

# Attention-Adjusted Discounted Utility

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## 1 Introduction

It has been a long tradition that researchers in economics and behavioral sciences use the discounted utility framework for modelling intertemporal choices

I develop a novel model of intertemporal choice, which I term “*attention-adjusted discounted utility*” (ADU). I postulate that the overall utility a decision maker can obtain from a reward sequence is decided by the weighted sum of utilities that can be obtained in each time period. The initial weight allocation is exponential, i.e. the decision maker is initially time-stationary. However, when evaluating the given reward sequence, she tends to assign more weights (pay more attention) to the time periods with larger rewards, in order to subjectively maximize her overall utility. This attention adjustment process incurs a cognitive cost; and the more the weight allocation deviates from the initial allocation, the greater the cost is. The decision maker optimally re-allocates the weights across time periods. I term the discounting factors that can be represented by such weights as *attention-adjusted* discounting factors.

In this paper, I show that a set of intertemporal choice anomalies can be attributed to such attention adjustment processes (that is, can be explained by ADU to some extent), including common difference effect and magnitude effect (Loewenstein and Prelec, 1992), risk aversion over time lotteries (Onay and Öncüler, 2007; DeJarnette et al., 2020), non-additive time intervals (Read, 2001; Scholten and Read, 2006), intertemporal correlation aversion (Andersen et al., 2018), and dynamic inconsistency. The model can also offer insights on the

preferences for sequences of outcomes (Loewenstein and Prelec, 1993) and the formation of reference-dependent preferences (Koszegi and Rabin, 2006). In an empirical test, I find ADU outperforms a set of time discounting models in predicting human intertemporal choices. Therefore, I think there is a need to rethink the foundation of many behavioral phenomena. The remaining part of this document is organized as follows. Section 2 outlines the model of attention-adjusted discounted utility (ADU). Section 3 explains how the model can help explain some empirical findings in intertemporal choice.

## 2 The Model

### 2.1 Rational Inattention and Time Discounting

Consider a reward sequence  $x = [x_0, x_1, \dots, x_T]$  that yields reward  $x_t$  in time period  $t$ . The time length of this sequence, denoted by  $T$ , is finite. For any  $t \in \{0, 1, \dots, T\}$ , the reward level  $x_t$  is a random variable defined on  $R_+$ . The support of  $x$  is  $X$ , which is a subset of  $R_+^T$ .

Suppose a decision maker evaluates reward sequence  $x$  by three steps: At first, she randomly draws some potential realizations of  $x$  from  $X$ . Then, from each drawn realization of  $x$ , she draws some time periods at random, taking the rewards of these periods into a sample. Finally, she uses the mean utility of sampled rewards as a value representation of  $x$ . Let  $s = [s_0, s_1, \dots, s_T]$  be a potentially realized outcome of  $x$  and  $p(s)$  be the probability that  $s$  is drawn. I use  $w(\cdot)$  and  $u(\cdot)$  to denote the decision maker's weight function and utility function, where  $w(s_t)$  is the probability that the reward of the  $t$ -th period in a potentially realized sequence  $s$  is sampled,  $u(s_t)$  is the utility obtained by reward  $s_t$  ( $t \in \{0, 1, \dots, T\}$ ),  $u' > 0$ ,  $u'' < 0$ .

The sampling process is sequential, and the decision maker wants to find a sampling strategy, denoted by function  $w(\cdot)$ , that maximizes her overall utility. In a given potentially realized sequence  $s$ , the periods with larger reward levels should be sampled more frequently. However, at the very beginning, the decision maker has no information about which period in  $s$  has a larger reward – she learns such information gradually in the process of sampling. This

learning process triggers a cognitive cost. Hence, her overall utility is the mean utility of sampled rewards minus the cognitive cost of learning.

Suppose when having no information, the weight on period  $t$  across each potentially realized sequence is equal ( $\equiv w_t^0$ ). Let  $W$  and  $P$  be the minimal sets that contain all available function  $w$  and  $p$  respectively. We can use an optimization problem to represent the described evaluation procedure:

$$\begin{aligned} \max_{w \in W} \quad & \sum_{s \in X} \sum_{t=0}^T w(s_t) u(s_t) - C(w; \theta) \\ \text{s.t.} \quad & \sum_{s \in X} \sum_{t=0}^T w(s_t) = 1 \\ & w(s_t) > 0, \forall s \in X, t = 0, 1, \dots, T \end{aligned}$$

where  $C(\cdot)$  is a cognitive cost function with  $\theta$  as its parameters. To solve this optimization problem, I add two additional assumptions. The first is that the weight updating process is consistent with Bayes rule, that is,  $w_t^0 = \sum_{s \in X} w(s_t)$ . The second is that the cognitive cost function takes a form similar to Shannon mutual information, that is

$$C(\mathbf{w}; \theta) = \lambda \sum_{s \in X} \sum_{t=0}^T w(s_t) \log \left( \frac{w(s_t)}{p(s)w_t^0} \right)$$

where  $p(s)w_t^0$  is the probability of  $s_t$  being sampled when no information is learned,  $w(s_t)$  is the probability of that after learning the information about  $x$ . Shannon mutual information quantifies the amount of information gain when learning about which time period has a larger reward in any initially unknown  $s$ . Consistent with Matějka and McKay (2015), I set  $C(\mathbf{w}; \theta)$  linear to that. Parameter  $\lambda$  denotes unit cost of information ( $\lambda > 0$ ).

Define  $w(s_t|s) = \frac{w(s_t)}{p(s)}$ . As is shown in Matějka and McKay (2015), the optimization problem can be easily solved by Lagrangian method. The solution is

$$w(s_t|s) = \frac{w_t^0 e^{u(s_t)/\lambda}}{\sum_{t=0}^T w_t^0 e^{u(s_t)/\lambda}} \quad (1)$$

Note  $w(s_t|s)$  reveals how the decision maker weights the utility of time period  $t$  in a drawn sequence  $s$ . It can naturally represent the discounting factor.  $w(s_t|s)$  is increasing in  $s_t$ ,

which implies the decision maker exhibit more patience for a larger reward.

While building the model, I was mainly inspired by the theories of rational inattention (Matějka and McKay, 2015; Jung et al., 2019; Maćkowiak et al., 2023). In Matějka and McKay (2015)’s theory of rational inattention, the decision maker makes choices between discrete alternatives; she evaluates each alternative via a costly information acquisition process, then decides the optimal choice strategy. The theory deduces the probability of each alternative being chosen should follow a logistic-like distribution. In ADU, I assume the discounting factors are generated by a similar process; hence, she subjectively weights each time period according to a logistic-like distribution – as Equation (1) does – as well.

The reason why I use Shannon mutual information as the cognitive cost function is twofold. First, note that  $w(s_t|s) \propto w_t^0 e^{u(s_t)/\lambda}$ . Given a certain stream  $s$  and two time periods  $t_1$  and  $t_2$  ( $t_2 > t_1$ ), the relative weight between them  $\frac{w(s_{t_1}|s)}{w(s_{t_2}|s)}$  is only relevant to  $s_{t_1}$  and  $s_{t_2}$ . Therefore, changing the reward of a third period has no impact on how the reward in  $t_2$  should be discounted relative to that in  $t_1$ . Second, under such settings, the objective function can be rewritten as

$$\sum_{s \in X} p(s)[w(s_t|s)u(s_t) - \lambda D_{KL}]$$

where  $D_{KL}$  is the KL divergence between the initial weights over time periods and the weights updated given the stream  $s$  is drawn. Clearly, the determination of  $w(s_t|s)$  in each  $s$  can be separated from each other. In other words, given two potentially realized streams  $s$  and  $s'$ , the changes in  $s'$  has no impact on the determination of discounting factors in  $s$ . This property is consistent with many forms of optimal sequential learning (for example, Zhong (2022) ). Matějka and McKay (2015) and Caplin et al. (2022) show that the two properties are jointly satisfied if and only if the solution of  $w(s_t|s)$  follows Equation (1).

## 2.2 Axiomatic Characterization

## 2.3 Related Literature

In addition to ADU, there are other models that attempt to incorporate attention mechanism into the formation of time preferences. For example, Steiner et al. (2017) consider a decision maker adjusting the belief  $p(s)$  over time but holding the discounting factor  $w(s_t|s)$  constant. In each time period, given that her ability to learn new information is limited, the updated belief cannot deviate from that in a previous period by too much, which causes behavioral inertia. Instead, ADU assumes the decision maker re-allocates  $w(s_t|s)$  each time period. Thus, the process of attention adjustment not only affects dynamic decision-making but also affects the choices in “Money Earlier or Later” (MEL) tasks. Besides, Gabaix and Laibson (2017) assume the perception of future rewards is noisy and the decision maker infers the value of them by sampling from normal distributions; Gershman and Bhui (2020) allow the decision maker optimally chooses sample variance to minimize the mean sample squared error. Such theories, together with a certain specification on rate-distortion function, can lead to magnitude-increasing patience and hyperbolic-like discounting. Discounting factors in this style can be viewed as a special case of those in ADU. Noor and Takeoka (2022), Noor and Takeoka (2023) construct an optimization problem similar to ADU. However, they use a different cognitive cost function. I compare the performance of ADU with models of Gershman and Bhui (2020), Noor and Takeoka (2022) and some other papers in predicting human choices in MEL tasks.

# 3 Explaining Non-Stationary Time Preferences

## 3.1 Evaluating A Delayed Reward

Suppose there is a reward sequence with a time length of  $T$ . A decision maker receives a positive deterministic reward in time period  $j$  (and no reward in other periods), that is, for any  $t \in [0, 1, \dots, T]$  and  $t \neq j$ ,  $x_t = 0$ . The decision maker evaluates the reward sequence

by implementing the ADU evaluation procedure. For simplicity, I set  $v(0) = 0$ . I also set the decision maker initially holds stationary time preferences, i.e.  $w_t^0 = \delta^t$ , where  $\delta \in (0, 1]$ . When  $\delta = 1$ , we say that the initial attention is uniformly distributed across time periods. Given the reward is detereminsic, one can omit  $s$  in  $w(s_t|s)$  and directly represent the weight on each time period  $t$  by  $w_t$ .

$$w_j = \begin{cases} \delta^j \frac{1}{1 + \frac{\delta}{1-\delta}(1 - \delta^T)e^{-v(x_j)}} , & 0 < \delta < 1 \\ \frac{1}{1 + T \cdot e^{-v(x_j)}} , & \delta = 1 \end{cases}$$

When  $\delta = 1$ , the attention weight  $w_j$  takes a form similar to hyperbolic discounting.

Therefore, the value of this delayed reward is  $w_T u(x_T)$ . By Equation (1),

$$w_T = \frac{1}{1 + G(T) \cdot e^{-u(x_T)/\lambda}} \quad (2)$$

where

$$G(T) = \begin{cases} T , & \delta = 1 \\ \frac{1}{1-\delta}(\delta^{-T} - 1) , & 0 < \delta < 1 \end{cases}$$

Equation (2) implies  $w_T$  is increasing in  $x_T$  and decreasing in  $T$ . Hereafter, I use  $w_T(x_T)$  to represent  $w_T$ . I will show that Equation (2) can explain a series of experimental findings.

### 3.2 Common Difference Effect

Suppose there are a large later reward  $x_l$  arriving at period  $t_l$  (denoted by LL) and a small sooner reward  $x_s$  arriving at period  $t_s$  (denoted by SS), where  $x_l > x_s > 0$ ,  $t_l > t_s > 0$ . Assuming  $w_{t_l}(x_l)u(x_l) = w_{t_s}(x_s)u(x_s)$ , common difference effect implies  $w_{t_l+\Delta t}(x_l)u(x_l) > w_{t_s+\Delta t}(x_s)u(x_s)$  for any positive integer  $\Delta t$ , magnitude effect implies  $w_{t_l}(x_l)u(x_l + \Delta x) > w_{t_s}(x_s)u(x_s + \Delta x)$  for any positive real number  $\Delta x$  (Loewenstein and Prelec, 1992).

Proposition 1: The common difference effect

$$(1) \text{ holds when } v(x_l) - v(x_s) + \ln \frac{u(x_l)}{u(x_s)} > -(t_l - t_s) \ln \delta, \text{ if } T_l = t_l \text{ and } T_s = t_s$$

(2) always holds, if  $T_s = T_l$ .

When  $\delta = 1$ , ADU predicts that the decision maker always performs common difference effect. This is obvious because the discounting factor  $w_T$  takes a hyperbolic-like form. When  $\delta < 1$ , the decision maker performs common difference effect only when the difference between  $x_l$  and  $x_s$  are much larger than the difference between  $t_l$  and  $t_s$ .

The ADU's prediction on common difference effect can be understood as follows. Note that  $w_t \propto \delta^t e^{u(x_t)/\lambda}$ . If we omit the constraint that the sum of weights on each time period is fixed (i.e. attention is limited), then  $w_{t_l+\Delta t}(x_l) = \delta^{\Delta t} \cdot w_{t_l}$  and the same can be applied to  $w_{t_s+\Delta t}$ . Thus,  $w_{t_l+\Delta t}/w_{t_s+\Delta t}$  keeps constant for any  $\Delta t$ . However, given the decision maker's attention is limited, the change from  $w_{t_l}$  to  $w_{t_l+\Delta t}$  is not only driven by  $\delta^{\Delta t}$ , but also driven by the effect that the final period, with a positive reward, can naturally grab attention from the previous periods which has no reward. Since  $x_l > x_s$ , this attention-grabbing effect is greater for LL than for SS. Meanwhile, when extending the time length, the average attention that can be allocated to each period should shrink. The decision makers performs common difference effect only when the former effect exceeds the latter effect.

### 3.3 Magnitude Effect

### 3.4 Reference-Dependent Preferences

ADU predicts that the larger the unit cost of information  $\lambda$  or the smaller the magnitude of  $x_l$  and  $x_s$  is, the more likely it is that the decision maker performs magnitude effect.

First, note that the magnitude effect requires the decision maker's overall utility  $w_T(x_T)u(x_T)$  to be a convex function of  $x_T$ . Given that  $u(\cdot)$  is concave, whether the magnitude effect holds should depend on  $w_T$ . Then, set  $z = u(x_T) - \lambda \log G(T)$ . We can rewrite Equation (2) as a logistic function of  $z$ , i.e.  $w_T = 1/(1 + e^{-z/\lambda})$ . By the shape of logistic function,  $w_T$  is convex in  $u(x_T)$  if and only if  $u(x_T) < \lambda \log G(T)$  (that is, when  $x_T$  is small relative to  $T$  or when  $\lambda$  is large). Finally, it is notable that the given condition is necessary but not sufficient to yield magnitude effect.

In summary, holding the others equal, the decision makers' overall utility can be convex in a future reward when the level of it is under a certain threshold, and be concave when it is above the threshold. This is also consistent to the theories about reference-dependent preferences (Koszegi and Rabin, 2006).

### 3.5 Non-Additive Time Intervals

Both exponential and hyperbolic discounting models predict the decision maker is risk seeking over time lotteries. That is, suppose a deterministic reward of level  $c$  ( $c > 0$ ) is delivered in period  $t_s$  with probability  $\pi$  and is delivered in period  $t_l$  with probability  $1 - \pi$  ( $0 < \pi < 1$ ); another deterministic reward, of the same level, is delivered in a certain period  $\pi t_s + (1 - \pi)t_l$ . The decision maker should prefer the former case to the latter case. However, Onay and Öncüler (2007) find in experiments that people are only risk seeking over time lotteries when  $\pi$  is small and are risk averse over time lotteries when  $\pi$  is large. This finding can be explained by the convexity of  $w_T$ .

Let  $t_m = \pi t_s + (1 - \pi)t_l$ . By definition, the decision makers are risk seeking over time lotteries when  $\pi w_{t_s}(c) + (1 - \pi)w_{t_l}(c) > w_{t_m}(c)$ . First, note the LHS equals to the RHS when  $\pi = 0$  or  $\pi = 1$ . Fixing  $t_s$  and  $t_l$ , the inequality implies  $w_{t_m}(c)$  is convex in  $t_m$ . Second, it can be proved that  $w_T(c)$  is convex in  $T$  if and only if  $T$  is above a certain threshold. This is also consistent with Takeuchi (2011) that suggests the discount function should be inverse S-shaped with respect to time. By contrast, in many models such as exponential and hyperbolic discounting, discounting factors are typically decided by a convex function of  $T$ . Third, note  $t_m$  is linearly decreasing with  $\pi$ , thus the decision maker is more likely to be risk seeking over time lotteries when  $\pi$  is small. The same can be applied to the risk aversion case.

Now consider  $T$  is small enough to make  $w_T$  concave in  $T$ . In this case, adding an extension to  $T$  will increase the rate at which  $w_T$  declines with  $T$  – this property is termed “super-additive time intervals” by Read (2001). Moreover, ADU predicts intervals are sub-additive when the total time length  $T$  is large, and are super-additive when  $T$  is small, which is consistent with Scholten and Read (2006).



### 3.6 Non-Separable Sequences

Let  $x$  and  $y$  denote two 2-period risky reward sequences. For  $x$ , the realized sequence is  $[\pounds 100, \pounds 100]$  with probability  $1/2$ , and is  $[\pounds 3, \pounds 3]$  with probability  $1/2$ . For  $y$ , the realized sequence is  $[\pounds 3, \pounds 100]$  with probability  $1/2$ , and is  $[\pounds 100, \pounds 3]$  with probability  $1/2$ . Classical models of intertemporal choice, such as Fishburn and Rubinstein (1982), typically assume the separability of potentially realized sequences. This implies that the decision maker is indifferent between  $x$  and  $y$ . However, Andersen et al. (2018) find evidence of intertemporal correlation aversion, that is, people often prefer  $y$  to  $x$ . Such a property is also termed “weak separability” in Noor and Takeoka (2023).

ADU can naturally yield intertemporal correlation aversion. For simplicity, suppose the initial attention is uniformly distributed across the two periods. For  $x$ , under each potentially realized sequence, the decision maker equally weights each period. For  $y$ , decision maker tends to assign more weight to the period with a reward of  $\pounds 100$  (suppose that weight is  $w$ ). Then the value of  $x$  is  $\frac{1}{2}u(100) + \frac{1}{2}u(3)$  and the value of  $y$  is  $w \cdot u(100) + (1 - w) \cdot u(3)$ . Given that  $x > \frac{1}{2}$ , the decision makers should strictly prefer  $y$  to  $x$ .

## 4 Explaining Dynamic Inconsistent Behavior

Suppose a decision maker has budget  $m$  ( $m > 0$ ) and is considering how to spend it over different time periods. We can use a reward sequence  $x$  to represent this decision problem, where the decision maker’s spending in period  $t$  is  $x_t$ . In period 0, she wants to find a  $x$  such that

$$\max_x \sum_{t=0}^T w_t u(x_t) \quad s.t. \quad \sum_{t=0}^T x_t = m \quad (3)$$

where  $w_t$  is the attention-adjusted discounting factor in period  $t$ . I assume  $w_t = \delta^t e^{u(x_t)/\lambda} / \sum_{\tau=0}^T \delta^\tau e^{u(x_\tau)/\lambda}$  and there is no risk under this setting.

In models like exponential and hyperbolic discounting, the discounting factor of a future period is consistently smaller than that of the current period. Thus, the decision maker should spend more at the present than in the future. By contrast, in ADU, when increasing

the spending in a certain period, the discounting factor corresponding to that period should also increase. So it is possible that the decision maker spends more in the future and that a future period has a greater discounting factor than the current period. This is consistent with Loewenstein and Prelec (1993) that find people sometimes prefer improving sequences to declining sequences.

ADU suggests there are two mechanisms that can help explain why people may perform dynamically inconsistent behavior. The first is *attention-grabbing effect*, that is, keeping the others equal, when we increase  $x_t$  (which lead to an increase in  $w_t$ ), the discounting factor in any other period should decrease due to limited attention. After omitting a previous period from the decision problem in Equation (3), the decision maker can assign more weights to remaining periods; thus, the attention-grabbing effect is enhanced. The increased attention-grabbing effect will offset some benefit of increasing spending toward a certain period. Therefore, when the decision maker prefers improving sequences, the attention-grabbing effect will make her perform a present bias-like behavior (always feeling that she should spend more at the present than the original plan); when the decision maker prefers declining sequences, this effect will make her perform a future bias-like behavior (always feeling she should spend more in the future).

The second mechanism is *initial attention updating*. As is assumed above, in period 0, prior to evaluating each reward sequence, the decision maker's initial weight on period  $t$  is proportional to  $\delta^t$ ; after evaluation, the weight becomes being proportional to  $\delta^t e^{u(x_t)/\lambda}$ . In period 1, if she implements the evaluation based on the information attained in period 0, the initial weight should be updated to being proportional  $\delta^t e^{u(x_t)/\lambda}$ ; thus, the weight after evaluation should become being proportional to  $\delta e^{2u(x_t)/\lambda}$ . As a result, the benefit of increasing spending toward a certain period gets strengthened. The updated initial attention can make those who prefer improving sequences perform present bias and those who prefer declining sequences perform future bias.

Both the attention-grabbing effect and initial attention updating are affected by the curvature of utility function. They jointly decide which behavior pattern that people should perform in dynamics.

## 5 Consumption Over-Smoothness and Over-Sensitivity

## 6 Discussion

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