An Attentional Model of Time Discounting

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August 13, 2024

1 Introduction

2 Model Setting

Assume time is discrete. Let $s_{0\to T} \equiv [s_0, s_1, ..., s_T]$ denote a reward sequence that starts delivering rewards at period 0 and ends at period T. At each period t of $s_{0\to T}$, a specific reward s_t is delivered, where $t \in \{0, 1, \ldots, T\}$. Throughout this paper, we only consider non-negative rewards and finite length of sequence, i.e. $s_t \in \mathbb{R}_{\geq 0}$ and $1 \leq T < \infty$. The DM's choice set is constituted by a range of alternative reward sequences which start from period 0 and end at some finite period. To calculate the value of each reward sequence, we adopt the additive discounted utility framework. The value of $s_{0\to T}$ is defined as $U(s_{0\to T}) \equiv \sum_{t=0}^{T} w_t u(s_t)$, where $u(s_t)$ is the instantaneous utility of receiving s_t, and w_t is the decision weight (sometimes called discount factors, $0 < w_t < 1$) assigned to s_t. The function $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing, and for convenience, we set $u(0) = 0$. When making an intertemporal choice, the DM seeks to find the reward sequence of the highest value in her choice set.

The determination of w_t is central to this paper. We assume that when evaluating a reward sequence, the DM needs to divide her attention to each reward in the sequence, in order to acquire its value information. The more attention she puts on a reward, the greater decision weight is assigned to that reward. For instance, to evaluate reward s_t ($t > 0$), she may need to imagine how much pleasure she would feel on the occasion when she receives s_t. As more attention is paid to that specific occasion, her imagination of the occasion will become more vivid and more salient. In this case, the utility of reward s_t could be less discounted (w_t will be greater). We assume that if the DM is totally focused on that specific occasion, the value of reward s_t within the sequence will be equal to the value of it alone, i.e. $u(s_t)$. After each reward is evaluated, the DM aggregates the values of all rewards within the sequence to construct a value representation of the total sequence.

The division of attention is subject to all rewards in the sequence. We propose that the DM's attention allocation process should follow at least two principles. First, she tends to overweight large rewards and underweight small rewards. For example, suppose a reward sequence $\mathbb{S}_{0\to1}$ delivers "£5 today and £200 in 1 week". When processing $\mathbb{S}_{0\to1}$, the DM might pay more attention to the period in which she can receive £200, and relatively ignore £5. Second, when the DM has to process more rewards in a sequence, the attention allocated to each reward would decline. To illustrate, consider a reward sequence $\mathbb{S}_{0\to3}$ that delivers "£5 today, and £200 in 1 week, and £85 in 2 weeks, and £10 in 3 weeks". To evaluate $\mathbb{S}_{0\to3}$, the DM needs to take into account more occasions when she can get a positive reward (compared with $\mathbb{S}_{0\to1}$). Suppose she faces the same attention constraint when evaluating each sequence. When the DM processes $S_{0\to3}$, each occasion would be less vivid in her imagination than its counterpart in $\mathbb{S}_{0\to1}$. In Section 4, we show that these two principles can account for a wide range of anomalies relevant to intertemporal choice. Specifically, we suggest decision weight w_t follow a softmax function. We define any weight in this style as an *attention-modulated discount* (AMD) factor, as in Definition 1.

Definition 1: *Let $\mathcal{W} \equiv [w_0, ..., w_T]$ denote the decision weights for all specific rewards in $s_{0\to T}$. $\mathcal{W}$ is called attention-modulated discount (AMD) factors if for any $t \in \{0, 1, \ldots, T\}$,*

$$w_t = \frac{d_t e^{u(s_t)/\lambda}}{\sum_{\tau=0}^{T} d_\tau e^{u(s_\tau)/\lambda}} \tag{1}$$

where $d_t > 0$, $\lambda > 0$, $u(.)$ is the utility function.

In intuition, how Definition 1 reflects the role of attention in valuation of reward sequences can be explained in four points. First, we view each reward in a sequence as a separate information source and the DM allocates limited attentional resources across those information sources. The AMD factors capture this notion by normalizing the discount factors (the sum of decision weights is 1). Similar assumptions are typically used in recursive utility models, such as Weil (1990) and Epstein and Zin (1991) . In this paper, the implication of normalization assumption is twofold. First, increasing the decision weight of one reward would reduce the decision weights of other rewards in the sequence, implying that focusing on one reward would make DM insensitive to other rewards. Second, when there are more rewards in the sequence, DM needs to split attention across a wider range to process each of them, which may reduce the attention to, or decision weight of, each individual reward.

Second, w_t is strictly increasing in s_t, indicating that the DM would pay more attention to larger rewards. This is consistent with empirical findings about attention in many decision domains. For instance, in visual search, people often perform a "value-driven attentional capture" effect (Della Libera and Chelazzi, 2009; Hickey et al., 2010; Anderson et al., 2011; Chelazzi et al., 2013; Jahfari and Theeuwes, 2017): visual stimuli associated with large rewards naturally capture attention. In one study (Anderson et al., 2011), researchers recruit participants to do a series of visual search tasks. In each task, participants earn a reward after detecting a target object from distractors. When an object is set as the target and associated with a large reward, it can capture attention even for the succeeding tasks. Therefore, in one following task, presenting this object as a distractor can slow down target detection.[1] In addition, in financial decision making, people often perform an ostrich effect (Galai and Sade, 2006; Karlsson et al., 2009): they have a desire for good news and tend to avoid bad news. One relative evidence is that people are more likely to check their financial accounts when they get paid and less likely when they overdraw (Olafsson and Pagel, 2017).

Third, w_t is "anchored" in a factor d_t. If $0 < d_t < 1$, then d_t could represent the initial decision weight that the DM would assign to a reward delivered at period t without knowing its realization. The DM reallocates attention across the rewards when learning the realization of each reward. We term d_t as a *default discount factor*. The deviation of w_t from d_t is

[1] Some scholars may classify attention into two categories: "bottom-up control" and "top-down control". However, value-driven attentional capture does not fall into either of these categories (Awh et al., 2012). In this paper, instead, we view attention as a mechanism that selects information in order to maximize some type of utility. Our view of attention is close to Gottlieb (2012) and Gottlieb et al. (2013).

mediated by a parameter λ, which can represent the unit cost of reallocating attention. This restriction on the deviation between w_t and d_t implies that shifting attention across rewards is cognitively costly. The greater the parameter λ is , the closer w_t is to d_t. The size of λ might be relevant to the DM's belief about how much those default discount factors can reflect her true time preference in the given context. If the DM is highly certain that the default discount factors truly characterize her preferences, she may inhibit the learning process and therefore λ should be extremely large.[2]

Fourth, we adopt the idea of Gottlieb (2012) and Gottlieb et al. (2013) that attention can be understood as an active information-sampling mechanism that selects information to maximize some type of utility. As illustrated in Section 3.1, we assume the DM selectively samples value information from each information source (i.e. each reward) when processing a reward sequence, and we use the AMD model to represent an approximately optimal sampling strategy.

3 Interpretation

In this section, we provide two approaches to characterize AMD: the first is based on the information maximizing exploration framework; the second is based on the optimal discounting framework. These approaches are closely related to the idea proposed by Gottlieb (2012), Gottlieb et al. (2013) and Sharot and Sunstein (2020), that people tend to pay attention to information with high *instrumental utility* (helping identify the optimal action), *cognitive utility* (satisfying curiosity), or *hedonic utility* (inducing positive feelings). It is worth mentioning that the well-known rational inattention theories, originating from Sims (2003), and the classical Blackwell notion of information (Blackwell et al., 1951), are grounded in the instrumental utility of information. In this paper, we draw on the cognitive and hedonic utility of information to build our theory of time discounting.

[2] Enke et al. (2023) document that when people experience higher cognitive uncertainty (which in our paper, means that they are willing to learn more information before decision, and thus induce a higher λ), their pattern of discounting will be closer to hyperbolic discounting. This can be viewed as a supportive evidence for our argument, because in Section 4.2, we show that exponential discount factors can be distorted to a hyperbolic style through attention modulation.

Our first approach to characterizing AMD is relevant to cognitive utility: the DM's information acquisition process is curiosity-driven. Similar to Gottlieb (2012) and Gottlieb et al. (2013), we interpret the model setting with a reinforcement learning framework. Specifically, we assume the DM adopts the commonly-used softmax exploration strategy in information acquisition. Our second approach is relevant to hedonic utility: the DM wants to process as much pleasant information (from large rewards) as possible. She adjust the decision weights toward that direction under some cognitive cost. Noor and Takeoka (2022, 2024) provide a theoretical background for the second approach.

3.1 Information Maximizing Exploration

For the information maximizing exploration approach, we assume that before having any information of a reward sequence, the DM perceives it has no value. When evaluation begins, each reward in the sequence $s_{0\to T}$ is processed as a separate information source. The DM engages her attention to actively sample signals at each information source, and updates her belief about the sequence value accordingly. The signals are noisy.[3] For any $t \in \{0, 1, \dots, T\}$, the signal sampled at information source s_t could be represented by $x_t = u(s_t) + \epsilon_t$, where each ϵ_t is i.i.d. and $\epsilon_t \sim N(0, \sigma_\epsilon^2)$. The sampling weight for information source s_t is denoted by w_t.

The DM's belief about the sequence value $U(s_{0\to T})$ is updated as follows. At the beginning, she holds a prior U_0. Given she perceives no value from the reward sequence, the prior could be represented by $U_0 \sim N(0, \sigma^2)$. Second, she draws a series of signals at each information source s_t and each signal indicates some information about the sequence value. Note we define $U(s_{0\to T})$ as a weighted mean of instantaneous utilities. Let $\bar{x}$ denote the mean sample signal and U denote a realization of $U(s_{0\to T})$. If there are overall k signals being sampled,

[3] Each value signal represents an estimate of the pleasure that the DM would get from receiving the reward in a corresponding period. The noise term implies the DM's estimate is imprecise. To illustrate, when evaluating "£10 today and £20 in 1 week", the DM should think about how much "receive £10 today" is worth (s_0), and how much "receive £20 in 1 weeks" is worth (s_1). She might think about s_0 first, or s_1 first, but it is little likely that she can think both at the same time. So, to think about both occasions, she has to consciously shift attention between the rewards. Each time when she thinks about an occasion, she has to imagine the pleasure that she would achieve on that occasion, and the imagination is not a constant. This process can be described as a sequential sampling methodology.

we should have $\bar{x}|U, \sigma_\epsilon \sim N(U, \frac{\sigma_\epsilon^2}{k})$. Third, she uses the sampled signals to infer $U(s_{0\to T})$ in a Bayesian fashion. Let U_k denote the DM's posterior about the sequence value after receiving k signals. According to Bayes' rule, we have $U_k \sim N(\mu_k, \sigma_k^2)$ and

$$\mu_k = \frac{k^2\sigma_\epsilon^{-2}}{\sigma^{-2} + k^2\sigma_\epsilon^{-2}}\bar{x} \qquad , \qquad \sigma_k^2 = \frac{1}{\sigma^{-2} + k^2\sigma_\epsilon^{-2}} \tag{1}$$

We assume the DM takes μ_k as the valuation of reward sequence. As $k \to \infty$, μ_k will converge to $\bar{x}$. Besides, note σ_k depends only on k. This implies that drawing more samples can always increase the precision of the DM's estimate about $U(s_{0\to T})$.

The DM's goal of sampling is to maximize her information gain. The information gain is defined as the KL divergence from the prior U_0 to the posterior U_k. In intuition, acquiring more information should move the DM's posterior belief farther away from the prior. We let $p_0(U)$ and $p_k(U)$ denote the probability density functions of U_0 and U_k. Then, the information gain is

$$\begin{aligned} D_{KL}(U_k||U_0) &= \int_{-\infty}^{\infty} p_k(U)\log\left(p_k(U)/p_0(U)\right)dU \\ &= \frac{\sigma_k^2 + \mu_k^2}{2\sigma^2} - \log\left(\frac{\sigma_k}{\sigma}\right) - \frac{1}{2} \end{aligned} \tag{2}$$

In Equation (2), the DM's information gain is increasing in μ_k^2, and μ_k is proportional to $\bar{x}$. As a result, the objective of maximizing $D_{KL}(U_k||U_0)$ could be reduced to maximizing $\bar{x}$ (note each $u(s_t)$ is non-negative). The reason is, a larger $\bar{x}$ implies more "surprises" in comparison to the DM's initial perception that $s_{0\to T}$ contains no value.

The problem of maximizing mean sample signal $\bar{x}$ under a given sample size k is a multi-armed bandit problem (Sutton and Barto, 2018, Ch.2). The key to solve the multi-armed bandit problem is to choose a proper exploration strategy. On the one hand, the DM wants to draw more samples at the information source that is known to produce the greatest value signals. On the other hand, she wants to explore at other information sources. We assume the DM takes a softmax exploration strategy to solve this problem. That is,

$$w_t \propto d_t e^{\bar{x}_t/\lambda} \tag{2}$$

where λ controls the rate of exploration, $\bar{x}_t$ is the mean sample signal generated by information source s_t so far, and d_t is the initial sampling weight for s_t.[4] Note when we analyze the intertemporal choice data, $\bar{x}_t$ is a latent variable. In principle, researchers who want to calculate the sampling weight w_t should conduct a series of simulations to generate $\bar{x}_t$ under a fixed σ_ϵ, and then fit σ_ϵ to the choice data. This process could be computationally expensive. To solve this computational issue, we apply the weak law of large numbers: when the sample size k is large, $\bar{x}_t$ will be highly likely to fall into a neighborhood of $u(s_t)$. Thus, we can use $w_t \propto d_t e^{u(s_t)/\lambda}$, which is the AMD factor, as a fair approximation to the softmax exploration strategy.

Researchers familiar with reinforcement learning algorithms may notice that in some sense $u(s_t)$ can be generalized to an action-value function (considering the future signals produced by the given exploration strategy). The AMD model thus can be somehow generalized to the soft Q-learning or policy gradient algorithms (Haarnoja et al., 2017; Schulman et al., 2017). Such algorithms are widely used (and sample-efficient) in reinforcement learning. Moreover, one may argue that the specification of the AMD factors is subject to the form of information gain specified by Equation (2). We acknowledge this limitation and suggest researchers interested in modifying the AMD model consider different assumptions about noises. For example, if noises $\epsilon_0, ..., \epsilon_T$ do not follow an i.i.d. normal distribution, the information gain $D_{KL}(U_k||U_0)$ may be complex to compute; thus, one can use its variational bound as the objective of maximization(Houthooft et al., 2016). Compared to these more complex settings, our model specification aims to provide a simple benchmark for understanding the role of attention in mental valuation of a reward sequence.

Two strands of literature can help justify our key assumptions in this subsection. First, for the assumption that DM seeks to maximize the information gain between the posterior and the prior, similar models have been studied extensively in both psychology (Oaksford and Chater, 1994; Itti and Baldi, 2009; Friston et al., 2017) and machine learning (Settles, 2009; Ren et al., 2021). In one study, Itti and Baldi (2009) find this assumption has a strong pre-

[4] The classic softmax strategy assumes the initial probability of taking any action is given by an uniform distribution. We relax this assumption by importing d_t, so that the DM can hold an “initial” preference for sampling over the information sources.

dictive power for visual attention. Our assumption that the DM updates decision weights toward a greater $D_{KL}(U_k||U_0)$ is generally consistent with this finding. Second, the softmax exploration strategy is widely used by neuroscientists in studying human reinforcement learning (Daw et al., 2006; FitzGerald et al., 2012; Collins and Frank, 2014; Niv et al., 2015; Leong et al., 2017). For instance, Daw et al. (2006) find this strategy characterizes humans' exploration behavior better than other classic strategies (e.g. ϵ-greedy). Collins and Frank (2014) show that models based on this strategy exhibit a good performance in explaining activities of the brain's dopaminergic system (which is central in sensation of pleasure and learning of rewarding behaviors) in reinforcement learning.

3.2 Optimal Discounting

The second approach to characterize AMD is based on the optimal discounting model (Noor and Takeoka, 2022, 2024). In one version of that model, the authors assume that DM has a limited capacity of attention (or in their term, "empathy"). The instantaneous utility $u(s_t)$ represents the well-being that the DM's self of period t can obtain from the reward sequence. Before evaluating a reward sequence $s_{0\to T}$, the DM naturally focuses on the current period. When evaluating that, she needs to split attention over T periods to consider the well-being of each self. This attention reallocation process is cognitively costly. The DM seeks to balance between improving the overall well-being of multiple selves and reducing the incurred cognitive cost. Noor and Takeoka (2022, 2024) specify an optimization problem to capture this balancing decision. In this paper, we adopt a variant of their original model. The formal definition of the optimal discounting problem is given by Definition 2. [5]

Definition 2: *Given reward sequence $s_{0\to T} = [s_0, ..., s_T]$, the following optimization problem*

[5] There are three differences between Definition 2 and the original optimal discounting model (Noor and Takeoka, 2022, 2024). First, in our setting, shifting attention to future rewards may reduce the attention to the current reward, while this would never happen in Noor and Takeoka (2022, 2024). Second, the original model assumes $f_t'(w_t)$ must be continuous at 0 and w_t must be no larger than 1. We relax these assumptions since neither $w_t = 0$ nor $w_t \geq 1$ is included our solutions. Third, the original model assumes that $f_t'(w_t)$ is left-continuous in $[0, 1]$, and there exist $\underline{w}, \bar{w} \in [0, 1]$ such that $f_t'(w_t) = 0$ when $w_t \leq \underline{w}$, $f_t'(w_t) = \infty$ when $w_t \geq \bar{w}$, and $f_t'(w_t)$ is strictly increasing when $w_t \in [\underline{w}, \bar{w}]$. We simplify this assumption by setting $f_t'(w_t)$ is continuous and strictly increasing in $(0, 1)$, and similarly, we set $f_t'(w_t)$ can approach infinity near at least one border of $[0, 1]$. For convenience, we set $\lim_{w_t\to 0} f_t'(w_t) = -\infty$, but it will be fine to assume it can approach positive infinity near the other border.

is called an optimal discounting problem for $s_{0\to T}$*:*

$$
\begin{aligned}
\max_{\mathcal{W}} \quad & \sum_{t=0}^{T} w_t u(s_t) - C(\mathcal{W}) \\
s.t. \quad & \sum_{t=0}^{T} w_t \leq M \\
& w_t \geq 0 \text{ for all } t \in \{0, 1, ..., T\}
\end{aligned}
\tag{3}
$$

where $M > 0$, $u(s_t) < \infty$. $C(\mathcal{W})$ *is the cognitive cost function and is constituted by time-separable costs, i.e.* $C(\mathcal{W}) = \sum_{t=0}^{T} f_t(w_t)$*. For all* $w_t \in (0,1)$*,* $f_t(w_t)$ *is differentiable,* $f_t'(w_t)$ *is continuous and* stric*tly increasing, and* $\lim_{w_t \to 0} f_t'(w_t) = -\infty$.

Here w_t reflects the attention paid to consider the well-being of t-period self. The DM's objective function is the attention-weighted sum of utilities, obtained by the multiple selves, minus the cost of attention reallocation. As Noor and Takeoka (2022, 2024) illustrate, a key feature of Equation (3) is that decision weight w_t increases with s_t, which implies the DM tends to pay more attention to larger rewards. It is easy to validate that if the following two conditions are satisfied, solution to the optimal discounting problem will take the AMD form:

(i) The constraint on the sum of decision weights is always tight, i.e. $\sum_{t=0}^{T} w_t = M$. Without loss of generality, we can set $M = 1$.

(ii) There exists a realization of decision weights $\mathcal{D} = [d_0, ..., d_T]$, such that the cognitive cost is proportional to the KL divergence from $\mathcal{D}$ to $\mathcal{W}$. That is, $C(\mathcal{W}) = \lambda \cdot D_{KL}(\mathcal{W}||\mathcal{D})$, where $\lambda > 0$, $d_t > 0$ for all $t \in \{0, \ldots, T\}$.

Here d_t sets a reference for determining the decision weight w_t, and the parameter λ controls how costly the attention reallocation process is. By definition, $D_{KL}(\mathcal{W}||\mathcal{D}) = \sum_{t=0}^{T} w_t \log(\frac{w_t}{d_t})$. Under condition (i)-(ii), the solution to the optimal discounting problem can be derived in exactly the same way as Theorem 1 in Matějka and McKay (2015). Also, note this specification of solution is equivalent to a kind of bounded rationality model (Todorov, 2009): the DM wants to find a $\mathcal{W}$ that maximizes $\sum_{t=0}^{T} w_t u(s_t)$ but can only search for solutions within a KL neighborhood of $\mathcal{D}$.

We interpret the implications of condition (i)-(ii) with four behavioral axioms. Note if each s_t is an independent option for choice and $\mathcal{W}$ is simply the DM's choice strategy, these conditions can be directly characterized by a rational inattention theory (Caplin et al., 2022). However, here $\mathcal{W}$ is a component in sequence value, and the DM is assumed to choose the option with the highest sequence value. Thus, we should derive the behavioral implications of condition (i)-(ii) in a different way from Caplin et al. (2022). To see how we derive these, let $\succsim$ denote the preference relation between two reward sequences.[6] For any reward sequence $s_{0\to T} = [s_0, ..., s_T]$, we define $s_{0\to t} = [s_0, ..., s_t]$ as a sub-sequence of it, where $1 \leq t \leq T$.[7] We first introduce two axioms for $\succsim$:

Axiom 1: $\succsim$ *has the following properties:*

(a) *(complete order)* $\succsim$ *is complete and transitive.*

(b) *(continuity) For any reward sequences* s, s' *and reward* $c \in \mathbb{R}_{\geq 0}$*, the sets* $\{\alpha \in (0,1) | \alpha \cdot s + (1-\alpha) \cdot c \succsim s'\}$ *and* $\{\alpha \in (0,1) | s' \succsim \alpha \cdot s + (1-\alpha) \cdot c\}$ *are closed.*

(c) *(state-independent) For any reward sequences* s, s' *and reward* $c \in \mathbb{R}_{\geq 0}$*,* $s \succsim s'$ *implies for any* $\alpha \in (0,1)$*,* $\alpha \cdot s + (1-\alpha) \cdot c \sim \alpha \cdot s' + (1-\alpha) \cdot c$*.*

(d) *(reduction of compound alternatives) For any reward sequences* s, s', q *and rewards* $c_1, c_2 \in \mathbb{R}_{\geq 0}$*, if there exist* $\alpha, \beta \in (0,1)$ *such that* $s \sim \alpha \cdot q + (1-\alpha) \cdot c_1$*, then* $s' \sim \beta \cdot q + (1-\beta) \cdot c_2$ *implies* $s' \sim \beta\alpha \cdot q + \beta(1-\alpha) \cdot c_1 + (1-\beta) \cdot c_2$*.*

Axiom 2: *For any* $s_{0\to T}$ *and any* $\alpha_1, \alpha_2 \in (0,1)$*, there exists* $c \in \mathbb{R}_{\geq 0}$ *such that* $\alpha_1 \cdot s_{0\to T-1} + \alpha_2 \cdot s_T \sim c$*.*

The two axioms are almost standard in decision theories. The assumption of complete order implies preferences between reward sequences can be characterized by an utility function. Continuity and state-independence ensure that in a stochastic setting where the DM can receive one reward sequence under some states and receive a single reward under other

[6] If $a \succsim b$ and $b \succsim a$, we say $a \sim b$. If $a \succsim b$ does not hold, we say $b \succ a$. $\succsim$ can also characterize the preference relation between single rewards as any single reward can be viewed as a one-period sequence.

[7] Unless otherwise specified, every sub-sequence is set to starts from period 0.

states, her preference can be characterized by the expected utility theorem (Herstein and Milnor, 1953). Reduction of compound alternatives ensures that the valuation of a certain reward sequence is constant over states. Axiom 2 is an extension of the Constant-Equivalence assumption in Bleichrodt et al. (2008). It implies there always exists a constant that can represent the value of a convex combination of sub-sequence $s_{0\to T}$ and the end-period reward s_T.

For a given $s_{0\to T}$, the optimal discounting model can generate a sequence of decision weights $[w_0, ..., w_T]$. Furthermore, the model assumes the DM's preference for $s_{0\to T}$ can be characterized by the preference for $w_0 \cdot s_0 + w_1 \cdot s_1 + ... + w_T \cdot s_T$. We use Definition 3 to capture this assumption.[8]

Definition 3: *Given reward sequence $s_{0\to T} = [s_0, ..., s_T]$ and $s'_{0\to T'} = [s'_0, ..., s'_{T'}]$, the preference relation $\succsim$ has an optimal discounting representation if*

$$s_{0\to T} \succsim s'_{0\to T'} \quad \Longleftrightarrow \quad \sum_{t=0}^{T} w_t \cdot s_t \succsim \sum_{t=0}^{T'} w'_t \cdot s'_t \tag{3}$$

where $\{w_t\}_{t=0}^{T}$ and $\{w'_t\}_{t=0}^{T'}$ are solutions to the optimal discounting problems for $s_{0\to T}$ and $s'_{0\to T'}$ respectively.

Furthermore, if Definition 3 is satisfied and both $\{w_t\}_{t=0}^{T}$ and $\{w'_t\}_{t=0}^{T'}$ take the AMD form, we say $\succsim$ has an *AMD representation*. Now we specify two additional behavioral axioms that are key to characterize AMD.

Axiom 3 (Sequential Outcome-Betweenness): *For any $s_{0\to T}$, there exists $\alpha \in (0,1)$ such that $s_{0\to T} \sim \alpha \cdot s_{0\to T-1} + (1-\alpha) \cdot s_T$.*

Axiom 4 (Sequential Bracket-Independence): *Suppose $T \geq 2$. For any $s_{0\to T}$, if there exist $\alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2 \in (0,1)$ such that $s_{0\to T} \sim \alpha_1 \cdot s_{0\to T-1} + \alpha_2 \cdot s_T$ and $s_{0\to T} \sim \beta_0 \cdot s_{0\to T-2} + \beta_1 \cdot s_{T-1} + \beta_2 \cdot s_T$, then we must have $\alpha_2 = \beta_2$.*

Axiom 3 implies that for a reward sequence $s_{0\to T-1}$, if we add a new reward s_T at the end of the sequence, then the value of the new sequence should lie between the original

[8] Noor and Takeoka (2022) refer the term "optimal discounting representation" as Costly Empathy representation.

sequence $s_{0\to T-1}$ and the newly added reward s_T. Notably, Axiom 3 is consistent with the empirical evidence about *violation of dominance* (Scholten and Read, 2014; Jiang et al., 2017) in intertemporal choice. To illustrate, suppose the DM is indifferent between a small-sooner reward (SS) "receive £75 today" and a large-later reward (LL) "receive £100 in 52 weeks". Scholten and Read (2014) find when we add a tiny reward after the payment in SS, e.g. changing SS to "receive £75 today and £5 in 52 weeks", the DM would be more likely to prefer LL over SS. Jiang et al. (2017) find the same effect can apply to LL. That is, if we add a tiny reward after the payment in LL, e.g. changing LL to "receive £100 in 52 weeks and £5 in 53 weeks", the DM may be more likely to prefer SS over LL.

Axiom 4 implies that no matter how the DM brackets a stream of rewards into sub-sequences, or how the sub-sequences get decomposed, the decision weights for rewards outside them should not be affected. Specifically, suppose we decompose reward sequence $s_{0\to T}$ and find its value is equivalent to a linear combination of $s_{0\to T-1}$ and s_T. Also, we further decompose the value of $s_{0\to T-1}$ to a linear combination of $s_{0\to T-2}$ and s_{T-1} (and we can recursively do this to $s_{0\to T-2}$). In any cases, as long as the decomposition is carried out inside $s_{0\to T-1}$, the weight of s_T in the valuation of $s_{0\to T}$ will remain the same. This axiom is analogous to the Independence of Irrelevant Alternatives principle in discrete choice analysis, while the latter is a key feature of softmax choice function.

We show in Proposition 1 that the optimal discounting model plus Axiom 1-4 can produce the AMD model.

Proposition 1: *Suppose $\succsim$ has an optimal discounting representation, then it satisfies Axiom 1-4 if and only if has an AMD representation.*

The necessity ("only if") is easy to see. We present the proof of sufficiency ("if") in Appendix A. The sketch of the proof is as follows. First, by recursively applying Axiom 3 and Axiom 1 to each sub-sequence of $s_{0\to T}$, we can obtain that there is a sequence of decision weights $\{w_t\}_{t=0}^T$ such that $s_{0\to T} \sim w_0 \cdot s_0 + ... + w_T \cdot s_T$, and $\sum_{t=0}^T w_t = 1$, $w_t > 0$. Second, by the FOC of the optimal discounting problem, we have $f_t'(w_t) = u(s_t) + \theta$, where θ is the Lagrangian multiplier. Given $f_t'(.)$ is continuous and strictly increasing, we define its inverse function as $\phi_t(.)$ and set $w_t = \phi_t(u(s_t)+\theta)$. Third, Axiom 4 indicates that the decision weight for a reward outside a sub-sequence is irrelevant to the decision weights inside it. Imagine that we add a new reward s_{T+1} to the end of $s_{0\to T}$ and denote the decision weights for $s_{0\to T+1}$ by $\{w'_t\}_{t=0}^{T+1}$. Doing this should not change the relative difference between the decision weights inside $s_{0\to T}$. That is, for all $1 \leq t \leq T$, the relative difference between w'_t and w'_{t-1} should be the same as that between w_t and w_{t-1}. So, by applying Axiom 4 jointly with Axiom 1-3, we should obtain $w_0/w'_0 = w_1/w'_1 = ... = w_T/w'_T$. Suppose $w'_t = \phi_t(u(s_t) - \eta)$, we have $w_t \propto e^{\ln \phi_t(u(s_t)-\eta)}$. Fourth, we can adjust s_{T+1} arbitrarily to get different realizations of η. Suppose under some s_{T+1}, we have $w'_t = \phi_t(u(s_t))$, which indicates $w_t \propto e^{\ln \phi_t(u(s_t))}$. Combining this with the proportional relation obtained in the third step, we can conclude that for some $\kappa > 0$, there must be $\ln \phi_t(u(s_t)) = \ln \phi_t(u(s_t) - \eta) + \kappa\eta$. This indicates $\ln \phi_t(.)$ is linear in a given range of η. Finally, we show that the linearity condition can hold when $\eta \in [0, u_{\max} - u_{\min}]$, where $u_{\max}, u_{\min}$ are the maximum and minimum instantaneous utilities in $s_{0\to T}$. Therefore, we can rewrite $\ln \phi_t(u(s_t))$ as $\ln \phi_t(u_{\min})+\kappa[u(s_t)-u_{\min}]$. Setting $d_t = \phi_t(u_{\min})$, $\lambda = 1/\kappa$, and re-framing the utility function, we obtain $w_t \propto d_t e^{u(s_t)/\lambda}$, which is the AMD factor.

4 Implications for Decision Making

4.1 Hidden Zero Effect

Empirical research suggests time discounting is influenced by the framing of sequences. One prominent evidence in this field is the hidden zero effect (Magen et al., 2008; Radu et al., 2011; Read et al., 2017). Studies about the hidden zero effect typically relate this effect to "temporal attention".

To illustrate the hidden zero effect, suppose the DM is indifferent between "receive £100 today" (SS) and "receive £120 in 25 weeks" (LL). The hidden zero effect suggests that people exhibit more patience when SS is framed as a sequence rather than as a single-period reward. In other words, if we frame SS as "receive £100 today and £0 in 25 weeks" (SS1), the DM would prefer LL to SS1. Similar to the violation of dominance, this phenomenon can be viewed as a justification for the Sequential Outcome-Betweenness axiom in Section 3.2. Moreover, Read et al. (2017) find the effect is asymmetric, that is, framing LL as "receive £0 today and £120 in 25 weeks" (LL1) has no effect on preference.

The AMD model provides a formal account for the effect. When the DM evaluates a reward sequence $s_{0\to T}$, the AMD model assumes she would splits a fixed amount of attention over T periods. In the given example, the DM may perceive the time length of SS as "today" and perceives the time length of SS1 as "25 weeks". For option SS, she can focus her attention on the current period in which she can get £100. For option SS1, she has to spend some of her attention to future periods in which no reward is delivered, and this also reduces the decision weight assigned to the current period. As a result, she values SS1 lower than SS. By contrast, the DM perceives the time length of both LL and LL1 as "25 weeks". Both LL and LL1 are sequences that deliver zero rewards from the current period to the period before "25 weeks". According to the AMD model, when she evaluates LL, she has already paid some attention to all periods earlier than "25 weeks". So, changing LL to LL1 does not affect her choice.

4.2 Relation to Hyperbolic Discounting

Most intertemporal choice studies only involve comparisons between single-period rewards (SS and LL). Here, we derive the formula of the AMD factor for SS/LL and use that to illustrate how attention modulation can account for some anomalies in this decision setting. For simplicity, we assume the DM initial discount factor (before attention modulation) is exponential, that is, the default discount factor $d_t = \delta^t$, where $\delta \in (0, 1]$.[9]

Consider a reward sequence $s_{0\to T}$ in which $u(s_t) = 0$ for all $t \leq T$, and only $u(s_T) > 0$. This implies the DM receives nothing until period T. In this case, the DM's valuation of $s_{0\to T}$ is $U(s_{0\to T}) = w_T u(s_T)$. Let $v(x) = u(x)/\lambda$. By Definition 1, we can derive that w_T is a function of s_T:

$$w_T = \frac{1}{1 + G(T)e^{-v(s_T)}} \tag{4}$$

[9] Strotz (1955) shows that, for any reward delivered at period t, if the DM's discount factor can be written as δ^t, then her preference will be stationary and consistent over time.

where

$$G(T) = \begin{cases} \frac{1}{1-\delta}(\delta^{-T} - 1)\ , & 0 < \delta < 1 \\ T\ , & \delta = 1 \end{cases} \tag{4}$$

The w_T in Equation (4) can represent the discount function for a single reward s_T, delivered at period T. Interestingly, when $\delta = 1$, $w_T(s_T)$ takes a form similar to hyperbolic discounting. In recent years, several studies have attempted to provide a rational account for hyperbolic discounting. For instance, Gabaix and Laibson (2017) propose a model with similar assumptions to our information maximizing exploration approach to AMD: the perception of instantaneous utility is noisy and the DM use Bayes' rule on the perceived signals to update her belief about the utility. Nevertheless, Gabaix and Laibson (2017) account for hyperbolic discounting with an additional assumption that the variance of signals is proportional to the corresponding reward delay. The account we propose via AMD is that the variance is constant but DM seeks to maximize the information gain when learning from signals. Besides, Gershman and Bhui (2020) propose an alternative model based on the work of Gabaix and Laibson (2017). We note under a certain specification of instantaneous utility, Equation (4) will generate a discount function similar to the function proposed by Gershman and Bhui (2020).[10] In a nutshell, such accounts for hyperbolic discounting can somehow be generated by variants or special cases of the AMD model.

In the following three subsections, we use Equation (4) to explain three decision anomalies: the common difference effect (and its reverse), risk aversion over time lotteries, and S-shaped value function.

4.3 Common Difference Effect

The common difference effect (Loewenstein and Prelec, 1992) suggests that, when the DM faces a choice between LL and SS, adding a common delay to both options can increase her preference for LL. For example, suppose the DM is indifferent between "receive £120 in 25

[10] Gershman and Bhui (2020) propose that the discount function for a single reward s_T is $1/[1 + (\beta s_T)^{-1}T]$, where $\beta > 0$. In Equation (4), if we set $\delta = 1$ and $v(s_T) = \ln(\beta s_T + 1)$, we will obtain $w_T = 1/[1 + (\beta s_T + 1)^{-1}T]$, which takes a similar form to Gershman and Bhui (2020).

weeks" (LL) and "receive £100 today" (LL). Then, she would prefer "receive £120 in 40 weeks" to "receive £100 in 15 weeks".

Let (v_l, t_l) denote a reward of utility v_l delivered at period t_l and (v_s, t_s) denote a reward of utility v_s delivered at period t_s. We set $v_l > v_s > 0$, $t_l > t_s > 0$. So, (v_l, t_l) can represent a LL and (v_s, t_s) can represents a SS. We denote the discount factors for LL and SS by $w_{t_l}(v_l)$ and $w_{t_s}(v_s)$. Suppose $w_{t_l}(v_l) \cdot v_l = w_{t_s}(v_s) \cdot v_s$. The common difference effect implies that $w_{t_l+\Delta t}(v_l) \cdot v_l > w_{t_s+\Delta t}(v_s) \cdot v_s$, where $\Delta t > 0$. We assume that the discount factors are derived from Equation (4), and describe the conditions that make the common difference effect hold in Proposition 2.

Proposition 2: *The following statements are true for AMD:*

(a) *If $\delta = 1$, the common difference effect always holds.*

(b) *If $0 < \delta < 1$, i.e. the DM is initially impatient, the common difference effect holds when and only when $v_l - v_s + \ln(v_l/v_s) > (t_l - t_s)\ln(1/\delta)$.*

The proof of Proposition 2 is in Appendix B. The part (b) of Proposition 2 yields a novel prediction about the common difference effect. That is, for any impatient DM, to make the common difference effect hold, the relative and absolute differences in reward utility between LL and SS must be significantly larger than their absolute difference in time delay. In the opposite, if the difference in delay is significantly larger than the difference in reward utility, we may observe a reverse common difference effect.[11]

When the DM is impatient, adding a common delay would naturally make v_l and v_s more discounted; so, less attention is paid to the two corresponding rewards. Suppose the sum of decision weights is not changed, this implies that the DM can "free up" some attention that is originally captured by these rewards and reallocate it to other periods in each option. There are three mechanisms jointly determining whether we could observe the common difference effect under this circumstance.

[11] Also, it is easy to validate that if we make the "hidden zeros" explicit in LL and SS, adding a common delay under the AMD model would always yield a the common difference effect.

First, in each option, the existing periods which have no reward delivered could grab some attention. That is, the DM would attend more to some rewards of zero utility, delivered in duration $[0, t_l)$ for LL and in duration $[0, t_s]$ for SS. Given $t_l > t_s$, the correponding duration in LL would naturally capture more attention than that in SS. In other words, adding the common delay makes the DM focus more on the original waiting time in LL than in SS, which decreases her preference for LL.

Second, the newly added time intervals could also grab some attention. That is, the DM needs to pay some attention to rewards (of zero utility as well) delivered in duration $(t_l, t_l + \Delta t)$ in LL and in duration $(t_s, t_s + \Delta t)$ in SS. For LL, there have been already a lot of periods to which DM has to attend before we add the delay Δt. So, the duration $(t_l, t_l + \Delta t)$ in LL can capture less attention than its counterpart in SS. This increases the DM's preference for LL.

Third, the only positive reward, delivered in t_l for LL and in t_s for SS, could draw some attention back. Given that the DM in general attends more to larger rewards, the positive reward in LL can capture more of the "freed-up" attention than that in SS. This also increases the preference for LL. If the latter two mechanisms override the first mechanism, we would observe a common difference effect in DM's choices.

Figure 1 demonstrates an example for the reverse common difference effect. In the figure, we set $v_s = x_s^{0.6}$, $v_l = x_l^{0.6}$, $\delta = 0.75$, $\lambda = 2$. For each level of the delay t_s, we identify the longer delay t_l that makes the value of LL equivalent to SS. If the common different effect is valid, increasing t_s and t_l by the same level would make the DM prefer LL. Under this condition, for one unit increase in t_s, to make LL and SS valued equally, the identified t_l should be increased by a time greater than one unit. On the contrary, if the reverse common difference effect is valid, for one unit increase in t_s, the identified t_l should be increased by a time smaller. In Figure 1, the blue line (above) reflects the common different effect, while the red line (below), which has a lower $v_l - v_s$ and v_l/v_s, reflects the reverse of it.

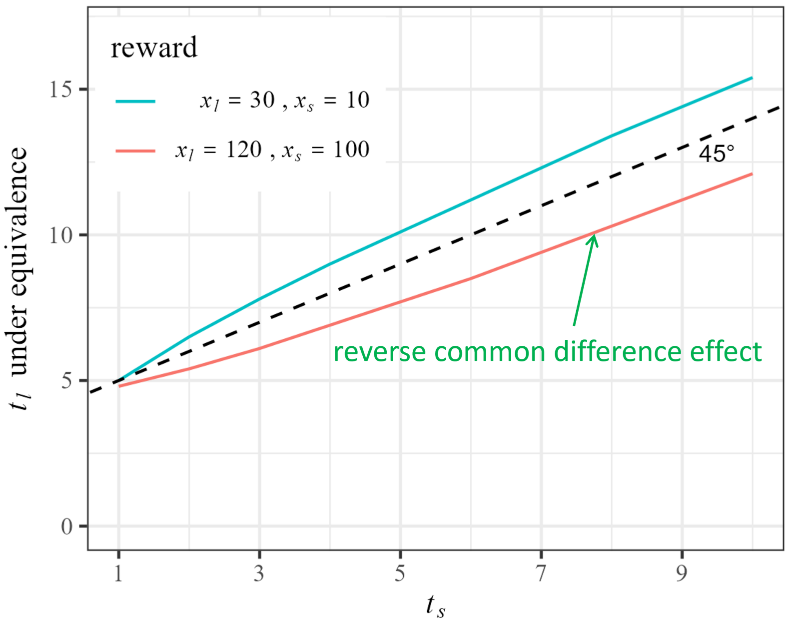


Figure 1: The common difference effect and its reverse

Note: x_l and x_s are the positive reward levels for LL and SS. The values of LL and SS are calculated based on Equation (4). $d_t = 0.75^t$, $u(x) = x^{0.6}$, $\lambda = 2$. For each level of t_s, we identify the delay t_l that makes the value of LL equivalent to SS. The blue line (above) demonstrates the common difference effect, and the red line (below) demonstrates the reverse common difference effect.

4.4 Concavity of Discount Function

Many time discounting models, such as exponential and hyperbolic discounting, assume the discount function is convex in time delay. This type of discount function predicts DM is *risk seeking over time lotteries*. To illustrate, suppose a reward of level x is delivered at period t_l with probability π and is delivered at period t_s with probability $1 - \pi$, where $0 < \pi < 1$. Meanwhile, another reward of the same level is delivered at period t_m, where $t_m = \pi t_l + (1-\pi)t_s$. Under such discount functions, the DM should prefer the former reward to the latter reward. For instance, she may prefer receiving an amount of money today or in 20 weeks with equal chance, rather than receiving it in 10 weeks with certainty. However, experimental studies suggest that people are often *risk averse over time lotteries*, i.e. they prefer the reward to be delivered at a certain time (Onay and Öncüler, 2007; DeJarnette et al., 2020).

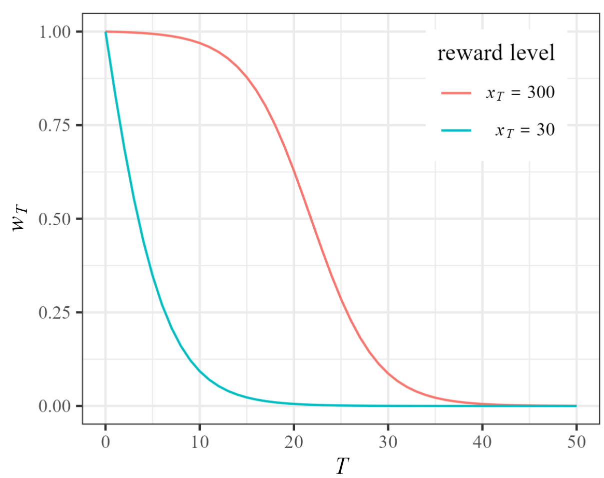
One way to accommodate risk aversion over time lotteries is to make the discount function concave in terms of delay. Notably, Onay and Öncüler (2007) find that people are more likely to be risk averse over time lotteries when π is small, and to be risk seeking when π is large. Given that t_m is increasing in π, we can claim that the discount function should be concave in delay for the near future but convex for the far future. Takeuchi (2011) find the supportive evidence for this shape of discount function. In Proposition 3, we apply Equation (4) and show that the AMD model produces this shape of discount function when the DM is impatient and the reward level x is large enough.

Proposition 3: *Suppose a single reward x* is *delivered at period T. Let w_T denote the AMD factor for this reward. If $\delta = 1$, then w_T is convex in T. If $0 < \delta < 1$, there exist a reward threshold $\underline{x} > 0$ and a time threshold $\underline{T} > 0$ such that:*

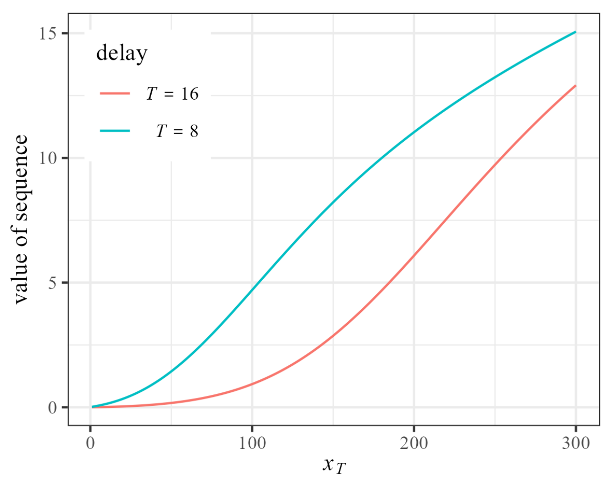
(a) *when $x \leq \underline{x}$, w_T* is *convex in T;*

(b) *when $x > \underline{x}$, w_T* is *convex in T given $T \geq \underline{T}$, and it is concave in T given $0 < T < \underline{T}$.*

The proof of Proposition 3 is in Appendix C. Figure 2(a) demonstrates the convex discount function (blue line, below) and the inverse-S shaped discount function (red line, above) that could be yielded by Equation (4).



(a) discount function



(b) value function

Figure 2: Discount function and value function for a delayed reward

Note: A reward of level x_T is delivered at period T. The discount function and value function are calculated based on Equation (4). $d_t = 0.75^t$, $u(x) = x^{0.6}$, $\lambda = 2$.

4.5 S-Shaped Value Function

A common assumption in decision theories for the instantaneous utility function $u(.)$ is $u'' < 0$. Usually, this implies the value function of a reward is concave. However, empirical evidence suggests that the value functions are often S-shaped. Such S-shaped value functions can be generated by various sources, such as reference dependence (Kahneman and Tversky, 1979) and efficient coding of numbers (Louie and Glimcher, 2012). Through the AMD model, we provide a novel account for S-shaped value function based on the insight that larger rewards capture more attention.

Consider a reward of level x delivered at period T. Its value function can be represented by $U(x,T) = w_T(x)u(x)$. We assume $u' > 0$, $u'' < 0$, and w_T is determined by Equation (4). w_T is increasing with x as the DM tends to pay more attention to larger rewards. Both functions $u(x)$ and $w_T(x)$ are concave in x; so when x is small, they both grow fast. At some conditions, it is possible that the product of the two functions is convex in x when x is small enough. We derive the conditions for the S-shaped value function in Proposition 4.

Proposition 4: *Suppose* $T \geq 1$, $\frac{d}{dx}\left(\frac{1}{v'(x)}\right)$ *is continuous in* $(0, +\infty)$*, then:*

(a) *There exists a threshold* $\bar{x} \in \mathbb{R}_{\geq 0}$ *such that* $U(x,T)$ *is strictly concave in* x *when* $x \in [\bar{x}, +\infty)$.

(b) *If* $\frac{d}{dx}\left(\frac{1}{v'(x)}\right)$ *is right-continuous at* $x = 0$ *and* $\frac{d}{dx}\left(\frac{1}{v'(0)}\right) < 1$*, there exists* $x^* \in (0, \bar{x})$ *such that, for any* $x \in (0, x^*)$*,* $U(x,T)$ *is strictly convex in* x.

(c) *There exists a threshold* λ^* *and an interval* (x_1, x_2) *such that, if* $\lambda < \lambda^*$*, for any* $x \in (x_1, x_2)$*,* $U(x,T)$ *is strictly convex in* x*, where* $\lambda^* > 0$ *and* $(x_1, x_2) \subset (0, \bar{x})$.

The proof of Proposition 4 is in Appendix D. Proposition 4 implies, if the derivative of $\frac{1}{v'(x)}$ converges to a small number when $x \to 0^+$, or the unit cost of attention reallocation λ is small enough, the value function $U(x,T)$ will be an S-shaped in some interval of x. Figure 2(b) demonstrates two examples of this S-shaped value function.

4.6 Intertemporal Correlation Aversion

Consider a DM facing two lotteries, $L1$ and $L2$. For lottery $L1$, she can receive £100 today and £100 in 30 weeks with probability 1/2, and receive £3 today and £3 in 20 weeks with probability 1/2. For lottery $L2$, she can receive £3 today and £100 in 30 weeks with probability 1/2, and receive £100 today and £3 in 20 weeks with probability 1/2. In lottery $L1$, rewards delivered at the two different periods are positively correlated; in lottery $L2$, those rewards are negatively correlated. The expected discounted utility theory predicts the DM is indifferent between the two lotteries. However, recent studies find the evidence of *intertemporal correlation aversion* (Andersen et al., 2018; Rohde and Yu, 2023). That is, people often prefer lottery $L2$ to $L1$.[12]

For the above example, intertemporal correlation aversion can be explained by the AMD model as follows. The model assumes the allocation of decision weights is within each certain reward sequence, which implies the DM would first aggregate values over time in each state and then solve the certainty equivalence. For simplicity, suppose there are only two periods. In the state that the DM receives £3 in two periods, suppose she allocates decision weight w to the first period and $1-w$ to the second period. Note when $u(s_0) = u(s_1) = ... = u(s_T)$, the AMD factor for every period t remains the same as its default discount factor d_t. So, in the state that the DM receives £100 in two periods, the decision weights is also the same as w and $1-w$. In the state that the DM can receive £100 in the first period and £3 in the second period, the reward of £100 can capture more attention so that its decision weight, say w', is greater than w. Similarly, in the state that the DM receives £3 earlier and then £100, the decision weight for the later reward £100, say $1-w''$, is greater than $1-w$. Therefore, the value of the lottery in which rewards are positively correlated, can be represented by $0.5 \cdot u(3) + 0.5 \cdot u(100)$. Whereas, for the lottery in which rewards are negatively correlated, the value can be represented by $0.5(1-w'+w'') \cdot u(3) + 0.5(1-w''+w') \cdot u(100)$. Given $(1-w'')+w' > 1-w+w = 1$, the decision weight assigned to $u(100)$, which is $0.5(1-w''+w')$,

[12] For theoretical analysis about intertemporal correlation aversion, please see Epstein (1983), Epstein and Zin (1989), Weil (1990), Bommier (2005), and Bommier et al. (2017). The AMD model takes a similar form to the class of models defined by Epstein (1983). A key feature of such models is that the discount factor for future utilities is dependent on the utility achieved in the current period.

should be greater than 0.5. As a result, the DM prefer the latter lottery than the former lottery.

In a more general setting, whether the AMD model can robustly produce intertemporal correlation aversion is influenced by λ. To see this, we adopt the same definition of intertemporal correlation aversion as Bommier (2005). Let (s_1, s_2) denote the result of a lottery in which the DM can receive reward s_1 in period t_1 and then reward s_2 in period t_2, where $t_2 > t_1 \geq 0$. The results of each lottery is of the same length of sequence. $L1$ generates (x_s, y_s) and (x_l, y_l) with equal chance, L_2 generates (x_s, y_l) and (x_l, y_s) with equal chance, $x_l > x_s > 0$, $y_l > y_s > 0$. By Proposition 5, we show that in this setting, we can always find a λ that makes the DM intertemporal correlation averse.

Proposition 5: *Suppose $U(L1), U(L2)$ are the values of lotteries $L1$ and $L2$ calculated based on the AAD model. For any $x_l > x_s > 0$, $y_l > y_s > 0$, any default discount factors, and any time length of lottery results, there exists a threshold λ^{**}* such that for all unit co*st of attention reallocation $\lambda \in (\lambda^{**}, +\infty)$, we have $U(L1) < U(L2)$, i.e. the DM performs intertemporal correlation aversion.*

The proof of Proposition 5 is in Appendix E. The threshold λ^{**} is jointly determined by x_l, y_l, y_s, as well as the default discount factors for rewards delivered at t_1 and t_2. Notably, when $\lambda \leq \lambda^{**}$, the DM may be intertemporal correlation seeking under some conditions.[13] This suggests a potentially new mechanism for intertemporal correlation aversion, that is, DM performs intertemporal correlation aversion because she attends more to larger rewards while attention reallocation is very costly.

4.7 Concentration Bias

In the existing literature, one approach to applying attention in explaining intertemporal chocies is the focus-weighted utility model, proposed by Kőszegi and Szeidl (2013). In their model, Kőszegi and Szeidl assume that within a reward sequence, the decision weight for a

[13] To validate, one can set $u(x_s) = 5$, $u(x_l) = 10$, $u(y_s) = 1$, $u(y_l) = 3$. Suppose the results of each lottery contain only two periods, t_1 and t_2, and the default discount factors are uniformly distributed, i.e. $d_{t_1} = d_{t_2}$. In this case, setting $\lambda = 1$ would generate intertemporal correlation seeking, while setting $\lambda = 100$ would generate intertemporal correlation aversion.

reward is increasing with the difference of that reward from its reference point. If we take zero reward as the reference point for every period, then this assumption is also true for the AMD model. The key difference between the AMD model and the focus-weighted utility model is that, in the latter, the decision weights are uncorrelated with each other and depend only on the choice set, whereas we assume increasing one decision weight can reduce some other decision weights within the sequence. The focus-weighted utility model predicts that people may perform a *concentration bias* and Dertwinkel-Kalt et al. (2022) find supportive evidence for this prediction. In this subsection, we show that the AMD model provides an alternative way to generate the concentration bias. Furthermore, we identify the conditions in terms of impatience and attention reallocation cost (which are beyond the predictions of Kőszegi and Szeidl (2013)) for concentration bias under the AMD model.

To illustrate the concentration bias, consider a DM with a consumption budget £100 to spend over four days (from period 0 to period 3). Suppose the DM has two options: concentrating all consumption at period 0, or splitting the consumption evenly over four periods. The concentration bias implies that she would prefer the first option to the second. We denote the first option as sequence [100,0,0,0] and the second option as sequence [25,25,25,25]. For convenience, we assume the default discount factor for any period t is $d_t = \delta^t$ and $0 < \delta < 1$. According to the AMD model, the DM prefers the first option if and only if

$$\frac{e^{u(100)/\lambda}}{e^{u(100)/\lambda} + \delta + \delta^2 + \delta^3} \cdot u(100) > u(25) \tag{5}$$

Obviously, this inequality holds only when δ or λ is small enough, which implies the DM should be very impatient or she can reallocate attention at a very low cost. Notably, under the AMD model, it is also possible that the DM prefers concentrating all consumption at the final period, i.e. the sequence [0,0,0,100], to the second option [25,25,25,25]. In this case, the decision weight multiplied by $u(100)$ in the inequality would become $\frac{\delta^3 \exp\{u(100)/\lambda\}}{1+\delta+\delta^2+\delta^3 \exp\{u(100)/\lambda\}}$. Then, the inequality holds only when both δ is large enough and λ is small enough. Both cases are in line with the claim in Kőszegi and Szeidl (2013) and Dertwinkel-Kalt et al. (2022) that the concentration bias can make people behave too impatiently or too patiently.

Next, we derive the conditions for concentration bias in a general optimal-decision setting.

From the above example we can draw an intuition that, in a general case, to observe the concentration bias we require the unit cost of attention reallocation λ to be small. We show in Proposition 6 that this intuition is true. Suppose the DM has a consumption budget m ($m > 0$) to spend over T periods. Let reward sequence $s_{0\to T}$ represent her consumption plan at period 0, and let $A \subset \mathbb{R}^{T+1}_{\geq 0}$ denote her alternative space. In period 0, DM wants to find a $s_{0\to T}$ to solve the optimization problem:

$$\max_{s_{0\to T}\in A} \sum_{t=0}^{T} w_t u(s_t) \tag{5}$$

where

$$A = \left\{ s_{0\to T} \middle| \sum_{t=0}^{T} s_t = m,\ \forall t : s_t \geq 0 \right\} \tag{6}$$

and w_t is the AMD factor for consumption in period t, subject to default discount factor d_t. For $s \in [0, m]$, we have $0 < u'(s) < \infty$, $-\infty < u''(s) < 0$. Henceforth, we denote the optimization problem in Equation (5) by $\mathcal{O}(m, A, \{d_t\}_{t=0}^T)$. By Proposition 6, we know as long as the DM is impatient (for all period $t < T$, we have $d_t > d_{t+1} > 0$) and λ is small enough, her optimal consumption plan is to consume all of m immediately. The proof of Proposition 6 is in Appendix F.

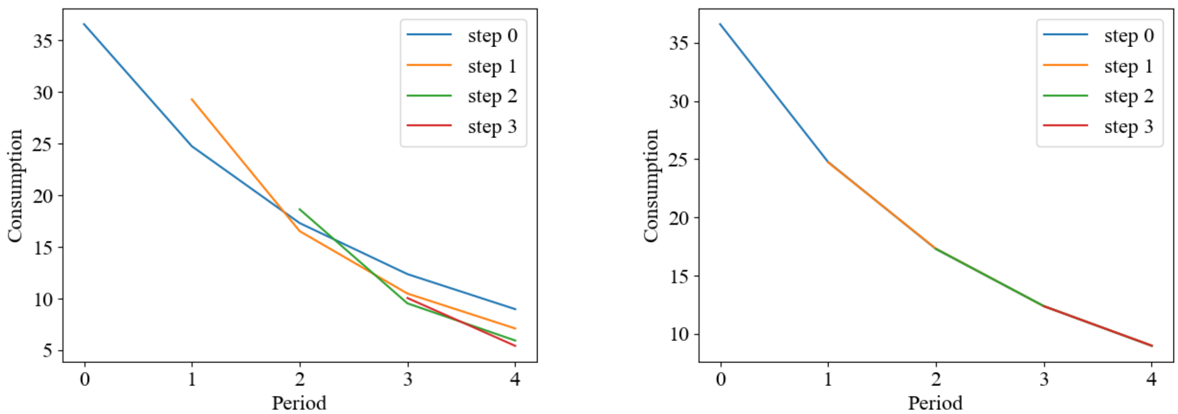
In Section 2, we state that the unit cost of attention reallocation λ has a potential link to cognitive uncertainty. If the DM is highly certain that the default discount factors $\{d_t\}_{t=0}^T$ truly capture her preference in the local context, she may inhibit the learning about value signals and thus λ should be high. This link is also helpful for understanding the relationship between λ and concentration bias: when allocating a budget over time, if the DM is totally uncertain about what to do (so λ is very small), she may simply concentrate her budget into one period and consume it all.

Proposition 6: *Suppose the DM faces the planning problem $\mathcal{O}(m, A, \{d_t\}_{t=0}^T)$, and for all period $t < T$*, we have $d_t > d_{t+1} > 0$. *There exists a threshold $\underline{\lambda} > 0$ such that for any $\lambda \leq \underline{\lambda}$, her optimal consumption plan is to concentrate all consumption at period 0.*

4.8 Inconsistent Planning and Learning

An extensive amount of research evidence suggests that people often exhibit time-inconsistent behaviors in their daily lives (Ericson and Laibson, 2019). For example, they often consume more than they originally planned, and procrastinate on effortful tasks. In a general sense, such behaviors can be termed present-biased behaviors. Several theories have been proposed for explaining the present-biased behaviors, such as dual-system preferences (Laibson, 1997), naivete (O'donoghue and Rabin, 1999), reference dependence (Kőszegi and Rabin, 2009), and optimistic beliefs (Brunnermeier et al., 2017). Based on the AMD model, we can provide an alternative explanation for these behaviors: in dynamic decision-making, people update their default discount factors over time. In more intuitive terms, during each decision step, people will reference their past experiences when allocating attention. If, in the last step, they allocate too much attention to a particular period, they may then take this as a given or default status in the following step and continue to add attention to it. Compared with the existing theoretical explanations, our explanation is built on learning and memory. To perform present-biased behaviors, the DM should recall her mental status at the end of the last step and learn how to allocate attention accordingly. A memoryless DM may perform the reverse behavior. We analyse this with the consumption planning problem in the last subsection.

Again, we suppose the DM has a budget m for consumption. She needs to allocate it over T periods ($T \geq 2$) and the end of sequence is a fixed date. In period 0, her default discount factors $\{d_t\}_{t=0}^{T}$ satisfy $d_t > d_{t+1} > 0$, where $t < T$. When making consumption plan, she would initially weight consumption of period 1 higher than consumption of any other future period. So, she may naturally plan to consume more in period 1 than in period $t = 2, \ldots, T$. This in turn, makes her relatively attend more to period 1. Her optimal consumption plan of period 0 should result in $w_1/w_t > d_1/d_t$ for each $t = 2, \ldots, T$. When arriving in period 1, we assume the DM will use the AMD factors determined in the last step as the new default discount factors. This will lead her to initially weight consumption of period 1 even higher, and therefore create a motive for over-consumption. In the end, her actual consumption in period 1 will be higher than what has been planned in the last step. Such a trend could continue until she reaches the final period or runs out of money.



(a) with learning

(b) without learning

Figure 3: Optimal Consumption Plan Over Time

Note: In step 0, the DM allocates consumption budget $m = 100$ over periods 0-4. In step 1, she allocates the remaining consumption over periods 1-4, and so on. "with learning" means the DM updates default discount factors per step; "without learning" means the default discount factors are constant over time. $d_t^0 = 0.9^t$, $u(x) = x^{0.6}$, $\lambda = 70$. Each optimization problem is solved by projection gradient descent method.

Figure 3 illustrates the relationship between time inconsistency and learning. In the figure, we set $u(x) = x^{0.6}$, $T = 4$, $\lambda = 70$. At the first step of planning, the DM has a budget of $m = 100$ for consumption and her default discount factors are exponential: $d_t^0 = 0.9^t$. Under the condition "with learning", which means the DM takes AMD factors of the optimal consumption plan as default discount factors for the next step, we observe a tendency for over-consumption. Under the condition "without learning", which means the default discount factors are constant over time, the DM's behavior is closer to being time-consistent, and in each step, the actual consumption is slightly lower than the consumption planned one step earlier.[14]

We state these results formally in Proposition 7. We focus on the transition from period 0 to period 1, but the conclusions can be expanded to other periods. When the DM plans her consumption in period j ($j = 0, 1$), the default discount factor for period t is termed d_t^j and the corresponding AMD factor is termed w_t^j, where $t \geq j$. In the period 0's optimal

[14]In one simulation under the condition "without learning", in step 0, the DM plans to consume 24.76 in period 1, and she ends up consuming 24.74. Also, in step 1, she plans to consume 17.32 in period 2, and she ends up consuming 17.31.

consumption plan, the DM plans to consume s_t^* in period t, and in the period 1's optimal plan, she plans to consume s_t^{**}. The alternative space is A in period 0 and becomes A' in period 1. Notably, to make the time-inconsistency results hold, the DM should not concentrate all consumption into period 0. From Proposition 6, we know that it requires λ to be large enough. The proof of Proposition 7 is in Appendix G.

Proposition 7: *Suppose the DM faces the planning problem $\mathcal{O}(m, A, \{d_t^0\}_{t=0}^T)$ in period 0 and $\mathcal{O}(m - s_0^*, A', \{d_t^1\}_{t=1}^T)$ in period 1. There exists a threshold $\bar{\lambda} > \underline{\lambda}$ such that when $\lambda \in [\bar{\lambda}, +\infty)$, the following is true:*

(a) *In period 0, for any sequence $s_{0\to T}^* \in \{s_{0\to T} | s_{0\to T} \in A, \forall t < T : s_t > s_{t+1} > 0\}$, there exist a specification of default discount factors $\{d_t^0\}_{t=0}^T$,* where *for all $t < T$ we have $d_t^0 > d_{t+1}^0 > 0$, such that $s_{0\to T}^*$ is the optimal consumption plan.*

(b) *Given $s_{0\to T}^*$ as the period 0's optimal consumption plan, if in period 1, for all $t \geq 1$ we have $d_t^1 = d_t^0$, then there must be $s_1^{**} < s_1^*$; if instead we have $d_t^1 = w_t^0$, then $s_1^{**} > s_1^*$.*

In the part (a) of Proposition 7, we select an interior solution for the consumption planning problem in period 0. The part (b) suggests that if we do not take into account the updating of default discount factors, the DM would perform under-consumption behavior over time. In reverse, assuming the default discount factors are updated based on the AMD factors of the most recent consumption plan will result in over-consumption behavior. The former reflects the behavior of a DM "without learning", while the latter reflects how the DM would behave "with learning".

5 Discussion

5.1 Selection of Sequence Length

In our model, the attention a DM can allocate to each time period is affected by sequence length. When a sequence contains more periods, the average attention she can allocate to each period naturally decreases. In Section 4.1, we discuss how this can generate the hidden zero effect. Nevertheless, in reality, we usually cannot observe the actual sequence length perceived by the people. Researchers using our model need to make their own assumptions about sequence length. In this subsection, we discuss two issues related to setting these assumptions and provide recommendations.

The first issue is about the unit of time. A given duration can be represented as sequences of different lengths depending on the time unit used, such as months or days. For example, "receive £10 in 1 month" and "receive £10 in 30 days" essentially mean the same thing, but the latter seems to involve more units of time. When represented in sequence, the former can be represented by [0,10], and the latter can be [0,0,...,0,10], with 30 zeros before the 10. In "standard" discounting models, the number of zeros before the 10 has no effect on the valuation of the sequence. Whereas, the AMD model predicts that, when the reward sequence is described with more time units, the DM may perceive the waiting period for the £10 payment as longer, thereby discounting its value to a greater extent.

As an illustration, in the exponential discounting model, suppose each period represents a month and the discount factor for period t is δ^t. The value of sequence [0,10] could be written as $\delta \times u(10)$. For the sequence [0,0,...,0,10], which includes 30 zeros where each period represents a day, we can convert the monthly discount rate to a daily discount rate. Thus, the discount factor for period t becomes $\delta^{\frac{1}{30}\cdot t}$, and the value of sequence is still the same. However, this approach does not apply to the AMD model. In the AMD model, suppose the default discount factors are exponential, as we have stated. According to Equation (4), the discount factor for the £10 payment is $1/(1 + G(T)e^{-v(10)})$. And in the former case $G(T) = \delta^{-1}$, while in the latter case $G(T) = \delta^{-1} + \delta^{-\frac{29}{30}} + ... + \delta^{-\frac{1}{30}}$. If we use the same parameterization in both cases, then in the latter case, the value of the £10 payment should be more discounted.[15]

The AMD model's prediction that people perceive the sequence with more time units as longer is also consistent with the numerosity effect. This effect refers to the tendency to overestimate quantity based on the number of units presented (Pelham et al., 1994) and

[15]If, in reverse, we elicit the DM's time preference from experiments and model it using the sequence with 30 zeros, then the estimate for δ should be greater compared to the sequence with only one zero.

has been extensively studied in various fields, particularly in consumer behavior (Zhang and Schwarz, 2012; Monga and Bagchi, 2012).[16] To the best of our knowledge, there is currently no clear evidence of the numerosity effect in intertemporal choice. Nonetheless, this can be a potential direction for future research. To capture such an effect, we recommend that researchers choose sequence lengths that match the time units presented to participants. For the given example, if a reward sequence is expressed as "receive £10 in 1 month", researchers would better represent it as [0,10] rather than a sequence with 30 zeros.

The second issue is about the end of sequence. In our explanation of the model's implications, an implicit assumption is that each sequence terminates at the period when the final positive reward is delivered. For example, for a sequence described by "receive £10 in 1 month", we represent it as [0,10] rather than [0,10,0,0]. This assumption is sufficient for most of the well-established psychological effects in intertemporal choice. However, under this assumption the model is incapable of distinguishing between two constant sequences of different lengths. As an illustration, consider a choice between two options: (A) "receive £10 now"; and (B) "receive £10 now, plus £10 in 1 month, plus £10 in 2 months". According our implicit assumption, option (A) can be represented by a single-period sequence [10] and option (B) can be represented by [10,10,10]. In reality, people will definitely prefer option (B) to (A). But, as the AMD model assumes a constant sum of decision weights, such representations would result in both options being equally valued at $u(10)$.

We propose two remedies for this issue. The simplest approach is to assume that the DM always takes into account one additional period when processing each sequence. For example, she represents option (A) as [10,0] and option (B) as [10,10,10,0]. Then, in option (B), the attention jointly captured by the three 10s must be greater than the attention captured by the single 10 in option (A).[17] In Appendix H, we show that adding a period to the end of each sequence does not affect the implications of the model that we have discussed for choice between sequences. In addition, a more complex way to address the issue is to assume the

[16]Some of such studies, such as Monga and Bagchi (2012), state that the unit itself can also influence decisions. For example, people may perceive "month" as longer than "day". So, the sequence that takes each month as a period may correspond to a smaller δ.

[17]Set $u(x) = x^{0.6}$, $d_t = 0.9^t$, $\lambda = 2$. According to the AMD model, the value of option (A) is 3.55 and that of option (B) is 3.84.

sequences ends at a random position. Given that we do not know how people would actually perceive the sequence length, for option (B), we could assume the end of sequence follows a distribution, ranging from the third period to infinity. Researchers could set the class of distribution and then estimate its parameters from the data, which would allow them to accommodate a wider range of phenomena. We consider this as a potential direction for future research.

To understand why including a zero (or zeros) at the end of a sequence is psychologically plausible, we can examine how people would perceive information that is not explicitly mentioned in each option. Adding a certain period to a sequence implies the DM must pay some attention to the reward delivered in that period, in order to process its value. For option (B), the DM knows there will be no reward delivered after period 3, although this is not explicitly described. On the one hand, this prohibits the DM from allocating more attention to perceive the value of rewards delivered in the far future (such rewards are not included in the sequence). On the other hand, "no reward will be delivered after period 3" could be seen as additional information beyond what is provided in option (B). In other words, the DM needs to at least allocate some attention to processing such an information, and adding some periods of no reward is a simple way to capture this.

5.2 Relation to Other Intertemporal Choice Models

In the existing literature, the theory most similar to AMD is the salience theory, originally proposed by Bordalo et al. (2012). A recent review (Bordalo et al., 2022) summarizes the latest developments in the theory. According to that theory, the "salience" of an element in a sequence is increasing with its deviation from a reference point. Within the scope of this paper, we could set the reference point to zero. Thus, the salience theory also predicts that people pay more attention to large rewards and less attention to small rewards. Moreover, in Section 4.8, we consider the role of memory and learning in generating time-inconsistent behavior under the AMD model. This is also related to Bordalo et al. (2020), which incorporates a memory-based reference point in the salience theory.

Nevertheless, unlike the AMD model, the salience theory does not impose any restrictions on the sum of decision weights; it merely re-normalizes the decision weights. Through appying that theory to intertemporal choice, the value of a sequence could be represented by $U = \sum_{t=0}^{T} \pi_t w_t u(s_t)$, where π_t is some "standard" discount factor (e.g. the exponential discount factor $\pi_t = \delta^t$), and the re-normalization weight w_t satisfies $\sum_{t=0}^{T} w_t = 1$. As a result, given a sequence [100, 4], additional two additional 4s to the end to make it [100, 4, 4, 4], would have different consequences under salience theory compared to the AMD model. In the salience theory, this may make the element 100 more "salient", as the total of decision weights increases from $\pi_0 + \pi_1$ to $\pi_0 + \pi_1 + \pi_2 + \pi_3$, and the 100 can capture the majority of this total. Whereas, in the AMD model, this operation always reduces the attention allocated to each element.

Some other models also claim to incorporate attentional mechanisms in intertemporal choice. For example, the focus-weighted utility theory (Kőszegi and Szeidl, 2013) also assumes that people discount the value of a large reward less because they focus on it more (if taking zero as the reference point). Despite that, the focus-weighted utility theory does not perform any normalization of decision weights. Steiner et al. (2017) examine the role of rational inattention in dynamic risky decision-making, and use it to account for inertia and status quo bias. That theory is grounded in the instrumental utility of information while our model is grounded in its hedonic and cognitive utilities (see Section 3). In psychology, some researchers use the attentional drift-diffusion model (aDDM, see Krajbich et al., 2010) to analyse choices between SS and LL (Amasino et al., 2019). In the aDDM, the value of an option is subject to an evidence accumulation process. If the DM pays more attention to an option (or attribute), she can accumulate more evidence about its value, leading her to value it more. So far, studies about aDDM have yet to examine how attention shifts within a sequence, but they may relate to the cost of attention reallocation, which controls the rate of information acquisition in the AMD model.

Our model extends the theories of endogenous time preference, which can be traced back to Uzawa (1968). Early theories, such as Uzawa (1968) and Becker and Mulligan (1997), assume that increasing a reward can lead to all sebsequent rewards in the same sequence being discounted more. However, they do not clearly address how changes in future rewards might affect the weighting of the rewards delivered earlier – an issue crucial for explaining empirical findings like the violation of dominance and hidden-zero effect. Some theories (e.g. Fudenberg and Levine, 2006; Noor and Takeoka, 2022) assume that time preferences are mediated by the cognitive cost of paying attention to the future. We borrowed this idea in the axiomatic characterization of the AMD model by using a modified version of the optimal discounting framework (Noor and Takeoka, 2022, 2024). We demonstrated that AMD is the only the model that satisfies both Sequential Outcome-Betweenness and Sequential Bracket-Independence within this general framework.

Moreover, the trade-off models for intertemporal choice provides an alternative approach to model choices between sequences (Read and Scholten, 2012, Scholten et al. (2016), Scholten et al. (2024)). Such models assume that the DM trades off the average utility of receiving rewards against the average disutility of waiting. Yet, they can still be expressed in a form similar to endogenous time preference. For example, in Scholten et al. (2016), the value of a sequence can be represented by $U = \sum_{t=0}^{T} w_t u(s_t)$, where $w_t = 1 - \kappa \frac{T(T+1)(T-t+1)}{\sum_{\tau=0}^{T}(T-\tau+1)u(s_\tau)}$ and κ is a positive parameter. Under this specification of w_t, increasing s_t can increase the decision weights for all rewards in the sequence, while in the AMD model, this increases the deicison weight for s_t but reduces the decision weights for all the other rewards.

5.3 Possible Improvements

There are several ways to improve our model. First, the DM may not just allocate attention within a sequence, but also across sequences. As a result, within the same choice set, the sum of decision weights for one sequence may be smaller than that for another. Researchers interested in this direction might also refer to Manzini and Mariotti (2014) and Gossner et al. (2021). Second, as suggested by the aDMM, when the DM focuses more on a reward, she may not only assign it greater weight, but also accelerate the rate at which she learns about its value. In our model, the learning rate is controlled by a unified parameter λ. But in reality, this parameter might vary across different rewards as they capture different levels of attention. Leong et al. (2017) provide an example of analyzing both consequences of attention simultaneously. Third, in the existing literature, the softmax function is usually used to model stochastic choices. Our model examines how this function can be used to study mental representations of reward sequences. Future research could explore its use in risk perception, and integrate these fields to offer a unified behavioral economic framework for analyzing dynamic risky decision-making problems.

Reference

Amasino, D. R., Sullivan, N. J., Kranton, R. E., and Huettel, S. A. (2019). Amount and time exert independent influences on intertemporal choice. *Nature human behaviour*, 3(4):383–392.

Andersen, S., Harrison, G. W., Lau, M. I., and Rutström, E. E. (2018). Multiattribute utility theory, intertemporal utility, and correlation aversion. *International Economic Review*, 59(2):537–555.

Anderson, B. A., Laurent, P. A., and Yantis, S. (2011). Value-driven attentional capture. *Proceedings of the National Academy of Sciences*, 108(25):10367–10371.

Awh, E., Belopolsky, A. V., and Theeuwes, J. (2012). Top-down versus bottom-up attentional control: A failed theoretical dichotomy. *Trends in cognitive sciences*, 16(8):437–443.

Becker, G. S. and Mulligan, C. B. (1997). The endogenous determination of time preference. *The Quarterly Journal of Economics*, 112(3):729–758.

Blackwell, D. et al. (1951). Comparison of experiments. In *Proceedings of the second Berkeley symposium on mathematical statistics and probability*, volume 1, page 26.

Bleichrodt, H., Rohde, K. I., and Wakker, P. P. (2008). Koopmans' constant discounting for intertemporal choice: A simplification and a generalization. *Journal of Mathematical Psychology*, 52(6):341–347.

Bommier, A. (2005). Risk aversion, intertemporal elasticity of substitution and correlation aversion. Technical report, ETH Zurich.

Bommier, A., Kochov, A., and Le Grand, F. (2017). On monotone recursive preferences. *Econometrica*, 85(5):1433–1466.

Bordalo, P., Gennaioli, N., and Shleifer, A. (2012). Salience theory of choice under risk. *The Quarterly journal of economics*, 127(3):1243–1285.

Bordalo, P., Gennaioli, N., and Shleifer, A. (2020). Memory, attention, and choice. *The Quarterly journal of economics*, 135(3):1399–1442.

Bordalo, P., Gennaioli, N., and Shleifer, A. (2022). Salience. *Annual Review of Economics*, 14(1):521–544.

Brunnermeier, M. K., Papakonstantinou, F., and Parker, J. A. (2017). Optimal time-inconsistent beliefs: Misplanning, procrastination, and commitment. *Management Science*, 63(5):1318–1340.

Caplin, A., Dean, M., and Leahy, J. (2022). Rationally inattentive behavior: Characterizing and generalizing shannon entropy. *Journal of Political Economy*, 130(6):1676–1715.

Chelazzi, L., Perlato, A., Santandrea, E., and Della Libera, C. (2013). Rewards teach visual selective attention. *Vision research*, 85:58–72.

Collins, A. G. and Frank, M. J. (2014). Opponent actor learning (opal): modeling interactive effects of striatal dopamine on reinforcement learning and choice incentive. *Psychological review*, 121(3):337.

Daw, N. D., O'doherty, J. P., Dayan, P., Seymour, B., and Dolan, R. J. (2006). Cortical substrates for exploratory decisions in humans. *Nature*, 441(7095):876–879.

DeJarnette, P., Dillenberger, D., Gottlieb, D., and Ortoleva, P. (2020). Time lotteries and stochastic impatience. *Econometrica*, 88(2):619–656.

Della Libera, C. and Chelazzi, L. (2009). Learning to attend and to ignore is a matter of gains and losses. *Psychological science*, 20(6):778–784.

Dertwinkel-Kalt, M., Gerhardt, H., Riener, G., Schwerter, F., and Strang, L. (2022). Concentration bias in intertemporal choice. *The Review of Economic Studies*, 89(3):1314–1334.

Enke, B., Graeber, T., and Oprea, R. (2023). Complexity and hyperbolic discounting.

Epstein, L. G. (1983). Stationary cardinal utility and optimal growth under uncertainty. *Journal of Economic Theory*, 31(1):133–152.

Epstein, L. G. and Zin, S. E. (1989). Substitution, risk aversion, and the temporal behavior of consumption. *Econometrica*, 57(4):937–969.

Epstein, L. G. and Zin, S. E. (1991). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis. *Journal of Political Economy*, 99(2):263–286.

Ericson, K. M. and Laibson, D. (2019). Intertemporal choice. In *Handbook of behavioral economics: Applications and foundations 1*, volume 2, pages 1–67. Elsevier.

FitzGerald, T. H., Friston, K. J., and Dolan, R. J. (2012). Action-specific value signals in reward-related regions of the human brain. *Journal of Neuroscience*, 32(46):16417–16423.

Friston, K., FitzGerald, T., Rigoli, F., Schwartenbeck, P., and Pezzulo, G. (2017). Active inference: a process theory. *Neural computation*, 29(1):1–49.

Fudenberg, D. and Levine, D. K. (2006). A dual-self model of impulse control. *American economic review*, 96(5):1449–1476.

Gabaix, X. and Laibson, D. (2017). Myopia and discounting. Technical report, National bureau of economic research.

Galai, D. and Sade, O. (2006). The "ostrich effect" and the relationship between the liquidity and the yields of financial assets. *The Journal of Business*, 79(5):2741–2759.

Gershman, S. J. and Bhui, R. (2020). Rationally inattentive intertemporal choice. *Nature communications*, 11(1):3365.

Gossner, O., Steiner, J., and Stewart, C. (2021). Attention please! *Econometrica*, 89(4):1717–1751.

Gottlieb, J. (2012). Attention, learning, and the value of information. *Neuron*, 76(2):281–295.

Gottlieb, J., Oudeyer, P.-Y., Lopes, M., and Baranes, A. (2013). Information-seeking, curiosity, and attention: computational and neural mechanisms. *Trends in cognitive sciences*, 17(11):585–593.

Haarnoja, T., Tang, H., Abbeel, P., and Levine, S. (2017). Reinforcement learning with deep energy-based policies. In *International conference on machine learning*, pages 1352–1361. PMLR.

Herstein, I. N. and Milnor, J. (1953). An axiomatic approach to measurable utility. *Econometrica*, 21(2):291–297.

Hickey, C., Chelazzi, L., and Theeuwes, J. (2010). Reward changes salience in human vision via the anterior cingulate. *Journal of Neuroscience*, 30(33):11096–11103.

Houthooft, R., Chen, X., Duan, Y., Schulman, J., De Turck, F., and Abbeel, P. (2016). Vime: Variational information maximizing exploration. *Advances in neural information processing systems*, 29.

Itti, L. and Baldi, P. (2009). Bayesian surprise attracts human attention. *Vision research*, 49(10):1295–1306.

Jahfari, S. and Theeuwes, J. (2017). Sensitivity to value-driven attention is predicted by how we learn from value. *Psychonomic bulletin & review*, 24(2):408–415.

Jiang, C.-M., Sun, H.-M., Zhu, L.-F., Zhao, L., Liu, H.-Z., and Sun, H.-Y. (2017). Better is worse, worse is better: Reexamination of violations of dominance in intertemporal choice. *Judgment and Decision Making*, 12(3):253–259.

Kahneman, D. and Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, 47(2):263–292.

Karlsson, N., Loewenstein, G., and Seppi, D. (2009). The ostrich effect: Selective attention to information. *Journal of Risk and uncertainty*, 38:95–115.

Kőszegi, B. and Rabin, M. (2009). Reference-dependent consumption plans. *American Economic Review*, 99(3):909–936.

Kőszegi, B. and Szeidl, A. (2013). A model of focusing in economic choice. *The Quarterly journal of economics*, 128(1):53–104.

Krajbich, I., Armel, C., and Rangel, A. (2010). Visual fixations and the computation and comparison of value in simple choice. *Nature neuroscience*, 13(10):1292–1298.

Laibson, D. (1997). Golden eggs and hyperbolic discounting. *The Quarterly Journal of Economics*, 112(2):443–478.

Leong, Y. C., Radulescu, A., Daniel, R., DeWoskin, V., and Niv, Y. (2017). Dynamic interaction between reinforcement learning and attention in multidimensional environments. *Neuron*, 93(2):451–463.

Loewenstein, G. and Prelec, D. (1992). Anomalies in intertemporal choice: Evidence and an interpretation. *The Quarterly Journal of Economics*, 107(2):573–597.

Louie, K. and Glimcher, P. W. (2012). Efficient coding and the neural representation of value. *Annals of the New York Academy of Sciences*, 1251(1):13–32.

Magen, E., Dweck, C. S., and Gross, J. J. (2008). The hidden zero effect: Representing a single choice as an extended sequence reduces impulsive choice. *Psychological science*, 19(7):648.

Manzini, P. and Mariotti, M. (2014). Stochastic choice and consideration sets. *Econometrica*, 82(3):1153–1176.

Matějka, F. and McKay, A. (2015). Rational inattention to discrete choices: A new foundation for the multinomial logit model. *American Economic Review*, 105(1):272–298.

Monga, A. and Bagchi, R. (2012). Years, months, and days versus 1, 12, and 365: The influence of units versus numbers. *Journal of Consumer Research*, 39(1):185–198.

Niv, Y., Daniel, R., Geana, A., Gershman, S. J., Leong, Y. C., Radulescu, A., and Wilson, R. C. (2015). Reinforcement learning in multidimensional environments relies on attention mechanisms. *Journal of Neuroscience*, 35(21):8145–8157.

Noor, J. and Takeoka, N. (2022). Optimal discounting. *Econometrica*, 90(2):585–623.

Noor, J. and Takeoka, N. (2024). Constrained optimal discounting. *Available at SSRN 4703748*.

Oaksford, M. and Chater, N. (1994). A rational analysis of the selection task as optimal data selection. *Psychological review*, 101(4):608.

O'donoghue, T. and Rabin, M. (1999). Doing it now or later. *American economic review*, 89(1):103–124.

Olafsson, A. and Pagel, M. (2017). The ostrich in us: Selective attention to financial accounts, income, spending, and liquidity. Technical report, National Bureau of Economic Research.

Onay, S. and Öncüler, A. (2007). Intertemporal choice under timing risk: An experimental approach. *Journal of Risk and Uncertainty*, 34:99–121.

Pelham, B. W., Sumarta, T. T., and Myaskovsky, L. (1994). The easy path from many to much: The numerosity heuristic. *Cognitive Psychology*, 26(2):103–133.

Radu, P. T., Yi, R., Bickel, W. K., Gross, J. J., and McClure, S. M. (2011). A mechanism for reducing delay discounting by altering temporal attention. *Journal of the experimental analysis of behavior*, 96(3):363–385.

Read, D., Olivola, C. Y., and Hardisty, D. J. (2017). The value of nothing: Asymmetric attention to opportunity costs drives intertemporal decision making. *Management Science*, 63(12):4277–4297.

Read, D. and Scholten, M. (2012). Tradeoffs between sequences: Weighing accumulated outcomes against outcome-adjusted delays. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 38(6):1675.

Ren, P., Xiao, Y., Chang, X., Huang, P.-Y., Li, Z., Gupta, B. B., Chen, X., and Wang, X. (2021). A survey of deep active learning. *ACM computing surveys (CSUR)*, 54(9):1–40.

Rohde, K. I. and Yu, X. (2023). Intertemporal correlation aversion—a model-free measurement. *Management Science*.

Scholten, M. and Read, D. (2014). Better is worse, worse is better: Violations of dominance in intertemporal choice. *Decision*, 1(3):215.

Scholten, M., Read, D., and Sanborn, A. (2016). Cumulative weighing of time in intertemporal tradeoffs. *Journal of Experimental Psychology: General*, 145(9):1177.

Scholten, M., Walters, D. J., Fox, C. R., and Read, D. (2024). The unified tradeoff model. *Psychological Review*.

Schulman, J., Chen, X., and Abbeel, P. (2017). Equivalence between policy gradients and soft q-learning. *arXiv preprint arXiv:1704.06440*.

Settles, B. (2009). Active learning literature survey.

Sharot, T. and Sunstein, C. R. (2020). How people decide what they want to know. *Nature Human Behaviour*, 4(1):14–19.

Sims, C. A. (2003). Implications of rational inattention. *Journal of monetary Economics*, 50(3):665–690.

Steiner, J., Stewart, C., and Matějka, F. (2017). Rational inattention dynamics: Inertia and delay in decision-making. *Econometrica*, 85(2):521–553.

Strotz, R. (1955). Myopia and inconsistency in dynamic utility maximization. *The Review of Economic Studies*, 23(3):165–180.

Sutton, R. S. and Barto, A. G. (2018). *Reinforcement learning: An introduction*. MIT press.

Takeuchi, K. (2011). Non-parametric test of time consistency: Present bias and future bias. *Games and Economic Behavior*, 71(2):456–478.

Todorov, E. (2009). Efficient computation of optimal actions. *Proceedings of the national academy of sciences*, 106(28):11478–11483.

Uzawa, H. (1968). Time preference, the consumption function, and optimum asset holdings. In *Value, capital and Growth*, pages 485–504. Routledge.

Weil, P. (1990). Nonexpected utility in macroeconomics. *The Quarterly Journal of Economics*, 105(1):29–42.

Zhang, Y. C. and Schwarz, N. (2012). How and why 1 year differs from 365 days: A conversational logic analysis of inferences from the granularity of quantitative expressions. *Journal of Consumer Research*, 39(2):248–259.

Appendix

A. Proof of Proposition 1

We present the proof of sufficiency here. That is, if $\succsim$ has an optimal discounting representation and satisfies Axiom 1-4, then it has an AMD representation.

Lemma 1: *If Axiom 1 and 3 hold, for any* $s_{0\to T}$*, there exist* $w_0, w_1, \dots, w_T > 0$ *such that* $s_{0\to T} \sim w_0 \cdot s_0 + ... + w_T \cdot s_T$*, where* $\sum_{t=0}^{T} w_t = 1$.

Proof: If $T = 1$, Lemma 1 is a direct application of Axiom 3. If $T \geq 2$, for any $2 \leq t \leq T$, there should exist $\alpha_t \in (0, 1)$ such that $s_{0\to t} \sim \alpha_t \cdot s_{0\to t-1} + (1-\alpha_t) \cdot s_t$. By state-independence and reduction of compound alternatives, we can recursively apply such equivalence relations as follows:

$$\begin{aligned} s_{0\to T} &\sim \alpha_{T-1} \cdot s_{0\to T-1} + (1 - \alpha_{T-1}) \cdot s_T \\ &\sim \alpha_{T-1}\alpha_{T-2} \cdot s_{0\to T-2} + \alpha_{T-1}(1 - \alpha_{T-2}) \cdot s_{T-1} + (1 - \alpha_{T-1}) \cdot s_T \\ &\sim ... \\ &\sim w_0 \cdot s_0 + w_1 \cdot s_1 + ... + w_T \cdot s_T \end{aligned} \tag{7}$$

where $w_0 = \prod_{t=0}^{T-1} \alpha_t$, $w_T = 1 - \alpha_{T-1}$, and for $0 < t < T$, $w_t = (1 - \alpha_{t-1}) \prod_{\tau=t}^{T-1} \alpha_\tau$. It is easy to show the sum of $w_0, \dots, w_T$ is equal to 1. *QED*.

Therefore, if Axiom 1 and 3 hold, for any reward sequence $s_{0\to T}$, we can always find a convex combination of all its elements, such that the DM is indifferent between the reward sequence and this convex combination. If $s_{0\to T}$ is a constant sequence, i.e. all its elements are constant, then we can directly assume $\mathcal{W}$ is AMD-style. So henceforth, we discuss whether AMD can apply to non-constant sequences.

By Lemma 2, we show adding a new reward to the end of $s_{0\to T}$ has no impact on the relative decision weights of rewards in the original reward sequence.

Lemma 2: *For any* $s_{0\to T+1}$*, if* $s_{0\to T} \sim \sum_{t=0}^{T} w_t \cdot s_t$ *and* $s_{0\to T+1} \sim \sum_{t=0}^{T+1} w'_t \cdot s_t$*, where* $w_t, w'_t > 0$ and $\sum_{t=0}^{T} w_t = 1$, $\sum_{t=0}^{T+1} w'_t = 1$*, then when Axiom 1-4 hold, we can obtain*

$\frac{w_0'}{w_0} = \frac{w_1'}{w_1} = \ldots = \frac{w_T'}{w_T}$.

Proof: According to Axiom 3, for any $s_{0\to T+1}$, there exist $\alpha, \zeta \in (0,1)$ such that

$$\begin{aligned} s_{0\to T} &\sim \alpha \cdot s_{0\to T-1} + (1-\alpha)\cdot s_T \\ s_{0\to T+1} &\sim \zeta \cdot s_{0\to T} + (1-\zeta)\cdot s_{T+1} \end{aligned} \tag{A1}$$

On the other hand, we drawn on Lemma 1 and set

$$s_{0\to T+1} \sim \beta_0 \cdot s_{0\to T-1} + \beta_1 \cdot s_T + (1-\beta_0-\beta_1)\cdot s_{T+1} \tag{A2}$$

where $\beta_0, \beta_1 > 0$. According to Axiom 4, $1-\zeta = 1-\beta_0-\beta_1$. So, $\beta_1 = \zeta - \beta_0$. This also implies $\zeta > \beta_0$.

According to Axiom 2, we suppose there exists a reward sequence s such that $s \sim \frac{\beta_0}{\zeta}\cdot s_{0\to T-1} + (1-\frac{\beta_0}{\zeta})\cdot s_T$. By Equation (A2) and reduction of compound alternatives, we have $s_{0\to T+1} \sim \zeta\cdot s + (1-\zeta)\cdot s_{T+1}$. Combining Equation (A2) with the second line of Equation (A1) and applying transitivity and state-independence, we obtain $s_{0\to T} \sim \frac{\beta_0}{\zeta}\cdot s_{0\to T-1} + (1-\frac{\beta_0}{\zeta})\cdot s_1$.

We aim to prove that for any $s_{0\to T+1}$, we can obtain $\alpha = \frac{\beta_0}{\zeta}$. We show this by contradiction.

Given the symmetry of α and $\frac{\beta_0}{\zeta}$, we can assume that $\alpha > \frac{\beta_0}{\zeta}$. Consider the case that $s_{0\to T-1} \succ s_T$. By state-independence, for any $c \in \mathbb{R}_{\geq 0}$, we have $(\alpha - \frac{\beta_0}{\zeta})\cdot s_{0\to T-1} + (1-\alpha+\frac{\beta_0}{\zeta})\cdot c \succ (\alpha-\frac{\beta_0}{\zeta})\cdot s_T + (1-\alpha+\frac{\beta_0}{\zeta})\cdot c$. By Axiom 2, there exists $z \in \mathbb{R}_{\geq 0}$ such that $(1-\alpha)\cdot s_T + \frac{\beta_0}{\zeta}\cdot s_{0\to T-1} \sim z$. Given c is arbitrary, we can set $(1-\alpha+\frac{\beta_0}{\zeta})\cdot c \sim z$. By reduction of compound alternatives, we can derive that

$$(\alpha - \frac{\beta_0}{\zeta})\cdot s_{0\to T-1} + (1-\alpha)\cdot s_T + \frac{\beta_0}{\zeta}\cdot s_{0\to T-1} \succ (\alpha-\frac{\beta_0}{\zeta})\cdot s_T + (1-\alpha)\cdot s_T + \frac{\beta_0}{\zeta}\cdot s_{0\to T-1} \tag{8}$$

where the LHS can be rearranged to $\alpha\cdot s_{0\to T-1} + (1-\alpha)\cdot s_T$, and the RHS can be rearranged to $\frac{\beta_0}{\zeta}\cdot s_{0\to T-1} + (1-\frac{\beta_0}{\zeta})\cdot s_1$. They both should be indifferent from $s_{0\to T}$. This results in a contradiction.

Similarly, in the case that $s_T \succ s_{0\to T-1}$, we can also derive such a contradiction. Meanwhile, when $s_{0\to T} \sim s_T$, α and $\frac{\beta_0}{\zeta}$ can be any number within $(0,1)$. In that case, we can directly set $\alpha = \frac{\beta_0}{\zeta}$.

Thus, we have $\alpha = \frac{\beta_0}{\zeta}$ for any $s_{0\to T+1}$, which indicates $\frac{\beta_0}{\alpha} = \frac{\beta_1}{1-\alpha} = \zeta$. We can recursively apply this equality to any sub-sequence $s_{0\to t}$ ($t \leq T$) of $s_{0\to T+1}$, so that the lemma will be proved. *QED*.

Now we move on to prove Proposition 1. The proof contains six steps.

First, we add the constraints $\sum_{t=0}^{T} w_t = 1$ and $w_t > 0$ to the optimal discounting problem for $s_{0\to T}$ so that the problem can accommodate Lemma 1. According to the first-order condition (FOC) of its solution, for all $t = 0, 1, \ldots, T$, we have

$$f_t'(w_t) = u(s_t) + \theta \tag{A3}$$

where θ is the Lagrange multiplier. Given that $f_t'(w_t)$ is strictly increasing, w_t is increasing with $u(s_t) + \theta$. We define the solution as $w_t = \phi_t(u(s_t) + \theta)$.

Second, we add a new reward s_{T+1} to the end of $s_{0\to T}$ and apply Lemma 2 as a constraint on optimal discounting problem. Look at the optimal discounting problem for $s_{0\to T+1}$. For all $t \leq T$, its FOC should take the same form as Equation (A3). Hence, if the introduction of s_{T+1} changes some w_t to w_t' ($w_t' \neq w_t$, where w_t is the solution to optimal discounting problem for $s_{0\to T}$), the only way is through changing the multiplier θ. Suppose introducing s_{T+1} changes θ to $\theta - \Delta\theta$, we have $w_t' = \phi_t(u(s_t) + \theta - \Delta\theta)$.

By Lemma 2, we know $\frac{w_0}{w_0'} = \frac{w_1}{w_1'} = \ldots = \frac{w_T}{w_T'}$. In other words, for $t = 0, 1, \ldots, T$, we have $w_t \propto \phi_t(u(s_t) + \theta - \Delta\theta)$. We can rewrite w_t as

$$w_t = \frac{\phi_t(u(s_t) + \theta - \Delta\theta)}{\sum_{\tau=0}^{T} \phi_\tau(u(s_\tau) + \theta - \Delta\theta)} \tag{A4}$$

Third, we show that in $s_{0\to T}$, if we change each s_t to z_t such that $u(z_t) = u(s_t) + \Delta u$, the decision weights $w_0, \ldots, w_T$ will remain the same. Note $\sum_{t=0}^{T} \phi_t(u(s_t) + \theta) = 1$. It is clear that $\sum_{t=0}^{T} \phi_t(u(z_t) + \theta - \Delta u) = 1$. Suppose changing every s_t to z_t moves θ to θ' and $\theta' < \theta - \Delta u$. Then, we must have $\phi_t(u(z_t) + \theta') < \phi_t(u(z_t) + \theta - \Delta u)$ since $\phi_t(.)$ is strictly increasing. Summing all such decision weights up will result in $\sum_{t=0}^{T} \phi_t(u(z_t) + \theta') < 1$, which contradicts with the constraint that the sum of decision weights is 1. The same contradiction can apply to the case that $\theta' > \theta - \Delta u$. Therefore, changing every s_t to z_t must move θ to $\theta - \Delta u$, and each w_t can only be moved to $\phi_t(u(z_t) + \theta - \Delta u)$, which is exactly the same as the original decision weight.

A natural corollary of this step is that, subtracting or adding a common number to all instantaneous utilities within a reward sequence has no effect on decision weights. What actually matters for determining the decision weights is the difference between these instantaneous utilities. This indicates, for convenience, we can subtract or add an arbitrary number to the utility function.

In other words, for a given $s_{0\to T}$ and s_{T+1}, we can define a new utility function $v(.)$ such that $v(s_t) = u(s_t) + \theta - \Delta\theta$. So, Equation (A4) can be rewritten as

$$w_t = \frac{\phi_t(v(s_t))}{\sum_{\tau=0}^{T} \phi_\tau(v(s_\tau))} \tag{A5}$$

If w_t takes the AMD form under the utility function $v(.)$, i.e. $w_t \propto d_t e^{v(s_t)/\lambda}$, then it should also take the AMD form under the original utility function $u(.)$.

Fourth, we show that in Equation (A4), $\Delta\theta$ has two properties: (i) $\Delta\theta$ is strictly increasing with $u(s_{T+1})$; (ii) suppose $\Delta\theta = \underline{\theta}$ when $u(s_{T+1}) = \underline{u}$ and $\Delta\theta = \bar{\theta}$ when $u(s_{T+1}) = \bar{u}$, where $\underline{u} < \bar{u}$, then for any $l \in (\underline{\theta}, \bar{\theta})$, there exists $u(s_{T+1}) \in (\underline{u}, \bar{u})$ such that $\Delta\theta = l$.

The property (i) can be shown by contradiction. Let $\{w'_t\}_{t=0}^{T+1}$ denote a sequence of decision weights for $s_{0\to T+1}$. Suppose $u(s_{T+1})$ is increased but $\Delta\theta$ is constant. In this case, each of $w'_0, \dots, w'_T$ should also be constant. However, w'_{T+1} should increase as it is strictly increasing with $u(s_{T+1}) + \theta - \Delta\theta$ (as θ is determined only by the optimal discounting problem for $s_{0\to T}$, any operations on s_{T+1} should have no effect on θ). This contradicts with the constraint that $\sum_{t=0}^{T+1} w'_t = 1$. The only way to avoid such contradictions is to set $\Delta\theta$ strictly increasing with s_{T+1}, so that $w'_0, \dots, w'_T$ are decreasing with $u(s_{T+1})$.

For property (ii), note that given $s_{0\to T+1}$ and θ, $\Delta\theta$ is defined as the solution to $\sum_{t=0}^{T+1} \phi_t(u(s_t) + \theta - \Delta\theta) = 1$. For any arbitrary number $l \in (\underline{\theta}, \bar{\theta})$, the proof of property (ii) consists of two stages. First, for period $t = 0, 1, \dots, T$, we need to show $u(s_t) + \theta - l$ is in the domain of $\phi_t(.)$. Second, for period $T+1$, we need to show given any $\omega \in (0,1)$, there exists $u(s_{T+1}) \in \mathbb{R}$ such that $\phi_{T+1}(u(s_{T+1}) + \theta - l) = \omega$.

For the first stage, note $\phi_t(.)$ is the inverse function of $f_t'(.)$. Suppose when $\Delta\theta = \bar{\theta}$, we have $f_t'(w_t^a) = u(s_t) + \theta - \bar{\theta}$, and when $\Delta\theta = \underline{\theta}$, we have $f_t'(w_t^b) = u(s_t) + \theta - \underline{\theta}$. For any $l \in (\underline{\theta}, \bar{\theta})$, we have $u(s_t) + \theta - l \in (f_t'(w_t^a), f_t'(w_t^b))$. Given that $f_t'(.)$ is continuous and strictly increasing, there must be $w_t \in (w_t^a, w_t^b)$ such that $f_t'(w_t) = u(s_t) + \theta - l$. So, $u(s_t) + \theta - l$ is in the domain of $\theta_t(.)$. For the second stage, given an arbitrary $\omega \in (0,1)$, we can directly set $u(s_{T+1}) = f'(\omega) - \theta + l$, so that the target condition is satisfied.

A corollary of this step is that we can manipulate $\Delta\theta$ in Equation (A4) at any level between $[\underline{\theta}, \bar{\theta}]$ by changing a hypothetical s_{T+1}.

Fifth, we show $\ln \phi_t(.)$ is linear under some condition. To do this, let us add a hypothetical s_{T+1} to the end of s_T and let $w_t' = \phi_t(v(s_t))$ denote the decision weights for $s_{0\to T+1}$. We can change the hypothetical s_{T+1} within the set $\{s_{T+1} | v(s_{T+1}) \in [\underline{v}, \bar{v}]\}$ and see what will happen to the decision weights from period 0 to period T. Suppose this changes each w_t' to $\phi_t(v(s_t) - \eta)$. Set $\eta = \underline{\eta}$ when $u(s_{T+1}) = \underline{v}$ and $\eta = \bar{\eta}$ when $u(s_{T+1}) = \bar{v}$. By Equation (A5), we have

$$\frac{\phi_t(v(s_t))}{\sum_{\tau=0}^{T} \phi_\tau(v(s_\tau))} = \frac{\phi_t(v(s_t) - \eta)}{\sum_{\tau=0}^{T} \phi_\tau(v(s_\tau) - \eta)} \tag{A6}$$

For each $t = 0, 1, ..., T$, we can rewrite $\phi_t(v(s_t))$ as $e^{\ln \phi_t(v(s_t))}$. For the LHS of Equation (A6), multiplying both the numerator and the denominator by a same number will not affect the value. Therefore, Equation (A6) can be rewritten as

$$\frac{e^{\ln \phi_t(v(s_t)) - \kappa\eta}}{\sum_{\tau=0}^{T} e^{\ln \phi_\tau(v(s_\tau)) - \kappa\eta}} = \frac{e^{\ln \phi_t(v(s_t) - \eta)}}{\sum_{\tau=0}^{T} e^{\ln \phi_\tau(v(s_\tau) - \eta)}} \tag{9}$$

where κ can be any constant number. By properly selecting κ, for all $t = 0, 1, ..., T$, we can obtain

$$\ln \phi_t(v(s_t)) - \kappa\eta = \ln \phi_t(v(s_t) - \eta) \tag{A7}$$

as long as $\eta \in [\underline{\eta}, \bar{\eta}]$. Since $\ln \phi_t(.)$ is strictly increasing, for any $\eta \neq 0$, we have $\kappa > 0$.

Finally, we denote the maximum and minimum of $\{v(s_t)\}_{t=0}^{T}$ by $v_{\max}$ and $v_{\min}$, and show that Equation (A7) can hold if $\eta = v_{\max} - v_{\min}$. That implies $v_{\max} - v_{\min} \in [\underline{\eta}, \bar{\eta}]$, where $\underline{\eta}, \bar{\eta}$ are the realizations of η at the points of $v(s_{T+1}) = \underline{v}$ and $v(s_{T+1}) = \bar{v}$. Obviously, $\underline{\eta}$ can take the value $\underline{\eta} = 0$. Thus, we focus on whether $\bar{\eta}$ can take a value $\bar{\eta} \geq v_{\max} - v_{\min}$.

The proof is similar with the fourth step and consists of two stages. First, for $t = 0, 1, \ldots, T$, we show $v(s_t) - v_{\max} + v_{\min}$ is in the domain of $\phi_t(.)$. That is, under some w_t, we have $f_t'(w_t) = v(s_t) - v_{\max} + v_{\min}$. Note in a non-constant reward sequence, $v_{\max} - v_{\min} \in (0, +\infty)$. On the one hand, Equation (A5) indicates that the equation $f_t'(\omega) = v(s_t)$ has a solution ω. On the other hand, by Definition 2, we know $\lim_{w_t \to 0} f_t'(w_t) = -\infty$. Given $f_t'(w_t)$ is continuous and strictly increasing, there must be a solution w_t lying in $(0, \omega)$ for equation $f_t'(w_t) = v(s_t) - v_{\max} + v_{\min}$. Second, we show that for any $\omega' \in (0, 1)$, there exists some $v(s_{T+1})$ such that $\phi_{T+1}(v(s_{T+1}) - v_{\max} + v_{\min}) = \omega'$. This can be achieved by setting $v(s_{T+1}) = f_{T+1}'(\omega') + v_{\max} - v_{\min}$.

As a result, for any period t in $s_{0 \to T}$, by Equation (A7), we have $\ln \phi_t(v(s_t)) = \ln \phi_t(v(s_t) - \eta) + \kappa\eta$ so long as $\eta \in [0, v_{\max} - v_{\min}]$, where $\kappa > 0$. We can rewrite each $\ln \phi_t(v(s_t))$ as $\ln \phi_t(v_{\min}) + \kappa[v(s_t) - v_{\min}]$. Therefore, we have

$$w_t \propto \phi_t(v_{\min}) \cdot e^{\kappa[v(s_t) - v_{\min}]} \tag{A8}$$

and $\sum_{t=0}^{T} w_t = 1$. In Equation (A8), setting $\phi_t(v_{\min}) = d_t$, $\lambda = 1/\kappa$, and apply the corollary of the third step, we can conclude that $w_t \propto d_t e^{u(s_t)/\lambda}$, which is of the AMD form.

B. Proof of Proposition 2

Note the instantaneous utilities of LL and SS are v_l and v_s, and the delays for LL and SS are t_l and t_s. According to Equation (4), the common difference effect implies that, if

$$\frac{v_s}{1 + G(t_s)e^{-v_s}} = \frac{v_l}{1 + G(t_l)e^{-v_l}} \tag{B1}$$

then for any $\Delta t \geq 0$, we have

$$\frac{v_s}{1+G(t_s+\Delta t)e^{-v_s}} < \frac{v_l}{1+G(t_l+\Delta t)e^{-v_l}} \tag{B2}$$

If $G(T) = T$, we have $G(t+\Delta t) = G(t) + \Delta t$. In this case, combining Equation (B1) and (B2), we can obtain

$$\frac{\Delta t e^{-v_s}}{v_s} > \frac{\Delta t e^{-v_l}}{v_l} \tag{B3}$$

Given that function $\psi(v) = e^{-v}/v$ is decreasing with v so long as $v > 0$, Equation (B3) is valid.

If $G(T) = \frac{1}{1-\delta}(\delta^{-T} - 1)$, we have

$$1 + G(t+\Delta t)e^{-v} = \delta^{-\Delta t}[1 + G(t)e^{-v}] + (\delta^{-\Delta t} - 1)(\frac{e^{-v}}{1-\delta} - 1) \tag{B4}$$

Thus, combining Equation (B1) and (B2), we can obtain

$$(\delta^{-\Delta t} - 1)\frac{\frac{e^{-v_s}}{1-\delta} - 1}{v_s} > (\delta^{-\Delta t} - 1)\frac{\frac{e^{-v_l}}{1-\delta} - 1}{v_l} \tag{B5}$$

Given that $0 < \delta < 1$, we have $\delta^{-\Delta t} > 1$. So, Equation (B5) is valid if and only if

$$\frac{1}{v_s} - \frac{1}{v_l} < \frac{1}{1-\delta}(\frac{e^{-v_s}}{v_s} - \frac{e^{-v_l}}{v_l}) \tag{B6}$$

By Equation (B1), we know that

$$\frac{1}{v_s} - \frac{1}{v_l} = \frac{1}{1-\delta}\left[\frac{(\delta^{-t_l} - 1)e^{-v_l}}{v_l} - \frac{(\delta^{-t_s} - 1)e^{-v_s}}{v_s}\right] \tag{B7}$$

Combining Equation (B6) and (B7), we have

$$\delta^{-t_l}\frac{e^{-v_l}}{v_l} < \delta^{-t_s}\frac{e^{-v_s}}{v_s} \Longleftrightarrow v_l - v_s + \ln\left(\frac{v_l}{v_s}\right) > -(t_l - t_s)\ln\delta \tag{10}$$

C. Proof of Proposition 3

Suppose a positive reward x is delivered at period T. By Equation (4), if w_T is convex in T, we should have $\frac{\partial^2 w_T}{\partial T^2} \geq 0$. This implies

$$2G'(T)^2 \geq (G(T) + e^{v(x)})G''(T) \tag{C1}$$

If $\delta = 1$, then $G(T) = T$. We have $G'(T) = 1$, $G''(T) = 0$. Thus, Equation (C1) is always valid.

If $0 < \delta < 1$, then $G(T) = (1-\delta)^{-1}(\delta^{-T} - 1)$. We have $G'(T) = (1-\delta)^{-1}(-\ln\delta)\delta^{-T}$, $G''(t) = (-\ln\delta)G'(T)$. Thus, Equation (C1) is valid when

$$\delta^{-T} \geq (1-\delta)e^{v(x)} - 1 \tag{C2}$$

Given $T > 0$, Equation (C2) holds true in two cases. The first case is $1 \geq (1-\delta)e^{v(x)} - 1$, which implies that $v(x)$ is no greater than a certain threshold $v(\underline{x})$, where $v(\underline{x}) = \ln(\frac{2}{1-\delta})$. The second case is that $v(x)$ is above $v(\underline{x})$ and T is above a threshold $\underline{t}$. In the second case, we can take the logarithm on both sides of Equation (C2). It yields $\underline{t} = \frac{\ln[(1-\delta)\exp\{v(x)\}-1]}{\ln(1/\delta)}$.

D. Proof of Proposition 4

For convenience, we use v to represent $v(x) \equiv u(x)/\lambda$, and use U to represent $U(x,T)$. Set $g = G(T)$. The first-order derivative of U with respect to x can be written as

$$\frac{\partial U}{\partial x} = v' \frac{e^v + U}{e^v + g} \tag{D1}$$

If U is strictly concave in x, we should have $\frac{\partial^2 U}{\partial x^2} < 0$. By Equation (D1), we calculate the second-order derivative of U with respect to x, and rearrange this second-order condition to

$$2\zeta(v) + \frac{1}{1 + v\zeta(v)} - 1 < \frac{-v''}{(v')^2} \equiv \frac{d}{dx}\left(\frac{1}{v'}\right) \tag{D2}$$

where $\zeta(v) = g/(g + e^v)$. Since $v'' < 0$, the RHS of Equation (D2) is clearly positive.

To prove the first part of Proposition 4, we can show that when x is large enough, the LHS of Equation (D2) will be non-positive. To make the LHS non-positive, we require

$$\zeta(v) + \frac{1}{v} \leq \frac{1}{2} \tag{D3}$$

hold true. Note that $\zeta(v)$ is decreasing in v, and v is increasing in x. Hence, $\zeta(v) + \frac{1}{v}$ is decreasing in x. Besides, it approaches $+\infty$ when $x \to 0$ and approaches 0 when $x \to +\infty$. When $\frac{d}{dx}\left(\frac{1}{v'(x)}\right)$ is continuous, there must be a unique realization of x in $(0, +\infty)$, say $\bar{x}$, making the equality in Equation (D3) valid. Moreover, when $x \geq \bar{x}$, Equation (D3) is always valid. In such cases, $U(x, T)$ is concave in x.

To prove the second part, first note that when $x = 0$, the LHS of Equation (D2) will become $\frac{2g}{g+1}$. If $\frac{d}{dx}\left(\frac{1}{v'(0)}\right)$ is smaller than this number, then the LHS of Equation (D2) should be greater than the RHS at the point of $x = 0$. Meanwhile, from the first part of the current proposition, we know the LHS is smaller than the RHS at the point of $x = \bar{x}$. Thus, given $\frac{d}{dx}\left(\frac{1}{v'(x)}\right)$ is continuous in $[0, \bar{x}]$, there must also be a point within $[0, \bar{x}]$, such that the LHS equals the RHS. Let x^* denote the minimum of x that makes the equality valid. Then, for any $x \in (0, x^*)$, we must have that the LHS of Equation (D2) is greater than the RHS, which implies $U(x, T)$ is convex in x. Given that $T \geq 1$, we have $g \geq 1$ and thus $\frac{2g}{g+1} \geq 1$. Therefore, when $\frac{d}{dx}\left(\frac{1}{v'(0)}\right) < 1$, $U(x, t)$ can be convex in x for any $x \in (0, x^*)$, regardless of g.

The prove the third part, note $v(x) = u(x)/\lambda$. So,

$$\frac{d}{dx}\left(\frac{1}{v'}\right) = \lambda \frac{d}{dx}\left(\frac{1}{u'}\right) \tag{11}$$

We arbitrarily draw a point from $(0, \bar{x})$ and derive the range λ relative to this point. For simplicity, we choose $x = \ln g$. In this case, the LHS of Equation (D2) becomes $\frac{2}{2+\ln g}$. Define a function $\xi(x)$, where ξ is the value of the LHS of Equation (D2) minus its RHS. Note $\xi(x)$ is continuous at $x = \ln g$. Therefore, for any positive real number b, there must exist a positive real number c such that, when $x \in (\ln g - c, \ln g + c)$, we have

$$\xi(\ln g) - b < \xi(x) < \xi(\ln g) + b \tag{D4}$$

If $\xi(\ln g) - b \geq 0$, then $\xi(x)$ will keep positive for all $x \in (\ln g - c, \ln g + c)$, which implies the LHS of Equation (D2) is always greater than its RHS.

Now we derive the condition for $\xi(\ln g) - b \geq 0$. Suppose when $x = \ln g$, $\frac{d}{dx}\left(\frac{1}{u'}\right) = a$ (note at this point we have $\frac{d}{dx}\left(\frac{1}{u'}\right) < +\infty$). Combining with Equation (D3), we know that $\xi(\ln g) - b = \frac{2}{2+\ln g} - \lambda a - b$. Letting this value be non-negative, we obtain

$$\lambda \leq \frac{2}{a(2 + \ln g)} - \frac{b}{a} \tag{D5}$$

Given that $T \geq 1$, we have $g \geq 1$ and thus $\frac{2}{2+\ln g}$ should be positive. Meanwhile, given that $u' > 0$ and $u'' < 0$, a should also be positive. Since b can be any positive number, Equation (D5) holds if $\lambda < \frac{2}{a(2+\ln g)}$. That is, when λ is positive but smaller than a certain threshold, there must be an interval $(\ln g - c, \ln g + c)$ such that the LHS of Equation (D2) is greater than the RHS. Set $x_1 = \max\{0, \ln g - c\}$, $x_2 = \min\{\bar{x}, \ln g + c\}$. When $x \in (x_1, x_2)$, function $U(x, T)$ must be convex in x.

E. Proof of Proposition 5

The proof consists of four steps. First, we write the expressions for $U(L1)$ and $U(L2)$. Suppose the time length of each lottery result is T. For a period τ at which no reward is delivered, the instantaneous utility is zero. Let Ω denote the set of all such period τ, then $\Omega = \{\tau \,|\, 0 \leq \tau \leq T,\ \tau \neq t_1, t_2\}$. For any $j, k \in \{s, l\}$, we define $\phi_j = d_{t_1} e^{v(x_j)}$ and $\eta_k = d_{t_2} e^{v(y_k)}$, where $v(s) = u(s)/\lambda$, and d_t represents the default discount factor for reward delivered at period t.

For a given lottery result (s_1, s_2), we denote the decision weight of each positive reward by w_{t_1} and w_{t_2}. By the definition of AMD, we have

$$w_{t_1} = \frac{\phi_j}{\phi_j + \eta_k + D} \quad , \quad w_{t_2} = \frac{\eta_k}{\phi_j + \eta_k + D} \tag{12}$$

where $j, k \in \{s, l\}$, $D = \sum_{\tau \in \Omega} d_\tau \geq 0$. The value of a lottery L can be written as $U(L) = w_{t_1} u(s_1) + w_{t_2} u(s_2)$. Hence,

$$\begin{aligned} U(L1) &= 0.5 \frac{\phi_s u(x_s) + \eta_s u(y_s)}{\phi_s + \eta_s + D} + 0.5 \frac{\phi_l u(x_l) + \eta_l u(y_l)}{\phi_l + \eta_l + D} \\ U(L2) &= 0.5 \frac{\phi_s u(x_s) + \eta_l u(y_l)}{\phi_s + \eta_l + D} + 0.5 \frac{\phi_l u(x_l) + \eta_s u(y_s)}{\phi_l + \eta_s + D} \end{aligned} \tag{E1}$$

We observe that, when $x_l = x_s$, we have $U(L1) = U(L2)$.

Second, suppose we increase x_l from x_s by an increment. This increases both $U(L1)$ and $U(L2)$ (either by a positive or a negative number). To make $U(L1) < U(L2)$, this increment should increase $U(L2)$ by a greater number than $U(L1)$. Specifically, we assume $U(L2)$ is increasing faster than $U(L1)$ at any level of x_l. That is, the partial derivative of $U(L2)$ in terms of x_l is always greater than that of $U(L1)$. Given ϕ_l is increasing in x_l, to see this, we can take partial derivatives in terms of ϕ_l.

In each line of Equation (E1), note only the second term contains x_l. Thus, we focus on the difference between the second terms. The second term of the $U(L1)$ is influenced by y_l, while that of the $U(L2)$ is influenced by y_s, where $y_l > y_s$. Thus, we can construct a function ξ such that

$$\xi(\phi_l, \eta) = \frac{\phi_l \cdot v(x_l) + \eta \cdot v(y)}{\phi_l + \eta + D} \tag{13}$$

where $\eta = d_{t_2} e^{v(y)}$. In reverse, we can define $v(x_l) = \ln(\phi_l / d_{t_1})$ and $v(y) = \ln(\eta / d_{t_2})$. The function ξ is similar to the second term of each line, but note we replace $u(.)$ by $v(.)$. When $y = y_l$, ξ is proportional to the second term of $U(L1)$. When $y = y_s$, ξ is proportional to the second term of $U(L2)$ (by the same proportion). Thus, to show that the partial derivative of $U(L2)$ in terms of x_l is greater than that of $U(L1)$, we just need to show $\partial \xi / \partial \phi_l$ is decreasing with y (or η).

Third, we take the first- and second-order partial derivatives of $\xi(\phi_l, \eta)$. The partial derivative of ξ in terms of ϕ_l is

$$\frac{\partial \xi}{\partial \phi_l} = \frac{(v(x_l)+1)\eta - v(y)\eta + \phi_l + D(v(x_l)+1)}{(\phi_l+\eta+D)^2} \tag{14}$$

We need to show that for $y \in [y_s, y_l]$, we can obtain $\partial^2\xi/\partial\phi_l\partial\eta < 0$. This implies

$$(v(x_l)+v(y)+2)D - (\phi_l-\eta)(v(x_l)-v(y)) + 2(\phi_l+\eta) > 0 \tag{E2}$$

We want Equation (E2) to hold for any $D \geq 0$. Given the LHS is increasing with D, this can only be achieved when

$$2(\phi_l+\eta) > (\phi_l-\eta)(v(x_l)-v(y)) \tag{E3}$$

Define $\kappa = d_{t_2}/d_{t_1}$, $\alpha = v(x_l) - v(y)$. Note $\kappa \in \mathbb{R}_{>0}$, $\alpha \in \mathbb{R}$. Equation (E3) can be rewritten as

$$(\alpha-2)\kappa^{-1}e^{\alpha} - \alpha - 2 < 0 \tag{E4}$$

Fourth, based on Equation (E4), we construct a function $h(\alpha) = (\alpha-2)\kappa^{-1}e^{\alpha} - \alpha - 2$. We aim to examine whether there exists some $\alpha \in \mathbb{R}$ that makes $h(a) < 0$. Obviously, $\alpha = -2$ and $\alpha = 2$ satisfy this condition. Moreover, note $h(\alpha)$ is decreasing in α when $(\alpha-1)e^{\alpha} \leq \kappa$ and is increasing in α otherwise. And when either $\alpha \to -\infty$ or $\alpha \to +\infty$, we have $h(\alpha) \to +\infty$. Thus, there must be a limited interval (α_1, α_2) such that $h(a) < 0$ so long as $\alpha \in (\alpha_1, \alpha_2)$, and obviously $[-2, 2] \subset (\alpha_1, \alpha_2)$. Since $v(s) = u(s)/\lambda$, this implies $\frac{u(x_l)-u(y)}{\lambda} \in (\alpha_1, \alpha_2)$.

For a given positive number κ, the points α_1, α_2 are determined by the solution to $\frac{\alpha-2}{\alpha+2}e^{\alpha} = \kappa$. In other words, for any x_l and $y \in [y_s, y_l]$, we can always achieve $U(L1) < U(L2)$ as long as $u(x_l)-u(y_l) \geq \lambda\alpha_1$ and $u(x_l)-u(y_s) \leq \lambda\alpha_2$. So, we can conclude that for any $x_l > x_s > 0$, $y_l > y_s > 0$, any time length of lottery results and default discount factor (which determines D and κ), there exists some λ that makes DM intertemporal correlation averse. Specifically, all $\lambda > \lambda^{**} = \max\{\frac{u(x_l)-u(y_l)}{\alpha_1}, \frac{u(x_l)-u(y_s)}{\alpha_2}\}$ satisfy the target condition.

Notably, if $\lambda \leq \lambda^{**}$, we have $h(a) \geq 0$, which by Equation (E2)(E3), indicates that under some conditions such as $D = 0$, there will be $\partial^2 \xi / \partial \phi_l \partial \eta \geq 0$ for all $y \in [y_s, y_l]$. In that case, at each level of x_l, the partial derivative of $U(L1)$ in terms of x_l is greater than that of $U(L2)$. So, increasing x_l by an increment from x_s can induce a greater increase in $U(L1)$ than in $U(L2)$. This makes it possible that $U(L1) > U(L2)$. In short, DM may perform intertemporal correlation seeking when $\lambda \leq \lambda^{**}$.

F. Proof of Proposition 6

Before proving the proposition, we first show that in the DM's optimal consumption plan $s_{0\to T}$, the largest consumption must be s_0. To show this, suppose the largest consumption is s_τ ($\tau > 0$). By Lemma 3 below, we can obtain that if we exchange the consumption planned in τ with the consumption planned in period 0, the total value of consumption will be non-decreasing. So, the largest consumption must be placed in period 0.

For convenience, henceforth we use u_t to represent $u(s_t)$.

Lemma 3: *Suppose in $s_{0\to T}$, we have $s_\tau = \max\{s_0, s_1, ..., s_T\}$ and $\tau > 0$. Set $u_0/\lambda = v_1$ and $u_\tau/\lambda = v_2$. If we change u_0/λ to v_2 and u_τ/λ to v_1, $U(s_{0\to T})$ will be non-decreasing.*

Proof: Let V/λ denote the total value of consumption before we exchange consumption between period 0 and τ, where

$$V = \frac{\sum_{t=0}^{T}(u_t/\lambda) \cdot d_t e^{u_t/\lambda}}{\sum_{t=0}^{T} d_t e^{u_t/\lambda}} \tag{15}$$

Set $d_0 = \delta_1$, $d_\tau = \delta_2$. We denote the numinator of V by $v_1\delta_1 e^{v_1} + v_2\delta_2 e^{v_2} + P$ and denote its denominator by $\delta_1 e^{v_1} + \delta_2 e^{v_2} + Q$.

Note $v_2 \geq V$. If changing u_t/λ to v_2 as well as $u_{t+\tau}/\lambda$ to v_1 does not decrease V, we should have

$$\frac{v_1\delta_1 e^{v_1} + v_2\delta_2 e^{v_2} + P}{\delta_1 e^{v_1} + \delta_2 e^{v_2} + Q} \leq \frac{v_2\delta_1 e^{v_2} + v_1\delta_2 e^{v_1} + P}{\delta_1 e^{v_2} + \delta_2 e^{v_1} + Q} \tag{F1}$$

where $\delta_1 > \delta_2 > 0$, $v_2 \geq v_1 > 0$. By rearranging Equation (F1), we can obtain

$$-(\delta_1 + \delta_2)e^{v_1+v_2}(v_2 - v_1) \leq e^{v_2}(Qv_2 - P) - e^{v_1}(Qv_1 - P) \tag{F2}$$

Clearly, Equation (F2) holds if $e^v(Qv - P)$ is increasing with v when $v \in [v_1, v_2]$, and the latter implies $v_1 \geq \frac{P}{Q} - 1$.

Notably, V is a weighted mean of v_1, v_2 and $\frac{P}{Q}$, and we have $v_2 \geq \max\{v_1, \frac{P}{Q}\}$. If $v_1 \geq V$, we must have $v_1 \geq \frac{P}{Q}$. In this case, Equation (F2) clearly holds. If $v_1 < V$, note that $\frac{\partial U}{\partial d_t} \propto u_t - U$. So, we will have $\frac{\partial U}{\partial d_0} < 0$ and $\frac{\partial U}{\partial d_\tau} > 0$. Both decreasing d_0 to δ_2 and increasing d_τ to δ_1 would increase the total value of consumption. In summary, for either case, after changing u_t/λ to v_2 and $u_{t+\tau}/\lambda$ to v_1, V should be non-decreasing. *QED.*

Denote the value of $s_{0\to T}$ by $U = \sum_{t=0}^{T} w_t u(s_t)$. Notably, if the optimal consumption plan is an interior solution, then the solution must satisfy $\frac{\partial U}{\partial s_{t+1}} / \frac{\partial U}{\partial s_t} = 1$. Suppose $\frac{\partial U}{\partial s_t} > \frac{\partial U}{\partial s_{t+1}} > 0$, then the DM can transfer an incremental consumption from s_{t+1} to s_t, as increasing s_t by an increment will lead to a greater improvement in U compared to the decrease in U caused by reducing s_{t+1} by the same amount. The DM will keep transfer consumption between periods until $\frac{\partial U}{\partial s_t} = \frac{\partial U}{\partial s_{t+1}}$. Nevertheless, if the DM keeps reducing s_{t+1} and still has $\frac{\partial U}{\partial s_t} > \frac{\partial U}{\partial s_{t+1}} > 0$ even when s_{t+1} is reduced to 0, this optimization problem will reach a corner solution. We aim to show that when λ is small enough, for all $t > 0$, the DM would tend to reduce s_t to zero.

The partial derivatives of U in terms of s_t is $\frac{\partial U}{\partial s_t} = w_t u_t'(u_t + \lambda - U)$. Therefore, we have

$$\frac{\partial U}{\partial s_{t+1}} / \frac{\partial U}{\partial s_t} = \frac{d_{t+1}}{d_t} \exp\{\frac{u_{t+1} - u_t}{\lambda}\} \cdot \frac{u_{t+1}'}{u_t'} \cdot \frac{u_{t+1} + \lambda - U}{u_t + \lambda - U} \tag{F3}$$

Drawing on Equation (F3), we construct a function $\rho(s_t; \mathcal{U}) = e^{u_t/\lambda} u_t'(u_t + \lambda - \mathcal{U})$. Its partial derivative in terms of s_t is

$$\frac{\partial \rho(s_t; \mathcal{U})}{\partial s_t} = e^{u_t/\lambda}[(u_t + \lambda - \mathcal{U})(\frac{(u_t')^2}{\lambda} + u_t'') + (u_t')^2] \tag{F4}$$

To prove Proposition 6, note that if

$$\lambda \leq \min_{0 \leq t \leq T} \inf_{s_{0\to T} \in A} \{-(u_t')^2 / u_t''\} \tag{F5}$$

we will always have $\frac{(u_t')^2}{\lambda} + u_t'' \geq 0$. Set $\underline{\lambda}$ as the RHS of Equation (F5). Note by Lemma 3, we know that s_0 must be the largest consumption in the DM's consumption plan. It can be proved that if $\lambda \leq \underline{\lambda}$, in an arbitrary sequence $s_{0\to T}$ that satisfies this condition, it is always beneficial for the DM to transfer consumption from the future periods to the current period, until all consumption is concentrated at the current period. We discuss this in two cases.

First, suppose for all $t > 0$, we have $u_t + \lambda - U > 0$. As s_0 is the largest consumption, we must have $u_0 > U$; so, we also have $u_0 + \lambda - U > 0$. In this case, if $\lambda \leq \underline{\lambda}$, according to Equation (F4), we can obtain $\frac{\partial \rho(s_t;U)}{\partial s_t} > 0$. By Equation (F3), for all $t > 0$, we have $\frac{\partial U}{\partial s_t} / \frac{\partial U}{\partial s_0} = \frac{d_t}{d_0} \frac{\rho(s_t;U)}{\rho(s_0;U)}$. Since $d_0 > d_t$ and $s_0 > s_t$, we can obtain $\frac{\partial U}{\partial s_0} > \frac{\partial U}{\partial s_t} > 0$. Therefore, it is beneficial for the DM to transfer an incremental consumption from s_t to s_0.

Second, suppose for some period $\tau > 0$, we have $u_\tau + \lambda - U < 0$. For this period τ, there must be $\frac{\partial U}{\partial s_\tau} < 0$. A reduction in s_τ will increase U. So, it is also beneficial to reduce it and transfer the consumption to the current period.

In conclusion, in any sequence, as long as s_0 is the largest consumption, transferring consumption from the future to the present must improve U. If in such a sequence, a future period has a positive consumption, the DM will keep transfer it to the current period until she concentrate all consumption into the current period.

G. Proof of Proposition 7

We use the same notation as in the proof of Proposition 6. Before proving Proposition 7, we list a set of conditions that, taken together, are sufficient to derive the resulting behavior in the proposition. We then prove that all these conditions are satisfied as long as λ is large enough.

First, suppose the consumption planing problem has an interior solution. Again, let U denote the value of a consumption sequence. Then, for any period t in the optimal plan, we should have $\frac{\partial U}{\partial s_{t+1}} / \frac{\partial U}{\partial s_t} = 1$ and $\frac{\partial^2 U}{\partial s_t^2} < 0$. The second-order condition implies

$$(1 - 2w_t)\frac{1}{\lambda} + \frac{1}{u_t + \lambda - U} < -\frac{u_t''}{(u_t')^2} \tag{G1}$$

Second, to analyze how a change in s_t will affect $\frac{\partial U}{\partial s_t}$, we construct a function $\tilde{\rho}_t(s_0, s_1, ..., s_T) \equiv \rho(s_t; U)$. By calculating the partial derivative of $\tilde{\rho}_t$ in terms of s_t, when $u_t + \lambda - U > 0$, we can obtain

$$\frac{\partial \tilde{\rho}_t}{\partial s_t} < 0 \iff (1 - w_t)\frac{1}{\lambda} + \frac{1}{u_t + \lambda - U} < -\frac{u_t''}{(u_t')^2} \tag{G2}$$

Third, when $u_t + \lambda - U > 0$, according to Equation (F4), we can also obtain that $\rho(s_t; U)$ decreases in s_t if and only if

$$\frac{1}{\lambda} + \frac{1}{u_t + \lambda - U} < -\frac{u_t''}{(u_t')^2} \tag{G3}$$

Clearly, if Equation (G3) holds, both Equation (G1) and (G2) will also hold. Note $u(m) > U$; so, if $\lambda - u(m) \geq 0$, we will always have $u_t + \lambda - U > 0$. Under this interval of λ, the LHS of Equation (G3) is continuous and decreasing in λ, and it converges to 0 with $\lambda \to 0$. Hence, there must be some $\bar{\lambda}_1 \geq u(m)$ such that, when $\lambda \geq \bar{\lambda}_1$, for any given consumption sequence, Equation (G3) is valid.

Fourth, we construct a new function $\Gamma(s_t; \mathcal{U}) = e^{u_t/\lambda} \cdot \left(\frac{u_t + \lambda}{u_t + \lambda - \mathcal{U}}\right)$. By calculating the partial derivative of Γ in terms of s_t, when $u_t + \lambda - U > 0$, we can obtain

$$\frac{\partial \Gamma(s_t; U)}{\partial s_t} > 0 \iff (\lambda + u_t - U)^2 + U(u_t - U) > 0 \tag{G4}$$

If λ is large enough, Equation (G4) must hold. Again, we can conclude that there is some $\bar{\lambda}_2 \geq u(m)$, such that when $\lambda \geq \bar{\lambda}_2$, for any given sequence, Equation (G4) is valid.

We define $\bar{\lambda}$ as $\max\{\bar{\lambda}_1, \bar{\lambda}_2\}$. It can be shown that any $\lambda \in [\bar{\lambda}, +\infty)$ satisfies the proposition.

For the part (a) of Proposition 7, note by Equation (F3), we have $\frac{\partial U}{\partial s_{t+1}} / \frac{\partial U}{\partial s_t} = \frac{d_{t+1}}{d_t} \frac{\rho(s_{t+1}; U)}{\rho(s_t; U)}$. Given an arbitrary sequence in the set $\{s_{0\to T} | s_{0\to T} \in A, \forall t < T : s_t > s_{t+1} > 0\}$ (the sequence is decreasing and available for choice, and all rewards are positive), when $\lambda \geq \bar{\lambda}$, for each period $t < T$ in the sequence, we have $\rho(s_{t+1}; U) > \rho(s_t; U)$, as $\rho(s_t; U)$ is decreasing in s_t.

Denote the given sequence by $s^*_{0\to T}$. To make $s^*_{0\to T}$ become the interior solution for the planning problem in period 0, according to the FOC, we need $\frac{d^0_{t+1}}{d^0_t} = \frac{\rho(s_t; U)}{\rho(s_{t+1}; U)}$. Thus, we can define d_t^0 as

$$d_t^0 = \frac{\rho(s_t;U)^{-1}}{\rho(s_0;U)^{-1} + ... + \rho(s_T;U)^{-1}} \tag{16}$$

According to this definition, it can be easily validated that $\frac{d_{t+1}^0}{d_t^0} = \frac{\rho(s_t;U)}{\rho(s_{t+1};U)}$ and $d_t^0 > d_{t+1}^0 > 0$. Under such $\{d_t^0\}_{t=0}^T$, the given sequence $s_{0\to T}^*$ is the optimal consumption plan.

For the part (b), we prove it in two steps. First, we show if $d_t^1 = d_t^0$, the DM will under-consume in period 1. In this case, for any $t \geq 1$ we set $d_t^0 = d_t^1 = d_t$. Given $s_{0\to T}^*$ as the period 0's optimal consumption plan, when moving to period 1, the DM will need to allocate budget over period $t = 1, \dots, T$, and the largest consumption s_0^* will be moved out from the sequence. Keeping everything equal, this will reduce the sequence value U, thereby (under the condition $\lambda \geq \bar{\lambda}$) increasing $\frac{u_{t+1}+\lambda-U}{u_t+\lambda-U}$. Thus, according to Equation (F3), $\frac{\partial U}{\partial s_{t+1}}/\frac{\partial U}{\partial s_t}$ will increase to greater than 1. To optimize the consumption plan, the DM needs to adjust s_t and s_{t+1} toward re-achieving $\frac{\partial U}{\partial s_{t+1}}/\frac{\partial U}{\partial s_t} = \frac{d_{t+1}}{d_t}\frac{\tilde{\rho}_{t+1}}{\tilde{\rho}_t} = 1$. Specifically, she is required to reduce $\frac{\tilde{\rho}_{t+1}}{\tilde{\rho}_t}$. According to Equation (G2), when $\lambda \geq \bar{\lambda}$, $\tilde{\rho}_t$ is decreasing in s_t, which implies that the DM needs to increase s_{t+1} in relative to s_t. In other words, she needs to transfer consumption from an earlier period to a later period. So, for the optimal consumption plan in period 1, we have $s_1^{**} < s_1^*$.

Finally, we discuss the case that $d_t^1 = w_t^0$. When moving to period 1, how the DM will adjust consumption is affected by two mechanisms. On the one hand, as $\frac{d_{t+1}^1}{d_t^1} = \frac{d_{t+1}^0}{d_t^0}\exp\{\frac{u_{t+1}-u_t}{\lambda}\}$, the DM will initially pay more attention to an earlier period than a later period; on the other hand, as the largest consumption s_0^* is moved out, keeping everything equal, the sequence value U will decrease. We suppose it will decrease from U_0 to U_1. The former mechanism drives the DM to transfer consumption from a later period to an earlier period, while the latter mechanism drives her to do the opposite. We can show that the former mechanism overrides the latter.

To show this, note for the period 0's optimal consumption plan, we have $\frac{\partial U}{\partial s_{t+1}}/\frac{\partial U}{\partial s_t} = \frac{d_{t+1}^0}{d_t^0}\frac{\rho(s_{t+1}^*;U_0)}{\rho(s_t^*;U_0)} = 1$. Substituting it into the equation of $\frac{\partial U}{\partial s_{t+1}}/\frac{\partial U}{\partial s_t}$ in period 1 (keeping everything equal), we have

$$\frac{\partial U}{\partial s_{t+1}}/\frac{\partial U}{\partial s_t} = \exp\{\frac{u(s^*_{t+1}) - u(s^*_t)}{\lambda}\}\frac{u_{t+1} + \lambda - U_1}{u_t + \lambda - U_1}\frac{u_t + \lambda - U_0}{u_{t+1} + \lambda - U_0} < \frac{\Gamma(s^*_{t+1}; U_0)}{\Gamma(s^*_t; U_0)} \tag{G5}$$

Equation (G5) implies that, uner this situation, if we reduce U_1 to 0, the value of $\frac{\partial U}{\partial s_{t+1}}/\frac{\partial U}{\partial s_t}$ will be increased to $\Gamma(s^*_{t+1}; U_0)/\Gamma(s^*_t; U_0)$. Since (under the condition $\lambda \geq \bar{\lambda}$) $s^*_t > s^*_{t+1}$ and $\Gamma(s_t; U_0)$ is increasing in s_t, we can obtain that $\Gamma(s^*_{t+1}; U_0)/\Gamma(s^*_t; U_0) < 1$. As a result, to optimize the consumption plan in period 1, the DM will need to increase $\frac{\partial U}{\partial s_{t+1}}/\frac{\partial U}{\partial s_t} = \frac{d^1_{t+1}}{d^1_t}\frac{\tilde{\rho}_{t+1}}{\tilde{\rho}_t}$. Specifically, she is required to increase $\frac{\tilde{\rho}_{t+1}}{\tilde{\rho}_t}$. Thus, she has to transfer consumption from a later period to a later period; that is, reducing s_{t+1} in relative to s_t. For period 1, we will have $s^{**}_1 > s^*_1$.

H. Impact of Adding a Zero to the Sequence End

We discuss the impact of adding a period of zero reward to the end of each sequence here. This technical trick primarily affects the contexts involving choices between sequences of different lengths. For intertemporal correlation aversion, S-shaped value function, and the anomalies related to allocating resources over a certain time (e.g. concentration bias, present bias), it doesn't require special consideration. For example, in our proof about intertemporal correlation aversion (see Appendix E), we import a parameter D to capture the effect incurred by all periods with zero reward delivered, and assume that D is arbitrary. Changing the scale of D has no effect on our proposition. Besides, the reason why this does not affect the hidden zero effect is immediately apparent. We thus focus on the following anomalies: (1) common difference effect; (2) concavity of discount function.

Given a sequence $s_{0\to T+1} = [0, 0, ..., 0, s_T, 0]$, where $s_T > 0$ and all the other periods have no reward delivered, we can obtain an equation similar to Equation (4). The discount function for s_T will be $w_T = \frac{1}{1+G(T)e^{-v_T}}$, where $v_T = u(s_T)/\lambda$ and

$$G(T) = \begin{cases} \dfrac{\delta^{-T} - 1}{1 - \delta} + \delta\ , & 0 < \delta < 1 \\ T + 1\ , & \delta = 1 \end{cases} \tag{H1}$$

For common difference effect, we follow the same notation and the process of proof in Appendix B. If $G(T) = T + 1$, Equation (B3) is still valid. Thus, the common difference effect must hold. If $G(T) = \frac{1}{1-\delta}(\delta^{-T} - 1) + \delta$, we will have

$$1 + G(t + \Delta t)e^{-v} = \delta^{-\Delta t}(1 + G(t)e^{-v}) + (\delta^{-\Delta t} - 1)[(\frac{1}{1-\delta} - \delta)e^{-v} - 1] \tag{H2}$$

which is similar to Equation (B4). Combining it with Equation (B1)(B2), we can obtain that the common difference effect holds when and only when

$$v_l - v_s + \ln\left(\frac{v_l}{v_s}\right) > \ln\left(\frac{\delta^{-t_l} - \delta(1-\delta)}{\delta^{-t_s} - \delta(1-\delta)}\right) \tag{17}$$

That is to say, to observe the common difference effect, the absolute and relative differences in reward utility should be significantly larger than the difference in reward delay (the implication of Proposition 2).

For concavity of discount function, we follow Appendix C. Again, if $\delta = 1$, Equation (C1) must be valid. If $0 < \delta < 1$, Equation (C1) is valid when

$$\delta^{-T} \geq (1-\delta)e^{v(x)} - 1 + \delta(1-\delta) \tag{H3}$$

To make Equation (H3) holds, we should consider two cases. The first case is the RHS is no greater than 1. This implies that $v(x) \leq v(\underline{x})$, where $v(\underline{x}) = \ln(\frac{2}{1-\delta} - \delta)$. If the first case does not hold, we should consider the second case: T is above a threshold $\underline{t}$, where $\underline{t} = \frac{\ln((1-\delta)e^{v(x)} - 1 + \delta(1-\delta))}{\ln(1/\delta)}$. So, when the reward is large enough (greater than $\underline{x}$), the discount function can be concave in the near future, which is the implication of Proposition 3.