An Attentional Model of Time Discounting

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1 Introduction

decision maker (DM)

Kullback–Leibler (KL) divergence (also called relative entropy)

hard attention

information avoidance

endogenous time preferences

optimal expectation

we present an axiomatic characterization of AAD with the optimal discounting framework

2 Model Setting

Assume time is discrete. Let $s_{0\to T} \equiv [s_0, s_1, ..., s_T]$ denote a reward sequence that starts delivering rewards at period 0 and ends at period T. At each period t of $s_{0\to T}$, a specific reward s_t is delivered, where $t \in \{0, 1, \dots, T\}$. Throughout this paper, we only consider non-negative rewards and finite length of sequence, i.e. we set $s_t \in \mathbb{R}_{\geq 0}$ and $1 \leq T < \infty$. The DM's choice set is constituted by a range of alternative reward sequences which start from period 0 and end at some finite period. When making an intertemporal choice, the DM seeks to find a reward sequence $s_{0\to T}$ in her choice set, which has the highest value among all alternative reward sequences. To calculate the value of each reward sequence, we admit the additive discounted utility framework. The value of $s_{0\to T}$ is defined as $U(s_{0\to T}) \equiv \sum_{t=0}^{T} w_t u(s_t)$, where $u(s_t)$ is the instantaneous utility of receiving s_t, and w_t is the decision weight (sometimes called discount factors) assigned to s_t. We assume the function $u(.)$ is twice differentiable, $u' > 0$, $u'' < 0$.

The determination of w_t is central to this paper. We believe that, due to the DM's limited attention and demand for information, the DM tends to overweight the large rewards and underweight the small rewards within the sequence. Specifically, we suggest w_t follow a (generalized) softmax function. We define any decision weight in this style as an *attention-adjusted discount* factors (AAD), as in Definition 1.

Definition 1: *Let $\mathcal{W} \equiv [w_0, ..., w_T]$ denote the decision weights for all specific rewards in $s_{0\to T}$. $\mathcal{W}$ is called attention-adjusted discount factors (AADs) if for any $t \in \{0, 1, \dots, T\}$,*

$$w_t = \frac{d_t e^{u(s_t)/\lambda}}{\sum_{\tau=0}^{T} d_\tau e^{u(s_\tau)/\lambda}} \tag{1}$$

where $d_t > 0$, $\lambda > 0$, $u(.)$ is the utility function.

In intuition, how Definition 1 reflects the role of attention in valuating reward sequences can be explained with four points. First, each reward in a sequence could be viewed as an information source and we assume the DM allocates limited information-processing resources across those information sources. The AADs capture this notion by normalizing the discount factors, i.e. fixing the sum of w_t at 1. As a result, increasing the decision weight of one reward would reduce the decision weights of other rewards in the sequence, implying that focusing on one reward would make DM insensitive to the values of other rewards. Meanwhile, when there are more rewards in the sequence, DM needs to split attention across a wider range to process each of them, which may reduce the attention to, or decision weight of, each individual reward.

Second, w_t is strictly increasing with s_t, indicating that DM would pay more attention to larger rewards. This is consistent with many empirical studies that suggest people tend to pay more attention to information associated with larger rewards. For instance, people perform a "value-driven attentional capture" effect in visual search (Della Libera and Chelazzi, 2009; Hickey et al., 2010; Anderson et al., 2011; Chelazzi et al., 2013; Jahfari and Theeuwes, 2017). In one study (Hickey et al., 2010), researchers recruit participants to do a series of visual search trials, in each of which participants earn a reward after detecting a target object from distractors. If a target object is associated with a large reward in previous trials, it can naturally capture more attention. Therefore, in the next trial, presenting the object as a distractor slows down the target detection.[1] In addition, in financial decision making, investors usually perform an ostrich effect (Galai and Sade, 2006; Karlsson et al., 2009). One relevant evidence is that stock traders log in their brokerage accounts less frequently after market declines (Sicherman et al., 2016).

Third, w_t is "anchored" in a reference weight d_t. For a certain sequence of rewards, d_t could denote the initial weight that the DM would assign to a reward delivered at period t without knowing its realization. The determination of d_t is mediated by the difficulty to mentally represent a future event (Trope and Liberman, 2003) and the frequency of time delays in a global context (Stewart et al., 2006). The constraint on the deviation between w_t and d_t indicates that reallocating attention or acquiring new information is costly. The deviation of w_t from d_t depends on parameter λ, which as we discuss in the next section, can reflect the unit cost of information acquisition. A large λ implies a low learning rate and a high cognitive cost in adapting the decision weights to the local context.

Fourth, we adopt the idea of Gottlieb (2012) and Gottlieb et al. (2013) that attention can be understood as an active information-sampling mechanism which selects information based on the perceived utility of information. For intertemporal choices, we assume the DM would selectively sample value information from each reward (information source) when processing a reward sequence, and the AAD can represent an approximately optimal sampling strategy. Note that the AADs follow a softmax function. Matějka and McKay (2015) and Maćkowiak

[1] Some scholars may classify attention into two categories: "bottom-up control" and "top-down control". However, the evidence about value-driven attentional capture does not fall into either of these categories. Thus, in this paper, we do not describe attention with this dichotomy. Instead, we view attention as a mechanism that seeks to maximize the utility of information.

et al. (2023) claim that if a behavioral strategy conforms to this type of function, then it can be interpreted as a solution to some optimization problem under information constraints.

3 Interpretation

3.1 Information Maximizing Exploration

In this section, we provide two approaches to characterize AAD: the first is based on information maximizing exploration, and the second is based on optimal discounting. These approaches are closely related to the idea proposed by Gottlieb (2012), Gottlieb et al. (2013) and Sharot and Sunstein (2020), that people tend to pay attention to information with high *instrumental utility* (help identifying the optimal action), *cognitive utility* (satisfying curiosity), or *hedonic utility* (inducing positive feelings). It is worth mentioning that the well-known rational inattention theories are grounded in the instrumental utility of information.[2] Instead, in this paper, we draw on the cognitive and hedonic utility of information to build our theory of time discounting. Our first approach to characterizing AAD is relevant to the cognitive utility: the DM's information acquisition process is curiosity-driven. The model setting of this approach, similar with Gottlieb (2012) and Gottlieb et al. (2013), is based on a reinforcement learning framework. Specifically, we assume the DM seeks to maximize the information gain with a commonly-used exploration strategy. Our second approach is relevant to the hedonic utility: the DM consider the feelings of multiple selves and seeks to maximize their total utility under some cognitive cost. The theoretical background for the second approach is Noor and Takeoka (2022, 2024). We describe the first approach in this subsection and the second approach in Section 3.2.

For the information maximizing exploration approach, we assume that before having any information of a reward sequence, the DM perceives it has no value. Then, each reward in the sequence $s_{0\to T}$ is processed as an individual information source. The DM engages her attention to actively sample signals at each information source and update her belief about

[2] The rational inattention theory assumes the DM learns information about different options in order to find the best option. For details, see Sims (2003), Matějka and McKay (2015), and Maćkowiak et al. (2023).

the sequence value accordingly. The signals are nosiy. For any $t \in \{0, 1, \ldots, T\}$, the signal sampled at information source s_t could be represented by $x_t = u(s_t) + \epsilon_t$, where each ϵ_t is i.i.d. and $\epsilon_t \sim N(0, \sigma_\epsilon^2)$. The sampling weight for information source s_t is denoted by w_t.

The DM's belief about the sequence value $U(s_{0\to T})$ is updated as follows. At first, she holds a prior U_0, and given she perceives no value from the reward sequence, the prior could be represented by $U_0 \sim N(0, \sigma^2)$. Second, she draws a series of signals at each information source s_t. Note we define $U(s_{0\to T})$ as a weighted mean of instantaneous utilities, i.e. $U(s_{0\to T}) = \sum_{t=0}^{T} w_t u(s_t)$. Let $\bar{x}$ denote the mean sample signal and U denote a realization of $U(s_{0\to T})$. If there are k signals being sampled in total, we should have $\bar{x}|U, \sigma_\epsilon \sim N(U, \frac{\sigma_\epsilon^2}{k})$. Third, she uses the sampled signals to infer $U(s_{0\to T})$ in a Bayesian fashion. Let U_k denote the valuer's posterior about the sequence value after receiving k signals. According to the Bayes' rule, we have $U_k \sim N(\mu_k, \sigma_k^2)$ and

$$\mu_k = \frac{k^2\sigma_\epsilon^{-2}}{\sigma^{-2} + k^2\sigma_\epsilon^{-2}}\bar{x} \qquad , \qquad \sigma_k^2 = \frac{1}{\sigma^{-2} + k^2\sigma_\epsilon^{-2}}$$

We assume the DM takes μ_k as the valuation of reward sequence. It is clear that as $k \to \infty$, the sequence value will converge to the mean sample signal, i.e. $\mu_k \to \bar{x}$.

The DM's goal of sampling signals is to maximize her information gain. The information gain is defined as the KL divergence from the prior U_0 to the posterior U_k. In intuition, the KL divergence provides a measure for distance between distributions. As the DM acquires more information about $s_{0\to T}$, her posterior belief should move farther away from the prior. We let $p_0(U)$ and $p_k(U)$ denote the probability density functions of U_0 and U_k. Then, the information gain is

$$\begin{aligned} D_{KL}(U_k||U_0) &= \int_{-\infty}^{\infty} p_k(U) \log\left(p_k(U)/p_0(U)\right) dU \\ &= \frac{\sigma_k^2 + \mu_k^2}{2\sigma^2} - \log\left(\frac{\sigma_k}{\sigma}\right) - \frac{1}{2} \end{aligned} \tag{2}$$

Notably, in Equation (2), σ_k depends only on sample size k and μ_k is proportional to $\bar{x}$. Therefore, the problem of maximizing $D_{KL}(U_k||U_0)$ could be reduced to maximizing $\bar{x}$ (as each $u(s_t)$ is non-negative). The reason is that, drawing more samples can always increase the precision of the DM's estimate about $U(s_{0\to T})$, and a larger $\bar{x}$ implies more "surprises" in comparison to the DM's initial perception that $s_{0\to T}$ contains no value.

Maximizing the mean sample signal $\bar{x}$ under a limited sample size k is actually a multi-armed bandit problem (Sutton and Barto, 2018, Ch.2). On the one hand, the DM wants to draw more samples at information sources that are known to produce greater value signals (exploit). On the other hand, she wants to learn some value information from other information sources (explore). We assume the DM would take a softmax exploration strategy to solve this problem. That is,

$$w_t \propto d_t e^{\bar{x}_t/\lambda}$$

where $\bar{x}_t$ is the mean sample signal generated by information source s_t so far, $1/\lambda$ is the learning rate, and d_t is the initial sampling weight for s_t.[3] Note $\bar{x}_t$ cannot be calculated without doing simulations under a certain σ_ϵ. For researchers, modelling an intertemporal choice in this way requires conducting a series of simulations and then calibrating σ_ϵ for every choiceable option, which could be computationally expensive. Fortunately, according to the weak law of large numbers, as the sample size k gets larger, $\bar{x}_t$ is more likely to fall into a neighborhood of $u(s_t)$. Thus, the AAD which assumes $w_t \propto d_t e^{u(s_t)/\lambda}$ could be viewed as a proper approximation to the softmax exploration strategy.

Those who familiar with reinforcement learning algorithms may notice that here $u(s_t)$ is a special case of action-value function (assuming that the learner only cares about the value of current reward in each draw of sample). The AAD thus can be viewed as a specific version of the soft Q-learning or policy gradient method for solving the given multi-armed bandit problem (Haarnoja et al., 2017; Schulman et al., 2017). Such methods are widely used (and sample-efficient) in reinforcement learning. Moreover, one may argue that the applicability of softmax exploration strategy is subject to our model assumptions. Under alternative assumptions, the strategy may not be ideal. We acknowledge this limitation and suggest that researchers interested in modifying our model consider different objective functions or different families of noises. For example, if the DM aims to minimize the regret rather than

[3] The softmax strategy is popular in reinforcement learning. Classic softmax strategy assumes the initial probability of taking an action follows an uniform distribution. We relax this assumption by importing d_t, so that the DM can hold an (initial) preference over the dated rewards.

maximizing $\bar{x}$, the softmax exploration strategy can produce suboptimal actions and one remedy is to use the Gumbel–softmax strategy (Cesa-Bianchi et al., 2017). If noises $\epsilon_0, ..., \epsilon_T$ do not follow an i.i.d normal distribution, the information gain $D_{KL}(U_k||U_0)$ may be complex to compute, thus one can use its variational bound as the objective (Houthooft et al., 2016). Compared to these complex settings, the model setting in this subsection aims to provides a simple benchmark for understanding the role of attention in mental valuation of a reward sequence.

Two strands of literature can justify the information maximizing approach to characterizing AAD. First,

Second, the softmax exploration strategy is widely used by neuroscientists in fitting human actions in reinforcement learning tasks (Daw et al., 2006; Niv et al., 2012; FitzGerald et al., 2012; Niv et al., 2015; Leong et al., 2017). For instance, Daw et al. (2006) find the softmax strategy can characterize humans' exploration behavior better than other classic strategies (e.g. ϵ-greedy). Besides, Collins and Frank (2014) show that models based on this strategy exhibit a good performance in explaining behaviors of the striatal dopaminergic system (which is central in brain's sensation of pleasure and learning of rewarding actions) in reinforcement learning.

3.2 Optimal Discounting

The second approach to characterize AAD is based on the optimal discounting model (Noor and Takeoka, 2022, 2024). In one version of this model, the authors assume that the DM has a limited capacity of attention (or in their term, "empathy"), and before evaluating a reward sequence $s_{0\rightarrow T}$, she naturally focuses on the current period. The instantaneous utility $u(s_t)$ represents the well-being that the DM's self of period t can obtain from the reward sequence. For valuating $s_{0\rightarrow T}$, the DM needs to split attention over T time periods to consider the feeling of each self. This re-allocation of attention is cognitive costly. The DM seeks to find a balance between improving the overall well-being of multiple selves and reducing the incurred cognitive cost. Noor and Takeoka (2022, 2024) specify an optimization problem to capture this decision. In this paper, we adopt a variant of their original model.

The formal definition of the optimal discounting problem is given by Definition 2. [4]

Definition 2: *Given reward sequence $s_{0\to T} = [s_0, ..., s_T]$, the following optimization problem is called an optimal discounting problem for $s_{0\to T}$:*

$$\begin{aligned} \max_{\mathcal{W}} \quad & \sum_{t=0}^{T} w_t u(s_t) - C(\mathcal{W}) \\ s.t. \quad & \sum_{t=0}^{T} w_t \leq M \\ & w_t \geq 0 \text{ for all } t \in \{0, 1, ..., T\} \end{aligned}$$

where $M > 0$, $C(\mathcal{W}) \geq 0$. For any $t \in \{0, 1, \ldots, T\}$, $u(s_t) \in \mathbb{R}$. *$C(\mathcal{W})$ is the cognitive cost function and is constituted by time-separable costs, i.e. $C(\mathcal{W}) = \sum_{t=0}^{T} f_t(w_t)$, where for all $w_t \in (0, 1)$, $f_t(w_t)$ is differentiable and $f_t'(w_t)$ is continuous and* stric*tly increasing.*

Here w_t reflects the attention paid to consider the feeling of t-period self. The DM's objective function is the attention-weighted sum of utilities obtained by the multiple selves minus the cognitive cost of attention re-allocation. As is illustrated by Noor and Takeoka (2022, 2024), a key feature of this model is that decision weight w_t is increasing with s_t, indicating the DM tends to pay more attention to larger rewards. Moreover, it is easy to validate that if the following three conditions are satisfied, the solution to the optimal discounting problem will take an AAD form:

(i) The constraint on sum of decision weights is always tight. That is, $\sum_{t=0}^{T} w_t = M$. Without loss of generality, we can set $M = 1$.

(ii) There exists a realization of decision weights $\mathcal{D} = [d_0, ..., d_T]$ such that $d_t > 0$ for all $t \in \{0, \ldots, T\}$ and the cognitive cost is proportional to the KL divergence from $\mathcal{D}$ to the DM's strategy $\mathcal{W}$ where applicable. That is, $C(\mathcal{W}) = \lambda \cdot D_{KL}(\mathcal{W}||\mathcal{D})$, where $\lambda > 0$.

[4] There are three differences between our Definition 2 and the original optimal discounting model (Noor and Takeoka, 2022, 2024). First, in our setting, shifting attention to future rewards may reduce the attention to the current reward, while this would never happen in Noor and Takeoka (2022, 2024). Second, the original model assumes $f'(w_t)$ must be continuous at 0 and w_t must be no larger than 1. We relax these assumptions since $w_t = 0$ or $w_t \geq 1$ is not in our solutions. Third, the original model assumes that $f'(w_t)$\$ is left-continuous and $f'(w_t) = 0$ when w_t is under a lower bound $\underline{w}$, $f'(w_t) = \infty$ when w_t is above a upper bound $\bar{w}$, and $f'(w_t)$ is strictly increasing when $w_t \in [\underline{w}, \bar{w}]$. To keep simplicity, this paper restricts $f'(w_t)$ to be continuous and strictly increasing. Our assumption can apply to many commonly used cost functions, including the power cost function discussed in Noor and Takeoka (2022, 2024).

Here d_t sets a reference for determining the decision weight w_t, the parameter λ indicates how costly the attention re-allocation process is, and $D_{KL}(\mathcal{W}||\mathcal{D}) = \sum_{t=0}^{T} w_t \log(\frac{w_t}{d_t})$. The solution to the optimal discounting problem under condition (i)-(ii) can be derived in the same way as Theorem 1 in Matějka and McKay (2015). Note this solution is equivalent to that of a bounded rationality model: assuming the DM wants to find a $\mathcal{W}$ that maximizes $\sum_{t=0}^{T} w_t u(s_t)$ but can only search for solutions within a KL neighborhood of $\mathcal{D}$. Related models can be found in Todorov (2009).

We interpret the implications of condition (i)-(ii) with behavioral axioms. Note if each s_t is an independent option and $\mathcal{W}$ simply represents the DM's choice strategy across options, then these condition can be characterized by rational inattention theories, e.g. Caplin et al. (2022). However, here $\mathcal{W}$ is a component of sequence value $U(s_{0\to T})$, and the DM is assumed to choose the option with highest sequence value. Thus, the behavioral implications of condition (i)-(ii) should be derived in different ways. To illustrate, let $\succsim$ denote the preference relation between two reward sequences.[5] For any reward sequence $s_{0\to T} = [s_0, ..., s_T]$, we define $s_{0\to t} = [s_0, ..., s_t]$ as a sub-sequence of it, where $1 \leq t \leq T$.[6] We first introduce two axioms for $\succsim$:

Axiom 1: $\succsim$ *has the following properties:*

(a) *(complete order)* $\succsim$ *is complete and transitive.*

(b) *(continuity) For any reward sequences* s, s' *and reward* $c \in \mathbb{R}_{\geq 0}$*, the sets* $\{\alpha \in (0,1) | \alpha \cdot s + (1-\alpha) \cdot c \succsim s'\}$ *and* $\{\alpha \in (0,1) | s' \succsim \alpha \cdot s + (1-\alpha) \cdot c\}$ *are closed.*

(c) *(state-independent) For any reward sequences* s, s' *and reward* $c \in \mathbb{R}_{\geq 0}$*,* $s \succsim s'$ *implies for any* $\alpha \in (0,1)$*,* $\alpha \cdot s + (1-\alpha) \cdot c \sim \alpha \cdot s' + (1-\alpha) \cdot c$*.*

(d) *(reduction of compound alternatives) For any reward sequences* s, s', y *and rewards* $c_1, c_2 \in \mathbb{R}_{\geq 0}$*, if there exist* $\alpha, \beta \in (0,1)$ *such that* $s \sim \alpha \cdot y + (1-\alpha) \cdot c_1$*, then* $s' \sim \beta \cdot s + (1-\beta) \cdot c_2$ *implies* $s' \sim \beta\alpha \cdot y + \beta(1-\alpha) \cdot c_1 + (1-\beta) \cdot c_2$*.*

[5] If $a \succsim b$ and $b \succsim a$, we say $a \sim b$ ("a is the same good as b"). If $a \succsim b$ does not hold, we say $b \succ a$ ("b is better than a"). $\succsim$ can also characterize the preference relation between single rewards as the single rewards can be viewed as one-period sequences.

[6] Notably, every sub-sequence starts with period 0.

Axiom 2: *For any $s_{0\to T}$ and any $\alpha_1, \alpha_2 \in (0,1)$, there exists $c \in \mathbb{R}_{\geq 0}$ such that $\alpha_1 \cdot s_{0\to T-1} + \alpha_2 \cdot s_T \sim c$.*

The two axioms are almost standard in decision theories. The assumption of complete order implies preferences between reward sequences can be characterized by an utility function. Continuity and state-independence ensure that in a stochastic setting where the DM can receive one reward sequence under some states and receive a single reward under other states, her preference can be characterized by expected utility (Herstein and Milnor, 1953). Reduction of compound alternatives ensures that the DM's valuation on a reward sequence is constant across states. Axiom 2 is an extension of the Constant-Equivalence assumption in Bleichrodt et al. (2008). It implies there always exists a constant that can represent the value of a linear combination of sub-sequence $s_{0\to T}$ and the end-period reward s_T, as long as the weights lie in $(0,1)$.

For a given $s_{0\to T}$, the optimal discounting model can generate a sequence of decision weights $[w_0, ..., w_T]$. Furthermore, the model assumes the DM's preference for $s_{0\to T}$ can be characterized by the preference for $w_0 \cdot s_0 + w_1 \cdot s_1 + ... + w_T \cdot s_T$. We use Definition 3 to capture this assumption.[7]

Definition 3: *Given reward sequence $s_{0\to T} = [s_0, ..., s_T]$ and $s'_{0\to T'} = [s'_0, ..., s'_{T'}]$, the preference relation $\succsim$ has an optimal discounting representation if*

$$s_{0\to T} \succsim s'_{0\to T'} \quad \Longleftrightarrow \quad \sum_{t=0}^{T} w_t \cdot s_t \succsim \sum_{t=0}^{T'} w'_t \cdot s'_t$$

where $\{w_t\}_{t=0}^{T}$ and $\{w'_t\}_{t=0}^{T'}$ are solutions to the optimal discounting problems for $s_{0\to T}$ and $s'_{0\to T'}$ respectively.

Furthermore, if Definition 3 is satisfied and $\{w_t\}_{t=0}^{T}$ as well as $\{w'_t\}_{t=0}^{T'}$ takes the AAD form, we say $\succsim$ has an *AAD representation.* Now we specify two behavioral axioms that are key to characterize the AAD functions.

Axiom 3 (sequential outcome-betweenness): *For any $s_{0\to T}$, there exists $\alpha \in (0,1)$ such that*

[7] In (Noor and Takeoka, 2022), it is claimed that $\succsim$ satisfying Definition 3 has a Costly Empathy representation.

$s_{0\to T} \sim \alpha \cdot s_{0\to T-1} + (1-\alpha) \cdot s_T$.

Axiom 4 (sequential bracket-independence): *Suppose $T \geq 2$. For any $s_{0\to T}$, if there exist $\alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2 \in (0,1)$ such that $s_{0\to T} \sim \alpha_1 \cdot s_{0\to T-1} + \alpha_2 \cdot s_T$ and $s_{0\to T} \sim \beta_0 \cdot s_{0\to T-2} + \beta_1 \cdot s_{T-1} + \beta_2 \cdot s_T$, then we must have $\alpha_2 = \beta_2$.*

Axiom 3 implies that for a reward sequence $s_{0\to T-1}$, if we add a new reward s_T at the end of the sequence, then the value of the new sequence should lie between the original sequence $s_{0\to T-1}$ and the newly added reward s_T. This characterizes condition (i), i.e. the sum of decision weights is bounded tightly at 1. Notably, Axiom 3 is consistent with the empirical evidence about *violation of dominance* (Scholten and Read, 2014; Jiang et al., 2017) in intertemporal choice. Suppose the DM is indifferent between a small-sooner reward (SS) "receive \$75 today" and a large-later reward (LL) "receive \$100 in 52 weeks". Scholten and Read (2014) find when we add a tiny reward after the payment in SS, e.g. changing SS to "receive \$75 today and \$5 in 52 weeks", the DM would be more likely to prefer LL over SS. Jiang et al. (2017) find the same effect can apply to LL. That is, if we add a tiny reward after the payment in LL, e.g. changing LL to "receive \$100 in 52 weeks and \$5 in 53 weeks", the DM may be more likely to prefer SS over LL.

Axiom 4 implies that no matter how the DM brackets the rewards into sub-sequences (or how the sub-sequences get further decomposed), the decision weights for rewards outside them should not be affected. Specifically, suppose we decompose reward sequence $s_{0\to T}$ and find its value is equivalent to a linear combination of $s_{0\to T-1}$ and s_T. We also can further decompose $s_{0\to T-1}$ to a linear combination of $s_{0\to T-2}$ and s_{T-1}. But no matter how we operate, as long as the decomposition is carried out inside $s_{0\to T-1}$, the weight of s_T in the valuation of $s_{0\to T}$ will always remain the same. This axiom is an analog to independence of irrelevant alternatives in discrete choice problems (which is a key feature of softmax choice function). We show in Proposition 1 that the optimal discounting model plus Axiom 1-4 can exactly produce AAD.

Proposition 1: *Suppose $\succsim$ has an optimal discounting representation, then it has an AAD representation if and only if it satisfies Axiom 1-4.*

The proof of Proposition 1 is in Appendix A.

4 Implications for Decision Making

hidden-zero effect

common-difference effect

concavity of discount function

S-shaped value function

Intertemporal correlation aversion

Learning and inconsistent planning

To illustrate how ADU with Shannon cost function can account for a broad set of anomalies about time preferences, imagine that a DM receives a positive detereminstic reward in period j (and no reward in other periods). That is, she receives a sequence of rewards $X_T = [x_0, x_1, \ldots, x_T]$, where $x_j > 0$ and is certain, and $x_t = 0$ for all $t \neq j$ (both j and t are in $\{0, 1, ..., T\}$).

For the convenience of illustration, I assume the DM holds stationary time preferences before acquiring any information, that is, $d_t = \delta^t$. Meanwhile, $\delta \in (0, 1]$, where $\delta = 1$ implies the initial attention is uniformly distributed across periods. For simplicity, I define $v(x_t) = u(x_t)/\lambda$, and set $v(0) = 0$. Let $w_t(X_T)$ denote the discounting factor for period t. From the formula of ADUS we can infer that

$$w_j(X_T) = \begin{cases} \delta^j \cdot \dfrac{1}{1 + \frac{\delta}{1-\delta}(1 - \delta^T)e^{-v(x_j)}} \ , & 0 < \delta < 1 \\ \dfrac{1}{1 + T \cdot e^{-v(x_j)}} \ , & \delta = 1 \end{cases}$$

Clearly, w_j is decreasing in T. This offers an account for a phenomenon called *hidden zero effect*.

4.1 Hidden Zero Effect

The most direct evidence that could support the ADUS model is likely the hidden zero effect (**?**). The hidden zero effect means, supposing people face a small sooner reward (SS) and a large later reward (LL), they tend to exhibit more patience when SS and LL are framed as sequences rather than being framed as single-period rewards. For instance, suppose SS is "receive £100 today" and LL is "receive £120 in 6 months", and we have

SS_0: "receive £100 today and £0 in 6 months"

LL_0: "receive £0 today and £120 in 6 months"

people will be more likely to prefer LL_0 over SS_0 than preferring LL over SS. Subsequent research (e.g. **?**) suggests that the hidden zero effect is asymmetric. That is, shifting SS to SS_0 and keeping LL unchanged leads to an increase in patience, whereas shifting LL to LL_0 and keeping SS unchanged cannot increase patience. ADUS assumes that, within a sequence, attention is limited and the weight assigned to each period is anchored in an initial positive weight. These properties naturally explain the hidden zero effect. To illustrate, in SS, the DM perceives the length of sequence as "today" and allocate no attention to future. Whereas, in SS_0, she perceives the length as "6 months". This makes some attention be paid to future periods with no reward, and decreases the attention paid to the only period with positive reward (given attention is limited); thus, the overall utility of sequence decreases. By contrast, shifting from LL to LL_0 does not change the length of sequence, thus does not change overall utility.

The existence of hidden zero effect also provides a hint in selection of time length T. When evaluating a reward delivered in period j, the range of T is $[j, +\infty)$. Any increase in T will reduce the overall utility. Thus, when comparing SS and LL, the DM may tend to set $T = j$ (the minimum length she can set), in order to maximize the overall utility. Any period out of this length can be perceived as irrelevant to the decision; so, she does not need to sample from the periods after j, when evaluating the given reward. Though, explicitly mentioning the periods after j will direct her attention to those periods, and lead to the hidden zero effect. By setting $T = j$, we have

$$w_T(x_T) = \frac{1}{1 + G(T)e^{-v(x_T)}}$$

where

$$G(T) = \begin{cases} \dfrac{1}{1-\delta}(\delta^{-T} - 1)\ , & 0 < \delta < 1 \\ T\ , & \delta = 1 \end{cases}$$

Given period T is now the only period with a non-zero reward within the sequence, I use x_T to directly represent the whole sequence, and let $w_T(x_T)$ denote the discounting factor for period T. Interestingly, when $\delta = 1$, $w_T(x_T)$ takes a form similar with hyperbolic discounting.

4.2 Common Difference Effect

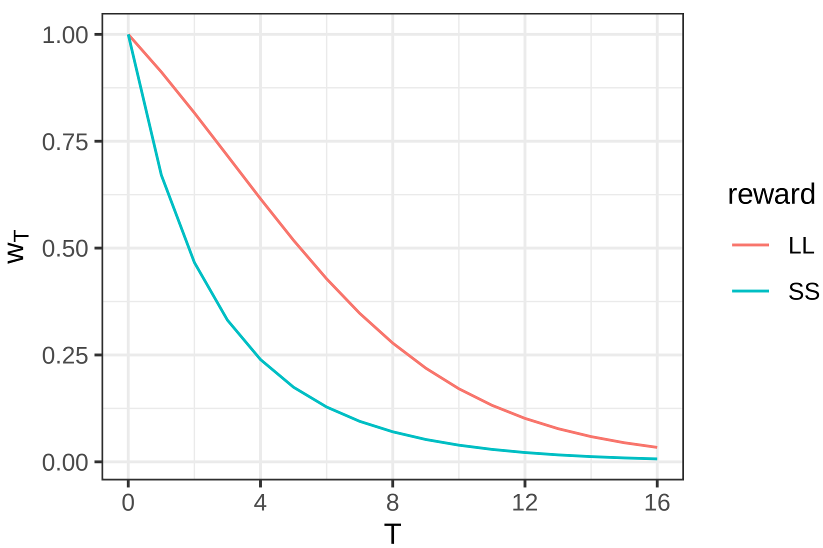
A well-known anomaly about time preferences is *common difference effect*, firstly defined by **?**. Suppose there are a large later reward x_l arriving at period t_l (denoted by LL) and a small sooner reward x_s arriving at period t_s (denoted by SS), where $x_l > x_s > 0$, $t_l > t_s > 0$. Define $V(x,t) = w_t(x_t)v(x_t)$. The common difference effect means, supposing$V(x_l, t_l) = V(x_s, t_l)$, we must have $V(x_l, t_l + \Delta t) > V(x_s, t_s + \Delta t)$ for any positive integer Δt.

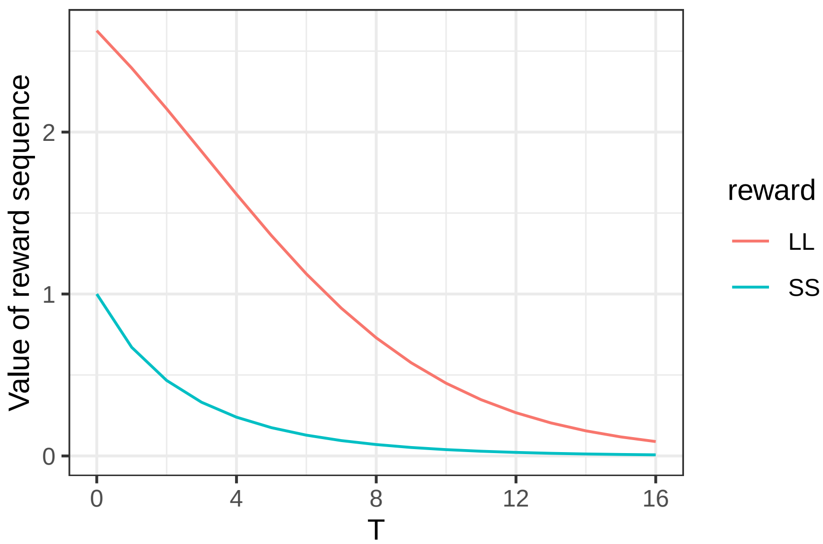
ADUS predicts that, if people are impatient, to observe the common difference effect, the difference between SS and LL in reward level must be set significantly larger than the difference in time delay. This is shown in Proposition 2.

Proposition 2: *In ADUS, if the initial weights are uniformly distributed, then the common difference effect always holds; if the initial weights exponentially declines over time, the common difference effect holds when* $v(x_l) - v(x_s) + \ln \frac{v(x_l)}{v(x_s)} > -(t_l - t_s) \ln \delta$.

Proposition 2 is interpreted as follows. When $\delta = 1$, ADUS predicts the DM always performs the common difference effect. This is obvious because discounting factor $w_T(x_T)$ takes a hyperbolic-like form. When $\delta < 1$, there are four factors jointly deciding whether we could observe the common difference effect or not. First, without considering attentional mechanism, when we extend time delay, each of $w_{t_l}(x_l)$ and $w_{t_s}(x_s)$, i.e. the discounting factor for (and attention paid to) the only period with positive reward, declines in an exponential fashion. Second, without considering newly added time interval, due to the decline of $w_{t_l}(x_l)$ and $w_{t_s}(x_s)$, the DM frees up some attention and can reallocate it across periods. Given that in LL, the DM has to wait longer for reward, the periods where she wait can grab more attention from the released capacity of attention, compared with those in SS. In other words, an extension of delay makes she focus more on the waiting time in LL than in SS, which decreases the preference for LL. Third, the newly added time interval also grabs some attention from other periods. Note the time delay is extended by $[t_l, t_l + \Delta t]$ in LL and by $[t_s, t_s + \Delta t]$ in SS; given $t_l > t_s$, if people are impatient, the newly added time interval will receive less attention in LL than in SS, without considering other factors. This increases the preference for LL. Fourth, ADU generally assumes that the DM tends to pay more attention to periods with larger rewards. Given $x_l > x_s$, the newly added interval grabs less attention from the period where x_l is positioned (in LL) than from the period where x_s is positioned (in SS). That is, the DM focuses comparatively more on reward level in LL than in SS, which mitigates the impact of discounting factor declining. This also increases the preference for LL. When the impact of the later two factors succeeds that of the second factor, the DM will perform the common difference effect.

Notably, if we explicit mention the zeros in LL and SS, extending time delay always lead to the common difference effect.





4.3 Magnitude Effect

The *magnitude effect* is another well-known anormaly about time preferences. Assuming we have t_l, t_s, x_s fixed, and want to find a x_l such that $V(x_l, t_l) \equiv V(x_s, t_s)$, the magnitude effect implies that, if we increase x_s, then the x_l/x_s that makes the equality valid will decrease.

In standard discounted utility model, the magnitude effect requires the elasticity of utility function to increase with the reward level (**?**). This requirement might be too restrictive, so that many commonly used utility functions (such as power or CARA utility function) does not satisfy it. By contrast, in ADU model, DM is generally assumed to attend more to periods with larger rewards. This implies that when comparing SS and LL, she exhibits more patience towards larger reward level, which is naturally compatible with the magnitude effect (**??**). By Proposition 3, I focus on ADU with Shannon cost function, and show how this requirement for curvature of utility function can be relaxed in this setting.

Proposition 3: *Define $v(x) \equiv u(x)/\lambda$ as the utility function. In ADUS, the magnitude effect always holds true when function $v(x)$ satisfies*

$$RRA_v(x) \leq 1 - \frac{e_v(x)}{v(x)+1}$$

where $RRA_v(x)$ is the relative risk aversion coefficient of $v(x)$, $e_v(x)$ is the elasticity of $v(x)$

to x.

Note that Proposition 3 is a very broad condition. In Corollary 1 and Corollary 2, I show that power utility function and CARA utility function both satisfy this condition in most cases.

Corollary 1: Suppose $v(x) = x^\gamma/\lambda$, where $0 < \gamma < 1$ and $\lambda > 0$. Then magnitude effect holds true for any $x \in \mathbb{R}_{>0}$.

Corollary 2: Suppose $v(x) = (1 - e^{-\gamma x})/\lambda$, where $\gamma > 0$ and $\lambda > 0$. The magnitude effect holds true for any $x \geq \frac{1+\eta}{\gamma}$, where $\eta > 0$ and $\eta e^{1+\eta} - \eta = 1$ (it can be calculated that $\eta \approx 0.35$).

4.4 Concavity of Time Discounting

Many time discounting models assumes discount function is convex in time delay, e.g. exponential and hyperbolic discounting. This style of discount function predicts DM is *risk seeking over time lotteries*. That is, suppose a deterministic reward of level x is delivered in period t_l with probability π and delivered in period t_s with probability $1 - \pi$ ($0 < \pi < 1$, $c > 0$); while another deterministic reward, of the same level, is delivered in a certain period t_m, where $t_m = \pi t_l + (1-\pi)t_s$. The DM should prefer the former reward to the latter reward. However, some experimental studies, such as **?** and **?**, suggest that people are often *risk averse over time lotteries*, i.e. preferring the reward delivered in a certain period.

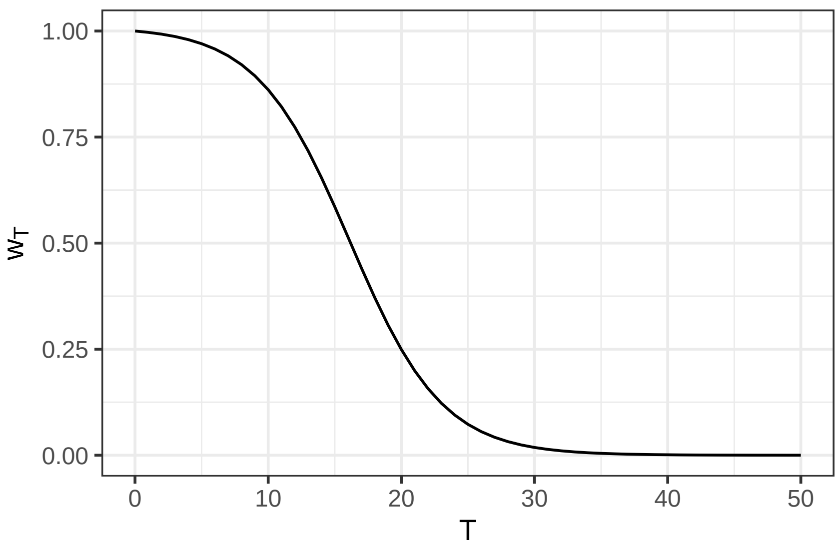
One way to accommodate the evidence about risk aversion over time lotteries, as is suggested by **?**, is to modify the convexity (concavity) of discount function. Under a general EDU framework, DM is risk averse over time lotteries when $\pi w_{t_l}(x) + (1-\pi) w_{t_s}(x) < w_{t_m}(x)$. Fixing t_s and t_l, the inequality suggests $w_{t_m}(c)$ is concave in t_m. In reverse, being risk seeking over time lotteries suggests $w_{t_m}(x)$ is convex in t_m. Notably, **?** find that people are more likely to be risk averse over time lotteries when π is small, and to be risk seeking over time lotteries when π is large. Given that when π gets larger, t_m is also larger, we can conclude that the discount function may be concave in delay for the near future but convex for the far future. Moreover, **?** also find evidence that support this shape of discount function.

In Proposition 4, I show that ADUS can produce such a shape of discount function as long as the reward level x is large enough.

Proposition 4: In ADUS, *if* $\delta = 1$, *then the discount function is convex in* t. *If* $0 < \delta < 1$, *then there are a reward threshold* $\underline{x}$ *and a time threshold* $\underline{t}$ *such that*

1) *when* $x \leq \underline{x}$, *the discount function is convex in* t*;*
2) *when* $x > \underline{x}$, *the discount function is convex in* t *given* $t \geq \underline{t}$, *and it is concave in* t *given* $t < \underline{t}$.

It can be derived that $v(\underline{x}) = \ln(\frac{2}{1-\delta})$, *and* $\underline{t} = \frac{\ln[(1-\delta)e^{v(x)}-1]}{-\ln\delta}$.



4.5 S-Shaped Value Function

In prospect theory, **?** propose an S-shaped value function that is convex for losses and concave for gains. Since that, S-shaped value functions have been widely embraced by behavioral economists. More recent theories have provided further justifications for it, including reference-dependent utility in a broad sense (**?**), and efficient coding of values (**?**). Here, I provide an account based on selective attention to time periods.

Suppose a DM is faced with a choice between a risky lottery and a fixed amount of money. When making this choice, she does not obtain any money from either option. Thus, she perceives the outcome of each option as something that will happen in the future. She allocate her attention between the present period and the period when she may receive the money. Assume that she perceives the outcome will be realized in period t, and in a certain state, the option she chooses yields reward x, then we can use the attentional discounted utility $V(x,t)$ to represent the value function. I derive the conditions in which ADUS can produce a S-shaped value function in Proposition 5.

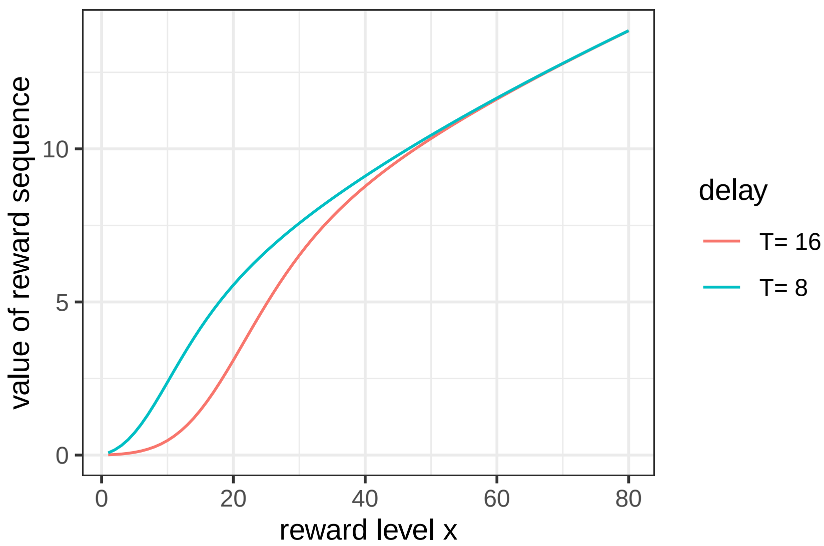
Proposition 5: *Suppose* $t \geq 1$, $\frac{d}{dx}\left(\frac{1}{v'(x)}\right)$ *is continuous in* $(0,+\infty)$, *in ADUS,*

1) *there exists a threshold* $\bar{x}$ *in* $(0,+\infty)$ *such that* $V(x,t)$ *is strictly concave in* x *when* $x \in [\bar{x},+\infty)$;

2) *if* $\frac{d}{dx}\left(\frac{1}{v'(x)}\right)$ *is right-continuous at* $x=0$, *and* $\frac{d}{dx}\left(\frac{1}{v'(0)}\right) < 1$, *then there exists a threshold* x^* *in* $(0,\bar{x})$ *such that, for any* $x \in (0,x^*)$, $V(x,t)$ *is strictly convex in* x;

3) *there exist a hyper-parameter* λ^* *and an interval* (x_1,x_2) *such that, if* $\lambda < \lambda^*$, *for any* $x \in (x_1,x_2)$, $V(x,t)$ *is strictly convex in* x, *where* $\lambda^* > 0$ *and* $(x_1,x_2) \subset (0,\bar{x})$.

Proposition 5 implies, if the derivative of $\frac{1}{v'(x)}$ converges to a small number when $x \to 0^+$, or the unit cost of information λ is small enough, value function $V(x,t)$ will perform an S shape in some interval of x. At the intuition level, note that $V(x,t) = w_t(x)v(x)$. When the level of reward x grows, both the instantaneous utility of it, i.e. $v(x)$, and the discounting factor assigned to it, i.e. $w_t(x)$, can increase. These functions are both concave in x: when the level of reward is small, they both grow fast. So, it is possible that their product is convex in this case. By contrast, when the level of reward is large, they grow slowly, so their product keeps concave.



4.6 Inseparability of Sequences

Let x and y denote two 2-period risky reward sequences. For x, the realized sequence is [£100,£100] with probability 1/2, and is [£3,£3] with probability 1/2. For y, the realized sequence is [£3,£100] with probability 1/2, and is [£100,£3] with probability 1/2. Classical models of intertemporal choice typically assume the separability of potentially realized sequences. This implies that the DM is indifferent between x and y. However, **?** find evidence of *intertemporal correlation aversion*, that is, people often prefer y to x.

ADU can naturally yield intertemporal correlation aversion. For simplicity, suppose the initial attention is uniformly distributed across the two periods. For x, under each potentially realized sequence, the DM equally weights each period. For y, DM tends to assign more weight to the period with a reward of £100 (suppose that weight is w). Then the value of x is $\frac{1}{2}u(100) + \frac{1}{2}u(3)$ and the value of y is $w \cdot u(100) + (1 - w) \cdot u(3)$. Given that $x > \frac{1}{2}$, the DMs should strictly prefer y to x.

- Other evidence related to inseparability: common sequence effect, (reverse) mere token effect, magnitude-increasing temporal sensitivity

5 The Role of Attention in Inconsistent Planning

5.1 Attention Grabbing and Updating

Suppose a DM has budget m ($m > 0$) and is considering how to spend it over different time periods. We can use a reward sequence x to represent this decision problem, where the DM's spending in period t is x_t. In period 0, she wants to find a x such that

$$\max_{x} \sum_{t=0}^{T} w_t u(x_t) \quad s.t. \sum_{t=0}^{T} x_t = m \tag{3}$$

where w_t is the attention-adjusted discounting factor in period t. I assume $w_t = \delta^t e^{u(x_t)/\lambda} / \sum_{t=\tau}^{T} \delta^{\tau} e^{u(x_{\tau})/\lambda}$ and there is no risk under this setting.

In models like exponential and hyperbolic discounting, the discounting factor of a future period is consistently smaller than that of the current period. Thus, the DM should spend more at the present than in the future. By contrast, in ADU, when increasing the spending in a certain period, the discounting factor corresponding to that period should also increase. So it is possible that the DM spends more in the future and that a future period has a greater discounting factor than the current period. This is consistent with **?** that find people sometimes prefer improving sequences to declining sequences.

ADU suggests there are two mechanisms that can help explain why people may perform dynamically inconsistent behavior. The first is *attention-grabbing effect*, that is, keeping the others equal, when we increase x_t (which lead to an increase in w_t), the discounting factor in any other period should decrease due to limited attention. After omitting a previous period from the decision problem in Equation (3), the DM can assign more weights to remaining periods; thus, the attention-grabbing effect is enhanced. The increased attention-grabbing effect will offset some benefit of increasing spending toward a certain period. Therefore, when the DM prefers improving sequences, the attention-grabbing effect will make her perform a present bias-like behavior (always feeling that she should spend more at the present than the original plan); when the DM prefers declining sequences, this effect will maker her perform a future bias-like behavior (always feeling she should spend more in the future).

The second mechanism is *initial attention updating*. As is assumed above, in period 0, prior to evaluating each reward sequence, the DM's initial weight on period t is proportional to δ^t; after evaluation, the weight becomes being proportional to $\delta^t e^{u(x_t)/\lambda}$. In period 1, if she implements the evaluation based on the information attained in period 0, the initial weight should be updated to being proportional $\delta^t e^{u(x_t)/\lambda}$; thus, the weight after evaluation should become being proportional to $\delta e^{2u(x_t)/\lambda}$. As a result, the benefit of increasing spending toward a certain period gets strengthened. The updated initial attention can make those who prefer improving sequences perform present bias and those who prefer declining sequences perform future bias.

Both the attention-grabbing effect and initial attention updating are affected by the curvature of utility function. They jointly decide which behavior pattern that people should perform in dynamics.

Proposition 6 (*spread-consistency correlation*) Suppose $\succsim$ has a ADU representation and satisfies Axiom 2-4. If there exist b and S_T such that, for any b' and S'_T, $bS_T \succsim b'S'_T$, where $b + \sum_{t=0}^{T} s_t = b' + \sum_{t=0}^{T} s'_t$, then for any S'_T, we have

$$S_T \succsim S'_T \Longleftrightarrow b \sim S_T$$

where $\sum_{t=0}^{T} s_t = \sum_{t=0}^{T} s'_t$.

Proposition 6 implies that, when allocating a consumption budget across time periods, the DM keeps her choice dynamically consistent if and only if she performs a strong preference for spread. Given that people are typically assumed to be impatient (preferring a declining sequence), one intuitive interpretation of Lemma 2 is that the less impatient a DM is in the present, the less inclined she is to deviate from the original choice in the future.

6 Discussion

6.1 Relation to Other Models of Intertemporal Choices

The theory most similar to AAD is the salience theory (Bordalo et al., 2012, 2013, 2020).

rational inattention

focus-weighted utility

bayesian updating and discounting

optimal precision

Relation with money/delay trade-off

6.2 Other relevant phenomena

6.3 Limitation

attention biases learning: learning rate is high for attended reward

7 Conclusion

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8 Appendix

8.1 A. Proof of Proposition 1

The sufficiency is easy to validate. We present the proof of necessity here. That is, if $\succsim$ has an optimal discounting representation and satisfies Axiom 1-4, then it has an AAD representation.

Lemma 1: *If Axiom 1 and 3 hold, for any $s_{0\to T}$, there exist $w_0, w_1, \dots, w_T > 0$ such that $s_{0\to T} \sim w_0 \cdot s_0 + ... + w_T \cdot s_T$, where $\sum_{t=0}^{T} w_t = 1$.*

Proof: If $T = 1$, Lemma 1 is a direct application of Axiom 3. If $T \geq 2$, for any $2 \leq t \leq T$, there should exist $\alpha_t \in (0, 1)$ such that $s_{0\to t} \sim \alpha_t \cdot s_{0\to t-1} + (1-\alpha_t) \cdot s_t$. By state-independence and reduction of compound alternatives, we can recursively apply this equivalence relation as follows:

$$
\begin{aligned}
s_{0\to T} &\sim \alpha_{T-1} \cdot s_{0\to T-1} + (1 - \alpha_{T-1}) \cdot s_T \\
&\sim \alpha_{T-1}\alpha_{T-2} \cdot s_{0\to T-2} + \alpha_{T-1}(1 - \alpha_{T-2}) \cdot s_{T-1} + (1 - \alpha_{T-1}) \cdot s_T \\
&\sim ... \\
&\sim w_0 \cdot s_0 + w_1 \cdot s_1 + ... + w_T \cdot s_T
\end{aligned}
$$

where $w_0 = \prod_{t=0}^{T-1} \alpha_t$, $w_T = 1 - \alpha_{T-1}$, and for $0 < t < T$, $w_t = (1 - \alpha_{t-1}) \prod_{\tau=t}^{T-1} \alpha_\tau$. It is easy to show the sum of $w_0, \dots, w_T$ is equal to 1. *QED*.

Therefore, if Axiom 1 and 3 hold, for any reward sequence $s_{0\to T}$, we can always find a convex combination of all its elements, such that the DM is indifferent between the reward sequence and this convex combination. If $s_{0\to T}$ is a constant sequence, i.e. all its elements are constant, then we can directly assume $\mathcal{W}$ is AAD-style. Henceforth, we discuss whether AAD can also apply to non-constant sequences.

By Lemma 2, we show adding a new reward to the end of $s_{0\to T}$ has no impact on the relative decision weights of rewards in the original reward sequence.

Lemma 2: *For any $s_{0\to T+1}$, if $s_{0\to T} \sim \sum_{t=0}^{T} w_t \cdot s_t$ and $s_{0\to T+1} \sim \sum_{t=0}^{T+1} w'_t \cdot s_t$, where*

$w_t, w'_t > 0$ and $\sum_{t=0}^{T} w_t = 1$, $\sum_{t=0}^{T+1} w'_t = 1$, *then when Axiom 1-4 hold, we can obtain* $\frac{w'_0}{w_0} = \frac{w'_1}{w_1} = \ldots = \frac{w'_T}{w_T}$.

Proof: According to Axiom 3, for any $s_{0\to T+1}$, there exist $\alpha, \zeta \in (0,1)$ such that

$$\begin{aligned} s_{0\to T} &\sim \alpha \cdot s_{0\to T-1} + (1-\alpha)\cdot s_T \\ s_{0\to T+1} &\sim \zeta \cdot s_{0\to T} + (1-\zeta)\cdot s_{T+1} \end{aligned} \tag{A1}$$

On the other hand, we drawn on Lemma 1 and set

$$s_{0\to T+1} \sim \beta_0 \cdot s_{0\to T-1} + \beta_1 \cdot s_T + (1-\beta_0-\beta_1)\cdot s_{T+1} \tag{A2}$$

where $\beta_0, \beta_1 > 0$. According to Axiom 4, $1-\zeta = 1-\beta_0-\beta_1$. So, $\beta_1 = \zeta - \beta_0$. This also implies $\zeta > \beta_0$.

According to Axiom 2, we suppose there exists a reward sequence s such that $s \sim \frac{\beta_0}{\zeta} \cdot s_{0\to T-1} + (1-\frac{\beta_0}{\zeta})\cdot s_1$. By Equation (A2) and reduction of compound alternatives, we have $s_{0\to T+1} \sim \zeta\cdot s + (1-\zeta)\cdot s_{T+1}$. Combining Equation (A2) with the second line of Equation (A1) and applying transitivity and state-independence, we obtain $s_{0\to T} \sim \frac{\beta_0}{\zeta}\cdot s_{0\to T-1} + (1-\frac{\beta_0}{\zeta})\cdot s_1$.

We aim to prove that for any $s_{0\to T+1}$, we can obtain $\alpha = \frac{\beta_0}{\zeta}$. To do this, we first assume (without loss of generality) that $\alpha > \frac{\beta_0}{\zeta}$.

Consider the case that $s_{0\to T-1} \succ s_T$. By state-independence, for any $c \in \mathbb{R}_{\geq 0}$, we have $(\alpha - \frac{\beta_0}{\zeta})\cdot s_{0\to T-1} + (1-\alpha+\frac{\beta_0}{\zeta})\cdot c \succ (\alpha-\frac{\beta_0}{\zeta})\cdot s_T + (1-\alpha+\frac{\beta_0}{\zeta})\cdot c$. By Axiom 2, there exists $z \in \mathbb{R}_{\geq 0}$ such that $(1-\alpha)\cdot s_T + \frac{\beta_0}{\zeta}\cdot s_{0\to T-1} \sim z$. Given c is arbitrary, we set $(1-\alpha+\frac{\beta_0}{\zeta})\cdot c \sim z$. By reduction of compound alternatives, we can derive that

$$(\alpha - \frac{\beta_0}{\zeta})\cdot s_{0\to T-1} + (1-\alpha)\cdot s_T + \frac{\beta_0}{\zeta}\cdot s_{0\to T-1} \succ (\alpha - \frac{\beta_0}{\zeta})\cdot s_T + (1-\alpha)\cdot s_T + \frac{\beta_0}{\zeta}\cdot s_{0\to T-1}$$

where the LHS can be rearranged to $\alpha\cdot s_{0\to T-1} + (1-\alpha)\cdot s_T$, and the RHS can be rearranged to $\frac{\beta_0}{\zeta}\cdot s_{0\to T-1} + (1-\frac{\beta_0}{\zeta})\cdot s_1$. They both should be indifferent from $s_{0\to T}$. This results in a contradiction. Similarly, in the case that $s_T \succ s_{0\to T-1}$, we can also derive a contradiction. Meanwhile, when $s_{0\to T} \sim s_T$, α and $\frac{\beta_0}{\zeta}$ can be any number within $(0,1)$. So, we can directly set $\alpha = \frac{\beta_0}{\zeta}$.

Thus, we have $\alpha = \frac{\beta_0}{\zeta}$ for any $s_{0\to T+1}$, which indicates $\frac{\beta_0}{\alpha} = \frac{\beta_1}{1-\alpha} = \zeta$. We can recursively apply this equality to any sub-sequence $s_{0\to t}$ ($t \leq T$) of $s_{0\to T+1}$, so that the lemma will be proved. *QED.*

Now we move on to prove Proposition 1. The proof contains five steps.

First, we add the constraints $\sum_{t=0}^{T} w_t = 1$ and $w_t > 0$ to the optimal discounting problem for $s_{0\to T}$ so that the problem is compatible with Lemma 1. According to the FOC of its solution, for all $t = 0, 1, \ldots., T$, we have

$$f_t'(w_t) = u(s_t) + \theta \tag{A3}$$

where θ is the Lagrange multiplier. Given that $f_t'(w_t)$ is strictly increasing, w_t is increasing with $u(s_t) + \theta$. We define the solution as $w_t = \phi_t(u(s_t) + \theta)$.

Second, we add a new reward s_{T+1} to the end of $s_{0\to T}$ and apply Lemma 2 as a constraint to optimal discounting problem. Look at the optimal discounting problem for $s_{0\to T+1}$. For all $t \leq T$, the FOC of its solution will take the same form as Equation (A3). So, if importing s_{T+1} changes some w_t to w_t' ($w_t' \neq w_t$, where w_t is the solution to optimal discounting problem for $s_{0\to T}$), the only way is through changing the multiplier θ. Suppose importing s_{T+1} changes θ to $\theta - \Delta\theta$, we have $w_t' = \phi_t(u(s_t) + \theta - \Delta\theta)$.

By Lemma 2, we know $\frac{w_0}{w_0'} = \frac{w_1}{w_1'} = \ldots = \frac{w_T}{w_T'}$. In other words, for $t = 0, 1, \ldots, T$, we have $w_t \propto \phi_t(u(s_t) + \theta - \Delta\theta)$. We can rewrite w_t as

$$w_t = \frac{\phi_t(u(s_t) + \theta - \Delta\theta)}{\sum_{\tau=0}^{T} \phi_\tau(u(s_\tau) + \theta - \Delta\theta)} \tag{A4}$$

Third, we show that in $s_{0\to T}$, if we change each s_t to z_t such that $u(z_t) = u(s_t) + \Delta u$, the decision weights $w_0, \ldots, w_T$ will remain the same. Note $\sum_{t=0}^{T} \phi_t(u(s_t) + \theta) = 1$. It is clear that $\sum_{t=0}^{T} \phi_t(u(z_t) + \theta - \Delta u) = 1$. Suppose changing every s_t to z_t moves θ to θ' and $\theta' < \theta - \Delta u$. Then, we must have $\phi_t(u(z_t) + \theta') < \phi_t(u(z_t) + \theta - \Delta u)$ since $\phi_t(.)$ is strictly increasing. This results in $\sum_{t=0}^{T} \phi_t(u(z_t) + \theta') < 1$, which contradicts with the constraint that the sum of all decision weights is 1. The same contradiction can apply to the case that $\theta' > \theta - \Delta u$. Therefore, changing every s_t to z_t must move θ to $\theta - \Delta u$, and each w_t can only be moved to $\phi_t(u(z_t) + \theta - \Delta u)$, which is exactly the same as the original decision weight.

A natural corollary of this step is that, subtracting or adding a common number to all intantaneous utilities in a reward sequence has no effect on decision weights. What actually matters for determining the decision weights is the difference between instantaneous utilities. This indicates, for convenience, we can subtract or add an arbitrary number to the utility function.

In other words, for a given $s_{0\to T}$ and s_{T+1}, we can define a new utility function $v(.)$ such that $v(s_t) = u(s_t) + \theta - \Delta\theta$. So, Equation (A4) can be re-written as

$$w_t = \frac{\phi_t(v(s_t))}{\sum_{\tau=0}^{T} \phi_\tau(v(s_\tau))}$$

If w_t takes the AAD form under the utility function $v(.)$, i.e. $w_t \propto d_t e^{v(s_t)/\lambda}$, then it should also take the AAD form under the original utility function $u(.)$.

Fourth, we show $\Delta\theta$ has two properties: (i) $\Delta\theta$ is strictly increasing with $u(s_{T+1})$; (ii) suppose $\Delta\theta = \underline{\theta}$ when $u(s_{T+1}) = \underline{u}$ and $\Delta\theta = \bar{\theta}$ when $u(s_{T+1}) = \bar{u}$, where $\underline{u} < \bar{u}$, then for any $l \in (\underline{\theta}, \bar{\theta})$, there exists $u(s_{T+1}) \in (\underline{u}, \bar{u})$ such that $\Delta\theta = l$.

The property (i) can be shown by contradiction. Given $w_0, \ldots, w_{T+1}$ a sequence of decision weights for $s_{0\to T+1}$. Suppose $u(s_{T+1})$ is increased but $\Delta\theta$ is constant. In this case, each of $w'_0, \ldots, w'_T$ should also be constant. However, w'_{T+1} must increase as it is stricly increasing with $u(s_{T+1}) + \theta - \Delta\theta$ (θ is determined by the optimal discounting problem for $s_{0\to T}$; thus, any operations on s_{T+1} should have no effect on θ). This contradicts with the constraint that $\sum_{t=0}^{T+1} w'_t = 1$. The only way to avoid such contradictions is to set $\Delta\theta$ strictly increasing with s_{T+1}, so that $w'_0, \ldots, w'_T$ are decreasing with $u(s_{T+1})$.

For property (ii), note that for any reward sequence $s_{0\to T+1}$ and a given θ, $\Delta\theta$ is defined as the solution to $\sum_{t=0}^{T+1} \phi_t(u(s_t) + \theta - \Delta\theta) = 1$. Given an arbitrary number $l \in (\underline{\theta}, \bar{\theta})$, the proof of property (ii) consists of two stages. First, for $t = 0, 1, \ldots, T$, we need to show that $u(s_t) + \theta - l$ is still in the domain of $\phi_t(.)$. Second, for period $T + 1$, we need to show for any $\omega \in (0,1)$, there exists $u(s_{T+1}) \in \mathbb{R}$ such that $\phi_{T+1}(u(s_{T+1}) + \theta - l) = \omega$.

For the first stage, note $\phi_t(.)$ is the inverse function of $f_t'(.)$. Suppose when $\Delta\theta = \underline{\theta}$, we have $f_t'(w_t^a) = u(s_t) + \theta - \underline{\theta}$, and when $\Delta\theta = \bar{\theta}$, we have $f_t'(w_t^b) = u(s_t) + \theta - \bar{\theta}$. For any $l \in (\underline{\theta}, \bar{\theta})$, we have $u(s_t) + \theta - l \in (f_t'(w_t^b), f_t'(w_t^a))$. Given that $f_t'(.)$ is continuous and strictly increasing, there must be $w_t \in (w_t^b, w_t^a)$ such that $f_t'(w_t) = u(s_t) + \theta - l$. So, $u(s_t) + \theta - l$ is in the domain of $\theta_t(.)$. For the second stage, given an arbitrary $\omega \in (0,1)$, we can set $u(s_{T+1}) = f'(\omega) - \theta + l$, so that the desired condition is satisfied.

A corollary of this step is that we can manipulate $\Delta\theta$ in Equation (A4) at any level in $[\underline{\theta}, \bar{\theta}]$ by changing a hypothetical s_{T+1}.

Fifth, we show $\ln \phi_t(.)$ is linear under some conditions. To do this, let us add a hypothetical s_{T+1} to the end of s_T and let $w_t' = \phi_t(v(s_t))$ denote the decision weights for $t = 0, 1, \ldots, T+1$ in $s_{0\to T+1}$. We change the hypothetical s_{T+1} within the set $\{s_{T+1} | u(s_{T+1}) \in [\underline{u}, \bar{u}]\}$ and see what will happen to the decision weights from period 0 to period T. Suppose this changes each w_t' to $\phi_t(v(s_t) + \theta')$. Set $\theta' = a$ when $u(s_{T+1}) = \underline{u}$ and $\theta' = b$ when $u(s_{T+1}) = \bar{u}$. By the second step, we have

$$\frac{\phi_t(v(s_t))}{\sum_{\tau=0}^{T} \phi_\tau(v(s_\tau))} = \frac{\phi_t(v(s_t) + \theta')}{\sum_{\tau=0}^{T} \phi_\tau(v(s_\tau) + \theta')} \tag{A5}$$

For each $t = 0, 1, ..., T$, we can rewrite $\phi_t(v(s_t))$ as $e^{\ln \phi_t(v(s_t))}$. For the LHS of Equation (A5), multiplying both the numerator and the denominator by a same number will not affect the value. Therefore, Equation (A5) can be rewritten as

$$\frac{e^{\ln \phi_t(v(s_t)) + \kappa\theta'}}{\sum_{\tau=0}^{T} e^{\ln \phi_\tau(v(s_\tau)) + \kappa\theta'}} = \frac{e^{\ln \phi_t(v(s_t) + \theta')}}{\sum_{\tau=0}^{T} e^{\ln \phi_\tau(v(s_\tau) + \theta')}}$$

where κ can be any real-valued constant. By properly selecting κ, we can obtain $\ln \phi_t(v(s_t)) + \kappa\theta' = \ln \phi_t(v(s_t) + \theta')$ for all $t = 0, 1, ..., T$. And if $\theta' > 0$, we must have $\kappa > 0$.

This indicates a property of function $\ln \phi_t(.)$: for any $v \in \mathbb{R}$, we have $\ln \phi_t(v) = \ln \phi_t(v - \theta') + \kappa\theta'$ as long as $\theta' \in [a, b]$.

So, we can write $\ln \phi_t(u(s_t))$ as $\ln \phi_t(u(0)) + \lambda u(s_t)$