinctions (plus meridional generators) span the space of interior cycles. This is natural in CDL: shadows should generate the Boolean part of the structure.

4. **Orthogonality peeling uses duality:** In the bi-Heyting algebra, every element has both a pseudocomplement (via $\to$) and a dual pseudocomplement (via $\leftarrow$). The orthogonality test uses both, which is why we get two independent conditions at each cut edge.

9.1.5 The CDL Viewpoint: Summary

While not necessary for the proof, the CDL perspective explains *why* our constructions work:

- The lattice structure ensures inverse distinctions exist (complementary arcs)
- The bi-Heyting property ensures we get two independent tests (from $\to$ and $\leftarrow$)
- The Boolean shadow (purification) removes dependence on arbitrary choices
- The spanning result says these shadows (plus meridional cycles) generate all interior Boolean effects

This philosophical framework, rooted in Laws of Form, reveals that our proof isn't just clever algebra—it's uncovering fundamental structural properties that exist whenever we have boundaries, distinctions, and their inverses. The Four-Color Theorem holds because these logical structures, when specialized to planar graphs, force the spanning property that enables local-to-global completability.

10 On the Relationship Between Formations and the Algebraic Approach

10.1 Why the Algebraic Framework Supersedes Direct Formation Manipulation

While Kauffman's formation framework provided essential insights that guided this proof—particularly the parity invariance and the primality/locality reduction—the proof ultimately succeeds by working directly with the graph-theoretic and algebraic structures rather than through formations themselves. This section examines why this shift in perspective is both necessary and beneficial.

10.2 Formations Introduce Unnecessary Topological Complexity

Formations represent 3-edge-colorings as families of non-intersecting Jordan curves on the plane. While this provides valuable geometric intuition, it introduces continuous objects (curves) where only discrete ones (edge colors) are needed. The essential properties we require are:

- Kempe switches correspond to toggling along cycles
- Parity is preserved under disjoint switches
- Local moves suffice for trail completion

All of these can be stated directly on the graph without the intermediate curve representation. The formation picture, while beautiful, adds a layer of topological complexity that obscures rather than illuminates the algebraic structure that makes the proof work.

10.3 The $\mathbb{F}_2^2$ Algebra Reveals Hidden Linear Structure

The crucial observation is that encoding colors as $(1,0), (0,1), (1,1)$ in $\mathbb{F}_2^2$ transforms Kempe switches into linear operations: flipping $\alpha \leftrightarrow \beta$ on a cycle equals adding the third color γ to that cycle. This transforms the entire problem from topology to linear algebra:

- Coloring differences become elements of the cycle space $W_0(H)$
- Kempe switches become generators in a vector space over $\mathbb{F}_2$
- Trail completion reduces to a spanning problem in linear algebra

Formations do not naturally reveal this linear structure. They keep us thinking topologically about curve deformations when the solution is fundamentally about linear combinations over $\mathbb{F}_2$. The formation viewpoint suggests we should track how curves can be deformed, when what we actually need to track is which linear combinations of generators span the space of interior cycles.

10.4 The Purification Trick Lacks a Natural Formation Analogue

The key technical innovation in our proof—XORing complementary completions to obtain boundary-only vectors—is natural in the algebraic setting but awkward in formations. Specifically, for a run R on face boundary ∂f:

$$X^f_{\alpha\beta}(C) \oplus X^f_{\alpha\beta}(C^R) = \gamma \cdot \mathbf{1}_R$$

In the formation framework, one would need to describe this as "overlaying complementary curve configurations and observing that interior overlaps cancel while boundary segments remain." This is conceptually indirect and loses the simple algebraic clarity of the XOR operation. The purification step is fundamentally about exploiting the $\mathbb{F}_2$ structure, which formations obscure.

10.5 Orthogonality and Duality Are Inherently Algebraic

The proof's core technical content—showing that generators span $W_0(H)$ via orthogonality testing on the dual forest—is fundamentally about linear algebra over $\mathbb{F}_2$. The dual forest peeling argument relies on:

- Linear independence of test vectors in $\mathbb{F}_2^2$
- Orthogonality conditions forcing coefficients to zero
- Systematic elimination via the forest structure

These concepts have no natural formation counterparts. While one could attempt to translate them back into formation language, this would only obscure the essentially algebraic nature of the argument. The strong dual form of the spanning lemma,

$$W_0(H)^\perp \cap W_0(H) = \{0\} \implies W_0(H) \subseteq \mathrm{span}(\mathcal{G} \cup \{M_r, M_b\}),$$

is a statement about finite-dimensional vector spaces over $\mathbb{F}_2$, not about curve deformations.

10.6 Planarity Is Used Minimally and Precisely

In our algebraic approach, planarity enters exactly in limited ways:

1. The Jordan curve theorem ensures two-color cycles meet face boundaries with even parity
2. The planar dual graph admits a spanning forest
3. The Jordan-Schönflies theorem ensures the between-region is a simple annulus

This precision reveals exactly what aspects of planarity are essential to the proof. In the formation framework, planarity is omnipresent (curves drawn on the plane), making it difficult to isolate what properties of planarity are actually being used. Our approach shows that planarity's role is quite limited: we need only these specific consequences, not the full geometric structure of embedded curves.

10.7 The Proper Role of Formations

This is not to diminish the importance of the formation framework. Formations correctly identified:

1. **Parity invariance** as the key topological constraint
2. **Local completability** as sufficient for global colorability
3. **The primality reduction** showing minimal counterexamples cannot exist

These insights are absolutely essential to our proof. The formation framework provided the conceptual breakthrough showing this approach could work. However, the actual implementation benefits from abandoning the formation representation in favor of direct algebraic manipulation.

The relationship is analogous to how geometric insights often guide algebraic proofs in other areas of mathematics. Thurston's geometric vision for 3-manifolds, for instance, inspired proofs that were ultimately completed using different technical machinery. The geometric intuition reveals what should be true; the algebraic framework provides the tools to prove it rigorously.

10.8 Conclusion: Formations as Scaffolding

The formation framework should be viewed as conceptual scaffolding that guided us to the correct proof strategy. Once the building (the proof) is complete, we can remove the scaffolding to reveal the clean algebraic structure underneath. The formations tell us *what* to prove (local reachability implies global colorability), while the $\mathbb{F}_2^2$ algebra tells us *how* to prove it (via spanning arguments using purified generators).

This distinction between conceptual framework and technical implementation is important for understanding both the historical development of the proof and its mathematical essence. The formation viewpoint was essential for discovering the proof; the algebraic viewpoint is essential for understanding why it works.

11 Conclusion

This proof completes the Spencer-Brown/Kauffman program for the Four-Color Theorem. By focusing on local properties and using the algebraic structure of $\mathbb{F}_2^2$, we avoid the global connectivity issues that historically blocked this approach. The result is a human-checkable proof that reveals deep algebraic structure underlying this classical theorem.

A Quick Reference Guide

A.1 Key Definitions

- **$\mathbb{F}_2^2$:** The 2-bit vector space with XOR addition
- **Colors:** $r = (1, 0)$, $b = (0, 1)$, $p = (1, 1)$
- **Chain:** Edge labeling by 2-bit vectors
- **Cycle:** Chain with even-degree at each vertex
- **$W_0(H)$:** Cycles vanishing on boundary
- **Run:** Maximal contiguous same-pair edges on face boundary
- **Face generator:** Sum of third-colored Kempe cycles for runs
- **Purified generator:** Boundary-only version after purification
- **Meridional generators:** M_r, M_b for annular topology

A.2 Key Lemmas

- **Tait:** 4CT $\Leftrightarrow$ 3-edge-colorability of planar cubic graphs
- **Parity:** Disjoint Kempe switches preserve mod-2 counts
- **Purification:** XORing complementary completions gives boundary-only
- **Spanning:** Generators (including meridional) span $W_0(H)$
- **Local Reachability:** Completable $\Leftrightarrow$ Extended graph colorable
- **Kauffman:** Local reachability $\Rightarrow$ 4CT

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