Proof

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January 05, 2024

Optimal discounting

$$\max_{\mathcal{W}} \quad \sum_{t=0}^{T} w_t u(s_t) - C(\mathcal{W})$$
$$s.t. \quad \sum_{t=0}^{T} w_t = 1$$
$$w_t > 0 \text{ for all } t \in \{0, 1, ..., T\}$$

separable information cost function

$$C(\mathcal{W}) = \sum_{t=0}^{T} f_t(w_t)$$

Axiom 1 (sequential outcome betweenness) For any $s_{0\to T}$, there exists a $\alpha \in (0,1)$ such that $s_{0\to T} \sim \alpha \cdot s_{0\to T-1} + (1-\alpha) \cdot s_T$.

Axiom 2 (sequential bracket independence) For any $s_{0\to T}$, if there exists non-negative real numbers α_1, α_2, β_0, β_1, β_2, such that $s_{0\to T} \sim \alpha_1 \cdot s_{0\to T-1} + \alpha_2 \cdot s_T$, and $s_{0\to T} \sim \beta_0 \cdot s_{0\to T-2} + \beta_1 \cdot s_{T-1} + \beta_2 \cdot s_T$, then we must have $\alpha_2 = \beta_2$.

Axiom 3 (state independence) $s_t \succ s'_t$ implies that for any $\alpha \in (0,1)$ and reward c, $\alpha \cdot s_t + (1-\alpha) \cdot c \succ \alpha \cdot s'_t + (1-\alpha) \cdot c$.

Axiom 4 (aggregate invariance to constant sequences) Consider two constant sequences, denoted as s_c and s'_c, where each element in s_c equals c and each element in s'_c equals c'. For any $s_{0\to T}$, $s'_{0\to T}$ and $\alpha \in (0,1)$, if $\alpha \cdot s_t + (1-\alpha)\cdot c \sim \alpha \cdot s'_t + (1-\alpha)\cdot c'$ holds for every t, then $\alpha \cdot s_{0\to T} + (1-\alpha)\cdot s_c \sim \alpha \cdot s'_{0\to T} + (1-\alpha)\cdot s'_c$.

Proof.

Lemma 1. If Axiom 1 holds, for any $s_{0\to T}$, there exist non-negative real numbers w_0, w_1,..., w_T such that $s_{0\to T} \sim w_0 \cdot s_0 + w_1 \cdot s_1 + ... + w_T \cdot s_T$ where $\sum_{t=0}^{T} w_t = 1$.

When $T = 1$, the lemma is a direct application of Axiom 1.

When $T \geq 2$, according to Axiom 1, for any $2 \leq t \leq T$, there should exist a real number $\alpha_t \in (0,1)$ such that $s_{0\to t} \sim \alpha_t \cdot s_{0\to t-1} + (1-\alpha_t)\cdot s_t$. For sequence $s_{0\to T}$, we can recursively apply such preference relations as follows:

$$
\begin{aligned}
s_{0\to T} &\sim \alpha_{T-1} \cdot s_{0\to T-1} + (1-\alpha_{T-1})\cdot s_T \\
&\sim \alpha_{T-1}\alpha_{T-2} \cdot s_{0\to T-2} + \alpha_{T-1}(1-\alpha_{T-2})\cdot s_{T-1} + (1-\alpha_{T-1})\cdot s_T \\
&\sim ... \\
&\sim w_0 \cdot s_0 + w_1 \cdot s_1 + ... + w_T \cdot s_T
\end{aligned}
$$

where $w_0 = \prod_{t=0}^{T-1} \alpha_t$, $w_T = 1 - \alpha_{T-1}$, and for $0 < t < T$, $w_t = (1-\alpha_{t-1})\prod_{\tau=t}^{T-1}\alpha_\tau$. It is easy to show the sum of all these weights, denoted by w_t ($0 \leq t \leq T$), equals 1.

Therefore, if Axiom 1 holds, for any sequence $s_{0\to T}$, we can always find a convex combination of all elements in it, such that the decision maker is indifferent between the sequence and the convex combination of its elements. By Lemma 2, I show this convex combination is unique.

Lemma 2. If Axiom 1-3 holds, suppose $s_{0\to T} \sim \sum_{t=0}^{T} w_t \cdot s_t$ and $s_{0\to T+1} \sim \sum_{t=0}^{T-1} w'_t \cdot s_t$, where $w_t > 0$, $w'_t > 0$, $\sum_{t=0}^{T} w_t = 1$, $\sum_{t=0}^{T+1} w'_t = 1$, we must have $\frac{w'_0}{w_0} = \frac{w'_1}{w_1} = \ldots = \frac{w'_T}{w_T}$.

When $T = 1$, according to Axiom 1, there exist $\alpha, \zeta \in (0,1)$ such that $s_{0\to 1} \sim \alpha \cdot s_0 + (1-\alpha)\cdot s_1$, $s_{0\to 2} \sim \zeta \cdot s_{0\to 1} + (1-\zeta)\cdot s_2$. Meanwhile, we set $s_{0\to 2} \sim w'_0 \cdot s_0 + w'_1 \cdot s_1 + (1 - w'_0 - w'_1)\cdot s_2$, where $w'_0, w'_1 > 0$.

According to Axiom 2, we must have $1 - \zeta = 1 - w'_0 - w'_1$. So, $w'_1 = \zeta - w'_0$.

According to Axiom 3, it can be derived that $s_{0\to 1} \sim \frac{w'_0}{\zeta}\cdot s_0 + (1 - \frac{w'_0}{\zeta})\cdot s_1$.

Given that $s_{0\to1} \sim \alpha \cdot s_0 + (1-\alpha) \cdot s_1$, suppose $\alpha > \frac{w'_0}{\zeta}$, we can rewrite this preference relation as $s_{0\to1} \sim (\alpha - \frac{w'_0}{\zeta}) \cdot s_0 + (1-\alpha) \cdot s_1 + \frac{w'_0}{\zeta} \cdot s_0$.

If $s_0 \succ s_1$, by applying Axiom 3, we can derive that $(\alpha - \frac{w'_0}{\zeta}) \cdot s_0 + (1-\alpha) \cdot s_1 + \frac{w'_0}{\zeta} \cdot s_0 \succ (\alpha - \frac{w'_0}{\zeta}) \cdot s_1 + (1-\alpha) \cdot s_1 + \frac{w'_0}{\zeta} \cdot s_0$, where the right-hand side, according to the above preference relation, is indifferent from $s_{0\to1}$. Thus, we get a contradiction.

Similarly, suppose $\alpha < \frac{w'_0}{\zeta}$, we will also get a contradiction.

Thus, $\alpha = \frac{w'_0}{\zeta}$, which indicates $\frac{w'_0}{\alpha} = \frac{w'_1}{1-\alpha} = \zeta$.

We can decompose $s_{0\to T+1}$ by

$$\begin{aligned} s_{0\to T+1} &\sim (1-\alpha) \cdot s_{0\to T} + \alpha \cdot s_{T+1} \\ &\sim (1-\alpha)\zeta \cdot s_{0\to T-1} + (1-\alpha)(1-\zeta) \cdot s_T + \alpha \cdot s_{T+1} \end{aligned}$$

Suppose there is another way to decompose $s_{0\to T+1}$ using a combination of $s_{0\to T-1}$, s_T, and s_{T+1}. We can denote this alternative decomposition as

$$s_{0\to T+1} \sim \beta_0 \cdot s_{0\to T-1} + \beta_1 \cdot s_T + \beta_2 \cdot s_{T+1}$$

According to Axiom 2, we must have $\alpha = \beta_2$.

Corollary 1.

Lemma 3. If Axiom 1 and Axiom 3-4 holds, then for any $s_{0\to T}$ and $s'_{0\to T}$, where $u(s_t) = u(s'_t) + \Delta u$ holds for any t and Δu is a constant real number, we have $w_t = w'_t$.

Suppose $\alpha \cdot s_t + (1-\alpha) \cdot c \sim \alpha \cdot s'_t + (1-\alpha) \cdot c'$

From Axiom 3, $\alpha \cdot u(s_t) + (1-\alpha) \cdot u(c) = \alpha \cdot u(s'_t) + (1-\alpha) \cdot u(c')$

This yields $u(s_t) - u(s'_t) = \Delta u$, where $\Delta u = \frac{1-\alpha}{\alpha}(u(c') - u(c))$.

By Lemma 1, if Axiom 1 holds, we have $V(s_c) = u(c)$. The same applies to $V(s'_c)$.

By Axiom 4, we have $V(s_{0\to T}) = V(s'_{0\to T}) + \Delta u$.

This yields $\sum_{t=0}^{T} w_t u(s_t) - w'_t u(s'_t) = \Delta u$

Replace Δu, we have $\sum_{t=0}^{T} w_t u(s_t) - w'_t u(s'_t) = \sum_{t=0}^{T} w_t (u(s_t) - u(s'_t))$

So, $\sum_{t=0}^{T} (w_t - w'_t) u(s'_t) = 0$

Given instantaneous utility can be any non-negative real number, we must have $w_t = w'_t$.

The FOC condition of the constrained optimal discounting problem is:

$$f'_t(w_t) = u(x_t) + \theta,\ \forall t \in \{0, 1, ..., T\}$$