Proof

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We define $s_{0\to T} = [s_0, s_1, ..., s_T]$ as a sequence of rewards, starting at period 0 and ending at period T. Similarly, $s_{0\to t} = [s_0, \ldots, s_t]$ is defined as a sub-sequence of it. Let $\mathcal{W} = [w_0, ..., w_T]$ denote the attention weights for all rewards in sequence $s_{0\to T}$, where $W \in [0,1]^{T+1}$. Let $C(\mathcal{W})$ denote the information cost function. We can construct the following constrained optimal discounting problem for $s_{0\to T}$:

$$\begin{aligned} \max_{\mathcal{W}} \quad & \sum_{t=0}^{T} w_t u(s_t) - C(\mathcal{W}) \\ s.t. \quad & \sum_{t=0}^{T} w_t = 1,\ w_t \geq 0 \text{ for all } t \in \{0, 1, \ldots, T\} \end{aligned} \tag{A.1}$$

We assume $C(\mathcal{W})$ is constituted by separable costs, that is, $C(\mathcal{W}) = \sum_{t=0}^{T} f_t(w_t)$, where $f_t(w_t)$ is an increasing and convex function.

Axiom 1 (sequential outcome-betweenness) For any $s_{0\to T}$, there exists a $\alpha \in (0,1)$ such that $s_{0\to T} \sim \alpha \cdot s_{0\to T-1} + (1-\alpha) \cdot s_T$.

Axiom 2 (sequential bracket-independence) For any $s_{0\to T}$, if there exists non-negative real numbers α_1, α_2, β_0, β_1, β_2, such that $s_{0\to T} \sim \alpha_1 \cdot s_{0\to T-1} + \alpha_2 \cdot s_T$, and $s_{0\to T} \sim \beta_0 \cdot s_{0\to T-2} + \beta_1 \cdot s_{T-1} + \beta_2 \cdot s_T$, then we must have $\alpha_2 = \beta_2$.

Axiom 3 (aggregate invariance to constant sequences) Consider two constant sequences, denoted as s_c and s'_c, where each element in s_c is equal to c and each element in s'_c is equal to c'. For any $s_{0\to T}$, $s'_{0\to T}$ and $\alpha \in (0,1)$, if $\alpha \cdot s_t + (1-\alpha) \cdot c \sim \alpha \cdot s'_t + (1-\alpha) \cdot c'$ holds for every t, then $\alpha \cdot s_{0\to T} + (1-\alpha) \cdot s_c \sim \alpha \cdot s'_{0\to T} + (1-\alpha) \cdot s'_c$.

Axiom 4 (state independence) If $s_t \succ s'_t$, then for any $\alpha \in (0,1)$ and reward c, $\alpha \cdot s_t + (1-\alpha) \cdot c \succ \alpha \cdot s'_t + (1-\alpha) \cdot c$.

Proposition: $\succsim$ admits an ADU representation if it admits a DU representation subject to the constrained optimal discounting problem, and satisfies Axiom 1-4.

Proof.

Lemma 1. If Axiom 1 holds, for any $s_{0\to T}$, there exist non-negative real numbers w_0, w_1,..., w_T such that $s_{0\to T} \sim w_0 \cdot s_0 + w_1 \cdot s_1 + ... + w_T \cdot s_T$ where $\sum_{t=0}^{T} w_t = 1$.

When $T = 1$, the claim of Lemma 1 is a direct application of Axiom 1. When $T \geq 2$, according to Axiom 1, for any $2 \leq t \leq T$, there should exist a real number $\alpha_t \in (0,1)$ such that $s_{0\to t} \sim \alpha_t \cdot s_{0\to t-1} + (1-\alpha_t) \cdot s_t$. For sequence $s_{0\to T}$, we can recursively apply these preference relations as follows:

$$
\begin{aligned}
s_{0\to T} &\sim \alpha_{T-1} \cdot s_{0\to T-1} + (1-\alpha_{T-1}) \cdot s_T \\
&\sim \alpha_{T-1}\alpha_{T-2} \cdot s_{0\to T-2} + \alpha_{T-1}(1-\alpha_{T-2}) \cdot s_{T-1} + (1-\alpha_{T-1}) \cdot s_T \\
&\sim ... \\
&\sim w_0 \cdot s_0 + w_1 \cdot s_1 + ... + w_T \cdot s_T
\end{aligned}
$$

where $w_0 = \prod_{t=0}^{T-1} \alpha_t$, $w_T = 1 - \alpha_{T-1}$, and for $0 < t < T$, $w_t = (1-\alpha_{t-1}) \prod_{\tau=t}^{T-1} \alpha_\tau$. It is easy to show the sum of all these weights, denoted by w_t $(0 \leq t \leq T)$, equals 1.

Thus, if Axiom 1 holds, for any sequence $s_{0\to T}$, we can always find a convex combination of all elements in it, such that the decision maker is indifferent between the sequence and the convex combination of its elements. By Lemma 2, I show this convex combination is unique.

Lemma 2. If Axiom 1-2 holds, then suppose $s_{0\to T} \sim \sum_{t=0}^{T} w_t \cdot s_t$ and $s_{0\to T+1} \sim \sum_{t=0}^{T+1} w'_t \cdot s_t$, where $w_t > 0$, $w'_t > 0$, $\sum_{t=0}^{T} w_t = 1$, and $\sum_{t=0}^{T+1} w'_t = 1$, we must have $\frac{w_0}{w'_0} = \frac{w_1}{w'_1} = \ldots = \frac{w_T}{w'_T}$.

Corollary 1.

Lemma 3. If Axiom 1 and Axiom 3-4 holds, then for any $s_{0\to T}$ and $s'_{0\to T}$, where $u(s_t) = u(s'_t) + \Delta u$ holds for any t and Δu is a constant real number, we have $w_t = w'_t$.

Suppose $\alpha \cdot s_t + (1-\alpha) \cdot c \sim \alpha \cdot s'_t + (1-\alpha) \cdot c'$

From Axiom 4, $\alpha \cdot u(s_t) + (1-\alpha) \cdot u(c) = \alpha \cdot u(s'_t) + (1-\alpha) \cdot u(c')$

This yields $u(s_t) - u(s'_t) = \Delta u$, where $\Delta u = \frac{1-\alpha}{\alpha}(u(c') - u(c))$.

By Lemma 1, if Axiom 1 holds, we have $V(s_c) = u(c)$. The same applies to $V(s'_c)$.

By Axiom 3, we have $V(s_{0\to T}) = V(s'_{0\to T}) + \Delta u$.

This yields $\sum_{t=0}^{T} w_t u(s_t) - w'_t u(s'_t) = \Delta u$

Replace Δu, we have $\sum_{t=0}^{T} w_t u(s_t) - w'_t u(s'_t) = \sum_{t=0}^{T} w_t (u(s_t) - u(s'_t))$

So, $\sum_{t=0}^{T} (w_t - w'_t) u(s'_t) = 0$

Given each instantaneous utility can be any non-negative real number, we must have $w_t = w'_t$.

The FOC condition of the constrained optimal discounting problem is:

$$f'_t(w_t) = u(s_t) + \theta,\ \forall t \in \{0, 1, ..., T\}$$