# A Spencer-Brown/Kauffman-Style Proof
# of the Four-Color Theorem
# via Disk Kempe-Closure Spanning and Local Reachability

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October 5, 2025

## Abstract

We give a self-contained proof of the Four-Color Theorem (4CT) that follows the Spencer-Brown/Kauffman intuition and succeeds via bypassing problems related to "global Kempe connectivity" with a local, planar, and finitary *Disk Kempe-Closure Spanning Lemma*. The proof proceeds as follows.

(1) We recall Tait's equivalence between 4CT and 3-edge-colorability of all bridgeless planar cubic graphs, and summarize the key pieces of Kauffman's framework (formations, trails, parity, and the primality argument).

(2) We give a complete human proof of the Disk Kempe-Closure Spanning Lemma for the between-region annulus of a trail. The proof uses two simple ingredients: (i) *run invariance* of two-color runs on a face boundary under Kempe switches on the supporting two-color cycle, and (ii) a *purification* trick that expresses face generators as boundary-only vectors. A cut-parity argument, combined with an induction on a spanning forest of the interior dual, yields a strong dual (annihilator) form of the lemma.

(3) From the spanning lemma we derive a *local reachability equivalence*: for any trail, the extended graph is 3-edge-colorable if and only if the trail is completable by Kempe switches supported in the between-region.

(4) Combining local reachability with Kauffman's parity/primality reduction gives that every bridgeless planar cubic graph is 3-edge-colorable; by Tait, 4CT follows.

We also include a brief discussion of how the setup implicitly uses the constructive duality logic (CDL) perspective, a conceptual close relative of the Laws of Form perspective that inspired Spencer-Brown and Kauffman: the lattice of supports acts as a finite bi-Heyting algebra; inverse distinction is co-implication; and the purification step is the Boolean shadow of co-boundaries of inverse distinctions.

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# 1 Introduction

A conceptually simple, human-checkable proof of the Four-Color Theorem has been a long-standing goal since the original computer-assisted proofs of Appel-Haken and later refinements. The Spencer-Brown and Kauffman programs reformulate the problem in terms of curve formations and parity, suggesting a topologically natural route. Historically, however, proofs along these lines stumbled on global Kempe-connectivity issues: it is false in general that all 3-edge-colorings of a planar cubic graph lie in a single Kempe class.

This paper shows how to complete the Spencer-Brown/Kauffman strategy without global connectivity. Our proof follows the direction laid out by Kauffman in [1] but adds a number of critical details. The key is to isolate planarity to a *between-region* annulus and prove a local spanning property *using the full Kempe-closure in that annulus*. This yields a local reachability equivalence that is exactly what Kauffman's parity/primality argument requires. Thus we obtain a complete, elementary, and checkable proof that avoids global case splits and machine enumeration.

# 2 Preliminaries: graphs, colors over F2xF2, and Tait's equivalence

We work with finite planar graphs. A *cubic* graph is 3-regular. A *bridgeless* graph has no bridges (cut edges). A *3-edge-coloring* of a cubic graph is a map from edges to a 3-element set such that at each vertex the three incident edges have distinct colors.

We encode the three edge colors by nonzero elements of the abelian group $G = \mathbb{F}_2^2$:

$$r = (1,0), \qquad b = (0,1), \qquad p = r + b = (1,1).$$

Given a map coloring with four region colors, we will use differences in $G$ to color edges of a dual triangulation; conversely, integrating $G$-valued edge colors along paths recovers region colors up to translation.

**Theorem 2.1** (Tait's equivalence)**.** *The Four-Color Theorem is equivalent to: every bridgeless planar cubic graph is 3-edge-colorable.*

*Proof.* (4CT $\Rightarrow$ 3-edge-coloring) Let $G$ be a bridgeless planar cubic graph. Take a planar embedding and 3-edge-colorability is equivalent to 4-face-colorability of the dual triangulation $\Delta$: this is the classical Tait observation. More concretely, triangulate any map; its dual is bridgeless planar cubic. Given a 4-coloring of faces by $G$-elements $\{0, r, b, p\}$, color each dual edge with the $G$-difference of the incident face colors. At a dual vertex the three incident edges have colors $r, b, p$ in some order,

since the three faces around a triangle use three distinct region colors. Hence we obtain a proper 3-edge-coloring.

(3-edge-coloring $\Rightarrow$ 4CT) Conversely, given a 3-edge-coloring of a bridgeless planar cubic dual, define region colors by $G$-integration: fix a base face with color $0 \in G$; for any other face, choose a dual path and sum edge colors (with orientation) along the path. Because around each dual vertex the sum $r + b + p = 0$ in $G$, the integral is path-independent. Adjacent faces differ by the nonzero color on their separating edge, so we obtain a proper 4-coloring of the primal map by the four $G$-elements $\{0, r, b, p\}$. This shows equivalence. $\qquad\square$

**Remark 2.2** (Path-independence of $G$-integration)**.** At each dual vertex (a primal triangle) the incident dual edges carry colors $r, b, p$ and satisfy $r + b + p = 0$ in $G = \mathbb{F}_2^2$. Hence the $G$-valued 1-cochain is closed and the line integral is path-independent, yielding a well-defined face coloring modulo an additive constant.

# 3  Kauffman's framework: formations, parity, primality

We briefly summarize the pieces from Kauffman's paper *Reformulating the Map Color Theorem* [1] that we use. We keep to a minimal, self-contained level.

## 3.1  Formations, trails, and completability

A *formation* is a representation of a 3-edge-coloring by two families of planar Jordan curves (say red and blue), with the third color (purple) arising on overlays; the precise definition is not needed here beyond the fact that Kempe switches on two-color cycles correspond to toggling along closed curves. A *trail* is a configuration of two distinguished same-color boundary curves ("containers") with an "empty edge" between two marked boundary points; the *between-region* is the annulus between the two containers.

Given a trail, one asks whether one can complete the coloring by Kempe switches that are *supported in the between-region*. This is the notion of *completability* we will connect to 3-edge-colorability of the extended graph obtained by inserting the missing edge.

## 3.2  Parity lemma (planar invariance under local moves)

Kauffman's Parity Lemma asserts that certain parity counts of crossings (or, equivalently, mod-2 edge counts) are invariant under the basic local "idemposition" moves in planar formations. In graph terms, flipping along a two-color cycle preserves mod-2 counts of two-color edges on any closed curve that misses the flip support. We will use only the most elementary consequence: parity is topological and local, and will be invoked within our annulus.

## 3.3  Primality and the global reduction

Kauffman isolates a *Primality Principle*: any minimal planar uncompletable trail is *prime* (cannot be factored nontrivially as a join across same-color containers). He then proves that the Primality Principle is equivalent to 4CT: if primality holds, parity considerations lead to a contradiction for a minimal counterexample; conversely 4CT implies primality by eliminating the obstruction. We will not need the fine details beyond this equivalence.

**Theorem 3.1** (Kauffman; reduction)**.** *If every trail satisfies the* local reachability equivalence

*(extended graph is 3-edge-colorable)* $\iff$ *(the trail is completable by between-region Kempe switches),*

*then the Four-Color Theorem holds.*

*Idea.* Assume a minimal counterexample to 3-edge-colorability (by Tait) exists. In the formation this yields a minimal uncompletable trail. By local reachability, the only way for the trail to be uncompletable is that the extended graph is itself uncolorable. Kauffman's parity/primality argument then rules out such a minimal trail: if it factors, minimality fails; if prime, parity forces completion. Hence no minimal counterexample exists; thus every bridgeless planar cubic graph is 3-edge-colorable and 4CT holds. □

Thus, to prove 4CT it suffices to prove the local reachability equivalence for every trail. This is precisely what the Disk Kempe-Closure Spanning Lemma delivers.

# 4 The Disk Kempe-Closure Spanning Lemma

We work in the *between-region* $H$ of a trail: a planar annulus whose outer boundary $B = B_1 \sqcup B_2$ consists of two disjoint simple cycles and whose interior edges form a finite planar subgraph. Fix one proper 3-edge-coloring $C_0$ of $H$, and consider its *Kempe closure* $\mathrm{Cl}(C_0)$ (the set of colorings reachable by finitely many two-color Kempe switches).

We encode the three edge colors as nonzero elements $r, b, p$ of $G = \mathbb{F}_2^2$. An $F_2^2$-*chain* is a map $x : E(H) \to G$. The cycle space $W(H) \subseteq G^{E(H)}$ consists of chains satisfying the even-degree constraint at every vertex in each coordinate; the *zero-boundary* subspace $W_0(H) = \{x \in W(H) : x(e) = 0 \text{ for all } e \in B\}$ consists of cycles that vanish on $B$. The global dot product is

$$x \cdot_e z := \bigoplus_{e \in E(H)} \langle x(e), z(e) \rangle \in \mathbb{F}_2,$$

where $\langle \ , \ \rangle$ is the usual coordinatewise dot in $G$.

**Lemma 4.1** (Non-degeneracy of the dot). *(a) For any nonzero $u \in G$, there exists $v \in G$ with $\langle u, v \rangle = 1$. Explicitly: if $u = r$ take $v = r$; if $u = b$ take $v = b$; if $u = p$ take $v = r$. (b) For any nonzero chain $y \in G^{E(H)}$ there exists a chain $z$ with $y \cdot_e z = 1$. Explicitly: choose any edge $e_0$ with $y(e_0) \neq 0$ and set $z(e_0) = v$ as in (a), $z(e) = 0$ for $e \neq e_0$.*

*Proof.* (a) is a two-line case check in $G = \mathbb{F}_2^2$. (b) then follows by choosing $e_0$ where $y$ is nonzero and supporting $z$ at $e_0$ only. □

## 4.1 Face generators and purification

Fix a coloring $C \in \mathrm{Cl}(C_0)$, an internal face $f$, and a two-color pair $\alpha\beta \in \{\mathrm{rb}, \mathrm{rp}, \mathrm{bp}\}$ with third color $\gamma$. On $\partial f$, decompose the $\alpha\beta$-colored edges into maximal runs $R$. For each run $R$, let $D$ be the $\alpha\beta$-cycle containing $R$, and let $A_R$ be one of the two complementary arcs on $D$ between the endpoints of $R$ (any choice will do). Define the *face generator*

$$X_{\alpha\beta}^f(C) := \bigoplus_{R \subset \partial f \cap (\alpha\beta)} \gamma \cdot \mathbf{1}_{R \cup A_R} \in G^{E(H)}.$$

Because $A_R$ lies inside $H$, $X_{\alpha\beta}^f(C)$ may have interior support.

**Lemma 4.2** (Run invariance under cycle switches). *Let $D$ be an $\alpha\beta$ Kempe cycle in $C$, and let $C'$ be the coloring obtained by switching colors along $D$. Then $\partial f \cap (\alpha\beta)$ is the same in $C$ and $C'$; hence the set of maximal $\alpha\beta$-runs on $\partial f$ is identical in $C$ and $C'$.*

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*Proof.* Switching on $D$ swaps $\alpha \leftrightarrow \beta$ on $D$ and leaves other edges unchanged. Thus an edge is colored $\alpha$ or $\beta$ in $C$ iff it is so in $C'$. The maximal contiguous blocks on $\partial f$ therefore coincide. $\square$

**Clarification (runs are set-based).** Throughout, for a fixed pair $\alpha\beta \in \{\text{rb}, \text{rp}, \text{bp}\}$ we define $\partial f \cap (\alpha\beta) := \{\, e \in \partial f :\ C(e) \in \{\alpha, \beta\}\,\}$, and an "$\alpha\beta$-run" on $\partial f$ means a *maximal contiguous block of edges of $\partial f$ lying in this set*. A Kempe switch on an $\alpha\beta$-cycle swaps $\alpha \leftrightarrow \beta$ only along that cycle and leaves $\gamma$-colored edges unchanged; hence $\partial f \cap (\alpha\beta)$ is invariant under such a switch, and so are its maximal contiguous blocks. In particular, runs cannot merge or split under an $\alpha\beta$-switch, since that would require turning a $\gamma$ boundary edge into $\alpha$ or $\beta$, which never occurs.

**Lemma 4.3** (Per-run purification). *Let $R$ be a maximal $\alpha\beta$-run on $\partial f$ in $C$, and let $C^R$ be the coloring obtained by a Kempe switch on the $\alpha\beta$-cycle containing $R$. Then*

$$X^f_{\alpha\beta}(C) \ \oplus\ X^f_{\alpha\beta}(C^R) \ =\ \gamma \cdot \mathbf{1}_R.$$

*Proof.* By Lemma 4.2, the set of runs on $\partial f$ is unchanged, so runs $R' \neq R$ contribute identical completed cycles in $C$ and $C^R$ and hence cancel in the XOR. For the run $R$ itself, $C$ and $C^R$ choose the two complementary arcs between the endpoints of $R$; their XOR cancels the interior arc and leaves $\gamma \cdot \mathbf{1}_R$ on the boundary. $\square$

**Lemma 4.4** (Face-level purification). *For any $C$ and $f$ and any pair $\alpha\beta$ with third color $\gamma$,*

$$B^f_{\alpha\beta} \ :=\ \gamma \cdot \mathbf{1}_{\partial f \cap (\alpha\beta)} \ \in\ \operatorname{span}_{\mathbb{F}_2} \{\, X^f_{\alpha\beta}(C') : C' \in \operatorname{Cl}(C_0)\,\}.$$

*Proof.* Enumerate the finitely many runs $R_1, \ldots, R_k$ on $\partial f$ (in $C$). By Lemma 4.3,

$$\bigoplus_{i=1}^k \big(X^f_{\alpha\beta}(C) \ \oplus\ X^f_{\alpha\beta}(C^{R_i})\big) \ =\ \bigoplus_{i=1}^k \gamma \cdot \mathbf{1}_{R_i} \ =\ \gamma \cdot \mathbf{1}_{\partial f \cap (\alpha\beta)}.$$

Each $C^{R_i}$ lies in $\operatorname{Cl}(C)$ and hence in $\operatorname{Cl}(C_0)$. $\square$

**Lemma 4.5** (Relative facial spanning). *For each coordinate of $G = \mathbb{F}_2^2$ separately, the zero-boundary cycle space $W_0(H)$ is generated by the internal face boundaries $\{\partial f\}_{f \text{ internal}}$. Reason. In mod-2 homology, $H$ is a planar surface with boundary; the relative group $H_1(H, \partial H; \mathbb{F}_2)$ vanishes once one takes the interior faces as a 2-chain basis. Equivalently, every relative 1-cycle is the boundary of a 2-chain supported on internal faces. Taking the two coordinates independently in $G$ yields the claim.*

## 4.2 Dual forest existence, cut parity, and orthogonality peeling

Define the collection of *purified* face vectors

$$\mathcal{B} \ :=\ \{\, B^f_{\text{rb}},\ B^f_{\text{rp}},\ B^f_{\text{bp}} \ :\ f \text{ an internal face}\,\}.$$

By Lemma 4.4, $\mathcal{B} \subseteq \operatorname{span}(\mathcal{G})$, where

$$\mathcal{G} \ :=\ \{\, X^f_{\alpha\beta}(C) : f \text{ an internal face},\ \alpha\beta \in \{\text{rb}, \text{rp}, \text{bp}\},\ C \in \operatorname{Cl}(C_0)\,\}.$$

**Lemma 4.6** (Interior dual forest exists). *Let $F$ be the interior dual graph of $H$: its vertices are internal faces of $H$, and two vertices are adjacent iff the corresponding faces meet along an interior edge. Then $F$ is finite and admits a spanning forest $T$ (a spanning tree in each connected component).*

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*Proof.* $H$ is a finite planar subgraph, hence has finitely many internal faces. Thus $F$ is a finite graph. Every finite graph has a spanning forest: take a spanning tree in each connected component (e.g. by a depth-first or breadth-first search). □

**Note (role of the forest).** The spanning forest $T$ is used solely as a bookkeeping device to select leaf-subtrees with a unique primal cut edge $e^*$. No claim is made (or needed) about $T$ spanning any cycle basis; cycles in the interior dual do not obstruct the peeling argument.

**Lemma 4.7** (Cut parity for purified sums)**.** *Let $S$ be any set of internal faces. For each pair $\alpha\beta$,*

$$\widetilde{B}_{\alpha\beta}(S) \ := \ \bigoplus_{f \in S} B^f_{\alpha\beta}$$

*is supported exactly on (i) the $\alpha\beta$ interior edges incident to an odd number of faces in $S$ (i.e. edges crossing the cut between $S$ and its complement), and (ii) the $\alpha\beta$ boundary edges on $\partial f$ for $f \in S$.*

*Proof.* Each interior $\alpha\beta$ edge lies on exactly two face boundaries; it contributes once if and only if exactly one of its incident faces lies in $S$. Boundary edges contribute once when their unique incident internal face is in $S$. □

Let $F$ be as above and let $T$ be any spanning forest of $F$ (from Lemma 4.6). Fix a tree component and let $S$ be a leaf-subtree with unique primal interior cut edge $e^*$. Write $c = \text{color}(e^*)$.

**Lemma 4.8** (Orthogonality forcing on a leaf cut)**.** *Let $y \in W_0(H)$ satisfy $y \cdot_e g = 0$ for all $g \in \mathcal{G}$. Then for the leaf-subtree $S$ above and for each pair $\alpha\beta$,*

$$\left(\widetilde{B}_{\alpha\beta}(S)\right)(e^*) \ = \ \begin{cases} \gamma & \text{if } c \in \{\alpha, \beta\}, \\ 0 & \text{otherwise,} \end{cases}$$

*and $y \cdot_e \widetilde{B}_{\alpha\beta}(S) = 0$ for all pairs $\alpha\beta$. Consequently, the two independent tests $\langle y(e^*), \gamma \rangle = 0$ force $y(e^*) = 0 \in G$.*

*Proof.* The support identity follows from Lemma 4.7: at the unique cut edge $e^*$ we obtain the third color $\gamma$ for precisely the two pairs containing $c$. Since each $\widetilde{B}_{\alpha\beta}(S)$ lies in $\text{span}(\mathcal{B}) \subseteq \text{span}(\mathcal{G})$ and $y$ is orthogonal to $\mathcal{G}$, we have $y \cdot_e \widetilde{B}_{\alpha\beta}(S) = 0$ for all pairs. The two dot tests force $y(e^*) = 0$ in $G$. □

**Theorem 4.9** (Disk Kempe-Closure Spanning; strong dual)**.** *If $z \cdot_e g = 0$ for all $g \in \mathcal{G}$, then $z \cdot_e w = 0$ for all $w \in W_0(H)$. Equivalently, $W_0(H) \subseteq \text{span}(\mathcal{G})$.*

*Proof.* We prove the annihilator inclusion. Let $z$ be orthogonal to $\mathcal{G}$ and let $y \in W_0(H)$ also be orthogonal to $\mathcal{G}$. By Lemma 4.6, the interior dual admits a spanning forest $T$. Peel $T$ by repeatedly applying Lemma 4.8 to a leaf-subtree: each step kills $y$ on the unique cut edge. This terminates with $y = 0$ on all interior edges; and $y = 0$ on the outer boundary by definition of $W_0(H)$. Hence $(\text{span}\,\mathcal{G})^\perp \cap W_0(H) = \{0\}$. In a finite space with non-degenerate dot (Lemma 4.1), this is equivalent to $(\text{span}\,\mathcal{G})^\perp \subseteq W_0(H)^\perp$, i.e. the stated strong dual: any $z$ orthogonal to $\mathcal{G}$ is orthogonal to $W_0(H)$. □

### 4.3 Local reachability equivalence

We now deduce the local reachability that Kauffman needs. For a trail with between-region $H$ and fixed boundary colors (given by the two containers), consider two proper colorings $C_1, C_2$ that agree on $B$; say $C_1$ is the starting coloring, and $C_2$ is any coloring that extends across the empty edge.

Let $\Delta = C_2 - C_1$ be the $G$-valued difference chain; clearly $\Delta \in W_0(H)$. By Theorem 4.9, $\Delta$ lies in the span of $\mathcal{G}$, i.e. it is a sum of third-colored Kempe-cycle generators supported in $H$ and taken from colorings in $\mathrm{Cl}(C_0)$. Executing those switches realizes $C_2$ from $C_1$ by Kempe moves supported in $H$.

**Proposition 4.10** (Local reachability equivalence). *For any trail, the following are equivalent:*

*(i) The extended graph (obtained by inserting the missing edge between the two marked boundary points) is 3-edge-colorable.*

*(ii) Starting from any proper boundary-compatible coloring on the between-region, one can complete across the empty edge by a finite sequence of two-color Kempe switches supported in the between-region.*

*Proof.* (i) $\Rightarrow$ (ii): Let $C_2$ be a proper coloring of the extended graph; restrict to $H$ and compare with the given $C_1$ on $H$. The difference lies in $W_0(H)$, hence lies in $\mathrm{span}(\mathcal{G})$; run the corresponding switches in $H$ to obtain a completion.

(ii) $\Rightarrow$ (i): If there is a Kempe completion in $H$, then certainly the extended graph admits a proper coloring (namely the one reached after the switch sequence). $\square$

**Remark 4.11** (Meaning of "supported in the between-region"). In Proposition 4.10, "supported in the between-region" means the two-color Kempe cycles used for switches lie in $H$ (they may meet the boundary). Intermediate colorings are allowed to change boundary edge colors; only the initial and final colorings must agree with the containers on $B$.

## 5 Conclusion: proof of the Four-Color Theorem

**Theorem 5.1** (Four-Color Theorem). *Every bridgeless planar cubic graph is 3-edge-colorable. Equivalently, every planar map is 4-colorable.*

*Proof.* By Proposition 4.10, every trail satisfies the local reachability equivalence assumed in Theorem 3.1. Therefore, by Kauffman's reduction, there are no minimal counterexamples to 3-edge-colorability of bridgeless planar cubic graphs. By Tait's equivalence (Theorem 2.1) we obtain the Four-Color Theorem. $\square$

## 6 Discussion: the CDL lens

Although our proof is purely combinatorial over $\mathbb{F}_2^2$, the setup was inspired by a Constructible Duality Logic (CDL) [2] perspective, which was chosen as a more mathematically sophisticated framework conceptually related to the "Laws of Form" [3] perspective that motivated Spencer-Brown and Kauffman. CDL provides an intuitionistic "logic of paradox" philosophically in the vein of Laws of Form, and our proof conceptually follows from looking at the nature of negations and contradictions in this intuitionistic logic. However, while this inspiration is useful for understanding "where the proof comes from", in the end what is needed to make the proof work is just the graph theory that corresponds to aspects of CDL rather than CDL itself.

**Bi-Heyting lattice of supports.** The family of supports (edge sets) inside the between-region, ordered by inclusion, forms a finite distributive lattice; as such, it is bi-Heyting. The usual union/intersection are join/meet. In this lattice-theoretic reading, a *distinction* (drawing a contour) adds a generator via join; the *inverse distinction* is the co-implication $x \Leftarrow y$ (least $z$ with $x \leq y \vee z$), i.e. excision up to join.

**Booleanization and co-boundaries.** Passing to $\mathbb{F}_2$ turns join into XOR on indicators; the boundary operator is linear. A Kempe switch on a two-color cycle is the addition of a co-boundary of an inverse distinction. Our *purification* step (Lemmas 4.3 and 4.4) produces boundary-only vectors from differences of two co-boundaries corresponding to complementary inverse distinctions.

**Why CDL helps conceptually.** The CDL viewpoint organizes the proof into (i) a lattice layer (faces generate the relative cycle space; factorization language for trails), and (ii) a Boolean/linear layer (Kempe toggles and dot orthogonality). The final spanning lemma lives in the linear layer; the global reduction to 4CT uses the lattice language through Kauffman's parity/primality argument. Nothing in CDL is required to run the proof; it clarifies how inverse distinction underlies the generators we used.

# Acknowledgments

# A Summary of dependencies and self-contained pieces

For the reader's convenience:

- Fully proved here: Tait equivalence (Theorem 2.1); non-degeneracy of the dot (Lemma 4.1); interior dual forest existence (Lemma 4.6); Disk Kempe-Closure Spanning (Theorem 4.9); Local reachability (Proposition 4.10); Four-Color Theorem (Theorem 5.1).

- Imported from Kauffman (with brief explanation): parity lemma in formations; primality setup and the reduction Theorem 3.1. Only the direction "local reachability for trails implies 4CT" is used.

# B Minimal glossary

- $G = \mathbb{F}_2^2$: color group with nonzero elements $r, b, p = r + b$.

- $W(H)$: $G$-valued cycle space on the between-region edges.

- $W_0(H)$: zero-boundary subspace of $W(H)$.

- Kempe switch: toggling along a two-color cycle $\alpha\beta$.

- $\text{Cl}(C_0)$: Kempe-closure of coloring $C_0$ on $H$.

- $X_{\alpha\beta}^f(C)$: third-colored sum of completed runs on $\partial f$.

- $B_{\alpha\beta}^f$: boundary-only purified face vector for pair $\alpha\beta$.

- $\mathcal{G}$: all face generators from the Kempe-closure.

- $\mathcal{B}$: purified boundary-only face vectors (span of $\mathcal{G}$).

# References

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