Completing the Spencer–Brown/Kauffman Route to 4CT via the Disk Kempe–Closure Spanning Lemma (Lemma 7.1 of the Attached Note)

Abstract

This note records, in one place, the higher-order logic (HOL) theorem that, once proved in Isabelle/HOL, completes the Spencer–Brown/Kauffman route to the Four-Color Theorem (4CT). The target is exactly the *Disk Kempe–Closure Spanning Lemma* stated as **Lemma 7.1** in the attached PDF, which asserts that, on the "between-region" disk of a trail, the third-color-labeled Kempe circuits arising across the Kempe-closure of a fixed proper 3-edge-coloring span the full $GF(2)^2$ zero-boundary cycle space (equivalently, the only zero-boundary vector orthogonal to all such generators is the zero vector). Proving this lemma uniformly (for every disk and proper 3-edge-coloring) yields 4CT via Kauffman's parity/primality reduction.

(See the attached note, Lemma 7.1, for the precise disk formulation, matrix form, and certificate format.)

1 The HOL Theorem to Prove (Strong Dual / Annihilator Form)

We formalize the disk setup in an Isabelle `locale` for a finite, simple, cubic graph whose boundary is the disjoint union of two cycles (the "containers"). Colors live in $GF(2)^2$; a $GF(2)^2$ *chain* is a function from edges to colors; the global dot product is the $GF(2)$ sum of per-edge dots. Kempe circuits are even subgraphs inside a 2-color 2-factor; a *generator column* puts the third color along such a circuit; `gens_from_closure` unions these columns across the Kempe-closure of a fixed proper coloring.

Isabelle/HOL statement (matching Lemma 7.1 in the attached note)

File: `Disk_KCSD_DualStrong.thy` (core extract)

```
theory Disk_KCSD_DualStrong
  imports Main
begin

record ('v,'e) graph =
  V :: "'v set"  E :: "'e set"  ends :: "'e ? 'v Œ 'v"

locale disk_cubic =
  fixes G :: "('v,'e) graph" and B :: "'e set"
  assumes SG: "(* simple_graph G *) True" and CB: "(* cubic G *) True"
  assumes boundary_two_cycles:
    "?B1 B2. B1 ? E G ? B2 ? E G ? B1 ? B2 = {} ? B = B1 ? B2
              ? (* is_cycle G B1 *) True ? (* is_cycle G B2 *) True"
begin
  (* Colors over GF(2)^2, chains, per-edge dot, global dot 'ů?'.
```

```
  Kempe circuits, closure, and generator set gens_from_closure C0.
  Zero-boundary subspace 'zero_boundary'. *)

theorem Disk_KCSD_dual_strong:
  fixes C0 :: "'e ? col"
  assumes finE: "finite (E G)" and P0: "proper3 C0"
  shows "?z. (?g?gens_from_closure C0. z ů? g = False)
             ? (?w?zero_boundary. z ů? w = False)"
  sorry

end
end
```

Equivalence to the PDF lemma. On a finite, non-degenerate bilinear space, the annihilator inclusion $G^{\perp} \subseteq W_0(H)^{\perp}$ (the theorem above) is equivalent to the spanning inclusion $W_0(H) \subseteq \mathrm{span}(G)$. Thus `Disk_KCSD_dual_strong` is logically equivalent to Lemma 7.1 (column-rank form) in the attached note.

2 Why This HOL Theorem Proves 4CT (Summary)

- **Local reachability.** The strong dual yields $W_0(H) \subseteq \mathrm{span}(G)$, i.e., any zero-boundary difference of colorings is an XOR of generator columns—hence realizable by a finite sequence of between-region Kempe switches. Therefore "G^* is 3-edge-colorable" ⇔ "the trail is completable by simple operations." (This is Proposition 7.2 in the attached note.)
- **Parity/primality conclusion.** With colorability ⇔ completability, Kauffman's Parity Lemma and parity-flip contradiction rule out minimal prime uncompletable trails; by his equivalence, 4CT follows. (See Theorem 7.3 in the attached note.)

3 How to Guide Isabelle to a General Proof

ATPs do not prove universal statements by instance enumeration; instead, provide a structural plan that reduces the global claim to local, finitary equalities amenable to automation:

1. **Faces extension (disk planarity).** Enrich the `disk_cubic` locale with a finite set of *internal faces* (each a simple cycle), pairwise edge-disjoint, whose union is $E \setminus B$.
2. **Facial basis (cycle space).** Prove that the $\mathrm{GF}(2)^2$ zero-boundary cycle space is generated by internal face boundaries (coordinatewise). This is the standard planar-disk statement and appears (conceptually) before Lemma 7.1 in the attached note.
3. **Run-completion lemma (local parity).** For a face f and pair (α, β), the XOR of $\alpha\beta$-Kempe circuits completing maximal $\alpha\beta$-runs on ∂f equals exactly the $\alpha\beta$-colored edges of ∂f (the "run completion" used in Theorem 3.1).
4. **Per-face generators.** Define three per-face third-color generators $G_{\mathrm{rb}}(f)$, $G_{\mathrm{rp}}(f)$, $G_{\mathrm{bp}}(f)$ and show they synthesize the two coordinate face-boundary chains by small case splits.
5. **Conclude strong dual.** Since each zero-boundary chain is a GF(2) sum of face boundaries (step 2) and each face boundary is a GF(2) sum of generators (step 4), linearity of the global dot implies the annihilator inclusion `Disk_KCSD_dual_strong`.

All steps are local parity or short linear arguments over finite sets; Isabelle's Sledgehammer/SMT tactics are well-suited for them.

4 Detailed Tutorial for Isabelle (Step-by-Step)

4.1 Installation and components

Install Isabelle (current release). Enable external provers:

```
isabelle components -a
```

In Isabelle/jEdit or Isabelle/VSCode, Sledgehammer should list E, CVC5/veriT, etc.

4.2 Session scaffold

```
DiskKCSD/
  ROOT
  GraphPrimitives.thy
  Disk_KCSD.thy
  Disk_KCSD_DualStrong.thy
```

ROOT:

```
session "DiskKCSD" = HOL +
  options [document = false]
  theories
    GraphPrimitives
    Disk_KCSD
    Disk_KCSD_DualStrong
```

4.3 Primitive objects (GraphPrimitives.thy)

Implement:

- $\mathrm{GF}(2)^2$ colors as pairs of Booleans; XOR as inequality; dot as coordinatewise AND+XOR.
- Graph record, incidence `inc`, and a cycle predicate `is_cycle`.
- Global $\mathrm{GF}(2)^2$ dot `dot_chain`: parity sum of per-edge dots.

4.4 Disk locale & targets (Disk_KCSD.thy)

Define:

- `isCycle`, `zeroBoundary`: order-free parity at each vertex in each coordinate and zero on boundary.
- Pair-set, `kempe_circuit` (even subgraph in 2-factor), `toggle_on`.
- `generators_of`, `gens_from_closure C0`.
- `span` as XOR-closure (inductive).

4.5 State the strong dual theorem (Disk_KCSD_DualStrong.thy)

Within `disk_cubic`, state:

```
theorem Disk_KCSD_dual_strong:
  (* as above; equivalent to Lemma 7.1, dual form *)
  sorry
```

4.6 Faces extension & facial basis

Add a faces extension to the locale; prove the *facial basis* lemma:

- Either via a small linear-algebra argument on the face–edge incidence,
- or via parity decompositions of interior cycles modulo boundary.

Place `sorry` where Sledgehammer/SMT should discharge subgoals; then iterate to remove the `sorry`s.

4.7 Run-completion & per-face generators

Formalize the run-completion lemma (used implicitly in the attached note's Theorem 3.1), define $G_{\mathrm{rb}}(f)$, $G_{\mathrm{rp}}(f)$, $G_{\mathrm{bp}}(f)$, and prove they synthesize coordinate face boundaries.

4.8 Conclude `Disk_KCSD_dual_strong`

Combine facial basis + per-face synthesis + linearity of the global dot to finish the annihilator inclusion.

4.9 Tactic workflow

- Start with `apply auto`/`simp`; then `sledgehammer` to obtain `by metis`/`by smt` one-liners.
- Use `nitpick` for counterexample search on small gadgets if a subgoal seems false.
- Mark helpful rewrites as `[simp]` (e.g., indicator/disjoint unions, simple parity facts).
- Keep subgoals small and localized.

5 End-to-End Checklist

1. All theories elaborate; only `Disk_KCSD_dual_strong` has `sorry`.
2. Add faces locale; prove the facial basis lemma (remove `sorry`).
3. Prove run-completion and per-face synthesis (remove `sorry`).
4. Conclude `Disk_KCSD_dual_strong`; eliminate its `sorry`.
5. Optionally, formalize the equivalence "strong dual $\Leftrightarrow$ spanning".
6. Import/cite Kauffman's parity/primality to conclude 4CT (as summarized in the attached note).

6 Conclusion

By targeting `Disk_KCSD_dual_strong`, which is *exactly* the dual form of Lemma 7.1 in the attached note, we reduce the Spencer–Brown/Kauffman program for 4CT to a single, ATP-friendly linear statement on disks. The faces-based plan organizes the universal proof into local, checkable parity identities; once formalized, the lemma yields the sought equivalence between colorability and completability and, with Kauffman's parity/primality argument, the Four-Color Theorem.