A Introduction to Cartesian Tensors

In this text book a certain knowledge of tensors has been assumed. We restrict ourselves to *Cartesian tensors*, since all equations in fluid mechanics can in principle be developed in Cartesian coordinate systems. The most important elements of Cartesian tensors are summarized in this chapter; otherwise the literature should be consulted.

A.1 Summation Convention

When dealing with quantities in index notation we make use of *Einstein's summation convention*, which states that all indices which appear twice within an expression are to be summed. In $\mathcal{R}^3$ the summation indices run from 1 to 3:

$$
\begin{aligned}
P &= F_i u_i = \sum_{i=1}^{3} F_i u_i \ , \\
t_i &= \tau_{ji} n_j = \sum_{j=1}^{3} \tau_{ji} n_j \ , \\
\vec{x} &= x_i \vec{e}_i = \sum_{i=1}^{3} x_i \vec{e}_i \ .
\end{aligned}
$$

Indices which appear twice are called *dummy indices*. Since they vanish after carrying out the summation, they may be arbitrarily named:

$$
\begin{aligned}
F_i u_i &= F_k u_k = F_j u_j \ , \\
x_i \vec{e}_i &= x_l \vec{e}_l = x_m \vec{e}_m \ .
\end{aligned}
$$

As well as the dummy indices, single indices can also appear in equations. These *free indices* must be identical in all terms of the same equation:

$$
\begin{aligned}
t_i &= \tau_{ji} n_j \ , \\
\vec{e}_i &= a_{ij} \vec{g}_j \ , \\
a_{ij} &= b_{ik} c_{kj} + d_{ijl} n_l \ .
\end{aligned}
$$

Otherwise they may be arbitrarily named:

$$t_m = \tau_{jm} n_j \ ,$$
$$t_k = \tau_{mk} n_m \ .$$

In order to by unambiguous, the summation convention requires that an index appears no more than twice within an expression. A forbidden expression would be

$$t_i = a_{ij} b_{ij} n_j \quad \text{(wrong!)} \ ,$$

but the following would be allowed

$$t_i = -p\,\delta_{ij} n_j + 2\eta\, e_{ij} n_j \ .$$

A.2 Cartesian Tensors

A tensor consists of *tensor components* and *basis vectors*. The number of linearly independent basis vectors gives the *dimension of the tensor space*. In three dimensional space $\mathcal{R}^3$, from which, in what follows, we shall always start from, there are three linearly independent vectors, which along with three linear factors are in the position to determine a point in space uniquely. Such a set of three vectors which span a (not necessarily orthogonal) *coordinate system* can be used as a set of basis vectors. If these basis vectors are functions of position, the coordinate system which they span is called a *curvilinear coordinate system*. (Think for example of polar coordinates where the direction of the basis vectors is a function of the polar angle.) As basis vectors we choose fixed, orthogonal *unit vectors*, which we denote by $\vec{e}_i$ $(i = 1, 2, 3)$. The coordinate system spanned by these is the *Cartesian coordinate system* with the coordinate axes x_i $(i = 1, 2, 3)$.

We differentiate between tensors of different orders. Tensors of order zero are *scalars*. Since a scalar is completely independent of the choice of coordinate system, no basis vector is needed to describe it. Tensors of order one are *vectors*. The example of the position vector,

$$\vec{x} = \sum_{i=1}^{3} x_i \vec{e}_i = x_i \vec{e}_i \ , \tag{A.1}$$

shows that each component of a tensor of order one appears along with one basis vector.

Tensors of order two (*dyadics*) can be thought of as being formed from two vectors $\vec{a}$ and $\vec{b}$ multiplied together, so that each term $a_i \vec{e}_i$ of the vector $\vec{a}$ is multiplied with each term $b_j \vec{e}_j$ of the vector $\vec{b}$:

$$\boldsymbol{T} = \vec{a}\vec{b} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \vec{e}_i \vec{e}_j = a_i b_j \vec{e}_i \vec{e}_j \ . \tag{1.2}$$

This product is called the *dyadic product*, and is not to be confused with the inner product $\vec{a}\cdot\vec{b}$ (whose result is a scalar), or the outer product $\vec{a}\times\vec{b}$ (whose result is a vector). Since the dyadic product is not commutative, the basis vectors $\vec{e}_i\vec{e}_j$ in (1.2) may not be interchanged, since $a_i b_j \vec{e}_j \vec{e}_i$ would correspond to the tensor $\vec{b}\vec{a}$. If we denote the components of the tensor $\mathbf{T}$ with t_{ij} in (1.2) we obtain

$$\mathbf{T} = t_{ij}\vec{e}_i\vec{e}_j \ . \tag{1.3}$$

Therefore to every component of a second order tensor there belong two basis vectors $\vec{e}_i$ and $\vec{e}_j$. In $\mathcal{R}^3$ nine of these basis vector pairs form the so called *basis* of the tensor.

Completely analogously tensors of any order may be formed: the dyadic product of a tensor of order n and one of order m forms a tensor of order $(m+n)$. The basis of an nth order tensor in $\mathcal{R}^3$ consists of 3^n products each of n basis vectors.

Since the basis vectors for Cartesian tensors (unit vectors $\vec{e}_i$) are constant, it suffices to give the components of a tensor if a Cartesian coordinate system has already been layed down. Therefore, for a vector $\vec{x}$ it is enough to state the components

$$x_i \qquad (i = 1, 2, 3) \ ,$$

and a second order tensor $\mathbf{T}$ is fully described by its components

$$t_{ij} \qquad (i, j = 1, 2, 3) \ .$$

Therefore, if we talk about the tensor t_{ij}, we shall tacitly mean the tensor given in (1.3).

The notation in which the mathematical relations between tensors are expressed solely by their components is the *Cartesian index notation*. Because we assume fixed and orthonormal basis vectors $\vec{e}_i$, Cartesian index notation is only valid for Cartesian coordinate systems. It is possible to develop this to general curvilinear coordinate systems, but we refer for this to the more advanced literature.

The components of tensors up to the second order may be written in the form of *matrices*, so for example

$$\mathbf{T} \hat{=} \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \ . \tag{1.4}$$

Note however that not every matrix is a tensor.

In order to derive some rules we shall digress from the pure index notation and carry the basis vectors along, using a *mixed notation*. First we shall deal with the *inner product (scalar product)*:

$$\vec{a}\cdot\vec{b} = (a_i\vec{e}_i)\cdot(b_j\vec{e}_j) = a_i b_j(\vec{e}_i\cdot\vec{e}_j) \ . \tag{1.5}$$

Because of the orthogonality of the unit vectors, the product $\vec{e}_i \cdot \vec{e}_j$ is different from zero only if $i = j$. If we expand (1.5) we can easily convince ourselves that it is enough to carry out the summation

$$\vec{a} \cdot \vec{b} = a_i b_i = a_j b_j \ . \tag{1.6}$$

Clearly within a summation, the product $\vec{e}_i \cdot \vec{e}_j$ will cause the index on one of the two vector components to be exchanged. We can summarize all possible products $\vec{e}_i \cdot \vec{e}_j$ into a second order tensor:

$$\delta_{ij} = \vec{e}_i \cdot \vec{e}_j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{1.7}$$

This tensor is called the *Kronecker delta*, or because of its properties stated above, the *exchange symbol*. Multiplying a tensor with the Kronecker delta brings about an exchange of index in this tensor:

$$a_{ij}\delta_{jk} = a_{ik} \ , \tag{1.8}$$

$$a_i b_j \delta_{ij} = a_i b_i = a_j b_j \ . \tag{1.9}$$

Applying the Kronecker delta in (1.5) therefore furnishes the inner product in Cartesian index notation

$$\vec{a} \cdot \vec{b} = a_i b_j \delta_{ij} = a_i b_i \ . \tag{1.10}$$

We now consider the *outer product (vector product)* of two vectors:

$$\vec{c} = \vec{a} \times \vec{b} = (a_i \vec{e}_i) \times (b_j \vec{e}_j) = a_i b_j (\vec{e}_i \times \vec{e}_j) \ . \tag{1.11}$$

Now the outer product of two orthogonal unit vectors is zero if $i = j$, since this is outer product of parallel vectors. If $i \neq j$, the outer product of the two unit vectors is the third unit vector, possibly with negative sign. It easily follows that the relation

$$\vec{e}_i \times \vec{e}_j = \epsilon_{ijk} \vec{e}_k \tag{1.12}$$

holds if we define ϵ_{ijk} as a third order tensor having the following properties:

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation (i.e. 123, 231, 312)} \\ -1 & \text{if } ijk \text{ is an odd permutation (i.e. 321, 213, 132)} \\ 0 & \text{if at least two indices are equal} \end{cases} \ . \tag{1.13}$$

We call ϵ_{ijk} the *epsilon tensor* or the *permutation symbol*. Inserting (1.12) into (1.11) leads to

$$\vec{c} = a_i b_j \epsilon_{ijk} \vec{e}_k \ . \tag{1.14}$$

We read off the components of $\vec{c}$ from this equation as

$$c_k = \epsilon_{ijk} a_i b_j \ , \tag{1.15}$$

where we have used the fact that the order of the factors is arbitrary; we are dealing with components, that is, just numbers.

We shall now examine the behavior of a tensor if we move from a Cartesian coordinate system with basis vectors $\vec{e}_i$ to another with basis vectors $\vec{e}_i{}'$. The "dashed" coordinate system arises from rotating (and possibly also from translating) the original coordinate system. If we are dealing with a zeroth order tensor, that is a scalar, it is clear that the value of this scalar (e.g. the density of a fluid particle) cannot depend of the coordinate system. The same holds for tensors of all orders. A tensor can only have a physical meaning if it is independent of the choice of coordinate system. This is clear in the example of the position vector of a point. If $\vec{x}$ and $\vec{x}'$ denote the same arrow (Fig. A.1) in the "dashed" and the "undashed" coordinate systems, then

$$\vec{x}' = \vec{x} \ , \tag{1.16}$$

that is,

$$x_i'\vec{e}_i{}' = x_i\vec{e}_i \ . \tag{1.17}$$

To decompose the vector $\vec{x}$ into its components relative to the dashed coordinate system, we form the scalar product with $\vec{e}_j{}'$ and obtain

$$x_i'\vec{e}_i{}' \cdot \vec{e}_j{}' = x_i\vec{e}_i \cdot \vec{e}_j{}' \ . \tag{1.18}$$

The scalar product of the unit vectors in the same (dashed) coordinate system $\vec{e}_i{}' \cdot \vec{e}_j{}'$, using (1.7), furnishes just δ_{ij}. The scalar product of the unit vectors of the dashed and undashed coordinate systems forms the matrix

$$a_{ij} = \vec{e}_i \cdot \vec{e}_j{}' \tag{A.19a}$$

or

$$a_{ij} = \cos(\angle\, x_i, x_j') \ . \tag{A.19b}$$

We call the matrix a_{ij} the *rotation matrix*. It is not associated with a basis and therefore is not a tensor. Inserting (A.19a) into (1.18) leads to the desired transformation law for the components of a vector:

$$x_j' = a_{ij}x_i \ . \tag{A.20}$$

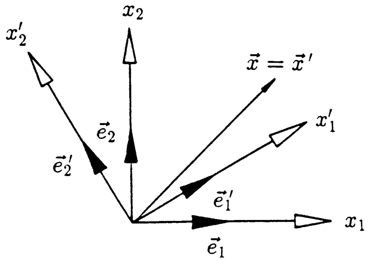


Fig. A.1. Rotation of the coordinate system

If we take the scalar product of (1.17) with $\vec{e}_j$ we decompose the vector $\vec{x}$ into its components relative to the undashed system and thus we obtain the inverse

$$x_j = a_{ji} x'_i \ . \tag{A.21}$$

The transformation and its inverse may look formally the same, but we note that in (A.20) we sum over the first index and in (A.21) over the second.

Knowing the transformation law for the components we can easily derive that for the basis vectors. To do this we relabel the dummy indices on the right-hand side of (1.17) as j so that we can insert (A.21). We obtain the equation

$$x'_i \vec{e}_i{}' = x'_i a_{ji} \vec{e}_j \ , \tag{A.22}$$

from which, using the fact that x'_i is arbitrary (independent variable), we can read off the transformation as $\vec{e}_i{}' = a_{ji}\vec{e}_j$. In order to be able to compare this with the components (A.20), we relabel the index i as j (and *vice versa*), and therefore write

$$\vec{e}_j{}' = a_{ij} \vec{e}_i \ . \tag{A.23}$$

We see that for Cartesian coordinate systems both the components and the basis vectors of a tensor obey the same transformation laws. Thus we take the inverse directly from (A.21) as

$$\vec{e}_j = a_{ji} \vec{e}_i{}' \ , \tag{A.24}$$

where we could also have obtained this formally be inserting (A.20) into (1.17).

Before we consider the transformation laws for tensors of a higher order we shall take note of one well known property of the rotation matrix. To do this we exchange the indices in the transformation (A.20) (e.g.: $x'_i = a_{ki} x_k$), insert this into (A.21) and thus obtain

$$x_j = a_{ji} a_{ki} x_k \ . \tag{A.25}$$

Since the vector components are independent variables we can read off the following identity from (A.25)

$$a_{ji} a_{ki} = \delta_{jk} \ , \tag{A.26a}$$

which reads

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I} \tag{A.26b}$$

in matrix notation. Since $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ is the equation which determines the inverse of $\mathbf{A}$, we conclude from (A.26a) that the transpose of the rotation matrix is equal to its inverse (orthogonal matrix).

The transformation for the components of a tensor of arbitrary order results from the transformations for the unit vectors (A.23) and (A.24). For clarity we shall restrict ourselves to a second order tensor whose basis we express in terms of the basis of the dashed coordinate system using the transformation (A.24) as

$$\mathbf{T} = t_{ij}\vec{e}_i\vec{e}_j = t_{ij}a_{ik}a_{jl}\vec{e}_k{}'\vec{e}_l{}' \; . \tag{A.27}$$

Because of $\mathbf{T} = \mathbf{T}' = t'_{kl}\vec{e}_k{}'\vec{e}_l{}'$ we can read off the components in the rotated system directly from (A.27) as

$$t'_{kl} = a_{ik}a_{jl}t_{ij} \; . \tag{A.28}$$

If in $\mathbf{T}'$ we replace the basis vectors using (A.23), we obtain

$$t_{kl} = a_{ki}a_{lj}t'_{ij} \; . \tag{A.29}$$

The same procedure is carried out for tensors of any order. The transformation behavior of tensor components is characteristic of them and therefore is used as the definition of a tensor. If we drop the basis vectors and use pure Cartesian index notation, the transformation behavior is the only criterion by which we can decide if a given expression is a tensor. Let us take an example: we shall examine whether the *gradient* of a scalar function is a scalar of order one. The equation $\vec{u} = \nabla\Phi$ reads in index notation

$$u_i = \frac{\partial\Phi}{\partial x_i} \; , \tag{A.30}$$

or in the rotated coordinate system

$$u'_j = \frac{\partial\Phi}{\partial x'_j} \; . \tag{A.31}$$

If $\vec{u}$ is a first order tensor, using the transformation (A.20) should transform (A.30) into (A.31)

$$u'_j = a_{ij}u_i = a_{ij}\frac{\partial\Phi}{\partial x_i} \; , \tag{A.32}$$

or using the chain rule,

$$u'_j = a_{ij}\frac{\partial\Phi}{\partial x'_k}\,\frac{\partial x'_k}{\partial x_i} \; . \tag{A.33}$$

By $x'_k = a_{jk}x_j$ we have

$$\frac{\partial x'_k}{\partial x_i} = a_{jk}\frac{\partial x_j}{\partial x_i} \; , \tag{A.34}$$

and since x_j and x_i are independent variables for $i \neq j$, we write

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij} \; , \tag{A.35}$$

so that we replace (A.34) with

$$\frac{\partial x'_k}{\partial x_i} = a_{ik} \; . \tag{A.36}$$

We should note that the result of (A.35) is the Kronecker delta and it is therefore a second order tensor, and should not be confused with (A.36), whose result is the rotation matrix and is therefore not a tensor. If we insert (A.36) into (A.33), we obtain

$$u'_j = a_{ij} a_{ik} \frac{\partial \Phi}{\partial x'_k} \ , \tag{A.37}$$

which, because of (A.26a), is identical to

$$u'_j = \delta_{jk} \frac{\partial \Phi}{\partial x'_k} = \frac{\partial \Phi}{\partial x'_j} \ . \tag{A.38}$$

This result corresponds to (A.31), and so the gradient of a scalar function is a second order tensor.

The gradient of a tensor of the nth order comes from forming the dyadic product with the Nabla operator and is therefore a tensor of the $(n+1)$th degree. An important example of this in fluid mechanics is the velocity gradient:

$$\nabla \vec{u} = \left(\vec{e}_i \frac{\partial}{\partial x_i} \right) (u_j \vec{e}_j) = \frac{\partial u_j}{\partial x_i} \vec{e}_i \vec{e}_j \ . \tag{A.39}$$

This is a second order tensor with the components

$$\nabla \vec{u} \mathrel{\hat{=}} t_{ij} = \frac{\partial u_j}{\partial x_i} \ . \tag{A.40}$$

The coordinate with respect to which we differentiate is given by the first index of t_{ij} (the row index in matrix representation) and the component of $\vec{u}$ is determined by the second index (the column index). In index notation we usually write the velocity gradient as $\partial u_i / \partial x_j$, that is in matrix representation as the transpose of (A.40). Although the matrix representation is not needed in index notation, in going from matrix equations to index notation (or *vice versa*), we should be aware of the sequence of indices determined by (A.39).

The *divergence* of the velocity vector (or of another first order tensor) reads $\partial u_i / \partial x_i$ in index notation, and formally corresponds with the scalar product of the Nabla operator with the vector $\vec{u}$. Thus symbolically the divergence reads $\nabla \cdot \vec{u}$ or else $\mathrm{div}\vec{u}$. The result is a scalar. In general, the divergence of an nth order tensor is an $(n-1)$th order tensor. Therefore the divergence of a scalar is not defined. An important quantity in fluid mechanics is the divergence of the stress tensor $\partial \tau_{ji} / \partial x_j$, which is a vector.

Every second order tensor can be decomposed into a symmetric and an antisymmetric part. From the identity

$$t_{ij} = \frac{1}{2}(t_{ij} + t_{ji}) + \frac{1}{2}(t_{ij} - t_{ji}) \tag{A.41}$$

we obtain the symmetric tensor

$$c_{ij} = \frac{1}{2}(t_{ij} + t_{ji}) \ , \tag{A.42}$$

and the antisymmetric tensor

$$b_{ij} = \frac{1}{2}(t_{ij} - t_{ji}) \ . \tag{A.43}$$

We can see that the symmetric part satisfies $c_{ij} = c_{ji}$ and the antisymmetric part satisfies $b_{ij} = -b_{ji}$. It follows immediately for the antisymmetric tensor that its diagonal elements (where $i = j$) must be zero. While a symmetric tensor has six independent components, an antisymmetric tensor is fully described by three components:

$$[b_{ij}] = \begin{bmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{bmatrix} \ . \tag{A.44}$$

In this connection we wish to refer to an important property of the ϵ tensor. To do this we multiply the decomposition of a second order tensor with the ϵ tensor:

$$p_k = \epsilon_{ijk} t_{ij} = \epsilon_{ijk} c_{ij} + \epsilon_{ijk} b_{ij} \ , \tag{A.45}$$

where c_{ij} and b_{ij} are again the symmetric and antisymmetric parts respectively of t_{ij}. We rewrite this equation as follows:

$$p_k = \frac{1}{2}(\epsilon_{ijk} c_{ij} + \epsilon_{ijk} c_{ji}) + \frac{1}{2}(\epsilon_{ijk} b_{ij} - \epsilon_{ijk} b_{ji}) \ , \tag{A.46}$$

which is allowable because of the properties of c_{ij} and b_{ij}. We now exchange the dummy indices in the second expression in brackets:

$$p_k = \frac{1}{2}(\epsilon_{ijk} c_{ij} + \epsilon_{jik} c_{ij}) + \frac{1}{2}(\epsilon_{ijk} b_{ij} - \epsilon_{jik} b_{ij}) \ . \tag{A.47}$$

From the definition of the ϵ tensor (1.13) it follows that $\epsilon_{ijk} = -\epsilon_{jik}$, so that the first bracket vanishes. We obtain the equation

$$p_k = \epsilon_{ijk} b_{ij} \ , \tag{A.48a}$$

which written in matrix form reads

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} b_{23} - b_{32} \\ b_{31} - b_{13} \\ b_{12} - b_{21} \end{bmatrix} = 2 \begin{bmatrix} b_{23} \\ -b_{13} \\ b_{12} \end{bmatrix} \ . \tag{A.48b}$$

Applying the ϵ tensor to an arbitrary second order tensor using (A.45) therefore leads to the three independent components of the antisymmetric part of the tensor (compare (A.48b) with (A.44)). From this we conclude that application of the ϵ tensor to a symmetric tensor furnishes the null vector:

$$\epsilon_{ijk} c_{ij} = 0 \ , \quad \text{if} \quad c_{ij} = c_{ji} \ . \tag{A.49}$$

Here follow four identities of the ϵ tensor, given without proof:

$$\epsilon_{ikm}\epsilon_{jln} = \det \begin{bmatrix} \delta_{ij} & \delta_{il} & \delta_{in} \\ \delta_{kj} & \delta_{kl} & \delta_{kn} \\ \delta_{mj} & \delta_{ml} & \delta_{mn} \end{bmatrix} \ . \tag{A.50}$$

Contraction by multiplication with δ_{mn} (setting $m = n$) leads to

$$\epsilon_{ikn}\epsilon_{jln} = \det \begin{bmatrix} \delta_{ij} & \delta_{il} \\ \delta_{kj} & \delta_{kl} \end{bmatrix} \ . \tag{A.51}$$

Contracting again by multiplying with δ_{kl} furnishes

$$\epsilon_{ikn}\epsilon_{jkn} = 2\delta_{ij} \ , \tag{A.52}$$

and finally for $i = j$

$$\epsilon_{ikn}\epsilon_{ikn} = 2\delta_{ii} = 6 \ . \tag{A.53}$$

Table A.1 contains a summary of the most important rules of calculation in vector and index notation.

Table A.1.

Operation	Symbolic Notation	Cartesian Index Notation
Scalar product	$c = \vec{a} \cdot \vec{b}$	$c = \delta_{ij} a_i b_j = a_i b_i$
	$\vec{c} = \vec{a} \cdot \mathbf{T}$	$c_k = \delta_{ij} a_i t_{jk} = a_i t_{ik}$
Vector product	$\vec{c} = \vec{a} \times \vec{b}$	$c_i = \epsilon_{ijk} a_j b_k$
Dyadic product	$\mathbf{T} = \vec{a}\vec{b}$	$t_{ij} = a_i b_j$
Gradient of a scalar field	$\vec{c} = \operatorname{grad} a = \nabla a$	$c_i = \dfrac{\partial a}{\partial x_i}$
Gradient of a vector field	$\mathbf{T} = \operatorname{grad} \vec{a} = \nabla \vec{a}$	$t_{ij} = \dfrac{\partial a_j}{\partial x_i}$
Divergence of a vector field	$c = \operatorname{div} \vec{a} = \nabla \cdot \vec{a}$	$c = \dfrac{\partial a_i}{\partial x_i}$
Divergence of a tensor field	$\vec{c} = \operatorname{div} \mathbf{T} = \nabla \cdot \mathbf{T}$	$c_i = \dfrac{\partial t_{ji}}{\partial x_j}$
Curl of a vector field	$\vec{c} = \operatorname{curl} \vec{a} = \nabla \times \vec{a}$	$c_i = \epsilon_{ijk} \dfrac{\partial a_k}{\partial x_j}$
Laplace operator on a scalar	$c = \triangle \varphi = \nabla \cdot \nabla \varphi$	$c = \dfrac{\partial^2 \varphi}{\partial x_i \partial x_i}$

B Curvilinear Coordinates

In applications it is often useful to use curvilinear coordinates. In order to derive the component equation for curvilinear coordinates we can start from general tensor calculus, which is valid in all coordinate systems. However, if we restrict ourselves to curvilinear but orthogonal coordinates, we can move relatively easily from the corresponding equations in symbolic notation to the desired component equations. Since it is orthogonal coordinate systems which are needed in almost all applications, we shall indeed restrict ourselves to these.

We consider the curvilinear orthogonal coordinates q_1, q_2, q_3, which can be calculated from the Cartesian coordinates x_1, x_2, and x_3:

$$\begin{aligned} q_1 &= q_1(x_1, x_2, x_3) \ , \\ q_2 &= q_2(x_1, x_2, x_3) \ , \\ q_3 &= q_3(x_1, x_2, x_3) \ , \end{aligned}$$

or in short:

$$q_i = q_i(x_j) \ . \tag{B.1}$$

We assume that (B.1) has a unique inverse:

$$x_i = x_i(q_j) \tag{B.2a}$$

or

$$\vec{x} = \vec{x}(q_j) \ . \tag{B.2b}$$

If q_2 and q_3 are kept constant, the vector $\vec{x} = \vec{x}(q_1)$ describes a curve in space which is the coordinate curve q_1. $\partial\vec{x}/\partial q_1$ is the tangent vector to this curve. The corresponding unit vector in the direction of increasing q_1 reads:

$$\vec{e}_1 = \frac{\partial\vec{x}/\partial q_1}{|\partial\vec{x}/\partial q_1|} \ . \tag{B.3}$$

If we set $|\partial\vec{x}/\partial q_1| = b_1$, we see that

$$\frac{\partial\vec{x}}{\partial q_1} = \vec{e}_1 \, b_1 \ , \tag{B.4}$$

and in the same way

$$\frac{\partial \vec{x}}{\partial q_2} = \vec{e}_2\, b_2 \ , \tag{B.5}$$

$$\frac{\partial \vec{x}}{\partial q_3} = \vec{e}_3\, b_3 \ , \tag{B.6}$$

with $b_2 = |\partial \vec{x}/\partial q_2|$ and $b_3 = |\partial \vec{x}/\partial q_3|$.

Because of $\vec{x} = \vec{x}(q_j)$ it follows that

$$\mathrm{d}\vec{x} = \frac{\partial \vec{x}}{\partial q_1}\,\mathrm{d}q_1 + \frac{\partial \vec{x}}{\partial q_2}\,\mathrm{d}q_2 + \frac{\partial \vec{x}}{\partial q_3}\,\mathrm{d}q_3 = b_1\,\mathrm{d}q_1\,\vec{e}_1 + b_2\,\mathrm{d}q_2\,\vec{e}_2 + b_3\,\mathrm{d}q_3\,\vec{e}_3 \ , \tag{B.7}$$

and, since the basis vectors are orthogonal to each other, the square of the line element is

$$\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} = b_1^2\,\mathrm{d}q_1^2 + b_2^2\,\mathrm{d}q_2^2 + b_3^2\,\mathrm{d}q_3^2 \ . \tag{B.8}$$

For the volume element $\mathrm{d}V$ (Fig. B.1) we have

$$\mathrm{d}V = b_1\,\mathrm{d}q_1\,\vec{e}_1\cdot(\,b_2\,\mathrm{d}q_2\,\vec{e}_2\times b_3\,\mathrm{d}q_3\,\vec{e}_3\,) = b_1\,b_2\,b_3\,\mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3 \ . \tag{B.9}$$

The q_1 surface element of the volume element $\mathrm{d}V$ (i.e. the surface element normal to the q_1 direction) is then

$$\mathrm{d}S_1 = |b_2\,\mathrm{d}q_2\,\vec{e}_2\times b_3\,\mathrm{d}q_3\,\vec{e}_3| = b_2\,b_3\,\mathrm{d}q_2\,\mathrm{d}q_3 \ . \tag{B.10}$$

In a similar manner we find for the remaining surface elements

$$\mathrm{d}S_2 = b_3\,b_1\,\mathrm{d}q_3\,\mathrm{d}q_1 \ , \tag{B.11}$$

$$\mathrm{d}S_3 = b_1\,b_2\,\mathrm{d}q_1\,\mathrm{d}q_2 \ . \tag{B.12}$$

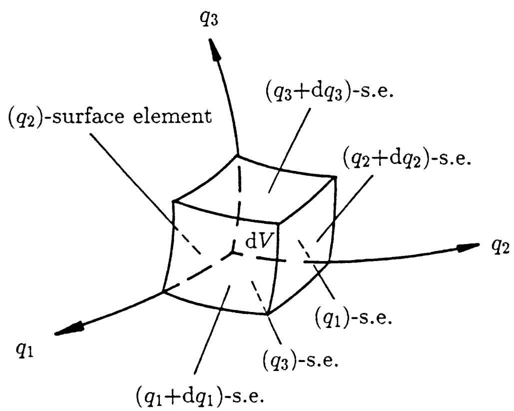


Fig. B.1. Volume element in the curvilinear orthogonal coordinate system

The continuity equation, Cauchy's equation of motion and the entropy equation read symbolically:

$$\frac{\partial \varrho}{\partial t} + \vec{u} \cdot \nabla \varrho + \varrho \nabla \cdot \vec{u} = 0 \ ,$$

$$\varrho \frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = \varrho \vec{k} + \nabla \cdot \mathbf{T} \ , \quad \text{and}$$

$$\varrho T \left[\frac{\partial s}{\partial t} + \vec{u} \cdot \nabla s \right] = \Phi + \nabla \cdot (\lambda \nabla T) \ .$$

In Cauchy's equation we write the material derivative in the form (1.78), as this is more useful for getting the equations in curvilinear coordinates:

$$\varrho \left[\frac{\partial \vec{u}}{\partial t} - \vec{u} \times (\nabla \times \vec{u}) + \nabla (\vec{u}^2/2) \right] = \varrho \vec{k} + \nabla \cdot \mathbf{T} \ . \tag{B.13}$$

Now in order to reach the component form of these equation, the Nabla operations ∇, $\nabla\cdot$ and $\nabla\times$ (gradient, divergence and curl) are given in curvilinear coordinates. The components of the vector $\nabla \Phi$ are:

$$\begin{aligned}
q_1 :& \quad (\nabla \Phi)_1 = \frac{1}{b_1} \frac{\partial \Phi}{\partial q_1} \ , \\
q_2 :& \quad (\nabla \Phi)_2 = \frac{1}{b_2} \frac{\partial \Phi}{\partial q_2} \ , \quad \text{and} \\
q_3 :& \quad (\nabla \Phi)_3 = \frac{1}{b_3} \frac{\partial \Phi}{\partial q_3} \ .
\end{aligned} \tag{B.14}$$

If u_1, u_2 and u_3 are the components of the vector $\vec{u}$ in the direction of increasing q_1, q_2 and q_3, we have:

$$\nabla \cdot \vec{u} = \frac{1}{b_1 b_2 b_3} \left[\frac{\partial}{\partial q_1} (b_2 b_3 u_1) + \frac{\partial}{\partial q_2} (b_3 b_1 u_2) + \frac{\partial}{\partial q_3} (b_1 b_2 u_3) \right] \ . \tag{B.15}$$

Since the basis vectors are orthonormal, the Laplace operator $\Delta = \nabla \cdot \nabla = \nabla^2$ can be easily calculated, if, in (B.15) we identify the components of $\vec{u}$ with the components of ∇:

$$\Delta = \frac{1}{b_1 b_2 b_3} \left\{ \frac{\partial}{\partial q_1} \left[\frac{b_2 b_3}{b_1} \frac{\partial}{\partial q_1} \right] + \frac{\partial}{\partial q_2} \left[\frac{b_3 b_1}{b_2} \frac{\partial}{\partial q_2} \right] + \frac{\partial}{\partial q_3} \left[\frac{b_1 b_2}{b_3} \frac{\partial}{\partial q_3} \right] \right\} \ . \tag{B.16}$$

$\nabla \times \vec{u}$ has the components

$$
\begin{aligned}
q_1 :& \qquad (\nabla \times \vec{u})_1 = \frac{1}{b_2\, b_3} \left[\frac{\partial}{\partial q_2}(b_3\, u_3) - \frac{\partial}{\partial q_3}(b_2\, u_2) \right] , \\
q_2 :& \qquad (\nabla \times \vec{u})_2 = \frac{1}{b_3\, b_1} \left[\frac{\partial}{\partial q_3}(b_1\, u_1) - \frac{\partial}{\partial q_1}(b_3\, u_3) \right] , \\
q_3 :& \qquad (\nabla \times \vec{u})_3 = \frac{1}{b_1\, b_2} \left[\frac{\partial}{\partial q_1}(b_2\, u_2) - \frac{\partial}{\partial q_2}(b_1\, u_1) \right] .
\end{aligned}
\tag{B.17}
$$

The components of the divergence of the stress tensor are:

$$
\begin{aligned}
q_1 :\ (\nabla \cdot \mathbf{T})_1 =& \frac{1}{b_1\, b_2\, b_3} \left[\frac{\partial}{\partial q_1}(b_2\, b_3\, \tau_{11}) + \frac{\partial}{\partial q_2}(b_3\, b_1\, \tau_{21}) + \frac{\partial}{\partial q_3}(b_1\, b_2\, \tau_{31}) \right] + \\
&+ \frac{\tau_{21}}{b_1\, b_2} \frac{\partial b_1}{\partial q_2} + \frac{\tau_{31}}{b_1\, b_3} \frac{\partial b_1}{\partial q_3} - \frac{\tau_{22}}{b_1\, b_2} \frac{\partial b_2}{\partial q_1} - \frac{\tau_{33}}{b_1\, b_3} \frac{\partial b_3}{\partial q_1} , \\
\\
q_2 :\ (\nabla \cdot \mathbf{T})_2 =& \frac{1}{b_1\, b_2\, b_3} \left[\frac{\partial}{\partial q_1}(b_2\, b_3\, \tau_{12}) + \frac{\partial}{\partial q_2}(b_3\, b_1\, \tau_{22}) + \frac{\partial}{\partial q_3}(b_1\, b_2\, \tau_{32}) \right] + \\
&+ \frac{\tau_{32}}{b_2\, b_3} \frac{\partial b_2}{\partial q_3} + \frac{\tau_{12}}{b_2\, b_1} \frac{\partial b_2}{\partial q_1} - \frac{\tau_{33}}{b_2\, b_3} \frac{\partial b_3}{\partial q_2} - \frac{\tau_{11}}{b_2\, b_1} \frac{\partial b_1}{\partial q_2} , \\
\\
q_3 :\ (\nabla \cdot \mathbf{T})_3 =& \frac{1}{b_1\, b_2\, b_3} \left[\frac{\partial}{\partial q_1}(b_2\, b_3\, \tau_{13}) + \frac{\partial}{\partial q_2}(b_3\, b_1\, \tau_{23}) + \frac{\partial}{\partial q_3}(b_1\, b_2\, \tau_{33}) \right] + \\
&+ \frac{\tau_{13}}{b_3\, b_1} \frac{\partial b_3}{\partial q_1} + \frac{\tau_{23}}{b_3\, b_2} \frac{\partial b_3}{\partial q_2} - \frac{\tau_{11}}{b_3\, b_1} \frac{\partial b_1}{\partial q_3} - \frac{\tau_{22}}{b_3\, b_2} \frac{\partial b_2}{\partial q_3} .
\end{aligned}
\tag{B.18}
$$

Here for example the stress component τ_{13} is the component in the direction of increasing q_3 which acts on the surface whose normal is in the direction of increasing q_1.

The Cauchy-Poisson law in symbolic form holds for the components of the stress :

$$\mathbf{T} = (-p + \lambda^* \nabla \cdot \vec{u})\,\mathbf{I} + 2\,\eta\,\mathbf{E} .$$

The components of the rate of deformation tensor are given by

$$
\begin{aligned}
e_{11} &= \frac{1}{b_1} \frac{\partial u_1}{\partial q_1} + \frac{u_2}{b_1\, b_2} \frac{\partial b_1}{\partial q_2} + \frac{u_3}{b_3\, b_1} \frac{\partial b_1}{\partial q_3} , \\
e_{22} &= \frac{1}{b_2} \frac{\partial u_2}{\partial q_2} + \frac{u_3}{b_2\, b_3} \frac{\partial b_2}{\partial q_3} + \frac{u_1}{b_1\, b_2} \frac{\partial b_2}{\partial q_1} , \\
e_{33} &= \frac{1}{b_3} \frac{\partial u_3}{\partial q_3} + \frac{u_1}{b_3\, b_1} \frac{\partial b_3}{\partial q_1} + \frac{u_2}{b_2\, b_3} \frac{\partial b_3}{\partial q_2} , \\
2\, e_{32} &= \frac{b_3}{b_2} \frac{\partial (u_3/b_3)}{\partial q_2} + \frac{b_2}{b_3} \frac{\partial (u_2/b_2)}{\partial q_3} = 2\, e_{23} ,
\end{aligned}
$$

$$2\,e_{13} = \frac{b_1}{b_3}\,\frac{\partial(u_1/b_1)}{\partial q_3} + \frac{b_3}{b_1}\,\frac{\partial(u_3/b_3)}{\partial q_1} = 2\,e_{31}\ , \quad \text{and}$$
$$2\,e_{21} = \frac{b_2}{b_1}\,\frac{\partial(u_2/b_2)}{\partial q_1} + \frac{b_1}{b_2}\,\frac{\partial(u_1/b_1)}{\partial q_2} = 2\,e_{12}\ . \tag{B.19}$$

As an example of how to apply this we consider spherical coordinates r, ϑ, φ with the velocity components u_r, u_ϑ, u_φ. The relation between Cartesian and spherical coordinates is given by the transformation (cf. Fig. B.4)

$$\begin{aligned} x &= r\,\cos\vartheta\ , \\ y &= r\,\sin\vartheta\,\cos\varphi\ , \\ z &= r\,\sin\vartheta\,\sin\varphi\ . \end{aligned} \tag{B.20}$$

The x axis is the polar axis and ϑ is the polar angle. With

$$q_1 = r\ , \quad q_2 = \vartheta\ , \quad \text{and} \quad q_3 = \varphi \tag{B.21}$$

it follows that

$$\begin{aligned} b_1 &= \left\{\cos^2\vartheta + \sin^2\vartheta\,(\sin^2\varphi + \cos^2\varphi)\right\}^{1/2} = 1\ , \\ b_2 &= \left\{r^2\,\sin^2\vartheta + r^2\,\cos^2\vartheta\,(\cos^2\varphi + \sin^2\varphi)\right\}^{1/2} = r\ , \\ b_3 &= \left\{r^2\,\sin^2\vartheta\,(\sin^2\varphi + \cos^2\varphi)\right\}^{1/2} = r\,\sin\vartheta\ . \end{aligned} \tag{B.22}$$

The line element reads

$$\mathrm{d}\vec{x} = \mathrm{d}r\,\vec{e}_r + r\,\mathrm{d}\vartheta\,\vec{e}_\vartheta + r\,\sin\vartheta\,\mathrm{d}\varphi\,\vec{e}_\varphi\ , \tag{B.23}$$

and the volume element is

$$\mathrm{d}V = r^2\,\sin\vartheta\,\mathrm{d}r\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\ . \tag{B.24}$$

For the surface elements we obtain

$$\begin{aligned} \mathrm{d}S_r &= r^2\,\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\ , \\ \mathrm{d}S_\vartheta &= r\,\sin\vartheta\,\mathrm{d}r\,\mathrm{d}\varphi\ , \\ \mathrm{d}S_\varphi &= r\,\mathrm{d}r\,\mathrm{d}\vartheta\ . \end{aligned} \tag{B.25}$$

The components of $\operatorname{grad}\Phi = \nabla\,\Phi$ are

$$\begin{aligned} r:&\quad (\nabla\,\Phi)_r = \frac{\partial\Phi}{\partial r}\ , \\ \vartheta:&\quad (\nabla\,\Phi)_\vartheta = \frac{1}{r}\,\frac{\partial\Phi}{\partial\vartheta}\ , \\ \varphi:&\quad (\nabla\,\Phi)_\varphi = \frac{1}{r\,\sin\vartheta}\,\frac{\partial\Phi}{\partial\varphi}\ . \end{aligned} \tag{B.26}$$

For $\operatorname{div}\vec{u} = \nabla\cdot\vec{u}$ it follows that

$$\nabla\cdot\vec{u} = (r^2\,\sin\vartheta)^{-1}\left[\frac{\partial}{\partial r}\,(r^2\,\sin\vartheta\,u_r) + \frac{\partial}{\partial\vartheta}\,(r\,\sin\vartheta\,u_\vartheta) + \frac{\partial}{\partial\varphi}\,(r\,u_\varphi)\right]\ . \tag{B.27}$$

The components of $\operatorname{curl}\vec{u} = \nabla\times\vec{u}$ are

$$\begin{aligned}
r\ :&\qquad (\nabla\times\vec{u})_r = (r^2\,\sin\vartheta)^{-1}\left[\frac{\partial}{\partial\vartheta}\,(r\,\sin\vartheta\,u_\varphi) - \frac{\partial}{\partial\varphi}\,(r\,u_\vartheta)\right]\ ,\\
\vartheta\ :&\qquad (\nabla\times\vec{u})_\vartheta = (r\,\sin\vartheta)^{-1}\left[\frac{\partial}{\partial\varphi}\,(u_r) - \frac{\partial}{\partial r}\,(r\,\sin\vartheta\,u_\varphi)\right]\ ,\\
\varphi\ :&\qquad (\nabla\times\vec{u})_\varphi = r^{-1}\left[\frac{\partial}{\partial r}\,(r\,u_\vartheta) - \frac{\partial}{\partial\vartheta}\,(u_r)\right]\ .
\end{aligned}\tag{B.28}$$

We now wish to calculate the rth component of the Navier-Stokes equations. To do this we require the rth component of $\vec{u}\times(\nabla\times\vec{u})$ and of $\nabla\cdot\mathbf{T}$:

$$\begin{aligned}
\{\vec{u}\times(\nabla\times\vec{u})\}_r = \frac{1}{r}\,u_\vartheta\left[\frac{\partial}{\partial r}\,(r\,u_\vartheta) - \frac{\partial}{\partial\vartheta}\,(u_r)\right] - \\
\frac{1}{r\,\sin\vartheta}\,u_\varphi\left[\frac{\partial}{\partial\varphi}\,(u_r) - \frac{\partial}{\partial r}\,(r\,\sin\vartheta\,u_\varphi)\right]\ ,
\end{aligned}\tag{B.29}$$

$$\begin{aligned}
(\nabla\cdot\mathbf{T})_r = \frac{1}{r^2\,\sin\vartheta}\left[\frac{\partial}{\partial r}\,(r^2\,\sin\vartheta\,\tau_{rr}) + \frac{\partial}{\partial\vartheta}\,(r\,\sin\vartheta\,\tau_{\vartheta r}) + \frac{\partial}{\partial\varphi}\,(r\,\tau_{\varphi r})\right] - \\
\frac{1}{r}\,(\tau_{\vartheta\vartheta} + \tau_{\varphi\varphi})\ ,
\end{aligned}\tag{B.30}$$

where, from (3.1b) for incompressible flow:

$$\begin{aligned}
\tau_{rr} &= -p + 2\,\eta\,e_{rr}\ ,\\
\tau_{\vartheta\vartheta} &= -p + 2\,\eta\,e_{\vartheta\vartheta}\ ,\\
\tau_{\varphi\varphi} &= -p + 2\,\eta\,e_{\varphi\varphi}\ ,\\
\tau_{\vartheta r} &= 2\,\eta\,e_{\vartheta r}\ ,\\
\tau_{\varphi r} &= 2\,\eta\,e_{\varphi r}\ ,\quad \text{and}\\
\tau_{\varphi\vartheta} &= 2\,\eta\,e_{\varphi\vartheta}\ .
\end{aligned}\tag{B.31}$$

The components of the rate of deformation tensor are

$$\begin{aligned}
e_{rr} &= \partial u_r/\partial r\ ,\\
e_{\vartheta\vartheta} &= \frac{1}{r}\,\{\partial u_\vartheta/\partial\vartheta + u_r\}\ ,\\
e_{\varphi\varphi} &= \frac{1}{r\,\sin\vartheta}\,(\partial u_\varphi/\partial\varphi) + \frac{1}{r}\,(u_r + u_\vartheta\,\cot\vartheta)\ ,
\end{aligned}$$

$$
\begin{aligned}
2\,e_{\varphi\vartheta} = 2\,e_{\vartheta\varphi} &= \sin\vartheta\,\frac{\partial}{\partial\vartheta}\left[\frac{1}{r\,\sin\vartheta}\,u_\varphi\right] + \frac{1}{\sin\vartheta}\,\frac{\partial}{\partial\varphi}\left[\frac{1}{r}\,u_\vartheta\right]\;, \\
2\,e_{r\varphi} = 2\,e_{\varphi r} &= \frac{1}{r\,\sin\vartheta}\,\partial u_r/\partial\varphi + r\,\sin\vartheta\,\frac{\partial}{\partial r}\left[\frac{1}{r\,\sin\vartheta}\,u_\varphi\right]\;, \quad \text{and} \\
2\,e_{\vartheta r} = 2\,e_{r\vartheta} &= r\,\frac{\partial}{\partial r}\left[\frac{1}{r}\,u_\vartheta\right] + \frac{1}{r}\,\partial u_r/\partial\vartheta\;. \qquad \text{(B.32)}
\end{aligned}
$$

By inserting these equations into Cauchy's equation, we obtain the rth component of the Navier-Stokes equations for incompressible flow

$$
\begin{aligned}
&\varrho\left\{\frac{\partial u_r}{\partial t} - \frac{u_\vartheta}{r}\left[\frac{\partial(r\,u_\vartheta)}{\partial r} - \frac{\partial u_r}{\partial\vartheta}\right] + \right. \\
&\left. + \frac{u_\varphi}{r\,\sin\vartheta}\left[\frac{\partial u_r}{\partial\varphi} - \frac{\partial(r\,\sin\vartheta\,u_\varphi)}{\partial r}\right] + \frac{1}{2}\,\frac{\partial(u_r^2 + u_\vartheta^2 + u_\varphi^2)}{\partial r}\right\} = \\
&= \varrho\,k_r + \frac{1}{r^2\,\sin\vartheta}\left\{\frac{\partial}{\partial r}\left[r^2\,\sin\vartheta\left\{-p + \underline{\underline{\eta\,\frac{\partial u_r}{\partial r}}} + \underline{\eta\,\frac{\partial u_r}{\partial r}}\right\}\right] + \right. \\
&+ \frac{\partial}{\partial\vartheta}\left[\underline{r^2\,\sin\vartheta\,\eta\,\frac{\partial(u_\vartheta/r)}{\partial r}} + \underline{\underline{\sin\vartheta\,\eta\,\frac{\partial u_r}{\partial\vartheta}}}\right] + \frac{\partial}{\partial\varphi}\left[\underline{\underline{\frac{\eta}{\sin\vartheta}\,\frac{\partial u_r}{\partial\varphi}}} + \right. \\
&\left.\left. + \underline{r^2\,\sin\vartheta\,\eta\,\frac{\partial}{\partial r}\left(\frac{1}{r\,\sin\vartheta}\,u_\varphi\right)}\right]\right\} + \frac{p}{r} - \frac{2\,\eta}{r^2}\left[\frac{\partial u_\vartheta}{\partial\vartheta} + \underline{u_r}\right] + \\
&+ \frac{p}{r} - \frac{2\,\eta}{r^2\,\sin\vartheta}\,\frac{\partial u_\varphi}{\partial\varphi} - \frac{2\,\eta}{r^2}\left(u_r + u_\vartheta\,\cot\vartheta\right)\;. \qquad \text{(B.33)}
\end{aligned}
$$

All terms containing p together result in $-\partial p/\partial r$. In spherical coordinates the Laplace operator reads

$$
\Delta = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left[r^2\,\frac{\partial}{\partial r}\right] + \frac{1}{r^2\,\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin\vartheta}\,\frac{\partial^2}{\partial\varphi^2}\right]\;. \qquad \text{(B.34)}
$$

We see that the doubly underlined terms can be written together as the differential operator $\eta\,\Delta u_r$. For the singly underlined terms we can write

$$
\eta\,\frac{\partial}{\partial r}\left\{\frac{1}{r^2\,\sin\vartheta}\left[\frac{\partial}{\partial r}\left(r^2\,\sin\vartheta\,u_r\right) + \frac{\partial}{\partial\vartheta}\left(r\,\sin\vartheta\,u_\vartheta\right) + \frac{\partial}{\partial\varphi}\left(r\,u_\varphi\right)\right]\right\}\;;
$$

we can convince ourselves of this by differentiating it out. The expression in curly brackets is, by (B.27) equal to $\nabla\cdot\vec{u}$, and in incompressible flow is zero.

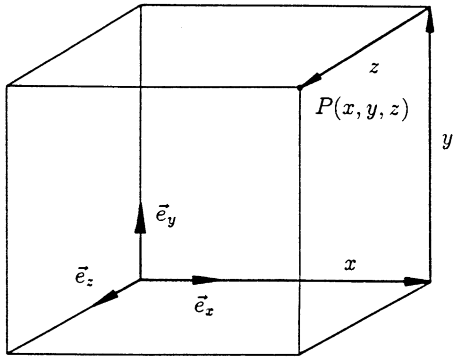


Fig. B.2. Cartesian coordinates

If we carry out all the differentiation on the left-hand side we find

$$\varrho \left\{ \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{1}{r} u_\vartheta \frac{\partial u_r}{\partial \vartheta} + \frac{1}{r \sin \vartheta} u_\varphi \frac{\partial u_r}{\partial \varphi} - \frac{u_\vartheta^2 + u_\varphi^2}{r} \right\} =$$

$$= \varrho k_r - \frac{\partial p}{\partial r} + \eta \left\{ \Delta u_r - \frac{2}{r^2} \left[u_r + \frac{\partial u_\vartheta}{\partial \vartheta} + u_\vartheta \cot \vartheta + \frac{1}{\sin \vartheta} \frac{\partial u_\varphi}{\partial \varphi} \right] \right\} \quad \text{(B.35)}$$

as the rth component of the Navier-Stokes equations. The remaining components are obtained in the same manner. We shall now summarize the results for Cartesian, cylindrical and spherical coordinates.

B.1 Cartesian Coordinates

a) Unit vectors:

$$\vec{e}_x\,,\ \vec{e}_y\,,\ \vec{e}_z$$

b) Position vector $\vec{x}$:

$$\vec{x} = x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z$$

c) Velocity vector $\vec{u}$:

$$\vec{u} = u\,\vec{e}_x + v\,\vec{e}_y + w\,\vec{e}_z$$

d) Line element:

$$\mathrm{d}\vec{x} = \mathrm{d}x\,\vec{e}_x + \mathrm{d}y\,\vec{e}_y + \mathrm{d}z\,\vec{e}_z$$

e) Surface elements:

$$
\begin{aligned}
\mathrm{d}S_x &= \mathrm{d}y\,\mathrm{d}z\\
\mathrm{d}S_y &= \mathrm{d}x\,\mathrm{d}z\\
\mathrm{d}S_z &= \mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

f) Volume element:

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

g) Gradient of the scalar $\varPhi$:

$$\operatorname{grad}\varPhi = \nabla\varPhi = \frac{\partial\varPhi}{\partial x}\,\vec{e}_x + \frac{\partial\varPhi}{\partial y}\,\vec{e}_y + \frac{\partial\varPhi}{\partial z}\,\vec{e}_z$$

h) Laplace operator on the scalar $\varPhi$:

$$\varDelta\varPhi = \nabla\cdot\nabla\varPhi = \frac{\partial^2\varPhi}{\partial x^2} + \frac{\partial^2\varPhi}{\partial y^2} + \frac{\partial^2\varPhi}{\partial z^2}$$

i) Divergence of the vector $\vec{u}$:

$$\operatorname{div}\vec{u} = \nabla\cdot\vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

j) Curl of the vector $\vec{u}$:

$$\operatorname{curl}\vec{u} = \nabla\times\vec{u} = \left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right]\,\vec{e}_x + \left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]\,\vec{e}_y + \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]\,\vec{e}_z$$

k) Laplace operator on the vector $\vec{u}$:

$$\varDelta\vec{u} = \nabla\cdot\nabla\,\vec{u} = \varDelta u\,\vec{e}_x + \varDelta v\,\vec{e}_y + \varDelta w\,\vec{e}_z$$

l) Divergence of the stress tensor $\mathbf{T}$:

$$
\begin{aligned}
\operatorname{div}\mathbf{T} = \nabla\cdot\mathbf{T} &= (\partial\tau_{xx}/\partial x + \partial\tau_{yx}/\partial y + \partial\tau_{zx}/\partial z)\,\vec{e}_x + \\
&+ (\partial\tau_{xy}/\partial x + \partial\tau_{yy}/\partial y + \partial\tau_{zy}/\partial z)\,\vec{e}_y + \\
&+ (\partial\tau_{xz}/\partial x + \partial\tau_{yz}/\partial y + \partial\tau_{zz}/\partial z)\,\vec{e}_z
\end{aligned}
$$

m) Rate of deformation tensor $\mathbf{E}$:

$$\begin{aligned} e_{xx} &= \partial u/\partial x \\ e_{yy} &= \partial v/\partial y \\ e_{zz} &= \partial w/\partial z \\ 2\,e_{xy} &= 2\,e_{yx} = \partial u/\partial y + \partial v/\partial x \\ 2\,e_{xz} &= 2\,e_{zx} = \partial u/\partial z + \partial w/\partial x \\ 2\,e_{yz} &= 2\,e_{zy} = \partial v/\partial z + \partial w/\partial y \end{aligned}$$

n) Continuity equation:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}(\varrho\, u) + \frac{\partial}{\partial y}(\varrho\, v) + \frac{\partial}{\partial z}(\varrho\, w) = 0$$

o) Navier-Stokes equations (with ϱ, $\eta = \text{const}$):

$$\begin{aligned} x:\ \ &\varrho\,(\partial u/\partial t + u\,\partial u/\partial x + v\,\partial u/\partial y + w\,\partial u/\partial z) &&= \varrho\,k_x - \partial p/\partial x + \eta\,\Delta u \\ y:\ \ &\varrho\,(\partial v/\partial t + u\,\partial v/\partial x + v\,\partial v/\partial y + w\,\partial v/\partial z) &&= \varrho\,k_y - \partial p/\partial y + \eta\,\Delta v \\ z:\ \ &\varrho\,(\partial w/\partial t + u\,\partial w/\partial x + v\,\partial w/\partial y + w\,\partial w/\partial z) &&= \varrho\,k_z - \partial p/\partial z + \eta\,\Delta w \end{aligned}$$

B.2 Cylindrical Coordinates

a) Unit vectors:

$$\begin{aligned} \vec{e}_r &= +\cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y \\ \vec{e}_\varphi &= -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y \\ \vec{e}_z &= \vec{e}_z \end{aligned}$$

b) Position vector $\vec{x}$:

$$\vec{x} = r\,\vec{e}_r + z\,\vec{e}_z$$

c) Velocity vector $\vec{u}$:

$$\vec{u} = u_r\,\vec{e}_r + u_\varphi\,\vec{e}_\varphi + u_z\,\vec{e}_z$$

d) Line element:

$$\mathrm{d}\vec{x} = \mathrm{d}r\,\vec{e}_r + r\,\mathrm{d}\varphi\,\vec{e}_\varphi + \mathrm{d}z\,\vec{e}_z$$

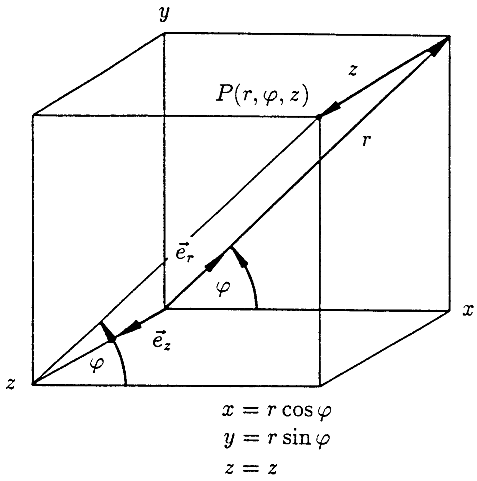


Fig. B.3. Cylindrical Coordinates

e) Surface elements:

$$\begin{aligned} \mathrm{d}S_r &= r\,\mathrm{d}\varphi\,\mathrm{d}z \\ \mathrm{d}S_\varphi &= \mathrm{d}r\,\mathrm{d}z \\ \mathrm{d}S_z &= r\,\mathrm{d}r\,\mathrm{d}\varphi \end{aligned}$$

f) Volume element:

$$\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z$$

g) Gradient of the scalar Φ:

$$\operatorname{grad}\Phi = \nabla\Phi = \frac{\partial\Phi}{\partial r}\,\vec{e}_r + \frac{1}{r}\,\frac{\partial\Phi}{\partial\varphi}\,\vec{e}_\varphi + \frac{\partial\Phi}{\partial z}\,\vec{e}_z$$

h) Laplace operator on the scalar Φ:

$$\Delta\Phi = \nabla\cdot\nabla\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

i) Divergence of the vector $\vec{u}$:

$$\operatorname{div}\vec{u} = \nabla\cdot\vec{u} = \frac{1}{r}\left\{\frac{\partial(u_r\,r)}{\partial r} + \frac{\partial u_\varphi}{\partial\varphi} + \frac{\partial(u_z\,r)}{\partial z}\right\}$$

j) Curl of the vector $\vec{u}$:

$$\operatorname{curl}\vec{u} = \nabla \times \vec{u} = \left\{\frac{1}{r}\frac{\partial u_z}{\partial \varphi} - \frac{\partial u_\varphi}{\partial z}\right\}\vec{e}_r + \left\{\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right\}\vec{e}_\varphi + \\ + \frac{1}{r}\left\{\frac{\partial (u_\varphi\, r)}{\partial r} - \frac{\partial u_r}{\partial \varphi}\right\}\vec{e}_z$$

k) Laplace operator on the vector $\vec{u}$:

$$\Delta\vec{u} = \nabla\cdot\nabla\vec{u} = \left\{\Delta u_r - \frac{1}{r^2}\left[u_r + 2\frac{\partial u_\varphi}{\partial \varphi}\right]\right\}\vec{e}_r + \\ + \left\{\Delta u_\varphi - \frac{1}{r^2}\left[u_\varphi - 2\frac{\partial u_r}{\partial \varphi}\right]\right\}\vec{e}_\varphi + \Delta u_z \vec{e}_z$$

l) Divergence of the stress tensor $\mathbf{T}$:

$$\operatorname{div}\mathbf{T} = \nabla\cdot\mathbf{T} = \left\{\frac{1}{r}\frac{\partial(\tau_{rr}\, r)}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\varphi r}}{\partial \varphi} + \frac{\partial \tau_{zr}}{\partial z} - \frac{\tau_{\varphi\varphi}}{r}\right\}\vec{e}_r + \\ + \left\{\frac{1}{r}\frac{\partial(\tau_{r\varphi}\, r)}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\varphi\varphi}}{\partial \varphi} + \frac{\partial \tau_{z\varphi}}{\partial z} + \frac{\tau_{r\varphi}}{r}\right\}\vec{e}_\varphi + \\ + \left\{\frac{1}{r}\frac{\partial(\tau_{rz}\, r)}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\varphi z}}{\partial \varphi} + \frac{\partial \tau_{zz}}{\partial z}\right\}\vec{e}_z$$

m) Rate of deformation tensor $\mathbf{E}$:

$$\begin{aligned} e_{rr} &= \frac{\partial u_r}{\partial r} \\ e_{\varphi\varphi} &= \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{1}{r}u_r \\ e_{zz} &= \frac{\partial u_z}{\partial z} \\ 2\,e_{r\varphi} = 2\,e_{\varphi r} &= r\,\frac{\partial(r^{-1}\,u_\varphi)}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \varphi} \\ 2\,e_{rz} = 2\,e_{zr} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \\ 2\,e_{\varphi z} = 2\,e_{z\varphi} &= \frac{1}{r}\frac{\partial u_z}{\partial \varphi} + \frac{\partial u_\varphi}{\partial z} \end{aligned}$$

n) Continuity equation:

$$\frac{\partial \varrho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\varrho\, u_r\, r) + \frac{1}{r}\frac{\partial}{\partial \varphi}(\varrho\, u_\varphi) + \frac{\partial}{\partial z}(\varrho\, u_z) = 0$$

o) Navier-Stokes equations (with ϱ, $\eta = \text{const}$):

$$r: \quad \varrho \left\{ \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} + \frac{1}{r} \left[u_\varphi \frac{\partial u_r}{\partial \varphi} - u_\varphi^2 \right] \right\} =$$

$$= \varrho k_r - \frac{\partial p}{\partial r} + \eta \left\{ \Delta u_r - \frac{1}{r^2} \left[u_r + 2 \frac{\partial u_\varphi}{\partial \varphi} \right] \right\}$$

$$\varphi: \quad \varrho \left\{ \frac{\partial u_\varphi}{\partial t} + u_r \frac{\partial u_\varphi}{\partial r} + u_z \frac{\partial u_\varphi}{\partial z} + \frac{1}{r} \left[u_\varphi \frac{\partial u_\varphi}{\partial \varphi} + u_r u_\varphi \right] \right\} =$$

$$= \varrho k_\varphi - \frac{1}{r} \frac{\partial p}{\partial \varphi} + \eta \left\{ \Delta u_\varphi - \frac{1}{r^2} \left[u_\varphi - 2 \frac{\partial u_r}{\partial \varphi} \right] \right\}$$

$$z: \quad \varrho \left\{ \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} + \frac{1}{r} u_\varphi \frac{\partial u_z}{\partial \varphi} \right\} = \varrho k_z - \frac{\partial p}{\partial z} + \eta \Delta u_z$$

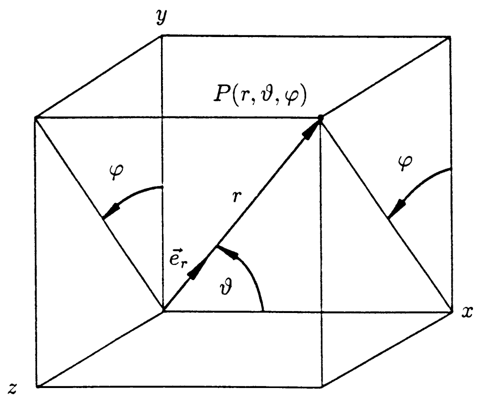
B.3 Spherical Coordinates

a) Unit vectors:

$$\begin{aligned}
\vec{e}_r &= \cos\vartheta\, \vec{e}_x + \sin\vartheta\, \cos\varphi\, \vec{e}_y + \sin\vartheta\, \sin\varphi\, \vec{e}_z \\
\vec{e}_\vartheta &= -\sin\vartheta\, \vec{e}_x + \cos\vartheta\, \cos\varphi\, \vec{e}_y + \cos\vartheta\, \sin\varphi\, \vec{e}_z \\
\vec{e}_\varphi &= -\sin\varphi\, \vec{e}_y + \cos\varphi\, \vec{e}_z
\end{aligned}$$

b) Position vector $\vec{x}$:

$$\vec{x} = r\, \vec{e}_r$$



$$\begin{aligned}
x &= r \cos\vartheta \\
y &= r \sin\vartheta \cos\varphi \\
z &= r \sin\vartheta \sin\varphi
\end{aligned}$$

Fig. B.4. Spherical coordinates

c) Velocity vector $\vec{u}$:

$$\vec{u} = u_r\,\vec{e}_r + u_\vartheta\,\vec{e}_\vartheta + u_\varphi\,\vec{e}_\varphi$$

d) Line element:

$$\mathrm{d}\vec{x} = \mathrm{d}r\,\vec{e}_r + r\,\mathrm{d}\vartheta\,\vec{e}_\vartheta + r\,\sin\vartheta\,\mathrm{d}\varphi\,\vec{e}_\varphi$$

e) Surface elements:

$$\begin{aligned} \mathrm{d}S_r &= r^2\,\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi \\ \mathrm{d}S_\vartheta &= r\,\sin\vartheta\,\mathrm{d}r\,\mathrm{d}\varphi \\ \mathrm{d}S_\varphi &= r\,\mathrm{d}r\,\mathrm{d}\vartheta \end{aligned}$$

f) Volume element:

$$\mathrm{d}V = r^2\,\sin\vartheta\,\mathrm{d}r\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$$

g) Gradient of the scalar Φ:

$$\operatorname{grad}\Phi = \nabla\Phi = \frac{\partial\Phi}{\partial r}\,\vec{e}_r + \frac{1}{r}\,\frac{\partial\Phi}{\partial\vartheta}\,\vec{e}_\vartheta + \frac{1}{r\,\sin\vartheta}\,\frac{\partial\Phi}{\partial\varphi}\,\vec{e}_\varphi$$

h) Laplace operator on the scalar Φ:

$$\Delta\Phi = \nabla\cdot\nabla\Phi = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left[r^2\,\frac{\partial\Phi}{\partial r}\right] + \frac{1}{r^2\sin\vartheta}\,\frac{\partial}{\partial\vartheta}\left[\sin\vartheta\,\frac{\partial\Phi}{\partial\vartheta}\right] + \frac{1}{r^2\sin^2\vartheta}\,\frac{\partial^2\Phi}{\partial\varphi^2}$$

i) Divergence of the vector $\vec{u}$:

$$\operatorname{div}\vec{u} = \nabla\cdot\vec{u} = \frac{1}{r^2\,\sin\vartheta}\left\{\frac{\partial(r^2\,\sin\vartheta\,u_r)}{\partial r} + \frac{\partial(r\,\sin\vartheta u_\vartheta)}{\partial\vartheta} + \frac{\partial(r\,u_\varphi)}{\partial\varphi}\right\}$$

j) Curl of the vector $\vec{u}$:

$$\begin{aligned} \operatorname{curl}\vec{u} = {} & \frac{1}{r^2\,\sin\vartheta}\left\{\frac{\partial(r\,\sin\vartheta\,u_\varphi)}{\partial\vartheta} - \frac{\partial(r\,u_\vartheta)}{\partial\varphi}\right\}\vec{e}_r + \\ & + \frac{1}{r\,\sin\vartheta}\left\{\frac{\partial u_r}{\partial\varphi} - \frac{\partial(r\,\sin\vartheta\,u_\varphi)}{\partial r}\right\}\vec{e}_\vartheta + \\ & + \frac{1}{r}\left\{\frac{\partial(r\,u_\vartheta)}{\partial r} - \frac{\partial u_r}{\partial\vartheta}\right\}\vec{e}_\varphi \end{aligned}$$

k) Laplace operator on the vector $\vec{u}$:

$$\begin{aligned} \Delta\vec{u} = {} & \left\{\Delta u_r - \frac{2}{r^2}\left[u_r + \frac{\partial u_\vartheta}{\partial\vartheta} + u_\vartheta\,\cot\vartheta + \frac{1}{\sin\vartheta}\,\frac{\partial u_\varphi}{\partial\varphi}\right]\right\}\vec{e}_r + \\ & + \left\{\Delta u_\vartheta + \frac{2}{r^2}\,\frac{\partial u_r}{\partial\vartheta} - \frac{1}{r^2\,\sin^2\vartheta}\left[u_\vartheta + 2\,\cos\vartheta\,\frac{\partial u_\varphi}{\partial\varphi}\right]\right\}\vec{e}_\vartheta + \\ & + \left\{\Delta u_\varphi - \frac{1}{r^2\,\sin^2\vartheta}\left[u_\varphi - 2\,\sin\vartheta\,\frac{\partial u_r}{\partial\varphi} - 2\,\cos\vartheta\,\frac{\partial u_\vartheta}{\partial\varphi}\right]\right\}\vec{e}_\varphi \end{aligned}$$

l) Divergence of the stress tensor $\mathbf{T}$:

$$
\begin{aligned}
\nabla\cdot\mathbf{T} = &\left\{\frac{1}{r^2\,\sin\vartheta}\left[\frac{\partial(r^2\,\sin\vartheta\,\tau_{rr})}{\partial r}+\frac{\partial(r\,\sin\vartheta\,\tau_{\vartheta r})}{\partial\vartheta}+\frac{\partial(r\,\tau_{\varphi r})}{\partial\varphi}\right]-\right.\\
&\left.-\frac{\tau_{\vartheta\vartheta}+\tau_{\varphi\varphi}}{r}\right\}\vec{e}_r+\\
&+\left\{\frac{1}{r^2\,\sin\vartheta}\left[\frac{\partial(r^2\,\sin\vartheta\,\tau_{r\vartheta})}{\partial r}+\frac{\partial(r\,\sin\vartheta\,\tau_{\vartheta\vartheta})}{\partial\vartheta}+\frac{\partial(r\,\tau_{\varphi\vartheta})}{\partial\varphi}\right]+\right.\\
&\left.+\frac{\tau_{r\vartheta}-\tau_{\varphi\varphi}\,\cot\vartheta}{r}\right\}\vec{e}_\vartheta+\\
&+\left\{\frac{1}{r^2\,\sin\vartheta}\left[\frac{\partial(r^2\,\sin\vartheta\,\tau_{r\varphi})}{\partial r}+\frac{\partial(r\,\sin\vartheta\,\tau_{\vartheta\varphi})}{\partial\vartheta}+\frac{\partial(r\,\tau_{\varphi\varphi})}{\partial\varphi}\right]+\right.\\
&\left.+\frac{\tau_{r\varphi}+\tau_{\vartheta\varphi}\,\cot\vartheta}{r}\right\}\vec{e}_\varphi
\end{aligned}
$$

m) Rate of deformation tensor $\mathbf{E}$:

$$
\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r}\\
e_{\vartheta\vartheta} &= \frac{1}{r}\,\frac{\partial u_\vartheta}{\partial\vartheta}+\frac{1}{r}\,u_r\\
e_{\varphi\varphi} &= \frac{1}{r\,\sin\vartheta}\,\frac{\partial u_\varphi}{\partial\varphi}+\frac{1}{r}\,(u_r+u_\vartheta\,\cot\vartheta)\\
2\,e_{\varphi\vartheta} &= 2\,e_{\vartheta\varphi}=\sin\vartheta\,\frac{\partial}{\partial\vartheta}\left[\frac{1}{r\,\sin\vartheta}\,u_\varphi\right]+\frac{1}{\sin\vartheta}\,\frac{\partial}{\partial\varphi}\left[\frac{1}{r}\,u_\vartheta\right]\\
2\,e_{r\varphi} &= 2\,e_{\varphi r}=\frac{1}{r\,\sin\vartheta}\,\frac{\partial u_r}{\partial\varphi}+r\,\sin\vartheta\,\frac{\partial}{\partial r}\left[\frac{1}{r\,\sin\vartheta}\,u_\varphi\right]\\
2\,e_{\vartheta r} &= 2\,e_{r\vartheta}=r\,\frac{\partial}{\partial r}\left[\frac{1}{r}\,u_\vartheta\right]+\frac{1}{r}\,\frac{\partial u_r}{\partial\vartheta}
\end{aligned}
$$

n) Continuity equation:

$$
\frac{\partial\varrho}{\partial t}+\frac{1}{r^2\,\sin\vartheta}\left[\frac{\partial}{\partial r}\,(r^2\,\sin\vartheta\,\varrho\,u_r)+\frac{\partial}{\partial\vartheta}\,(r\,\sin\vartheta\,\varrho\,u_\vartheta)+\frac{\partial}{\partial\varphi}\,(r\,\varrho\,u_\varphi)\right]=0
$$

o) Navier-Stokes equations (with ϱ, $\eta=\text{const}$):

$$
\begin{aligned}
r:\quad &\varrho\left\{\frac{\partial u_r}{\partial t}+u_r\,\frac{\partial u_r}{\partial r}+\frac{1}{r}\,u_\vartheta\,\frac{\partial u_r}{\partial\vartheta}+\frac{1}{r\,\sin\vartheta}\,u_\varphi\,\frac{\partial u_r}{\partial\varphi}-\frac{u_\vartheta^2+u_\varphi^2}{r}\right\}=\\
&=\varrho\,k_r-\frac{\partial p}{\partial r}+\eta\left\{\Delta u_r-\frac{2}{r^2}\left[u_r+\frac{\partial u_\vartheta}{\partial\vartheta}+u_\vartheta\,\cot\vartheta+\frac{1}{\sin\vartheta}\,\frac{\partial u_\varphi}{\partial\varphi}\right]\right\}
\end{aligned}
$$

$$\vartheta: \quad \varrho \left\{ \frac{\partial u_\vartheta}{\partial t} + u_r \frac{\partial u_\vartheta}{\partial r} + \frac{1}{r} u_\vartheta \frac{\partial u_\vartheta}{\partial \vartheta} + \frac{1}{r \sin\vartheta} u_\varphi \frac{\partial u_\vartheta}{\partial \varphi} + \frac{u_r u_\vartheta - u_\varphi^2 \cot\vartheta}{r} \right\} =$$
$$= \varrho k_\vartheta - \frac{1}{r} \frac{\partial p}{\partial \vartheta} + \eta \left\{ \Delta u_\vartheta + \frac{2}{r^2} \frac{\partial u_r}{\partial \vartheta} - \frac{1}{r^2 \sin^2\vartheta} \left[u_\vartheta + 2 \cos\vartheta \frac{\partial u_\varphi}{\partial \varphi} \right] \right\}$$
$$\varphi: \quad \varrho \left\{ \frac{\partial u_\varphi}{\partial t} + u_r \frac{\partial u_\varphi}{\partial r} + \frac{1}{r} u_\vartheta \frac{\partial u_\varphi}{\partial \vartheta} + \frac{1}{r \sin\vartheta} u_\varphi \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\varphi u_r + u_\vartheta u_\varphi \cot\vartheta}{r} \right\} =$$
$$= \varrho k_\varphi - \frac{1}{r \sin\vartheta} \frac{\partial p}{\partial \varphi} + \eta \left\{ \Delta u_\varphi - \frac{1}{r^2 \sin^2\vartheta} \left[u_\varphi - 2 \cos\vartheta \frac{\partial u_\vartheta}{\partial \varphi} - 2 \sin\vartheta \frac{\partial u_r}{\partial \varphi} \right] \right\}$$

C Tables and Diagrams for Compressible Flow

Table C.1

Pressure, density, temperature and area ratio as dependent on the Mach number for calorically perfect gas ($\gamma = 1.4$)

Subsonic

M	p/p_t	ϱ/ϱ_t	T/T_t	a/a_t	A^*/A
0.000	1.000000	1.000000	1.000000	1.000000	0.000000
0.010	0.999930	0.999950	0.999980	0.999990	0.017279
0.020	0.999720	0.999800	0.999920	0.999960	0.034552
0.030	0.999370	0.999550	0.999820	0.999910	0.051812
0.040	0.998881	0.999200	0.999680	0.999840	0.069054
0.050	0.998252	0.998751	0.999500	0.999750	0.086271
0.060	0.997484	0.998202	0.999281	0.999640	0.103456
0.070	0.996577	0.997554	0.999021	0.999510	0.120605
0.080	0.995533	0.996807	0.998722	0.999361	0.137711
0.090	0.994351	0.995961	0.998383	0.999191	0.154767
0.100	0.993032	0.995018	0.998004	0.999002	0.171767
0.110	0.991576	0.993976	0.997586	0.998792	0.188707
0.120	0.989985	0.992836	0.997128	0.998563	0.205579
0.130	0.988259	0.991600	0.996631	0.998314	0.222378
0.140	0.986400	0.990267	0.996095	0.998046	0.239097
0.150	0.984408	0.988838	0.995520	0.997758	0.255732
0.160	0.982284	0.987314	0.994906	0.997450	0.272276
0.170	0.980030	0.985695	0.994253	0.997122	0.288725
0.180	0.977647	0.983982	0.993562	0.996776	0.305071
0.190	0.975135	0.982176	0.992832	0.996409	0.321310
0.200	0.972497	0.980277	0.992064	0.996024	0.337437
0.210	0.969733	0.978286	0.991257	0.995619	0.353445
0.220	0.966845	0.976204	0.990413	0.995195	0.369330
0.230	0.963835	0.974032	0.989531	0.994752	0.385088
0.240	0.960703	0.971771	0.988611	0.994289	0.400711
0.250	0.957453	0.969421	0.987654	0.993808	0.416197
0.260	0.954085	0.966984	0.986660	0.993308	0.431539
0.270	0.950600	0.964460	0.985629	0.992789	0.446734

M	p/p_t	ϱ/ϱ_t	T/T_t	a/a_t	A^*/A
0.280	0.947002	0.961851	0.984562	0.992251	0.461776
0.290	0.943291	0.959157	0.983458	0.991695	0.476661
0.300	0.939470	0.956380	0.982318	0.991120	0.491385
0.310	0.935540	0.953521	0.981142	0.990526	0.505943
0.320	0.931503	0.950580	0.979931	0.989915	0.520332
0.330	0.927362	0.947559	0.978684	0.989285	0.534546
0.340	0.923117	0.944460	0.977402	0.988637	0.548584
0.350	0.918773	0.941283	0.976086	0.987971	0.562440
0.360	0.914330	0.938029	0.974735	0.987287	0.576110
0.370	0.909790	0.934700	0.973350	0.986585	0.589593
0.380	0.905156	0.931297	0.971931	0.985865	0.602883
0.390	0.900430	0.927821	0.970478	0.985128	0.615979
0.400	0.895614	0.924274	0.968992	0.984374	0.628876
0.410	0.890711	0.920657	0.967474	0.983602	0.641571
0.420	0.885722	0.916971	0.965922	0.982813	0.654063
0.430	0.880651	0.913217	0.964339	0.982008	0.666348
0.440	0.875498	0.909398	0.962723	0.981185	0.678424
0.450	0.870267	0.905513	0.961076	0.980345	0.690287
0.460	0.864960	0.901566	0.959398	0.979489	0.701937
0.470	0.859580	0.897556	0.957689	0.978616	0.713371
0.480	0.854128	0.893486	0.955950	0.977727	0.724587
0.490	0.848607	0.889357	0.954180	0.976821	0.735582
0.500	0.843019	0.885170	0.952381	0.975900	0.746356
0.510	0.837367	0.880927	0.950552	0.974963	0.756906
0.520	0.831654	0.876629	0.948695	0.974010	0.767231
0.530	0.825881	0.872279	0.946808	0.973041	0.777331
0.540	0.820050	0.867876	0.944894	0.972056	0.787202
0.550	0.814165	0.863422	0.942951	0.971057	0.796846
0.560	0.808228	0.858920	0.940982	0.970042	0.806260
0.570	0.802241	0.854371	0.938985	0.969012	0.815444
0.580	0.796206	0.849775	0.936961	0.967968	0.824398
0.590	0.790127	0.845135	0.934911	0.966908	0.833119
0.600	0.784004	0.840452	0.932836	0.965834	0.841609
0.610	0.777841	0.835728	0.930735	0.964746	0.849868
0.620	0.771639	0.830963	0.928609	0.963643	0.857894
0.630	0.765402	0.826160	0.926458	0.962527	0.865688
0.640	0.759131	0.821320	0.924283	0.961396	0.873249
0.650	0.752829	0.816443	0.922084	0.960252	0.880579
0.660	0.746498	0.811533	0.919862	0.959094	0.887678
0.670	0.740140	0.806590	0.917616	0.957923	0.894545
0.680	0.733758	0.801616	0.915349	0.956739	0.901182
0.690	0.727353	0.796612	0.913059	0.955541	0.907588
0.700	0.720928	0.791579	0.910747	0.954331	0.913765
0.710	0.714485	0.786519	0.908414	0.953107	0.919715
0.720	0.708026	0.781434	0.906060	0.951872	0.925437

M	p/p_t	ϱ/ϱ_t	T/T_t	a/a_t	A^*/A
0.730	0.701552	0.776324	0.903685	0.950624	0.930932
0.740	0.695068	0.771191	0.901291	0.949363	0.936203
0.750	0.688573	0.766037	0.898876	0.948091	0.941250
0.760	0.682071	0.760863	0.896443	0.946807	0.946074
0.770	0.675562	0.755670	0.893991	0.945511	0.950678
0.780	0.669050	0.750460	0.891520	0.944203	0.955062
0.790	0.662536	0.745234	0.889031	0.942885	0.959228
0.800	0.656022	0.739992	0.886525	0.941554	0.963178
0.810	0.649509	0.734738	0.884001	0.940214	0.966913
0.820	0.643000	0.729471	0.881461	0.938862	0.970436
0.830	0.636496	0.724193	0.878905	0.937499	0.973749
0.840	0.630000	0.718905	0.876332	0.936126	0.976853
0.850	0.623512	0.713609	0.873744	0.934743	0.979750
0.860	0.617034	0.708306	0.871141	0.933349	0.982443
0.870	0.610569	0.702997	0.868523	0.931946	0.984934
0.880	0.604117	0.697683	0.865891	0.930533	0.987225
0.890	0.597680	0.692365	0.863245	0.929110	0.989317
0.900	0.591260	0.687044	0.860585	0.927677	0.991215
0.910	0.584858	0.681722	0.857913	0.926236	0.992920
0.920	0.578476	0.676400	0.855227	0.924785	0.994434
0.930	0.572114	0.671079	0.852529	0.923325	0.995761
0.940	0.565775	0.665759	0.849820	0.921857	0.996901
0.950	0.559460	0.660443	0.847099	0.920380	0.997859
0.960	0.553169	0.655130	0.844366	0.918894	0.998637
0.970	0.546905	0.649822	0.841623	0.917400	0.999238
0.980	0.540668	0.644520	0.838870	0.915898	0.999663
0.990	0.534460	0.639225	0.836106	0.914389	0.999916
1.000	0.528282	0.633938	0.833333	0.912871	1.000000

Supersonic

M	p/p_t	ϱ/ϱ_t	T/T_t	a/a_t	A^*/A
1.000	0.528282	0.633938	0.833333	0.912871	1.000000
1.010	0.522134	0.628660	0.830551	0.911346	0.999917
1.020	0.516018	0.623391	0.827760	0.909813	0.999671
1.030	0.509935	0.618133	0.824960	0.908273	0.999263
1.040	0.503886	0.612887	0.822152	0.906726	0.998697
1.050	0.497872	0.607653	0.819336	0.905172	0.997975
1.060	0.491894	0.602432	0.816513	0.903611	0.997101
1.070	0.485952	0.597225	0.813683	0.902044	0.996077
1.080	0.480047	0.592033	0.810846	0.900470	0.994907
1.090	0.474181	0.586856	0.808002	0.898890	0.993593
1.100	0.468354	0.581696	0.805153	0.897303	0.992137
1.110	0.462567	0.576553	0.802298	0.895711	0.990543
1.120	0.456820	0.571427	0.799437	0.894113	0.988815
1.130	0.451114	0.566320	0.796572	0.892509	0.986953
1.140	0.445451	0.561232	0.793701	0.890899	0.984963
1.150	0.439829	0.556164	0.790826	0.889284	0.982845
1.160	0.434251	0.551116	0.787948	0.887664	0.980604
1.170	0.428716	0.546090	0.785065	0.886039	0.978242
1.180	0.423225	0.541085	0.782179	0.884409	0.975762
1.190	0.417778	0.536102	0.779290	0.882774	0.973167
1.200	0.412377	0.531142	0.776398	0.881134	0.970459
1.210	0.407021	0.526205	0.773503	0.879490	0.967643
1.220	0.401711	0.521292	0.770606	0.877842	0.964719
1.230	0.396446	0.516403	0.767707	0.876189	0.961691
1.240	0.391229	0.511539	0.764807	0.874532	0.958562
1.250	0.386058	0.506701	0.761905	0.872872	0.955335
1.260	0.380934	0.501888	0.759002	0.871207	0.952012
1.270	0.375858	0.497102	0.756098	0.869539	0.948597
1.280	0.370828	0.492342	0.753194	0.867867	0.945091
1.290	0.365847	0.487609	0.750289	0.866192	0.941497
1.300	0.360914	0.482903	0.747384	0.864514	0.937819
1.310	0.356029	0.478225	0.744480	0.862832	0.934057
1.320	0.351192	0.473575	0.741576	0.861148	0.930217
1.330	0.346403	0.468954	0.738672	0.859461	0.926299
1.340	0.341663	0.464361	0.735770	0.857771	0.922306
1.350	0.336971	0.459797	0.732869	0.856078	0.918242
1.360	0.332328	0.455263	0.729970	0.854383	0.914107
1.370	0.327733	0.450758	0.727072	0.852685	0.909905
1.380	0.323187	0.446283	0.724176	0.850985	0.905639
1.390	0.318690	0.441838	0.721282	0.849283	0.901310
1.400	0.314241	0.437423	0.718391	0.847579	0.896921
1.410	0.309840	0.433039	0.715502	0.845874	0.892474
1.420	0.305489	0.428686	0.712616	0.844166	0.887972
1.430	0.301185	0.424363	0.709733	0.842457	0.883416

M	p/p_t	ϱ/ϱ_t	T/T_t	a/a_t	A^*/A
1.440	0.296929	0.420072	0.706854	0.840746	0.878810
1.450	0.292722	0.415812	0.703978	0.839034	0.874154
1.460	0.288563	0.411583	0.701105	0.837320	0.869452
1.470	0.284452	0.407386	0.698236	0.835605	0.864706
1.480	0.280388	0.403220	0.695372	0.833889	0.859917
1.490	0.276372	0.399086	0.692511	0.832173	0.855087
1.500	0.272403	0.394984	0.689655	0.830455	0.850219
1.510	0.268481	0.390914	0.686804	0.828736	0.845315
1.520	0.264607	0.386876	0.683957	0.827017	0.840377
1.530	0.260779	0.382870	0.681115	0.825297	0.835405
1.540	0.256997	0.378896	0.678279	0.823577	0.830404
1.550	0.253262	0.374955	0.675448	0.821856	0.825373
1.560	0.249573	0.371045	0.672622	0.820135	0.820315
1.570	0.245930	0.367168	0.669801	0.818414	0.815233
1.580	0.242332	0.363323	0.666987	0.816693	0.810126
1.590	0.238779	0.359511	0.664178	0.814971	0.804998
1.600	0.235271	0.355730	0.661376	0.813250	0.799850
1.610	0.231808	0.351982	0.658579	0.811529	0.794683
1.620	0.228389	0.348266	0.655789	0.809808	0.789499
1.630	0.225014	0.344582	0.653006	0.808088	0.784301
1.640	0.221683	0.340930	0.650229	0.806368	0.779088
1.650	0.218395	0.337311	0.647459	0.804648	0.773863
1.660	0.215150	0.333723	0.644695	0.802929	0.768627
1.670	0.211948	0.330168	0.641939	0.801211	0.763382
1.680	0.208788	0.326644	0.639190	0.799494	0.758129
1.690	0.205670	0.323152	0.636448	0.797777	0.752869
1.700	0.202594	0.319693	0.633714	0.796061	0.747604
1.710	0.199558	0.316264	0.630987	0.794347	0.742335
1.720	0.196564	0.312868	0.628267	0.792633	0.737064
1.730	0.193611	0.309502	0.625555	0.790920	0.731790
1.740	0.190698	0.306169	0.622851	0.789209	0.726517
1.750	0.187824	0.302866	0.620155	0.787499	0.721245
1.760	0.184990	0.299595	0.617467	0.785791	0.715974
1.770	0.182195	0.296354	0.614787	0.784083	0.710707
1.780	0.179438	0.293145	0.612115	0.782378	0.705444
1.790	0.176720	0.289966	0.609451	0.780674	0.700187
1.800	0.174040	0.286818	0.606796	0.778971	0.694936
1.810	0.171398	0.283701	0.604149	0.777270	0.689692
1.820	0.168792	0.280614	0.601511	0.775571	0.684457
1.830	0.166224	0.277557	0.598881	0.773874	0.679230
1.840	0.163691	0.274530	0.596260	0.772179	0.674014
1.850	0.161195	0.271533	0.593648	0.770486	0.668810
1.860	0.158734	0.268566	0.591044	0.768794	0.663617
1.870	0.156309	0.265628	0.588450	0.767105	0.658436
1.880	0.153918	0.262720	0.585864	0.765418	0.653270

M	p/p_t	ϱ/ϱ_t	T/T_t	a/a_t	A^*/A
1.890	0.151562	0.259841	0.583288	0.763733	0.648118
1.900	0.149240	0.256991	0.580720	0.762050	0.642981
1.910	0.146951	0.254169	0.578162	0.760369	0.637859
1.920	0.144696	0.251377	0.575612	0.758691	0.632755
1.930	0.142473	0.248613	0.573072	0.757016	0.627668
1.940	0.140283	0.245877	0.570542	0.755342	0.622598
1.950	0.138126	0.243170	0.568020	0.753671	0.617547
1.960	0.135999	0.240490	0.565509	0.752003	0.612516
1.970	0.133905	0.237839	0.563006	0.750337	0.607504
1.980	0.131841	0.235215	0.560513	0.748674	0.602512
1.990	0.129808	0.232618	0.558030	0.747014	0.597542
2.000	0.127805	0.230048	0.555556	0.745356	0.592593
2.010	0.125831	0.227505	0.553091	0.743701	0.587665
2.020	0.123888	0.224990	0.550637	0.742049	0.582761
2.030	0.121973	0.222500	0.548192	0.740400	0.577879
2.040	0.120087	0.220037	0.545756	0.738753	0.573020
2.050	0.118229	0.217601	0.543331	0.737110	0.568186
2.060	0.116399	0.215190	0.540915	0.735469	0.563375
2.070	0.114597	0.212805	0.538509	0.733832	0.558589
2.080	0.112823	0.210446	0.536113	0.732197	0.553828
2.090	0.111075	0.208112	0.533726	0.730566	0.549093
2.100	0.109353	0.205803	0.531350	0.728937	0.544383
2.110	0.107658	0.203519	0.528983	0.727312	0.539699
2.120	0.105988	0.201259	0.526626	0.725690	0.535041
2.130	0.104345	0.199025	0.524279	0.724071	0.530410
2.140	0.102726	0.196814	0.521942	0.722456	0.525806
2.150	0.101132	0.194628	0.519616	0.720844	0.521229
2.160	0.099562	0.192466	0.517299	0.719235	0.516679
2.170	0.098017	0.190327	0.514991	0.717629	0.512157
2.180	0.096495	0.188212	0.512694	0.716027	0.507663
2.190	0.094997	0.186120	0.510407	0.714428	0.503197
2.200	0.093522	0.184051	0.508130	0.712832	0.498759
2.210	0.092069	0.182004	0.505863	0.711240	0.494350
2.220	0.090640	0.179981	0.503606	0.709652	0.489969
2.230	0.089232	0.177980	0.501359	0.708067	0.485617
2.240	0.087846	0.176001	0.499122	0.706485	0.481294
2.250	0.086482	0.174044	0.496894	0.704907	0.477000
2.260	0.085139	0.172110	0.494677	0.703333	0.472735
2.270	0.083817	0.170196	0.492470	0.701762	0.468500
2.280	0.082515	0.168304	0.490273	0.700195	0.464293
2.290	0.081234	0.166433	0.488086	0.698631	0.460117
2.300	0.079973	0.164584	0.485909	0.697071	0.455969
2.310	0.078731	0.162755	0.483741	0.695515	0.451851
2.320	0.077509	0.160946	0.481584	0.693963	0.447763
2.330	0.076306	0.159158	0.479437	0.692414	0.443705

M	p/p_t	ϱ/ϱ_t	T/T_t	a/a_t	A^*/A
2.340	0.075122	0.157390	0.477300	0.690869	0.439676
2.350	0.073957	0.155642	0.475172	0.689327	0.435677
2.360	0.072810	0.153914	0.473055	0.687790	0.431708
2.370	0.071681	0.152206	0.470947	0.686256	0.427769
2.380	0.070570	0.150516	0.468850	0.684726	0.423859
2.390	0.069476	0.148846	0.466762	0.683200	0.419979
2.400	0.068399	0.147195	0.464684	0.681677	0.416129
2.410	0.067340	0.145563	0.462616	0.680159	0.412309
2.420	0.066297	0.143950	0.460558	0.678644	0.408518
2.430	0.065271	0.142354	0.458510	0.677133	0.404758
2.440	0.064261	0.140777	0.456471	0.675626	0.401026
2.450	0.063267	0.139218	0.454442	0.674123	0.397325
2.460	0.062288	0.137677	0.452423	0.672624	0.393653
2.470	0.061326	0.136154	0.450414	0.671129	0.390010
2.480	0.060378	0.134648	0.448414	0.669638	0.386397
2.490	0.059445	0.133159	0.446425	0.668150	0.382814
2.500	0.058528	0.131687	0.444444	0.666667	0.379259
2.510	0.057624	0.130232	0.442474	0.665187	0.375734
2.520	0.056736	0.128794	0.440513	0.663712	0.372238
2.530	0.055861	0.127373	0.438562	0.662240	0.368771
2.540	0.055000	0.125968	0.436620	0.660772	0.365333
2.550	0.054153	0.124579	0.434688	0.659309	0.361924
2.560	0.053319	0.123206	0.432766	0.657849	0.358543
2.570	0.052499	0.121849	0.430853	0.656394	0.355192
2.580	0.051692	0.120507	0.428949	0.654942	0.351868
2.590	0.050897	0.119182	0.427055	0.653494	0.348573
2.600	0.050115	0.117871	0.425170	0.652051	0.345307
2.610	0.049346	0.116575	0.423295	0.650611	0.342068
2.620	0.048589	0.115295	0.421429	0.649176	0.338858
2.630	0.047844	0.114029	0.419572	0.647744	0.335675
2.640	0.047110	0.112778	0.417725	0.646316	0.332521
2.650	0.046389	0.111542	0.415887	0.644893	0.329394
2.660	0.045679	0.110320	0.414058	0.643474	0.326294
2.670	0.044980	0.109112	0.412239	0.642058	0.323222
2.680	0.044292	0.107918	0.410428	0.640647	0.320177
2.690	0.043616	0.106738	0.408627	0.639239	0.317159
2.700	0.042950	0.105571	0.406835	0.637836	0.314168
2.710	0.042295	0.104418	0.405052	0.636437	0.311204
2.720	0.041650	0.103279	0.403278	0.635042	0.308266
2.730	0.041016	0.102152	0.401513	0.633650	0.305355
2.740	0.040391	0.101039	0.399757	0.632263	0.302470
2.750	0.039777	0.099939	0.398010	0.630880	0.299611
2.760	0.039172	0.098851	0.396272	0.629501	0.296779
2.770	0.038577	0.097777	0.394543	0.628126	0.293972
2.780	0.037992	0.096714	0.392822	0.626755	0.291190

M	p/p_t	ϱ/ϱ_t	T/T_t	a/a_t	A^*/A
2.790	0.037415	0.095664	0.391111	0.625389	0.288435
2.800	0.036848	0.094626	0.389408	0.624026	0.285704
2.810	0.036290	0.093601	0.387714	0.622667	0.282999
2.820	0.035741	0.092587	0.386029	0.621312	0.280319
2.830	0.035201	0.091585	0.384352	0.619962	0.277663
2.840	0.034669	0.090594	0.382684	0.618615	0.275033
2.850	0.034146	0.089616	0.381025	0.617272	0.272426
2.860	0.033631	0.088648	0.379374	0.615934	0.269844
2.870	0.033124	0.087692	0.377732	0.614599	0.267286
2.880	0.032625	0.086747	0.376098	0.613268	0.264753
2.890	0.032134	0.085813	0.374473	0.611942	0.262242
2.900	0.031652	0.084889	0.372856	0.610619	0.259756
2.910	0.031176	0.083977	0.371248	0.609301	0.257293
2.920	0.030708	0.083075	0.369648	0.607986	0.254853
2.930	0.030248	0.082183	0.368056	0.606676	0.252436
2.940	0.029795	0.081302	0.366472	0.605370	0.250043
2.950	0.029349	0.080431	0.364897	0.604067	0.247672
2.960	0.028910	0.079571	0.363330	0.602768	0.245323
2.970	0.028479	0.078720	0.361771	0.601474	0.242997
2.980	0.028054	0.077879	0.360220	0.600183	0.240693
2.990	0.027635	0.077048	0.358678	0.598897	0.238412
3.000	0.027224	0.076226	0.357143	0.597614	0.236152

Table C.2

Pressure, density, temperature, total pressure and Mach number M_2 behind a normal shock as dependent on the Mach number M_1 in front of the shock for calorically perfect gas ($\gamma = 1.4$).

M_1	p_2/p_1	ϱ_2/ϱ_1	T_2/T_1	p_{t2}/p_{t1}	M_2
1.000	1.000000	1.000000	1.000000	1.000000	1.000000
1.010	1.023450	1.016694	1.006645	0.999999	0.990132
1.020	1.047133	1.033442	1.013249	0.999990	0.980520
1.030	1.071050	1.050240	1.019814	0.999967	0.971154
1.040	1.095200	1.067088	1.026345	0.999923	0.962026
1.050	1.119583	1.083982	1.032843	0.999853	0.953125
1.060	1.144200	1.100921	1.039312	0.999751	0.944445
1.070	1.169050	1.117903	1.045753	0.999611	0.935977
1.080	1.194133	1.134925	1.052169	0.999431	0.927713
1.090	1.219450	1.151985	1.058564	0.999204	0.919647
1.100	1.245000	1.169082	1.064938	0.998928	0.911770
1.110	1.270783	1.186213	1.071294	0.998599	0.904078
1.120	1.296800	1.203377	1.077634	0.998213	0.896563
1.130	1.323050	1.220571	1.083960	0.997768	0.889219
1.140	1.349533	1.237793	1.090274	0.997261	0.882042
1.150	1.376250	1.255042	1.096577	0.996690	0.875024
1.160	1.403200	1.272315	1.102872	0.996052	0.868162
1.170	1.430383	1.289610	1.109159	0.995345	0.861451
1.180	1.457800	1.306927	1.115441	0.994569	0.854884
1.190	1.485450	1.324262	1.121719	0.993720	0.848459
1.200	1.513333	1.341615	1.127994	0.992798	0.842170
1.210	1.541450	1.358983	1.134267	0.991802	0.836014
1.220	1.569800	1.376364	1.140541	0.990731	0.829987
1.230	1.598383	1.393757	1.146816	0.989583	0.824083
1.240	1.627200	1.411160	1.153094	0.988359	0.818301
1.250	1.656250	1.428571	1.159375	0.987057	0.812636
1.260	1.685533	1.445989	1.165661	0.985677	0.807085
1.270	1.715050	1.463413	1.171952	0.984219	0.801645
1.280	1.744800	1.480839	1.178251	0.982682	0.796312
1.290	1.774783	1.498267	1.184557	0.981067	0.791084
1.300	1.805000	1.515695	1.190873	0.979374	0.785957
1.310	1.835450	1.533122	1.197198	0.977602	0.780929
1.320	1.866133	1.550546	1.203533	0.975752	0.775997
1.330	1.897050	1.567965	1.209880	0.973824	0.771159
1.340	1.928200	1.585379	1.216239	0.971819	0.766412
1.350	1.959583	1.602785	1.222611	0.969737	0.761753
1.360	1.991200	1.620182	1.228997	0.967579	0.757181
1.370	2.023050	1.637569	1.235398	0.965344	0.752692
1.380	2.055133	1.654945	1.241814	0.963035	0.748286
1.390	2.087450	1.672307	1.248245	0.960652	0.743959

M_1	p_2/p_1	ϱ_2/ϱ_1	T_2/T_1	p_{t2}/p_{t1}	M_2
1.400	2.120000	1.689655	1.254694	0.958194	0.739709
1.410	2.152783	1.706988	1.261159	0.955665	0.735536
1.420	2.185800	1.724303	1.267642	0.953063	0.731436
1.430	2.219050	1.741600	1.274144	0.950390	0.727408
1.440	2.252533	1.758878	1.280665	0.947648	0.723451
1.450	2.286250	1.776135	1.287205	0.944837	0.719562
1.460	2.320200	1.793370	1.293765	0.941958	0.715740
1.470	2.354383	1.810583	1.300346	0.939012	0.711983
1.480	2.388800	1.827770	1.306947	0.936001	0.708290
1.490	2.423450	1.844933	1.313571	0.932925	0.704659
1.500	2.458333	1.862069	1.320216	0.929786	0.701089
1.510	2.493450	1.879178	1.326884	0.926586	0.697578
1.520	2.528800	1.896258	1.333574	0.923324	0.694125
1.530	2.564383	1.913308	1.340288	0.920003	0.690729
1.540	2.600200	1.930327	1.347025	0.916624	0.687388
1.550	2.636250	1.947315	1.353787	0.913188	0.684101
1.560	2.672533	1.964270	1.360573	0.909697	0.680867
1.570	2.709050	1.981192	1.367384	0.906151	0.677685
1.580	2.745800	1.998079	1.374220	0.902552	0.674553
1.590	2.782783	2.014931	1.381081	0.898901	0.671471
1.600	2.820000	2.031746	1.387969	0.895200	0.668437
1.610	2.857450	2.048524	1.394882	0.891450	0.665451
1.620	2.895133	2.065264	1.401822	0.887653	0.662511
1.630	2.933050	2.081965	1.408789	0.883809	0.659616
1.640	2.971200	2.098627	1.415783	0.879920	0.656765
1.650	3.009583	2.115248	1.422804	0.875988	0.653958
1.660	3.048200	2.131827	1.429853	0.872014	0.651194
1.670	3.087050	2.148365	1.436930	0.867999	0.648471
1.680	3.126133	2.164860	1.444035	0.863944	0.645789
1.690	3.165450	2.181311	1.451168	0.859851	0.643147
1.700	3.205000	2.197719	1.458330	0.855721	0.640544
1.710	3.244783	2.214081	1.465521	0.851556	0.637979
1.720	3.284800	2.230398	1.472741	0.847356	0.635452
1.730	3.325050	2.246669	1.479991	0.843124	0.632962
1.740	3.365533	2.262893	1.487270	0.838860	0.630508
1.750	3.406250	2.279070	1.494579	0.834565	0.628089
1.760	3.447200	2.295199	1.501918	0.830242	0.625705
1.770	3.488383	2.311279	1.509287	0.825891	0.623354
1.780	3.529800	2.327310	1.516686	0.821513	0.621037
1.790	3.571450	2.343292	1.524117	0.817111	0.618753
1.800	3.613333	2.359223	1.531577	0.812684	0.616501
1.810	3.655450	2.375104	1.539069	0.808234	0.614281
1.820	3.697800	2.390934	1.546592	0.803763	0.612091
1.830	3.740383	2.406712	1.554146	0.799271	0.609931
1.840	3.783200	2.422439	1.561732	0.794761	0.607802

M_1	p_2/p_1	ϱ_2/ϱ_1	T_2/T_1	p_{t2}/p_{t1}	M_2
1.850	3.826250	2.438112	1.569349	0.790232	0.605701
1.860	3.869533	2.453733	1.576998	0.785686	0.603629
1.870	3.913050	2.469301	1.584679	0.781125	0.601585
1.880	3.956800	2.484815	1.592392	0.776548	0.599568
1.890	4.000783	2.500274	1.600138	0.771959	0.597579
1.900	4.045000	2.515680	1.607915	0.767357	0.595616
1.910	4.089450	2.531030	1.615725	0.762743	0.593680
1.920	4.134133	2.546325	1.623568	0.758119	0.591769
1.930	4.179049	2.561565	1.631444	0.753486	0.589883
1.940	4.224200	2.576749	1.639352	0.748844	0.588022
1.950	4.269583	2.591877	1.647294	0.744195	0.586185
1.960	4.315200	2.606949	1.655268	0.739540	0.584372
1.970	4.361050	2.621964	1.663276	0.734879	0.582582
1.980	4.407133	2.636922	1.671317	0.730214	0.580816
1.990	4.453450	2.651823	1.679392	0.725545	0.579072
2.000	4.500000	2.666667	1.687500	0.720874	0.577350
2.010	4.546783	2.681453	1.695642	0.716201	0.575650
2.020	4.593800	2.696181	1.703817	0.711527	0.573972
2.030	4.641049	2.710851	1.712027	0.706853	0.572315
2.040	4.688533	2.725463	1.720270	0.702180	0.570679
2.050	4.736249	2.740016	1.728548	0.697508	0.569063
2.060	4.784200	2.754511	1.736860	0.692839	0.567467
2.070	4.832383	2.768948	1.745206	0.688174	0.565890
2.080	4.880799	2.783325	1.753586	0.683512	0.564334
2.090	4.929450	2.797643	1.762001	0.678855	0.562796
2.100	4.978333	2.811902	1.770450	0.674203	0.561277
2.110	5.027450	2.826102	1.778934	0.669558	0.559776
2.120	5.076799	2.840243	1.787453	0.664919	0.558294
2.130	5.126383	2.854324	1.796006	0.660288	0.556830
2.140	5.176199	2.868345	1.804594	0.655666	0.555383
2.150	5.226249	2.882307	1.813217	0.651052	0.553953
2.160	5.276533	2.896209	1.821875	0.646447	0.552541
2.170	5.327050	2.910052	1.830569	0.641853	0.551145
2.180	5.377800	2.923834	1.839297	0.637269	0.549766
2.190	5.428783	2.937557	1.848060	0.632697	0.548403
2.200	5.480000	2.951220	1.856859	0.628136	0.547056
2.210	5.531450	2.964823	1.865693	0.623588	0.545725
2.220	5.583133	2.978365	1.874563	0.619053	0.544409
2.230	5.635050	2.991848	1.883468	0.614531	0.543108
2.240	5.687200	3.005271	1.892408	0.610023	0.541822
2.250	5.739583	3.018634	1.901384	0.605530	0.540552
2.260	5.792200	3.031937	1.910396	0.601051	0.539295
2.270	5.845049	3.045179	1.919443	0.596588	0.538053
2.280	5.898133	3.058362	1.928527	0.592140	0.536825
2.290	5.951449	3.071485	1.937645	0.587709	0.535612

M_1	p_2/p_1	ϱ_2/ϱ_1	T_2/T_1	p_{t2}/p_{t1}	M_2
2.300	6.005000	3.084548	1.946800	0.583294	0.534411
2.310	6.058783	3.097551	1.955991	0.578897	0.533224
2.320	6.112799	3.110495	1.965218	0.574517	0.532051
2.330	6.167049	3.123379	1.974480	0.570154	0.530890
2.340	6.221533	3.136202	1.983779	0.565810	0.529743
2.350	6.276249	3.148967	1.993114	0.561484	0.528608
2.360	6.331199	3.161671	2.002485	0.557177	0.527486
2.370	6.386383	3.174316	2.011892	0.552889	0.526376
2.380	6.441799	3.186902	2.021336	0.548621	0.525278
2.390	6.497449	3.199429	2.030815	0.544372	0.524192
2.400	6.553332	3.211896	2.040332	0.540144	0.523118
2.410	6.609450	3.224304	2.049884	0.535936	0.522055
2.420	6.665800	3.236653	2.059473	0.531748	0.521004
2.430	6.722383	3.248944	2.069098	0.527581	0.519964
2.440	6.779200	3.261175	2.078760	0.523435	0.518936
2.450	6.836250	3.273347	2.088459	0.519311	0.517918
2.460	6.893533	3.285461	2.098193	0.515208	0.516911
2.470	6.951050	3.297517	2.107965	0.511126	0.515915
2.480	7.008800	3.309514	2.117773	0.507067	0.514929
2.490	7.066783	3.321453	2.127618	0.503030	0.513954
2.500	7.125000	3.333333	2.137500	0.499015	0.512989
2.510	7.183449	3.345156	2.147418	0.495022	0.512034
2.520	7.242133	3.356922	2.157373	0.491052	0.511089
2.530	7.301049	3.368629	2.167365	0.487105	0.510154
2.540	7.360199	3.380279	2.177394	0.483181	0.509228
2.550	7.419583	3.391871	2.187460	0.479280	0.508312
2.560	7.479199	3.403407	2.197562	0.475402	0.507406
2.570	7.539049	3.414885	2.207702	0.471547	0.506509
2.580	7.599133	3.426307	2.217879	0.467715	0.505620
2.590	7.659449	3.437671	2.228092	0.463907	0.504741
2.600	7.719999	3.448980	2.238343	0.460123	0.503871
2.610	7.780783	3.460232	2.248631	0.456362	0.503010
2.620	7.841799	3.471427	2.258955	0.452625	0.502157
2.630	7.903049	3.482567	2.269317	0.448912	0.501313
2.640	7.964532	3.493651	2.279716	0.445223	0.500477
2.650	8.026249	3.504679	2.290153	0.441557	0.499649
2.660	8.088199	3.515651	2.300626	0.437916	0.498830
2.670	8.150383	3.526569	2.311137	0.434298	0.498019
2.680	8.212800	3.537431	2.321685	0.430705	0.497216
2.690	8.275450	3.548239	2.332270	0.427135	0.496421
2.700	8.338333	3.558991	2.342892	0.423590	0.495634
2.710	8.401449	3.569690	2.353552	0.420069	0.494854
2.720	8.464800	3.580333	2.364249	0.416572	0.494082
2.730	8.528383	3.590923	2.374984	0.413099	0.493317
2.740	8.592199	3.601459	2.385756	0.409650	0.492560

M_1	p_2/p_1	ϱ_2/ϱ_1	T_2/T_1	p_{t2}/p_{t1}	M_2
2.750	8.656249	3.611941	2.396565	0.406226	0.491810
2.760	8.720532	3.622369	2.407412	0.402825	0.491068
2.770	8.785049	3.632744	2.418296	0.399449	0.490332
2.780	8.849799	3.643066	2.429217	0.396096	0.489604
2.790	8.914783	3.653335	2.440176	0.392768	0.488882
2.800	8.980000	3.663552	2.451173	0.389464	0.488167
2.810	9.045449	3.673716	2.462207	0.386184	0.487459
2.820	9.111133	3.683827	2.473279	0.382927	0.486758
2.830	9.177049	3.693887	2.484388	0.379695	0.486064
2.840	9.243199	3.703894	2.495535	0.376486	0.485375
2.850	9.309583	3.713850	2.506720	0.373302	0.484694
2.860	9.376199	3.723755	2.517942	0.370140	0.484019
2.870	9.443048	3.733608	2.529202	0.367003	0.483350
2.880	9.510132	3.743411	2.540499	0.363890	0.482687
2.890	9.577449	3.753163	2.551834	0.360800	0.482030
2.900	9.644999	3.762864	2.563207	0.357733	0.481380
2.910	9.712782	3.772514	2.574618	0.354690	0.480735
2.920	9.780800	3.782115	2.586066	0.351670	0.480096
2.930	9.849050	3.791666	2.597552	0.348674	0.479463
2.940	9.917533	3.801167	2.609076	0.345701	0.478836
2.950	9.986250	3.810619	2.620637	0.342750	0.478215
2.960	10.055200	3.820021	2.632236	0.339823	0.477599
2.970	10.124383	3.829375	2.643874	0.336919	0.476989
2.980	10.193799	3.838679	2.655549	0.334038	0.476384
2.990	10.263450	3.847935	2.667261	0.331180	0.475785
3.000	10.333333	3.857143	2.679012	0.328344	0.475191

Table C.3

Prandtl-Meyer function and Mach angle as dependent on the Mach number for calorically perfect gas (stated for ν and μ in degrees).

M	ν	μ
1.000	0.0000	90.0000
1.010	0.0447	81.9307
1.020	0.1257	78.6351
1.030	0.2294	76.1376
1.040	0.3510	74.0576
1.050	0.4874	72.2472
1.060	0.6367	70.6300
1.070	0.7973	69.1603
1.080	0.9680	67.8084
1.090	1.1479	66.5534
1.100	1.3362	65.3800
1.110	1.5321	64.2767
1.120	1.7350	63.2345
1.130	1.9445	62.2461
1.140	2.1600	61.3056
1.150	2.3810	60.4082
1.160	2.6073	59.5497
1.170	2.8385	58.7267
1.180	3.0743	57.9362
1.190	3.3142	57.1756
1.200	3.5582	56.4427
1.210	3.8060	55.7354
1.220	4.0572	55.0520
1.230	4.3117	54.3909
1.240	4.5694	53.7507
1.250	4.8299	53.1301
1.260	5.0931	52.5280
1.270	5.3590	51.9433
1.280	5.6272	51.3752
1.290	5.8977	50.8226
1.300	6.1703	50.2849
1.310	6.4449	49.7612
1.320	6.7213	49.2509
1.330	6.9995	48.7535
1.340	7.2794	48.2682
1.350	7.5607	47.7945
1.360	7.8435	47.3321
1.370	8.1276	46.8803
1.380	8.4130	46.4387
1.390	8.6995	46.0070
1.400	8.9870	45.5847

M	ν	μ
2.000	26.3798	30.0000
2.010	26.6550	29.8356
2.020	26.9295	29.6730
2.030	27.2033	29.5123
2.040	27.4762	29.3535
2.050	27.7484	29.1964
2.060	28.0197	29.0411
2.070	28.2903	28.8875
2.080	28.5600	28.7357
2.090	28.8290	28.5855
2.100	29.0971	28.4369
2.110	29.3644	28.2899
2.120	29.6309	28.1446
2.130	29.8965	28.0008
2.140	30.1613	27.8585
2.150	30.4253	27.7177
2.160	30.6884	27.5785
2.170	30.9507	27.4406
2.180	31.2121	27.3043
2.190	31.4727	27.1693
2.200	31.7325	27.0357
2.210	31.9914	26.9035
2.220	32.2494	26.7726
2.230	32.5066	26.6430
2.240	32.7629	26.5148
2.250	33.0184	26.3878
2.260	33.2730	26.2621
2.270	33.5268	26.1376
2.280	33.7796	26.0144
2.290	34.0316	25.8923
2.300	34.2828	25.7715
2.310	34.5331	25.6518
2.320	34.7825	25.5332
2.330	35.0310	25.4158
2.340	35.2787	25.2995
2.350	35.5255	25.1843
2.360	35.7715	25.0702
2.370	36.0165	24.9572
2.380	36.2607	24.8452
2.390	36.5041	24.7342
2.400	36.7465	24.6243

M	ν	μ
1.410	9.2756	45.1715
1.420	9.5650	44.7670
1.430	9.8553	44.3709
1.440	10.1464	43.9830
1.450	10.4381	43.6028
1.460	10.7305	43.2302
1.470	11.0235	42.8649
1.480	11.3169	42.5066
1.490	11.6109	42.1552
1.500	11.9052	41.8103
1.510	12.1999	41.4718
1.520	12.4949	41.1395
1.530	12.7901	40.8132
1.540	13.0856	40.4927
1.550	13.3812	40.1778
1.560	13.6770	39.8683
1.570	13.9728	39.5642
1.580	14.2686	39.2652
1.590	14.5645	38.9713
1.600	14.8604	38.6822
1.610	15.1561	38.3978
1.620	15.4518	38.1181
1.630	15.7473	37.8428
1.640	16.0427	37.5719
1.650	16.3379	37.3052
1.660	16.6328	37.0427
1.670	16.9276	36.7842
1.680	17.2220	36.5296
1.690	17.5161	36.2789
1.700	17.8099	36.0319
1.710	18.1034	35.7885
1.720	18.3964	35.5487
1.730	18.6891	35.3124
1.740	18.9814	35.0795
1.750	19.2732	34.8499
1.760	19.5646	34.6235
1.770	19.8554	34.4003
1.780	20.1458	34.1802
1.790	20.4357	33.9631
1.800	20.7251	33.7490
1.810	21.0139	33.5377
1.820	21.3021	33.3293
1.830	21.5898	33.1237
1.840	21.8768	32.9207
1.850	22.1633	32.7204

M	ν	μ
2.410	36.9881	24.5154
2.420	37.2289	24.4075
2.430	37.4687	24.3005
2.440	37.7077	24.1945
2.450	37.9458	24.0895
2.460	38.1831	23.9854
2.470	38.4195	23.8822
2.480	38.6551	23.7800
2.490	38.8897	23.6786
2.500	39.1236	23.5782
2.510	39.3565	23.4786
2.520	39.5886	23.3799
2.530	39.8199	23.2820
2.540	40.0503	23.1850
2.550	40.2798	23.0888
2.560	40.5085	22.9934
2.570	40.7363	22.8988
2.580	40.9633	22.8051
2.590	41.1894	22.7121
2.600	41.4147	22.6199
2.610	41.6392	22.5284
2.620	41.8628	22.4377
2.630	42.0855	22.3478
2.640	42.3074	22.2586
2.650	42.5285	22.1702
2.660	42.7488	22.0824
2.670	42.9682	21.9954
2.680	43.1868	21.9090
2.690	43.4045	21.8234
2.700	43.6215	21.7385
2.710	43.8376	21.6542
2.720	44.0529	21.5706
2.730	44.2673	21.4876
2.740	44.4810	21.4053
2.750	44.6938	21.3237
2.760	44.9059	21.2427
2.770	45.1171	21.1623
2.780	45.3275	21.0825
2.790	45.5371	21.0034
2.800	45.7459	20.9248
2.810	45.9539	20.8469
2.820	46.1611	20.7695
2.830	46.3675	20.6928
2.840	46.5731	20.6166
2.850	46.7779	20.5410

M	ν	μ
1.860	22.4492	32.5227
1.870	22.7344	32.3276
1.880	23.0190	32.1349
1.890	23.3029	31.9447
1.900	23.5861	31.7569
1.910	23.8687	31.5714
1.920	24.1506	31.3882
1.930	24.4318	31.2072
1.940	24.7123	31.0285
1.950	24.9920	30.8519
1.960	25.2711	30.6774
1.970	25.5494	30.5050
1.980	25.8269	30.3347
1.990	26.1037	30.1664
2.000	26.3798	30.0000

M	ν	μ
2.860	46.9820	20.4659
2.870	47.1852	20.3914
2.880	47.3877	20.3175
2.890	47.5894	20.2441
2.900	47.7903	20.1713
2.910	47.9905	20.0990
2.920	48.1898	20.0272
2.930	48.3884	19.9559
2.940	48.5863	19.8852
2.950	48.7833	19.8149
2.960	48.9796	19.7452
2.970	49.1752	19.6760
2.980	49.3700	19.6072
2.990	49.5640	19.5390
3.000	49.7574	19.4712

Diagram C.1

Relation between wave angle Θ and deflection angle δ for an oblique shock, and calorically perfect gas ($\gamma = 1.4$)

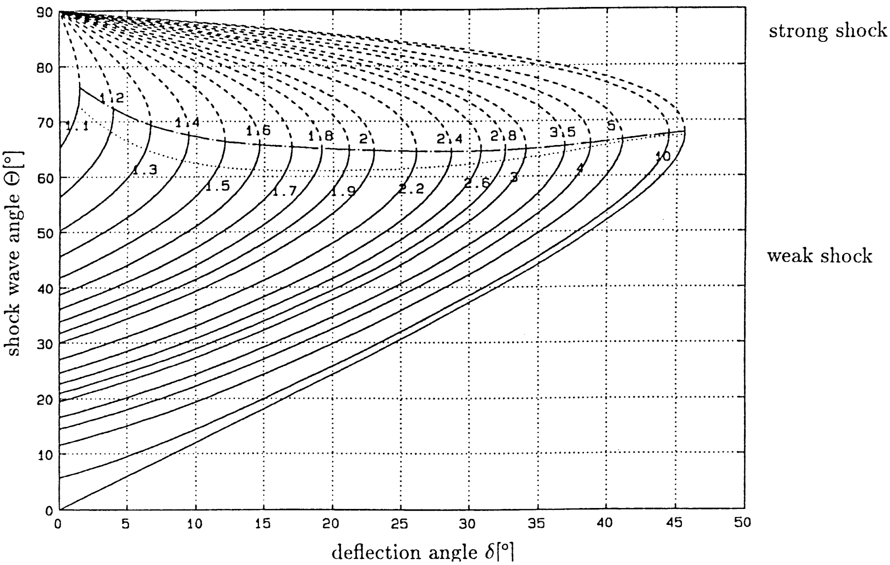
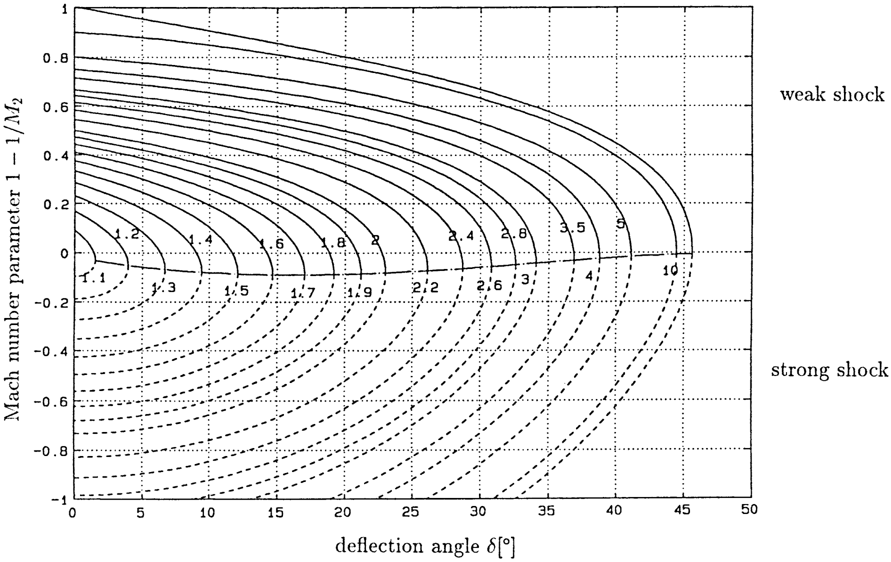


Diagram C.2

Relation between Mach number M_2 behind an oblique shock and deflection angle δ, for calorically perfect gas ($\gamma = 1.4$)



D Physical Properties of Air and Water

Table D.1. Dynamic viscosity η [in 10^{-6} kg/(m s)] of dry air

p (bar)	t (°C)								
	-50	0	25	50	100	200	300	400	500
1	14.55	17.10	18.20	19.25	21.60	25.70	29.20	32.55	35.50
5	14.63	17.16	18.26	19.30	21.64	25.73	29.23	32.57	35.52
10	14.74	17.24	18.33	19.37	21.70	25.78	29.27	32.61	35.54
50	16.01	18.08	19.11	20.07	22.26	26.20	29.60	32.86	35.76
100	18.49	19.47	20.29	21.12	23.09	26.77	30.05	33.19	36.04
200	25.19	23.19	23.40	23.76	24.98	28.03	31.10	34.10	36.69
300	32.68	27.77	27.25	27.28	27.51	29.67	32.23	34.93	37.39
400	39.78	32.59	31.41	30.98	30.27	31.39	33.44	35.85	38.15
500	46.91	37.29	35.51	34.06	32.28	33.15	34.64	36.86	38.96

Table D.2. Kinematic viscosity ν [in 10^{-8} m^2/s] of dry air

p (bar)	t (°C)								
	-50	0	25	50	100	200	300	400	500
1	931.1	1341.	1558.	1786.	2315.	3494.	4809.	6295.	7886.
5	186.1	268.5	312.2	358.1	464.2	700.5	964.1	1262.	1580.
10	93.03	134.5	156.5	179.6	232.8	351.4	483.6	632.8	792.1
50	19.11	27.74	32.39	37.19	48.13	72.43	99.35	129.5	161.8
100	10.53	14.82	17.23	19.72	25.34	37.75	51.48	66.77	83.15
200	7.402	9.140	10.33	11.57	14.33	20.68	27.83	35.74	44.00
300	7.274	7.916	8.615	9.455	11.15	15.34	20.11	25.42	31.03
400	7.633	7.687	8.112	8.693	9.825	12.84	16.38	20.38	24.64
500	8.188	7.762	8.005	8.273	8.962	11.44	14.21	17.45	20.87

Table D.3. Thermal conductivity λ [in 10^{-3} W/(m K)] of dry air

p (bar)	t (°C)								
	-50	0	25	50	100	200	300	400	500
1	20.65	24.54	26.39	28.22	31.81	38.91	45.91	52.57	58.48
5	20.86	24.68	26.53	28.32	31.89	38.91	45.92	52.56	58.42
10	21.13	24.88	26.71	28.47	32.00	38.94	45.96	52.57	58.36
50	24.11	27.15	28.78	30.26	33.53	40.34	46.86	53.41	58.98
100	28.81	30.28	31.53	32.75	35.60	42.00	48.30	54.56	60.07
200	41.96	38.00	37.90	38.21	39.91	45.18	50.69	56.62	61.96
300	54.84	46.84	45.38	44.56	44.81	48.54	53.06	58.70	63.74
400	65.15	55.30	52.83	51.29	49.97	52.59	55.91	60.95	65.56
500	73.91	62.92	59.80	57.40	54.70	55.66	58.60	62.86	67.24

Table D.4. Dynamic viscosity η [in 10^{-6} kg/(m s)] of water

p (bar)	t (°C)								
	0	20	50	100	150	200	300	400	500
1	1750.	1000.	544.0	12.11	14.15	16.18	20.25	24.30	28.40
10	1750.	1000.	544.0	279.0	181.0	15.85	20.22	24.40	28.50
50	1750.	1000.	545.0	280.0	182.0	135.0	20.06	25.00	28.90
100	1750.	1000.	545.0	281.0	183.0	136.0	90.50	25.80	29.50
150	1740.	1000.	546.0	282.0	184.0	137.0	91.70	26.90	30.30
200	1740.	999.0	546.0	283.0	185.0	138.0	93.00	28.60	31.10
300	1740.	998.0	547.0	285.0	188.0	141.0	95.50	45.70	32.70
400	1730.	997.0	548.0	287.0	190.0	143.0	98.10	62.80	36.90
500	1720.	996.0	549.0	289.0	192.0	145.0	101.0	69.30	42.20

Table D.5. Kinematic viscosity ν [in 10^{-6} m^2/s] of water

p (bar)	t (°C)								
	0	20	50	100	150	200	300	400	500
1	1.750	1.000	0.551	20.50	27.40	35.20	53.40	75.40	101.0
10	1.750	1.000	0.550	0.291	0.197	3.260	5.220	7.480	10.10
50	1.750	1.000	0.550	0.292	0.198	0.156	0.909	1.450	2.020
100	1.740	0.998	0.549	0.292	0.198	0.156	0.126	0.681	0.967
150	1.730	0.995	0.549	0.292	0.199	0.157	0.126	0.421	0.630
200	1.720	0.992	0.548	0.293	0.199	0.157	0.127	0.285	0.459
300	1.720	0.987	0.547	0.293	0.202	0.159	0.127	0.128	0.284
400	1.700	0.981	0.545	0.294	0.203	0.160	0.128	0.120	0.207
500	1.680	0.977	0.544	0.295	0.204	0.162	0.130	0.120	0.164

Table D.6. Thermal conductivity λ [in 10^{-3} W/(m K)] of water

p (bar)	t (°C)								
	0	20	50	100	150	200	300	400	500
1	569.0	604.0	643.0	24.80	28.60	33.10	43.30	54.50	66.60
10	570.0	604.0	644.0	681.0	687.0	35.00	44.20	55.20	67.20
50	573.0	608.0	647.0	684.0	690.0	668.0	52.10	59.30	70.50
100	577.0	612.0	651.0	688.0	693.0	672.0	545.0	67.40	75.70
150	581.0	616.0	655.0	691.0	696.0	676.0	559.0	81.80	82.50
200	585.0	620.0	659.0	695.0	700.0	681.0	571.0	106.0	91.50
300	592.0	627.0	666.0	701.0	706.0	689.0	592.0	263.0	117.0
400	599.0	634.0	672.0	707.0	713.0	697.0	609.0	388.0	153.0
500	606.0	640.0	678.0	713.0	720.0	704.0	622.0	437.0	202.0

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