Projecte Fi de Carrera

**Matemàtiques**

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# Physical Layer Network Coding:
## Design of Constellations over Rings

Irene de Zarza i Cubero

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Directora: María Ángeles Vázquez i Castro

Departament de Telecomunicació i Enginyeria de Sistemes

**Escola Tècnica Superior d'Enginyeria (ETSE)**

Tutor: Josep Maria Mondelo i González

Departament de Matemàtiques

**Facultat de Ciències**

**Universitat Autònoma de Barcelona (UAB)**


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UAB

El sotasignant, VÁZQUEZ I CASTRO María Ángeles, Professora de l'Escola Tècnica Superior d'Enginyeria (ETSE) de la Universitat Autònoma de Barcelona (UAB),

CERTIFICA:

Que el projecte presentat en aquesta memòria de Projecte Fi de Carrera ha estat realitzat sota la seva direcció per l'alumna DE ZARZA I CUBERO Irene.

I, perquè consti a tots els efectes, signa el present certificat.

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# Physical-Layer Network Coding:

# Design of Constellations over Rings

**Thesis**
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**Author: H. C. Zarza**
**Advisor: M. A. Vázquez**
**Tutor: J. M. Mondelo**

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# 1 Rationale and Objectives

The framework of the work of this project is the recently proposed communication scheme known as Physical-Layer Network Coding. A fundamental issue of the scheme is the geometrical properties of the set of transmission symbols. Such set is known in the field as signal constellation and is obtained making use of algebraic properties of commutative rings.

We have focused in designing constellations over EISENSTEIN Integers and GAUSSIAN Integers rings in a particular communication system of Physical-Layer Network Coding.

A survey of the mathematical tools needed is presented as well as a description of the system model under study. Moreover, we will explain the performance metrics needed to do a well-round analysis and proceed to propose a constellation design methodology.

MATLAB and PYTHON programming languages will be used throughout the project to link the theory with practice.

# 2 Introduction

A brief explanation about the main concepts related to this project follows. We will introduce a basic communication system and then proceed to explain what actually is Physical-Layer Network Coding, as well as present what is a constellation and why we focus in its design.

First we are going to introduce a basic communication system scheme based on three parts: Transmitter, Channel and Receiver.

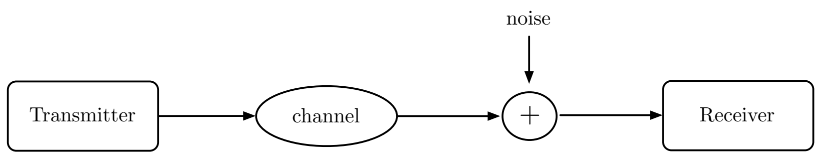



Figure 1: Communication System

The communication process involves the transmission of information from one point to another through a succession of processes. At the transmitter, the message signal (voice, music, picture, or computer data) is described by a set of symbols (signal constellation), in a form suitable for transmission over a physical medium of interest. These symbols are transmitted through the channel to the desired destination. Noise due to propagation and interferences is added in the process. Finally, at the receiver, the reconstruction of the original signal is done.

For the sake of simplicity, in this communication scheme we just consider a transmitter and a receiver. However, intermediate nodes can be added. These nodes would originally have the only function of forwarding the received packets.

## 2.1 Introduction to Physical-Layer Network Coding

The key idea of network coding was first proposed by Ahlswede et al in reference [18], who showed that, by allowing intermediate nodes to combine packets before forwarding them, a maximum information flow can be achieved in a network.

Network coding was first proposed ignoring the underlying physical nature of the communication channels. More recently, the principles of network coding have been applied at the Physical-Layer, by exploiting the natural superposition of electromagnetic waves that occurs in wireless communications.

It is a simple fact in physics that, when multiple electromagnetic waves come together within the same physical space, they add. This mixing of electromagnetic waves is a form of network coding on itself, performed by nature.

This superposition is generally regarded as an obstacle to reliable communication, where the recovery of individual signals is required. For example, in Wi-Fi networks, when multiple nodes transmit together, packet collisions occur, and none of the packets can be received correctly.

Physical-Layer Network Coding (PNC) (first proposed by Zhang et al in reference [19]) was an attempt to turn the situation around. By exploiting the network coding operation performed by nature, the “interference” could be put to good use and have a positive effect, enhancing communication efficiency.

We can see a PNC scheme in the next figure.

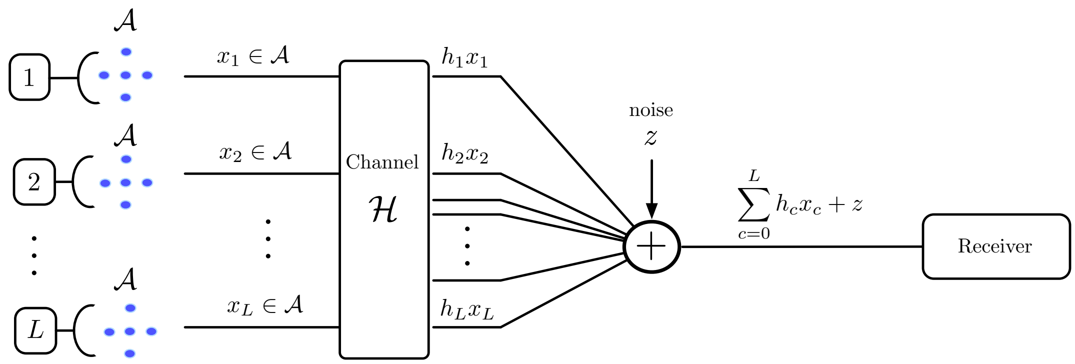



Figure 2: PNC Communication System

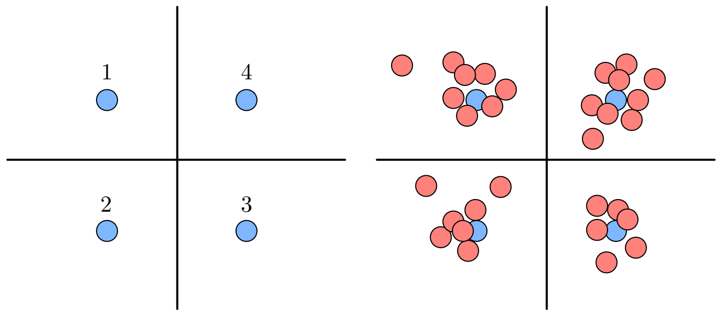
Each one of the $L$ transmitting antennas sends a symbol from the set $\mathcal{A}$ which is the set of all possible transmitted symbols, known as signal constellation. Each one of these symbols travels over the communication channel, which will be considered constant throughout the communication of a symbol, and can be understood as the product by a constant coefficient. We model the electromagnetic mixing property described as the sum of each component. Moreover, we add a noise $z$ to model the random errors that appear in the communication process. Finally, the receiver recovers the original symbols after proper processing.

## 2.2 The Role of a Signal Constellation in a Communication System

A signal constellation is a set of points in the complex plane used to describe all the possible symbols used by a system to transmit data. It is an aid to designing better communications systems. They help us design a transmission system that is less prone to errors and can possibly recover from transmission problems.

Between the transmitter and the receiver, signals can be corrupted. A signal's original output power can be reduced or attenuated by the environment, or the signal can be shifted out of phase. The signal can become so corrupted that it becomes unrecognizable. Designing a constellation that spreads the symbols apart in such a way that they are not easily confused for one another is just part of the improvement in a communication system.

Figure 3a is a signal constellation with four symbols. The blue dots represent the constellation points. Figure 3b shows what actually happens when the signal is transmitted.




(a) Signal Constellation (b) Points at the Receiver

Figure 3: Transmission and Reception Points

The red dots represent the value actually received. With this constellation arrangement, it is less likely that distortion or deterioration of the signal will result in a transmission error. Furthermore, it is possible for the receiver to detect errors and even correct them, if the transmission scheme is designed well enough. The red dots around the blue dot labeled 1 are automatically corrected to be the 1 value, because that is the symbol they most closely resemble. When a constellation is designed properly, it is less possible for one signal value to be confused with another.

The aim of this project is to design constellations in a such way that the geometry between the symbols induces a good performance in the system. The constellation performance is measured using the shape of the decision regions and error probability analysis, which will be studied in detail throughout the project.

# 3 Problem Statement and Mathematical Preliminaries

## 3.1 Problem Statement

This project aims to propose constellations for a PNC scheme. Performance metrics will be used to do the theoretical performance analysis comparing the proposed constellation with the most common constellations used nowadays.

A constellation is a set of points $\mathcal{A} = \{x_i\}$ in the complex plane. They correspond to the symbols that will be transmitted in a communication system. In designing constellations, we are interested in fixing a certain geometry between these points in order to improve the performance of the overall system. In order to do that, we will use modular arithmetic to define a set of mappings that will determine the points with an induced geometry of the proposed constellations.

## 3.2 Mathematical Preliminaries

In this work we are going to focus on the design of constellations over two different rings: GAUSSIAN Integers and EISENSTEIN Integers. Next, the theory needed will be introduced.

### 3.2.1 The Ring $\mathbb{Z}[i]$

Most of the material in this section is based on references [2], [17] and [11].

GAUSSIAN Integers are a subset of complex numbers which have integers as real and imaginary parts,

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}.$$

In $\mathbb{Z}$, magnitude is measured using the absolute value. In $\mathbb{Z}[i]$, we use the norm defined next.

**Definition 3.1.** *For $\alpha = a + bi \in \mathbb{Z}[i]$, its norm is the product*

$$N(\alpha) = \alpha\alpha^* = (a + bi)(a - bi) = a^2 + b^2$$

The reason to deal with norms in $\mathbb{Z}[i]$ instead of absolute values is that norms are integers rather than square roots.

We are going to prove one interesting property of the norm. It is a general property for both rings: GAUSSIAN Integers and EISENSTEIN Integers.

**Theorem 3.1.** *The norm is multiplicative, that is to say, for $\alpha$ and $\beta$ in $\mathbb{Z}[*]$, $N(\alpha\beta) = N(\alpha)N(\beta)$.*

*Proof.* Using the norm definition $N(\alpha\beta) = (\alpha\beta)(\alpha\beta)^* = \alpha\beta\alpha^*\beta^*$. Since we are in a commutative ring, $\mathbb{Z}[*]$, we can rewrite the expression as $N(\alpha\beta) = (\alpha\alpha^*)(\beta\beta^*) = N(\alpha)N(\beta)$.

□

The only GAUSSIAN Integers which are invertible in $\mathbb{Z}[i]$ are $\pm 1$ and $\pm i$. Invertible elements are called units.

#### 3.2.1.1 Division Theorem

The following theorem provides the analog result in $\mathbb{Z}[i]$ of the well-known division theorem with remainder in $\mathbb{Z}$.

**Theorem 3.2.** ***(Division Theorem)****. For $\alpha, \beta \in \mathbb{Z}[i]$ with $\beta \neq 0$, there are $\gamma, \rho \in \mathbb{Z}[i]$ such that $\alpha = \beta\gamma + \rho$ where $N(\rho) < N(\beta)$.*

*Proof.* Let $\alpha, \beta \in \mathbb{Z}[i]$ with $\beta \neq 0$. Then $\alpha/\beta \in \mathbb{C}$, so that $\alpha/\beta = u + iv$ for some real numbers $u$ and $v$. Let $a$ be an integer which is as close as possible to $u$. Then $|u - a| \leq 1/2$ so that $(u - a)^2 \leq 1/4$. Similarly, let $b$ be an integer which is as close as possible to $v$, so that $(v - b)^2 \leq 1/4$. Set $\gamma = a + ib$. Then $\gamma \in \mathbb{Z}[i]$. Set $\rho = \alpha - \gamma\beta$. Then $\rho \in \mathbb{Z}[i]$ because $\alpha, \gamma, \beta \in \mathbb{Z}[i]$.

It remains to prove that $N(\rho) < N(\beta)$. Note that $\beta \neq 0$, then $N(\beta) \neq 0$. By Theorem 3.1 we can assert that $N(\rho) = N((\rho/\beta)\beta) = N(\rho/\beta)N(\beta)$. So that $N(\rho) < N(\beta)$ if and only if $N(\rho/\beta) < 1$. We know that $\rho/\beta = (\alpha - \gamma\beta)/\beta = \alpha/\beta - \gamma = (u + iv) - (a + ib) = (u - a) + i(v - b)$, so that $N(\rho/\beta) = (u - a)^2 + (v - b)^2 \leq 1/4 + 1/4 = 1/2 < 1$. Therefore $\alpha = \gamma\beta + \rho$ with $N(\rho) < N(\beta)$.

□

The numbers $\gamma$ and $\rho$ are the quotient and remainder, and the remainder is bounded in size (according to its norm) by the size of the divisor $\beta$.

We note that there is a subtlety in trying to calculate $\gamma$ and $\rho$. This is best understood by working through an example.

**Example 3.1.** *Let $\alpha = 27 - 23i$ and $\beta = 8 + i$. The norm of $\beta$ is 65. We want to write $\alpha = \beta\gamma + \rho$ where $N(\rho) < 65$. The idea is to consider the ratio $\alpha/\beta$ and rationalize the denominator*

$$\frac{\alpha}{\beta} = \frac{\alpha\beta^*}{\beta\beta^*} = \frac{(27 - 23i)(8 - i)}{65} = \frac{193 - 211i}{65}.$$

*Since $193/65 = 2.969\ldots$ and $-211/65 = -3.246\ldots$ we replace each fraction with its closest integer from the left (as in the division theorem in $\mathbb{Z}$) and try $\gamma = 2 - 4i$. However:*

$$\alpha - \beta(2 - 4i) = 7 + 7i$$

*and using $\rho = 7 + 7i$ is a bad idea: $N(7 + 7i) = 98$ is larger than $N(\beta) = 65$. The usefulness of a division theorem is the smaller remainder. Therefore our choice of $\gamma$ and $\rho$ is not desirable. This is the subtlety referred to before we started our example.*

*To correct our approach, we have to think more carefully about the way we replace $193/65 = 2.969\ldots$ and $-211/65 = -3.246\ldots$ with nearby integers. Let's use the closest integer (as in the modified division theorem in $\mathbb{Z}$) rather than the closest integer from the left: try: $\gamma = 3 - 3i$. Then*

$$\alpha - \beta(3 - 3i) = -2i$$

*and $-2i$ has norm less than $N(\beta) = 65$. So we use $\gamma = 3 - 3i$ and $\rho = -2i$.*

Formally we can note the previous rounding operation in $\mathbb{Z}[i]$ as follows:

**Definition 3.2.** ***(Rounding of GAUSSIAN Integers)*** *$[a + ib] = [a] + i[b]$ where $[\cdot]$ denotes rounding to the closest integer.*

There is one interesting difference between the division theorem in $\mathbb{Z}[i]$ and the usual division theorem in $\mathbb{Z}$ (where the rounding is done to the closest integer from the left): the quotient and remainder are not unique in $\mathbb{Z}[i]$.

**Example 3.2.** *Now, we give an example where the division algorithm allows for two different outcomes. Let $\alpha = 1 + 8i$ and $\beta = 2 - 4i$. Then*

$$\frac{\alpha}{\beta} = \frac{\alpha\beta^*}{N(\beta)} = \frac{-30i + 20i}{20} = -\frac{3}{2} + i.$$

*Since $-3/2$ lies right in the middle between $-2$ and $-1$, we can use $\gamma = -1 + i$ or $\gamma = -2 + i$. Using the first choice, we obtain*

$$\alpha = \beta(-1 + i) - 1 + 2i.$$

*Using the second choice,*

$$\alpha = \beta(-2 + i) + 1 - 2i.$$

However, this lack of uniqueness in the quotient and remainder does not seriously limit the usefulness of division in $\mathbb{Z}[i]$. It is irrelevant for many important applications (such as Euclid's Algorithm).

#### 3.2.1.2 $\mathbb{Z}[i]$ as a Principal Ideal Domain

We will introduce two definitions to make the reading of this subsection easier.

**Definition 3.3.** ***(Integral Domain)****. An Integral Domain is a commutative ring without zero divisors. In other words, if $x$ and $y$ are nonzero elements of the ring, then $xy \neq 0$.*

**Definition 3.4.** ***(PID)****. A Principal Ideal Domain is an integral domain in which every ideal is principal, equivalently, every ideal can be generated by a single element.*

The structure of Principal Ideal Domain (PID) in the ring of GAUSSIAN Integers provides us an interesting framework in order to design the constellation mappings in sections (5.4) and (5.6).

That is because, in the design we are looking for structures of the form $R/\alpha R$ where $R$ is a ring and $\alpha R$ an ideal, in such a way that $R/\alpha R$ forms a field (a commutative ring in which every nonzero element has an inverse.). When $R$ is a PID, the theory that we will describe next allows to obtain it.

**Definition 3.5.** *An ideal $I$ is a maximal ideal of a ring $R$ if there are no other ideals contained between $I$ and $R$.*

The first theorem provides us a way of obtaining this structure using maximal ideals:

**Theorem 3.3.** *Let $R$ be a commutative ring with unity. Let $J$ be an ideal of $R$. Then, $J$ is a maximal ideal if and only if the quotient ring $R/J$ is a field.*

The next proposition explains us how a maximal ideal is defined in a PID:

**Proposition 3.1.** *Let $R$ be a PID and $a \in R$, $a \neq 0$. Then $aR$ is maximal if and only if $a$ is irreducible.*

**Definition 3.6.** ***(Irreducible)**. A GAUSSIAN Integer is called irreducible if its only divisors are units.*

Therefore, from the two results above we can deduce a straightforward way in order to obtain the desired structure in a PID:

**Corollary 3.1.** *If $R$ is a PID and $a \in R$ is irreducible, then $R/aR$ is a field.*

**Example 3.3.** *For a PID $R = \mathbb{Z}[i]$, $a = 2 + i$ is irreducible. Hence, $\mathbb{Z}[i]/(2 + i)\mathbb{Z}[i]$ is a field.*

Now, we are going to do some work in the proof that $\mathbb{Z}[i]$ is a PID.

First we are going to introduce the concept of Euclidean domain defined next.

**Definition 3.7.** ***(Euclidean Domain)**. An integral domain $R$ is said to be an Euclidean domain if there is a function $N$ from the set of non-zero elements of $R$ to the set of non-negative integers such that*

*(1) (Division Theorem) given $a, b \in R$ with $b \neq 0$ there exist $q, r \in R$ such that $a = bq+r$ where $N(r) < N(b)$, and*

*(2) for all non-zero elements $a$ and $b$ of $R$ we have $N(a) \leq N(ab)$.*

**Theorem 3.4.** *Euclidean domains are Principal Ideal Domains (PIDs).*

*Proof.* Let $C$ be any non-zero ideal of the Euclidean domain $R$, and let $d \in C$ be a nonzero element of minimum norm.

We claim $(d) = C$. Certainly, $(d) \subseteq C$.

Let $a \in C$. By the Division Theorem 3.2, $a = qd + r$, with $r = 0$ or $N(r) < N(d)$. Since $a - qd = r \in C$, by minimality of $N(d)$ we see $r = 0$ and $a = qd \in (d)$.

□

**Theorem 3.5.** *$\mathbb{Z}[i]$ is a Principal Ideal Domain (PID)*

*Proof.* We have to prove that $\mathbb{Z}[i]$ is an Euclidean Domain. In order to do that first we are going to prove that is an integral domain:

$\mathbb{Z}[i] \subseteq \mathbb{C}$ which is a field $\Rightarrow \mathbb{C}$ has no zero divisors $\Rightarrow \mathbb{Z}[i]$ is an integral domain.

We have proved the Division Theorem 3.2 for the ring $\mathbb{Z}[i]$ (part (1) of the Euclidean domain definition). Using that the norm is multiplicative (Theorem 3.1) we can prove part (2) of the definition. We have $N(a) \leq N(ab) = N(a)N(b)$ so $1 \leq N(b)$ it is true because $b$ is a positive integer. Therefore, $\mathbb{Z}[i]$ is an Euclidean domain. Theorem 3.4 ends the proof.

□

In a PID, every prime is irreducible. Therefore for every prime $p$ in a PID $R$ we have that $R/pR$ is a field. Now, the natural question is asking which are the primes in $\mathbb{Z}[i]$. We will answer it in the next subsection.

#### 3.2.1.3 Primes in $\mathbb{Z}[i]$

The classification of the factorization of prime integers in the ring of GAUSSIAN Integers can be summarized in the next theorem:

**Theorem 3.6.** *Let $p$ be a prime in $\mathbb{Z}^+$. The factorization of $p$ in $\mathbb{Z}[i]$ is determined by $p \bmod 4$:*

- $2 = (1+i)(1-i) = -i(1+i)^2$
- *if $p \equiv 1 \bmod 4$ then $p = \pi\pi^*$ is a product of two conjugate primes $\pi$, $\pi^*$ in $\mathbb{Z}[i]$ which are not unit multiples.*
- *if $p = 3 \bmod 4$ then $p$ stays prime in $\mathbb{Z}[i]$.*

### 3.2.2 The Ring $\mathbb{Z}[w]$

This section is based on reference [9].

EISENSTEIN Integers are a subset of complex numbers which have integer linear combination of unity and the primitive cube root of 1:

$$\mathbb{Z}[w] = \{a + bw : a, b \in \mathbb{Z}\}$$

with $w$ is a primitive cube root of 1:

$$w = e^{2\pi i/3} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$$

Magnitude is measured using the norm of $\mathbb{Z}[i]$, that has a different expression in terms of $a$ and $b$:

**Definition 3.8.** *For $\gamma = a + bw \in \mathbb{Z}[w]$, its norm is defined as*

$$N(\gamma) = \gamma\gamma^* = a^2 - ab + b^2$$

Note that if $\gamma = a_1 + b_1 w \in \mathbb{Z}[w]$, then $\gamma^* = a_1 + b_1 w^2$ is the conjugate of $\gamma$.

The only EISENSTEIN Integers which are invertible (unities) are the elements of $\mathbb{Z}[w]$ which have norm 1: $\pm 1, \pm w, \pm w^2$.

#### 3.2.2.1 Division Theorem

One reason we will be able to transfer a lot of results from $\mathbb{Z}[i]$ to $\mathbb{Z}[w]$ is the following analogue division theorem with remainder:

**Theorem 3.7.** ***(Division Theorem)**. For $\gamma, k \in \mathbb{Z}[w]$ with $k \neq 0$, there are $\gamma_1, \gamma_2 \in \mathbb{Z}[w]$ such that $\gamma = k\gamma_1 + \gamma_2$ where $N(\gamma_2) < N(k)$.*

*Proof.* Let $\gamma, k \in \mathbb{Z}[w]$ with $k \neq 0$. Then $\gamma/k \in \mathbb{C}$, so that $\gamma/k = u + wv$ for some real numbers $u$ and $v$. Let $a$ be an integer which is as close as possible to $u$. Then $|u - a| \leq 1/2$ so that $(u-a)^2 \leq 1/4$. Similarly, let $b$ be an integer which is as close as possible to $v$, so that $(v-b)^2 \leq 1/4$. Set $\gamma_1 = a + wb$. Then $\gamma_1 \in \mathbb{Z}[w]$. Set

$\gamma_2 = \gamma - \gamma_1 k$. Then $\gamma_2 \in \mathbb{Z}[w]$ because $\gamma, \gamma_1, k \in \mathbb{Z}[w]$.

It remains to prove that $N(\gamma_2) < N(k)$. Note that $k \neq 0$, then $N(k) \neq 0$. By Theorem 3.1 we can assert that $N(\gamma_2) = N((\gamma_2/k)k) = N(\gamma_2/k)N(k)$. So that $N(\gamma_2) < N(k)$ if and only if $N(\gamma_2/k) < 1$. We know that $\gamma_2/k = (\gamma - \gamma_1 k)/k = \gamma/k - \gamma_1 = (u + wv) - (a + wb) = (u - a) + w(v - b)$, so that $N(\gamma_2/k) = (u-a)^2 + (v-b)^2 - (u-a)(v-b) \leq 1/4 + 1/4 - 1/4 = 1/4 < 1$. Therefore $\gamma = \gamma_1 k + \gamma_2$ with $N(\gamma_2) < N(k)$.

□

The numbers $\gamma_1$ and $\gamma_2$ are the quotient and remainder, and the remainder is bounded in size (according to its norm) by the size of the divisor $k$.

For EISENSTEIN Integers we can define similarly a rounding operation:

**Definition 3.9.** ***(Rounding of EISENSTEIN Integers)*** *Let $z = x + wy$ with $x$, $y$ real numbers, then $[z]$ denotes the closest EISENSTEIN Integer to $z$, that is to say, $[z]$ is the EISENSTEIN Integer which gives the smallest value of $N(z - [z])$.*

#### 3.2.2.2 $\mathbb{Z}[w]$ as a Principal Ideal Domain

Equivalently to section (3.2.1.2) for GAUSSIAN Integers, we are interested in seeing that $\mathbb{Z}[w]$ is a PID.

**Theorem 3.8.** *$\mathbb{Z}[w]$ is a Principal Ideal Domain (PID)*

*Proof.* We have to prove that $\mathbb{Z}[w]$ is an Euclidean Domain. In order to do that first we are going to prove that is an integral domain:

$\mathbb{Z}[w] \subseteq \mathbb{C}$ which is a field $\Rightarrow \mathbb{C}$ has no zero divisors $\Rightarrow \mathbb{Z}[w]$ is an integral domain.

We have proved the Division Theorem 3.7 for the ring $\mathbb{Z}[w]$ (part (1) of the Euclidean domain definition). Using that the norm is multiplicative (Theorem 3.1) we can prove part (2) of the definition. We have $N(a) \leq N(ab) = N(a)N(b)$ so $1 \leq N(b)$ it is true because $b$ is a positive integer. Therefore, $\mathbb{Z}[w]$ is an Euclidean domain. Theorem 3.4 ends the proof.

□

#### 3.2.2.3 Primes in $\mathbb{Z}[w]$

There exists an analog classification theorem for primes in the ring of EISENSTEIN Integers:

**Theorem 3.9.** *There are three classes of EISENSTEIN primes:*

- *$1 - w$ and its unit multiples.*
- *Numbers of the form $a + wb$ where $b = 0$ and is a prime in $\mathbb{Z}$ congruent with $2$ modulo $3$. That is to say, if $p$ prime in $\mathbb{Z}$ with $p \equiv 2 \bmod 3$, $p$ stays prime in $\mathbb{Z}[w]$.*
- *Numbers of the form $\pi = a + bw$ or its conjugate $\pi^* = a + bw^2$, where $\pi\pi^* = (a + bw)(a + bw^2) = p$ prime in $\mathbb{Z}$ congruent with $1$ modulo $6$. Primes of this form can always be written as $p = u^2 + 3v^2$.*

### 3.2.3 Euclid's Algorithm and BÉZOUT Theorem

Euclid's Algorithm and BÉZOUT Theorem will provide us with the theoretical keys to build the mappings for the designed constellations.

#### Euclid's Algorithm

We begin by defining greatest common divisors in $\mathbb{Z}[i]$ and $\mathbb{Z}[w]$. We will denote both rings as $\mathbb{Z}[*]$.

**Definition 3.10.** *For non-zero $\alpha$ and $\beta$ in $\mathbb{Z}[*]$, a greatest common divisor of $\alpha$ and $\beta$ is a common divisor with maximal norm.*

This is analogous to the usual definition of greatest common divisor in $\mathbb{Z}$, except that the concept does not refer to a specific number. If $r$ is a greatest common divisor of $\alpha$ and $\beta$, so are its unit multiples. Therefore, we can speak of a greatest common divisor, but not the greatest common divisor.

**Definition 3.11.** *When $\alpha$ and $\beta$ only have unit factors in common, we call them* relatively prime*.*

**Theorem 3.10.** ***(Euclid's Algorithm)**. Let $\alpha, \beta \in \mathbb{Z}[*]$ be non-zero. Apply recursively the division theorem, starting with this pair, and make the divisor and remainder in one equation the new dividend and divisor in the next one, provided the remainder is not zero:*

$$\begin{aligned} \alpha &= \beta\gamma_1 + \rho_1, \quad N(\rho_1) < N(\beta) \\ \beta &= \rho_1\gamma_2 + \rho_2, \quad N(\rho_2) < N(\rho_1) \\ \rho_1 &= \rho_2\gamma_3 + \rho_3, \quad N(\rho_3) < N(\rho_2) \\ &\cdots \end{aligned}$$

*The last non-zero remainder is divisible by all common divisors of $\alpha$ and $\beta$, and is itself a common divisor, so it is a greatest common divisor of $\alpha$ and $\beta$.*

**Corollary 3.2.** *For non-zero $\alpha$ and $\beta$ in $\mathbb{Z}[*]$, let $\delta$ be a greatest common divisor produced by Euclid's Algorithm. Any greatest common divisor of $\alpha$ and $\beta$ is a unit multiple of $\delta$.*

Now, we are going to see some examples in the ring $\mathbb{Z}[i]$.

**Example 3.4.** *We compute a greatest common divisor of $\alpha = 32 + 9i$ and $\beta = 4 + 11i$.*

$$\begin{aligned} 32 + 9i &= (4 + 11i)(2 - 2i) + 2 - 5i \\ 4 + 11i &= (2 - 5i)(-2 + i) + 3 - i \\ 2 - 5i &= (3 - i)(1 - i) - i \\ 3 - i &= (-i)(1 + 3i) + 0 \end{aligned}$$

*The last non-zero remainder is $-i$ a greatest common divisor, so $\alpha$ and $\beta$ only have unit factors in common. They are relatively prime.*

**Example 3.5.** *Here is an example where the greatest common divisor is not a unit. Let $\alpha = 11+3i$ and $\beta = 1+8i$. Then*

$$\begin{aligned} 11+3i &= (1+8i)(1-i)+2-4i \\ 1+8i &= (2-4i)(-1+i)-1+2i \\ 2-4i &= (-1+2i)(-2)+0 \end{aligned}$$

*so a greatest common divisor of $\alpha$ and $\beta$ is $-1+2i$.*

*We could proceed in a different way in the second equation (due to the lack of uniqueness of the division theorem), and obtain a different non-zero remainder,*

$$\begin{aligned} 11+3i &= (1+8i)(1-i)+2-4i \\ 1+8i &= (2-4i)(-2+i)+1-2i \\ 2-4i &= (1-2i)(2)+0 \end{aligned}$$

*Therefore $1-2i$ is also a greatest common divisor. Our two different answers are not inconsistent: a greatest common divisor is defined at best only up to a unit multiple anyway, and $-1+2i$ and $1-2i$ are unit multiples of each other: $-1+2i = (-1)(1-2i)$.*

**BÉZOUT Theorem**

In $\mathbb{Z}$, BÉZOUT Theorem says for any non-zero $a$ and $b$ in $\mathbb{Z}$ that $gcd(a,b) = ax+by$ for some $x$ and $y$ in $\mathbb{Z}$ found by back-substitution in Euclid's Algorithm. The same idea works in $\mathbb{Z}[i]$ and $\mathbb{Z}[w]$ and gives us BÉZOUT Theorem there.

**Theorem 3.11.** ***(BÉZOUT Theorem)*** *Let $\delta$ be any greatest common divisor of two non-zero elements of $\mathbb{Z}[*]$, $\alpha$ and $\beta$. Then $\delta = \alpha x + \beta y$ for some $x, y \in \mathbb{Z}[*]$.*

**Corollary 3.3.** *The non-zero elements of $\mathbb{Z}[*]$, $\alpha$ and $\beta$, are relatively prime if and only if we can write*

$$1 = \alpha x + \beta y$$

*for some $x, y \in \mathbb{Z}[*]$.*

Now, we are going to see some examples in the ring $\mathbb{Z}[i]$:

**Example 3.6.** *We saw in the previous example that $\alpha = 32+9i$ and $\beta = 4+11i$ are relatively prime, since the last non-zero remainder in Euclid's Algorithm is $-i$. We can reverse the calculations in this example to express $-i$ as a $\mathbb{Z}[i]$-combination of $\alpha$ and $\beta$:*

$$\begin{aligned} -i &= 2-5i-(3-i)(1-i) \\ &= 2-5i-(\beta-(2-5i)(-2+i))(1-i) \end{aligned}$$

$$
\begin{aligned}
&= (2-5i)(1+(-2+i)(1-i)) - \beta(1-i) \\
&= (2-5i)(3i) - \beta(1-i) \\
&= (\alpha - \beta(2-2i)(3i)) - \beta(1-i) \\
&= \alpha(3i) - \beta(7+5i)
\end{aligned}
$$

*To write 1, rather than $-i$, as a combination of $\alpha$ and $\beta$, multiply by $i$:*

$$1 = \alpha(-3) + \beta(5-7i)$$

### 3.2.4 Finite Fields

Now we are going to do an introduction to finite fields. We are interested in working with fields because it will allow us to finish the communication process. We will explain this fact in detail in the design section.

**Theorem 3.12.** *A finite field or Galois Field (noted by $\mathbb{F}$) is a field that contains a finite number of elements and they are classified as follows:*

- *The number of elements of a finite field is of the form $p^n$ where $p$ is a prime number and $n$ is a positive integer.*
- *For every prime number $p$ and positive integer $n$, there exists a finite field with $p^n$ elements.*
- *Any two finite fields with the same number of elements are isomorphic.*

### 3.2.5 Homomorphism

Formally, the canonical projection presented in this subsection would be the first step in the constellation design.

In this work we are interested in the design of constellations with a finite number of points so we need to narrow the infinite number of points of the initial ring.

When we do the quotient of a set $X$ we are narrowing the initial set into a subset with the representative elements of each class and that is just we are looking for.

The set of all equivalence classes in $X$ given an equivalence relation $\sim$ is denoted as $X/\sim$ and called the quotient set of $X$ by $\sim$.

So we are interested in building a mapping between the initial ring $\mathbb{Z}[*]$ and a quotient set of it. Note that $\mathbb{Z}[*]$ refers to $\mathbb{Z}[i]$ and $\mathbb{Z}[w]$.

A mapping that takes an element to its equivalence class under a given equivalence relation is known as the **canonical projection**.

In this work the relation will be defined as the congruence relation so we can define the canonical projection as:

$$\begin{array}{ccc} \mathbb{Z}[i] & \xrightarrow{\quad\pi\quad} & \mathbb{Z}[i]/(C) \\ x & \longmapsto & [x]_C \end{array}$$

Figure 4: Canonical Projection

where $(C)$ is an ideal of the ring $\mathbb{Z}[*]$ and $[x]_C$ is the representative of the class $x$ congruent $C$, that is to say, $[x]_C \equiv x \bmod C \Leftrightarrow [x]_C - x \in C$.

It is easy to observe that $\pi$ corresponds to the modulo function and in this case it is a surjective mapping, every element in $\mathbb{Z}[*]/(C)$ has a corresponding element in $\mathbb{Z}[*]$.

The number of elements of a quotient ring using these two rings is defined as:

**Theorem 3.13.** *If $\alpha \neq 0$ in $\mathbb{Z}[i]$ or $\mathbb{Z}[w]$, then $n(\alpha) = N(\alpha)$, where $n(\alpha)$ denotes the number of GAUSSIAN Integers or EISENSTEIN Integers modulo $\alpha$. That is, the size of $\mathbb{Z}[*]/\alpha\mathbb{Z}[*]$ is $N(\alpha)$.*

There is an analogy with the absolute value on $\mathbb{Z}$, where $\#(\mathbb{Z}/m\mathbb{Z}) = |m|$, with $m \neq 0$ and now $\#(\mathbb{Z}[*]/\alpha\mathbb{Z}[*]) = N(\alpha)$ with $\alpha \neq 0$.

#### 3.2.6 Theorems Needed in the Design

Now we are going to introduce the theorems used in the design.

We have seen that we are interested in finding fields $R/\alpha R$ where $R$ is a PID and $\alpha R$ an ideal. We have introduced these ideas briefly at the beginning of section (3.2.1.2), however we are going to present the theorems referenced in the design section which follow from the theory already described.

**Definition 3.12.** *Let $R$ be an ideal. An ideal $P \neq R$ is said to be prime if for all $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$.*

The first theorem tells us a way to find maximal ideals in a PID:

**Theorem 3.14.** *Let $R$ be a commutative ring with unity.*
*If $R$ is a PID, then every nonzero prime ideal is maximal.*

We have seen in the previous theorem that prime ideals will help us to find maximal ideals in a PID. Now, we can see how to obtain prime ideals in the following theorem:

**Theorem 3.15.** *If $R$ is an integral domain and $\alpha \in R$ is a nonzero, non-unit element, then*
*$(\alpha)$ is a prime $R$-ideal if and only if $\alpha$ is a prime in $R$.*

Finally, From theorems 3.3, 3.14 and 3.15 we can deduce the next two theorems: the first one is a version for integer ring and the second one is the version for polynomial ring:

**Theorem 3.16.** *For every prime $p$ in $\mathbb{Z}$, $\mathbb{Z}/p\mathbb{Z}$ forms a field.*

In some steps in the design we will work with polynomial quotient ring in order to obtain the structure of field, in these cases we will use the next theorem:

**Theorem 3.17.** *For a prime $p$ and a monic irreducible $m(x)$ in $\mathbb{F}_p[x]$ of degree $n$, the ring $\mathbb{F}_p[x]/(m(x))$ is a field of order $p^n$.*

# 4 Performance Metrics and Mostly Used Constellations

## 4.1 Performance Metrics

We are going to introduce the theoretical concepts used in the analysis of proposed constellations. The performance analysis of constellations will be based on decision regions and the analysis of error probability. Finally, we will do the study of two key parameters needed for the error probability computation: the minimum distance and the average number of neighbours.

### 4.1.1 Decision Regions

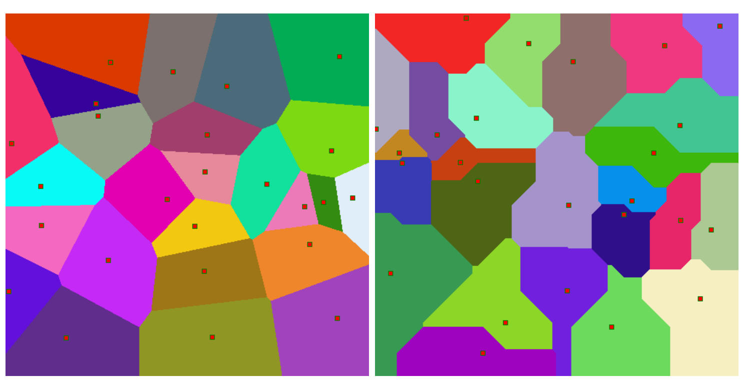
Decision regions are a fundamental concept in understanding the design of constellations. Moreover, they are of utmost importance in the computation of error probability. Most of the material of this section is based on references [8] and [22].

Let $X = \mathbb{C}$ be the complex space endowed with a distance $d$ and a constellation with $M$ points defined inside it. The decision region for constellation point $c_z$ is the set of all complex numbers that are closer to $c_z$ than to any other point of the signal constellation. Therefore $X$ is partitioned into $M$ decision regions $\mathcal{R}_z$, $1 \leq z \leq M$ one by each point of the constellation. More formally,

$$\mathcal{R}_z = \{x \in X : \quad d(x, P_z) \leq d(x, P_k) \quad \forall z \neq k\}$$

where $d(x, A) = inf\{d(x, a)|a \in A\}$ denotes the distance between the point $x$ and the subset $A$ and $P_z$ is the set of all points in whose distance to $x$ is not greater than their distance to the other sites $P_k$, where $k$ is any index different from $z$.

Decision regions defined in this way for a fixed set of points in the complex space are called VORONOI Regions. VORONOI Regions depend significantly on the metric used. We are going to show two different examples of VORONOI Regions in the complex plane choosing 25 random points with two different distances in order to see how the metric affects to the shape of the region.[1]



(a) Euclidean Distance (b) MANHATTAN Distance

Figure 5: Two Examples of VORONOI Regions Using Different Measures

[1]The code of this two examples can be found in Annex V

In Figure 5a it is shown the VORONOI Regions using the Euclidean distance defined as:

$$d(x_1, x_2) = \sqrt{(\text{Real}(x_1) - \text{Real}(x_2))^2 + (\text{Imaginary}(x_1) - \text{Imaginary}(x_2))^2} \quad \text{where} \quad x_1, x_2 \in \mathbb{C}.$$

In this case every side of a VORONOI Region is a segment of the perpendicular bisector of the line connecting two neighbors.

In Figure 5b it is used the MANHATTAN Distance defined as:

$$d(x_1, x_2) = |\text{Real}(x_1) - \text{Real}(x_2)| + |\text{Imaginary}(x_1) - \text{Imaginary}(x_2)| \quad \text{where} \quad x_1, x_2 \in \mathbb{C}.$$

In this project, we are going to focus on the Euclidean distance.

#### 4.1.2 Symbol Error Probability

Most of the material for this section can be found in references [13], [8] and [5].

In order to study the probability of error of a given constellation, we are going to assume an AWGN (Additive White Gaussian Noise) channel. This channel adds to the signal $x(t)$ an uncorrelated GAUSSIAN noise $n(t)$ to the output.

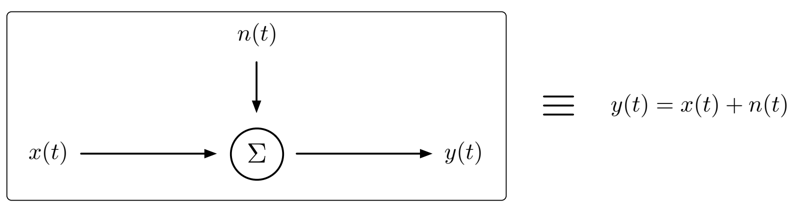



Figure 6: AWGN Channel

The noise $n(t)$ is a 1 dimensional GAUSSIAN random signal with zero mean, variance $\sigma^2$ and probability distribution:

$$p_n(u) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}u^2}. \tag{1}$$

From the hypothesis, the AWGN channel is equivalent to a vector channel with output given by

$$\mathbf{y} = \mathbf{x} + \mathbf{n}. \tag{2}$$

We can write the vector analogy of the AWGN noise explained above:

The noise vector **n** is an $N$ dimensional GAUSSIAN random vector with zero mean, equal-variance and uncorrelated components in each dimension. The noise distribution is

$$p_{\mathbf{n}}(\mathbf{u}) = (\pi\mathcal{N}_0)^{-N/2} \cdot e^{\frac{1}{\mathcal{N}_0}||\mathbf{u}||^2} = (2\pi\sigma^2)^{-N/2} \cdot e^{-\frac{1}{2\sigma^2}||\mathbf{u}||^2}. \tag{3}$$

The computation of $P_e$ assumes the inputs $\mathbf{x_c}$ equally likely, that is to say, $p_{\mathbf{x}}(c) = \frac{1}{M} \forall c$. Under this assumption and based on reference [13], we can assert that the optimum detector is the ML detector, which has decision rule

$$\hat{m} \Rightarrow m_c \quad \text{if} \quad ||v - x_c||^2 \leq ||v - x_z||^2 \quad \forall z \neq c. \tag{4}$$

ML takes the constellation point which the detected point is nearest to. The probability of error associated with this rule depends on the signal constellation $\mathbf{x}_c$ and the noise variance $\sigma^2 = \frac{N_0}{2}$.

The exact $P_e$ corresponds to the sum of probabilities of having an error when transmitting a given symbol. These probabilities are disjoint and can be expressed as: $P(e \cap c) = P(e|c) \cdot P(c)$, that is to say, having an error from the symbol c. Therefore,

$$\begin{aligned} P_e &= \sum_{c=0}^{M-1} P_{e|c} \cdot P(c), & (5) \\ &= \frac{1}{M} \sum_{c=0}^{M-1} P_{e|c}, & (6) \\ &= 1 - \frac{1}{M} \sum_{c=0}^{M-1} P_{r|c}. & (7) \end{aligned}$$

This may be difficult to compute: be aware that $P_{e|c}$ depends on the decision region geometry and therefore it consists on the computation of an integral of the distribution Gaussian on the complementary decision region. So convenient and accurate bounding procedures can approximate $P_e$.

#### 4.1.2.1 The Union Bound

We are going to propose a $P_e$ approximation that will be useful in our analysis. In order to do that we are going to start studying a first approximation: the Union Bound. Then, we are going to go further and use this first approximation to derive a tighter bound: The Nearest Neighbor Union Bound.

$P_e$ approximations:

1 Union Bound

2 The Nearest Neighbor Union Bound

First, we introduce an important metric that is of the utmost importance in constellation design and performance: the minimum distance.

**Definition 4.1.** *The minimum distance , $d_{\min}$ is defined as the minimum distance between any two points in a constellation $\{x_c\}_{c=0,\cdots,M-1}$:*

$$d_{\min} = \min_{c \neq z} ||x_c - x_z|| \quad \forall c, z. \tag{8}$$

We are going to study a simplified case that will help us in the derivation:

Suppose a system that has two possible constellation points in $N$ dimensions with an AWGN channel, as illustrated for $N = 1$ dimension in Figure 7. Then the probability of error for the ML detector is the probability that the component of the noise vector $\mathbf{n}$ along the line connecting the two data symbols is greater than half the distance along this line. In this case, the noisy received vector $\mathbf{y}$ lies in the incorrect decision region, resulting in an error.

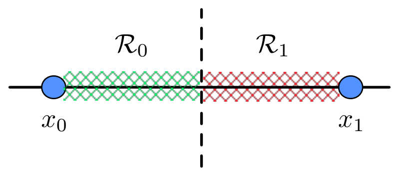



Figure 7: One Dimension Constellation with Two Symbols

Since the noise is white Gaussian, its projection in any dimension, in particular, the segment of the line connecting the two data symbols, is of variance $\sigma^2 = \frac{N_0}{2}$. This is because AWGN statistics are rotation invariant and since a projection is a particular case of rotation, the statistics remain unchanged (see reference[13]).

$$P_e = P\left(\langle n, \phi\rangle \geq \frac{d}{2}\right)$$

where $\phi$ is a unit norm vector along the line between $x_0$ and $x_1$ and $d = ||x_0 - x_1||$. This error probability is

$$P_e = \int_{\frac{d}{2}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}u^2} du = \int_{\frac{d}{2\sigma}}^{\infty} e^{-\frac{u^2}{2}} du = Q\left[\frac{d}{2\sigma}\right], \tag{9}$$

if we consider $\sigma^2 = \frac{N_0}{2}$

$$P_e = Q\left[\frac{d}{\sqrt{2N_0}}\right]. \tag{10}$$

This result is useful in the proof of the following Theorem.

**Theorem 4.1** (Union Bound)**.** *The probability of error for the ML detector on the AWGN channel, with an M-point signal constellation with minimum distance* $d_{\min}$ *is bounded by*

$$P_e \leq (M-1)Q\left[\frac{d_{\min}}{2\sigma}\right]. \tag{11}$$

*Proof.* We suppose an error $\epsilon_{cz}$ occurs when $x_c$ is transmitted and the ML detector chooses $\hat{x} = x_z$. We can express it as:

$$P_{e|c} = P\left(\bigcup_{z=0(z\neq c)}^{M-1} \epsilon_{cz}\right). \tag{12}$$

Since errors are mutually exclusive, the probability of the union is the sum of the probabilities

$$P_{e|c} = \sum_{z=0(z\neq c)}^{M-1} P(\epsilon_{cz}). \tag{13}$$

We bound $P(\epsilon_{cz})$ by the probability of a given point, $y$, to be closer to $x_z$ than to $x_c$, that is to say, $P(||y - x_z|| \leq ||y - x_c||) = P'(x_c, x_z)$:

$$P(\epsilon_{cz}) \leq P'(x_c, x_z) \tag{14}$$

and therefore

$$P_{e|c} \leq \sum_{z=0(z\neq c)}^{M-1} P'(x_c, x_z). \tag{15}$$

It is important to note that $P(\epsilon_{cz})$ is the probability that the received vector $y$ lies inside the decision region of the point $z$. However, $P'(x_c, x_z)$ is the probability that the received vector $y$ lies closer to $z$ than $c$, and that can be translated as it lies in the half plane closer to $z$. Therefore, $P'(x_c, x_z)$ includes the region for $P(\epsilon_{cz})$, as it can be seen on the next picture.

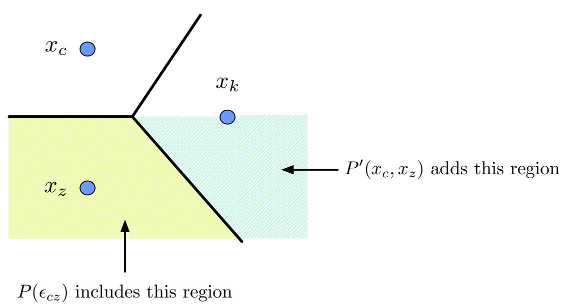



Figure 8: Probability of Error Regions

Using the result derived earlier in equation 10

$$P'(x_c, x_z) = P(||y - x_z|| \leq ||y - x_c||) = Q\left[\frac{||x_c - x_z||}{2\sigma}\right], \tag{16}$$

and if we substitute

$$P_{e|c} \leq \sum_{z=0(z\neq c)}^{M-1} P'(x_c, x_z) = \sum_{z=0(z\neq c)}^{M-1} Q\left[\frac{||x_c - x_z||}{2\sigma}\right]. \tag{17}$$

If we introduce that $Q(x)$ is monotonically decreasing with $x$, and since $d_{\min} \leq ||x_c - x_z||$

$$Q\left[\frac{||x_c - x_z||}{2\sigma}\right] \leq Q\left[\frac{d_{\min}}{2\sigma}\right]. \tag{18}$$

If we average over all the constellation points we get $P_e$

$$\begin{aligned} P_e &\leq \sum_{c=0}^{M-1} \sum_{z=0(z\neq c)}^{M-1} Q\left[\frac{d_{\min}}{2\sigma}\right] p_x(c), & (19)\\ &= \sum_{c=0}^{M-1} (M-1) Q\left[\frac{d_{\min}}{2\sigma}\right] p_x(c), & (20)\\ &= (M-1) Q\left[\frac{d_{\min}}{2\sigma}\right]. & (21) \end{aligned}$$

□

#### 4.1.2.2 The Nearest Neighbor Union Bound

The factor $(M-1)$ in the original Union Bound is often too large for accurate performance prediction. The Nearest Neighbor Union Bound gives a tighter bound on the probability of error for a signal constellation by lowering this factor of the Q-function.

We introduce a new performance metric:

**Definition 4.2** (Average Number of Nearest Neighbors)**.** *The average number of neighbors, $N_e$, for a signal constellation can be defined as*

$$N_e = \sum_{c=0}^{M-1} N_c p_x(c) \tag{22}$$

*where $N_c$ is the number of neighboring constellation points of the point $\mathbf{x}_c$, in other words the number of other signal constellation points sharing a common decision region boundary with $x_c$.*

Moreover, $N_e$ is often approximated by

$$N_e \approx \sum_{c=0}^{M-1} \tilde{N}_c p_x(c) \tag{23}$$

where $\tilde{N}_c$ is the set of points at minimum distance from $\mathbf{x}_c$. This approximation is often very tight and facilitates computation of $N_e$ when signal constellations are complicated. $N_e$ measures the average number of sides of the decision regions surrounding any point in the constellation.

**Theorem 4.2** (Nearest Neighbor Union Bound)**.** *The probability of error for the ML detector on the AWGN channel, with an M-point constellation with minimum distance $\mathrm{d}_{\min}$ is bounded by*

$$P_e \le N_e Q\left[\frac{d_{\min}}{2\sigma}\right]. \tag{24}$$

*Proof.* For each constellation point, the distance to each decision region boundary must be at least $\frac{d_{\min}}{2}$. The probability of error for point $\mathbf{x}_c$, $P_{e|c}$ is upper bounded by the union bound as

$$P_{e|c} \le N_c Q\left[\frac{d_{\min}}{2\sigma}\right].$$

Thus,

$$P_e = \sum_{c=0}^{M-1} P_{e|c} p_{\mathbf{x}(c)} \le Q\left[\frac{d_{\min}}{2\sigma}\right] \sum_{c=0}^{M-1} N_c p_{\mathbf{x}}(c) = N_e Q\left[\frac{d_{\min}}{2\sigma}\right].$$

□

#### 4.1.3 Computation of the Parameters

We are going to explain how the parameters involved in the error probability formula have been computed.

##### 4.1.3.1 Minimum Distance, $d_{\min}$

A fundamental parameter in order to do a well-round analysis of error probability is the minimum distance. Now we are going to study the implemented algorithm in order to compute it.

Compute the minimum distance in a constellation is the same that finding, in a set of points, the closest pair. So we can reformulate the problem as follows including the fact that all the constellations used are defined using Euclidean distance:

**Problem**: Given a set of points $\mathcal{A} = \{p_1, \cdots, p_n\}$ find the pair of points $\{p_c, p_z\}$ that are closest together, that is to say, which minimize the Euclidean distance.

**Brute-force search algorithm**

The most natural and straightforward way is using brute-force search that just check each pair of points and take the pair with minimum distance. In this case, let $n$ be the number of points in the set we need to check

$$\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2} \quad \text{pair of points.} \tag{25}$$

Here we have the pseudocode of the brute-force search algorithm taken and adapted from reference [15]:

**Algorithm 1** Brute-Force Closest Pair

1: **procedure** BRUTEFORCECLOSESTPAIR(of $p_1, p_2, \cdots, p_n$)
2:     **if** $n < 2$ **then**
3:         **return** $\infty$
4:     **else**
5:         $min\_distance \leftarrow |p_1 - p_2|$
6:         $min\_points \leftarrow \{p_1, p_2\}$
7:         **for** each $c \in [1, n-1]$ **do**
8:             **for** each $z \in [c+1, n]$ **do**
9:                 **if** $|p_c - p_z| < min\_distance$ **then**
10:                     $min\_distance \leftarrow |p_c - p_z|$
11:                     $min\_points \leftarrow \{p_c - p_z\}$
12:                 **end if**
13:             **end for**
14:         **end for**
15:         **return** $min\_distance, min\_points$
16:     **end if**
17: **end procedure**

It is easy to see, not only analytically by the formula (25) but also from the pseudocode: see the two for loops (lines 7 and 8 of the Algorithm), that the order of the algorithm is $O(n^2)$, where $n$ is the length of the set of points.

The MATLAB code implementation can be found in Annex II.

In order to improve the order of the brute-force search algorithm we could propose an alternative method: the recursive divide and conquer method applied to the problem of the closest pair using Euclidean distance. However, we are using constellations with a relative small number of points and therefore there is not a significant difference in the use of the brute force search algorithm or the recursive divide and conquer method.

#### 4.1.3.2 Average Number of Neighbors, $N_e$

The implementation of this section, which can be found in Annex II, is an answer to the problem proposed by professor J. Burkardt in reference [12].

In order to compute the Average Number of Neighbors, we need to compute the neighbors of each point given a constellation and its decision regions. There are different ways to compute the decision regions of a set of points. Within Matlab, there are two commands, voronoi and voronoin. It turns out that the command voronoin returns enough information to determine the number of neighbors of each point using Matlab and we can use this information to compute $N_e$.

The VORONOI diagram of a set of points in the plane divides the plane into polygons and a few infinite regions bounded by straight lines. Matlab adds a point at infinity, and pretends these infinite regions all include that point as a vertex, so from now on, we can pretend that every point is contained in a closed polygon defined by the VORONOI diagram. Two points are neighbors if their polygons share an edge. So our question becomes, how do we take the information that Matlab returns to describe a VORONOI diagram, and analyze it so that we can determine which points are neighbors?

Let's use the following array of 5 points (p=5) as an example:

```
x5 = [0,1,0,0,-1];
y5 = [0,0,-1,1,0];
```

For this small problem, we could answer our question by computing the diagram with MATLAB command:

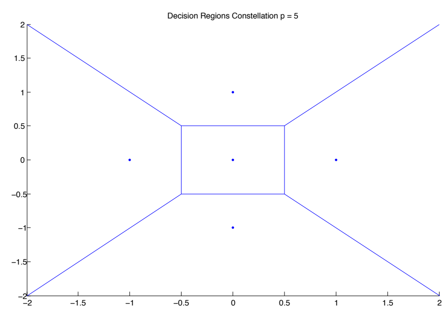



Figure 9: Decision Regions for $p = 5$

so that, for example, we see that point (0,0) has neighbors (1,0), (0,1), (0,-1) and (-1,0).

However, we wish to be able to extract this information computationally for larger problems, and using Matlab. Let us now consider how to proceed.

MATLAB command voronoin is called using this parameters

```
[ v, g ] = voronoin ( [x(:) y(:)])
```

The vertices $v$ are returned as

```
      Inf        Inf
  -0.5000     0.5000
   0.5000     0.5000
  -0.5000    -0.5000
   0.5000    -0.5000
```

But what is more interesting is the entries of the cell array $g$, which contain, for each node, the sequence of vertices that form its boundary.

For our data, the $g$ information is:

```
5      3      2      4

     5      1      3

     5      1      4

     3      1      2
```

Now two nodes are neighbors if they share an edge. The edges for node 1 are (5,3), (3,2), (2,4) and (4,5). If I rewrite these edges so the pairs of nodes are sorted, and then sort these edges by first element, I have the edges (3,5), (2,3), (2,4) and (4,5). Node 1 is a neighbor of node $I$ if and only if node $I$ also uses one of these edges to bound its polygonal region. So we can calculate a matrix with the points that share edges and therefore that they are neighbors.

We can see that the nodal neighbor array is:

```
      1     2     3     4     5
    +------------------------
  1 | 0     1     1     1     1
  2 | 1     0     1     1     0
  3 | 1     1     0     0     1
  4 | 1     1     0     0     1
  5 | 1     0     1     1     0
```

Finally, we compute $N_e$, the average number of neighbors, by summing columns an averaging.

## 4.2 Description of Mostly Used Constellations

We are going to introduce the mostly used constellations in order to compare them with the proposed constellations in subsequent sections. This section is based and adapted from reference [20].

### 4.2.1 M-PAM

We will start by looking at the simplest form constellation PAM, Pulse Amplitude Modulation.

We are going to define a PAM constellation with $M$ symbols, in order to do it we first define a sequence of integers $k[n]$:

$$k[n] \in \{0, 1, \cdots, M-1\}.$$

Now, PAM can be described as a sequence of symbols $a[n]$ defined as:

$$a[n] = A((-M+1) + 2k[n])$$

where we use $M-1$ odd integers around 0. So for instance, if $M = 4$ we have $a[n] \in \{-3A, -A, A, 3A\}$. Finally the PAM constellation with $M = 4$ is defined as the set of points:

$$\mathcal{A} = \{-3A, -A, A, 3A\}.$$

Here we have in Figure 10, the example of a PAM constellation with $M = 4$:

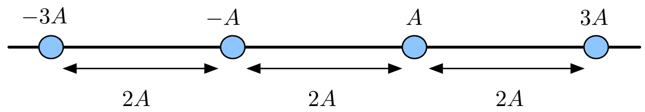



Figure 10: PAM Constellation with M=4

We can observe that distance between two transmitted symbols, is $2A$. Furthermore, the reason why we use the odd integers is because it creates a zero-mean sequence. If we assume that each symbol is equiprobable, the resulting mean is zero.

#### 4.2.2 M-QAM

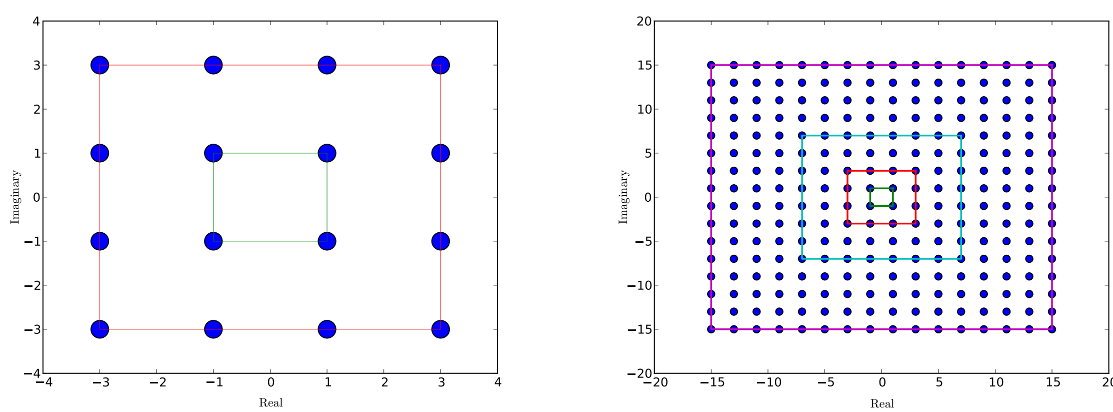
QAM (Quadrature Amplitude Modulation) is an improvement of the PAM constellation, where the main goal is to increase the throughput. In this case we use complex numbers and build a complex valued transmission system.

The QAM symbol sequence is a sequence of complex numbers with M symbols, where the real part is the a $\sqrt{M}$-PAM sequence and the imaginary part is also a $\sqrt{M}$-PAM sequence

$$a[n] = A(a_r[n] + i a_c[n]).$$

So the signal constellation $\mathcal{A}$ is given by points in the complex plane, with odd-valued coordinates around the origin.

Here in Figure 11 we have an example of QAM for different values of $M$:

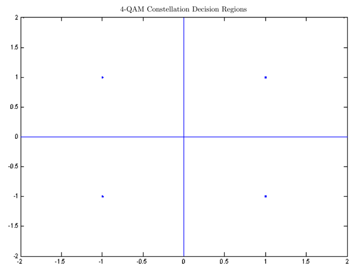



(a) QAM Constellation with $M = 16$

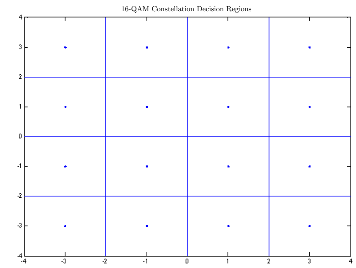
(b) QAM Constellation with $N = 256$

Figure 11: Two Examples of QAM Constellations Using Different Values of $M$ and $A = 1$

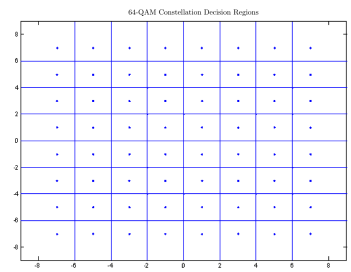
We enclose the decision regions corresponding to 4, 16, 64 and 256 QAM. In order to compute them we have used the function voronoi() implemented in Matlab.



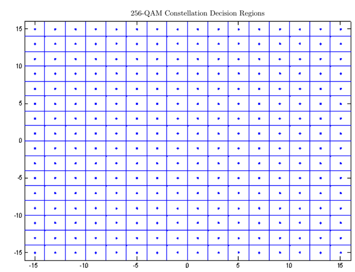

(a) 4-QAM Decision Regions Constellation




(b) 16-QAM Decision Regions Constellation




(c) 64-QAM Decision Regions Constellation




(d) 256-QAM Decision Regions Constellation

Figure 12: M-QAM Decision Regions Constellations with $M = 4, 16, 64, 256$

Decision regions will be used both as a visual aid to understand the geometrical structure induced in the constellation and as a tool in the computation of the average number of neighbors.

We compute the minimum distance $d_{\min}$ and the average number of neighbors $N_e$ using the algorithms described in earlier sections.

| | |
|---|---|
| 4-QAM Constellation | $d_{\min} = 2$ |
| 16-QAM Constellation | $d_{\min} = 2$ |
| 64-QAM Constellation | $d_{\min} = 2$ |
| 256-QAM Constellation | $d_{\min} = 2$ |

Table 1: $d_{\min}$ Numerical Results for M-QAM Constellations

| | |
|---|---|
| 4-QAM Constellation | $N_e = 2$ |
| 16-QAM Constellation | $N_e = 3$ |
| 64-QAM Constellation | $N_e = 3.5$ |
| 256-QAM Constellation | $N_e = 3.75$ |

Table 2: $N_e$ Numerical Results for M-QAM Constellations

Finally, based on [21] and [22], we are going to compare the exact error probability of M-QAM with the results obtained using the Union Bound and the Nearest Neighbor Union Bound.

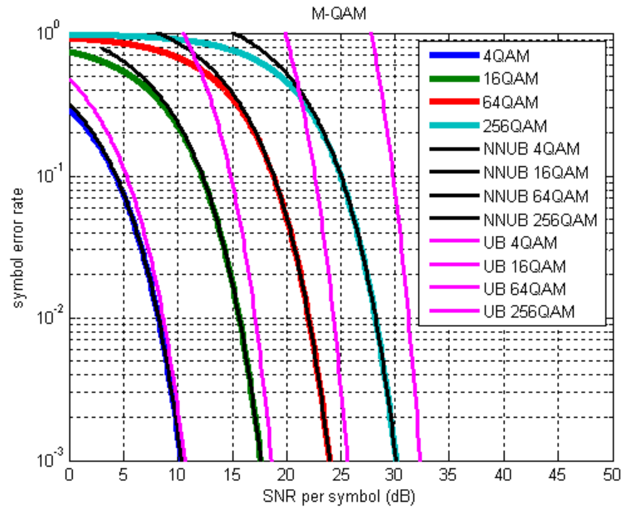



Figure 13: Comparison between NNUB, UB and the Exact Value of the Error Probability for M-QAM Constellations

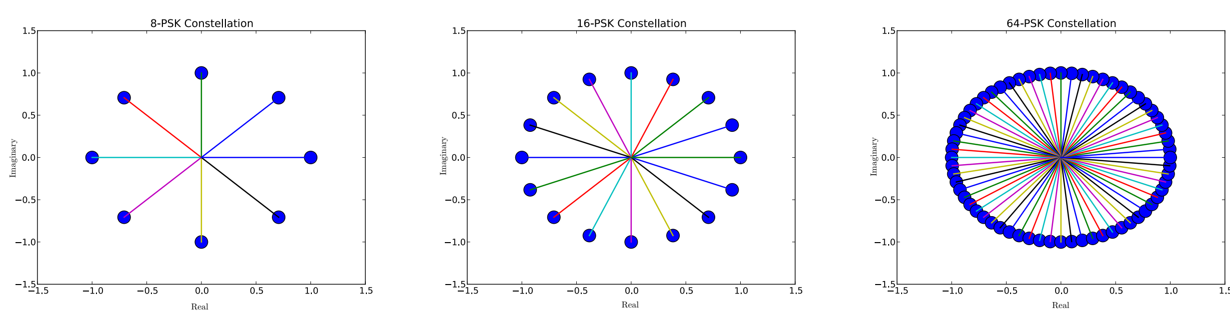
We can see how the Nearest Neighbor Union resembles almost perfectly the exact error probability whereas the Union Bound is loose when M is large. Therefore, from now on we are going to use the Nearest Neighbor Union bound in order to do our analysis.

### 4.2.3 M-PSK

The M-PSK constellation is the set:

$$\mathcal{A} = \{Ae^{j2k\pi/M}\}, \quad k = 1, 2, \cdots, M.$$

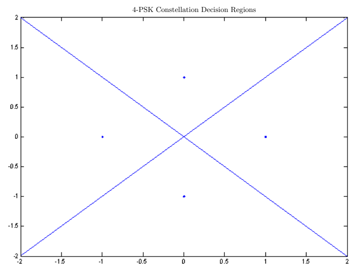
Here there are three examples for different values of $M$:



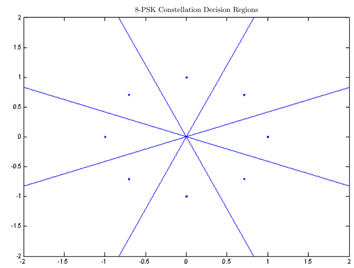

(a) PSK Constellation with $M = 8$ (b) PSK Constellation with $M = 16$ (c) PSK Constellation with $M = 64$

Figure 14: Three Examples of PSK Constellations Using Different Values of $M$ and $A = 1$

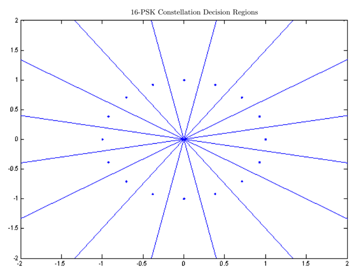
We enclose the decision regions corresponding to 4, 8, 16 and 64 PSK.



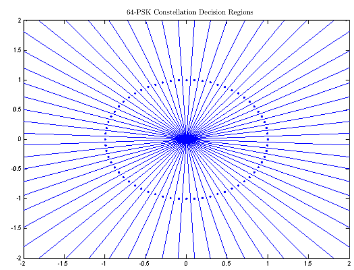

(a) 4-PSK Decision Regions Constellation




(b) 8-PSK Decision Regions Constellation




(c) 16-PSK Decision Regions Constellation




(d) 64-PSK Decision Regions Constellation

Figure 15: M-PSK Decision Regions Constellations with $M = 4, 8, 16, 64$

We compute the minimum distance $d_{\min}$ and the average number of neighbors $N_e$ using the algorithms described in earlier sections.

| | |
|---|---|
| 4-PSK Constellation | $d_{\min} = 1.4142$ |
| 8-PSK Constellation | $d_{\min} = 0.7654$ |
| 16-PSK Constellation | $d_{\min} = 0.3902$ |
| 64-PSK Constellation | $d_{\min} = 0.0981$ |

Table 3: $d_{\min}$ Numerical Results for M-PSK Constellations

| | |
|---|---|
| 4-PSK Constellation | $N_e = 2$ |
| 8-PSK Constellation | $N_e = 2$ |
| 16-PSK Constellation | $N_e = 2$ |
| 64-PSK Constellation | $N_e = 2$ |

Table 4: $N_e$ Numerical Results for M-PSK Constellations

Finally, we plot the comparison between Nearest Neighbor Union Bound and Union Bound using these values.

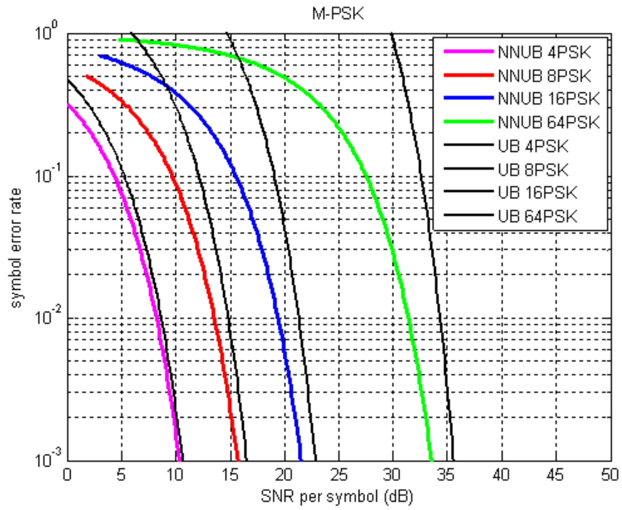



Figure 16: Comparison between NNUB and UB for M-PSK Constellations

Where we can see that for low M the Union Bound resembles the Nearest Neighbor Union Bound whereas the Union Bound is loose when M is large. In PSK, it is important to note that $M > 16$ PSK constellations are not of practical utility because of its bad performance.

# 5 Design of Proposed Constellations

In this section we are going to focus on designing constellations over a PNC communication system. First, we will study a case example of design in order to introduce the particular system model under study and understand how it works. Then, we have proposed a general design methodology which will be the basis of the following proposed constellations. Finally, an analysis of each proposed constellation will be done, using the performance metrics described in the previous section.

## 5.1 Introduction to Constellation Design

In this section we do a first approach to constellation design. This section builds up on references [1] and [2], and provides a theoretical description.

The first step is to study in detail the mappings that allow us to design this first proposed constellation.

In this section we are going to propose a constellation design using primes $p$ in $\mathbb{Z}^+$ congruent with $1$ modulo $4$. By Theorem 3.6 we know that this type of primes in $\mathbb{Z}^+$ can be written as a product of two relatively primes in $\mathbb{Z}[i]$, that is to say, $p = \pi\pi^*$.

Now, by theorems (3.16) and (3.3, 3.14, 3.15) we can assert that $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$ are fields with the same number of elements (theorem 3.13) and therefore exists an isomorphism between them.

As a first step, we are going to define a homomorphism, explained in section (3.2.5). Let $\mathbb{Z}[i]$ be the GAUSSIAN Integers and $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$ the residue class of $\mathbb{Z}[i]$ modulo $\pi$, where the modulo function $\psi : \mathbb{Z}[i] \to \mathbb{Z}[i]/\pi\mathbb{Z}[i]$ is defined according to

$$\psi(g) = g \bmod \pi.$$

We know that if $g$ is an element of $\mathbb{Z}[i]$, in order to find the corresponding element in $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$ we only need to find the remainder of $g/\pi$. Therefore, the natural idea to implement this function is to use the division theorem in $\mathbb{Z}[i]$, explained in section (3.2.1.1), and solve for the residue $\gamma$.

We first state the division theorem in $\mathbb{Z}[i]$

$$\begin{aligned}
&g = \lambda \cdot \pi + \gamma, \\
&\text{with } N(\gamma) < N(\pi), \\
&\text{where } \lambda = \left[\frac{g}{\pi}\right] = \left[\frac{g\pi^*}{\pi\pi^*}\right].
\end{aligned}$$

Note that in this equation we multiply up and down for $\pi^*$ in order to get the $N(\pi)$ in the denominator, and $[\cdot]$ is the GAUSSIAN Integer rounding defined in section (3.2.1.1).

And if we solve for gamma (the residue) we get

$$
\begin{aligned}
\gamma &= g - \lambda\pi, \\
\gamma &= g - \left[\frac{g\pi^*}{\pi\pi^*}\right]\pi.
\end{aligned}
$$

Therefore,

$$\psi(g) = g \bmod \pi = \gamma = g - \left[\frac{g\pi^*}{\pi\pi^*}\right]\pi.$$

Based on the previous modulo function, in order to design the constellation, we are going to define an isomorphism between the fields $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$.

$$
\begin{tikzcd}
\underset{\textstyle r}{\mathbb{Z}/p\mathbb{Z}} \arrow[rr, bend left, "\mu(r) = r\text{mod}\pi"] & & \underset{\textstyle z}{\mathbb{Z}[i]/\pi\mathbb{Z}[i]} \arrow[ll, bend left, "\mu^{-1}(z) = z\text{mod}p"]
\end{tikzcd}
$$

Our candidate is the modulo function $\mu : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}[i]/\pi\mathbb{Z}[i]$ defined before as follows:

$$\mu(g) = g \bmod \pi = \gamma = g - \left[\frac{g\pi^*}{\pi\pi^*}\right]\pi.$$

If we want to find the inverse function $\mu^{-1}$, that is to say, the mapping modulo-p $\mathbb{Z}[i]/\pi\mathbb{Z}[i] \to \mathbb{Z}/p\mathbb{Z}$ properly, first we need to remember that $\pi$ and $\pi^*$ are relatively primes and in terms of BÉZOUT Theorem, it can be translated as:

$$1 = u\pi + v\pi^* \tag{26}$$

where $u$ and $v$ can be computed using the Euclid's Algorithm.

We need to define the inverse mapping in such way that two conjugated elements in $\mathbb{Z}/p\mathbb{Z}$ will have the same image in $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$.

We need to bind one-to-one element. In order to do that let's think about the element $r$ of the field $\mathbb{Z}/p\mathbb{Z}$ related with $z$. We can write it as

$$r = k\pi + z \to r \bmod \pi = z \bmod \pi. \tag{27}$$

At the same time, we know that in $\mathbb{Z}$ an integer and its conjugate are the same number $r = r^*$

$$r = r^* = k^*\pi^* + z^* \to r \bmod \pi = (k^*\pi^* + z^*) \bmod \pi. \tag{28}$$

From the two equations above we know that $z = k^*\pi^* + z^*$ modulo $\pi$ because when we apply a function to the same element ($r = r^*$) the result must be the same.

Let's now take an element $z$ of the field $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$ and multiply it by 1, and use the BÉZOUT Theorem stated in equation (26)

$$\begin{aligned} z &= z \cdot 1, \\ &= z \cdot (u\pi + v\pi^*), \\ &= zu\pi + zv\pi^*. \end{aligned}$$

We now impose that two conjugated elements in $\mathbb{Z}/p\mathbb{Z}$ have the same image in $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$, which results in the condition $z = k^*\pi^* + z^*$ modulo $\pi$

$$\begin{aligned} z &= zu\pi + zv\pi^*, \\ &= (k^*\pi^* + z^*)u\pi + zv\pi, \\ &= k^*u\pi\pi^* + z^*u\pi + zv\pi^*, \end{aligned}$$

and apply $\bmod p$ to the equation above we get

$$\begin{aligned} z \bmod p &= (k^*up + z^*u\pi + zv\pi^*) \bmod p, \\ z \bmod p &= (z^*u\pi + zv\pi^*) \bmod p. \end{aligned}$$

Therefore, we can define the inverse function as the modulo-p function as follows:

$$\mu^{-1}(z) = z \bmod p = (z^*u\pi + zv\pi^*) \bmod p.$$

Finally, let's see that efectively this gives $z \bmod p = r \bmod p$.

If $r$ is an integer of $\mathbb{Z}/p\mathbb{Z}$ then $r$ and $r^*$ can be expressed as in equations (27) and (28). And using the modulo-$p$ function defined above:

$$\begin{aligned} z \bmod p &= (z(v\pi^*) + z^*(u\pi)) \bmod p = ((r - k\pi)(v\pi^*) + (r - k^*\pi^*)(u\pi)) \bmod p, \\ &= (rv\pi^* - kv\pi\pi^* + ru\pi - k^*u\pi\pi^*) \bmod p = r(v\pi^* + u\pi) \bmod p. \\ &= r \bmod p \end{aligned}$$

Thus, we have defined the inverse function.

Now, we will see an example in order to show exactly how the defined mappings are building a constellation design. We are going to use $p = 5$ and $\pi = 2 + i$.

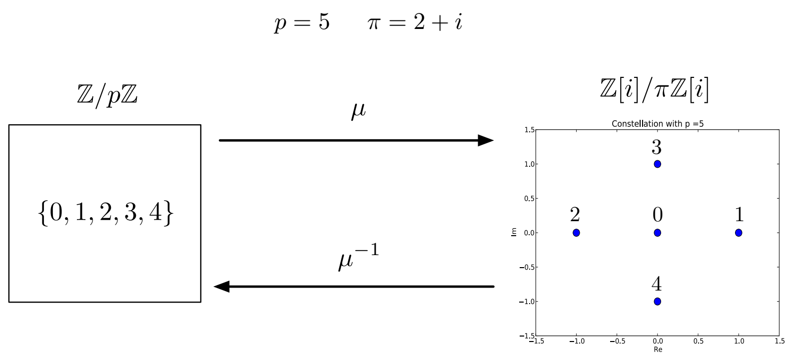



Figure 17: Constellation Design

**Example 5.1.**

Once we have the constellation designed we are prepared to study the system model.

## 5.2 System Model Description

We consider the following system model based on references [1] and [4], which is a PNC scheme with $L$ sources, a relay and a destination, based on the same principles as described in section (2.1).

For the sake of understanding, we are going to explain how the system works for the proposed constellation in the earlier subsection. The constellations proposed in sections (5.4.1) and (5.6.1) can be used in the system straightforward, note that it is just necessary to change the isomorphism used. The other two proposed constellations are designed in order to check the proposed design methodology.

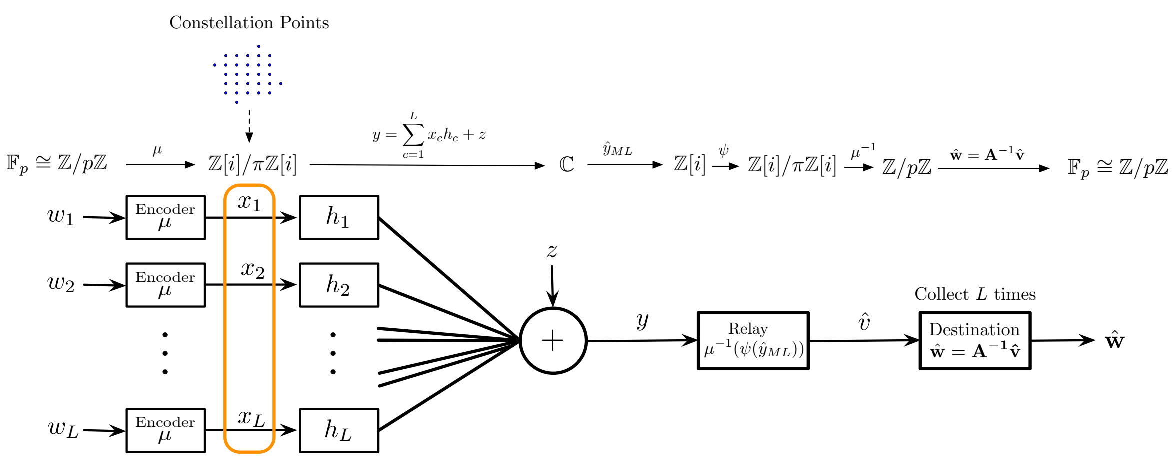



Figure 18: System Model

Let $\omega_l \in \mathbb{F}_p$ be the message to be transmitted by the $l$-th source chosen from a finite field $\mathbb{F}_p$. The vector of all source messages is given by $\mathbf{w} = [w_1 \dots w_L]$. Each source encodes the message $w_l$ into a complex signal constellation point using the encoder $\mu : \mathbb{F}_p \rightarrow \mathbb{Z}[i]/\pi\mathbb{Z}[i]$ to obtain $x_l = \mu(w_l)$, where $\mu$ is the function defined earlier as:

$$\mu(w_l) = w_l \bmod \pi = w_l - \left[\frac{w_l \pi^*}{\pi\pi^*}\right]\pi.$$

The signals are transmitted across the channel to the relay. We assume that the channel undergoes slow fading, that is to say, remains constant throughout the transmission of each signal.

The signal obtained at the relay is given by

$$y = h_1 x_1 + h_2 x_2 + \ldots + h_L x_L + z \in \mathbb{C}$$

where $h_l \in \mathbb{Z}[i]$ is the channel coefficient between transmitter $l$ and the relay node and $z \in \mathbb{C}$ is i.i.d GAUSSIAN Noise given by $z \sim \mathcal{CN}(0, \sigma^2)$.

The aim of the relay is to compute a linear combination of source messages in the original message space $v \in \mathbb{Z}/p\mathbb{Z}$ given by

$$v = a_1\omega_1 \oplus a_2\omega_2 \oplus \ldots \oplus a_L\omega_L$$

where $\oplus$ denotes summation over finite field and $a_l \in \mathbb{Z}/p\mathbb{Z}$ can be computed as follows:

$$a_l = \mu^{-1}(\psi(h_l))$$

where $\psi : \mathbb{Z}[i] \rightarrow \mathbb{Z}[i]/\pi\mathbb{Z}[i]$

$$\psi(h_l) = h_l \bmod \pi = h_l - \left[\frac{h_l \pi^*}{\pi\pi^*}\right]\pi$$

and $\mu^{-1} : \mathbb{Z}[i]/\pi\mathbb{Z}[i] \rightarrow \mathbb{Z}/p\mathbb{Z}$

$$\mu^{-1}(\psi(h_l)) = \psi(h_l) \bmod p = (\psi(h_l)^* u\pi + \psi(h_l) v\pi^*) \bmod p$$

where $u$ and $v$ can be computed using the Euclid's Algorithm, as stated in the previous section.

In order to decode the linear combination $v$, the relay obtains a Maximum Likelihood (ML) estimate, $\phi : \mathbb{C} \rightarrow \mathbb{Z}[i]$, of the received signal $y$ to remove the noise and obtain the closest GAUSSIAN Integer to $y$

$$\phi(y) = \hat{y}_{ML} = argmin_{t \in \mathbb{Z}[i]} ||y - t||^2 \in \mathbb{Z}[i].$$

Further, this signal is mapped to $\mathbb{Z}/p\mathbb{Z}$. Therefore, the decoder at the relay is given by

$$\hat{v} = \mu^{-1}(\psi(\hat{y}_{ML})).$$

The estimate of the linear combination $\hat{v}$ is transmitted to the destination. We assume this transmission between relay and destination to be error free, that is to say, the linear combination is obtained at the destination exactly as estimated at the relay. This procedure gives us a linear combination. However, in order to decode the $L$ transmitted messages $w_l$ from $\hat{v}$, we need to collect $L$ times such linear combinations. Therefore, the $L$ linear combinations obtained at the destination can be written as

$$\begin{bmatrix} \hat{v}^1 \\ \vdots \\ \hat{v}^L \end{bmatrix} = \begin{bmatrix} a_1^1 & \cdots & a_L^1 \\ \vdots & \ddots & \vdots \\ a_1^L & \cdots & a_L^L \end{bmatrix} \begin{bmatrix} \hat{w}_1 \\ \vdots \\ \hat{w}_L \end{bmatrix}.$$

The decoder at the destination inverts the matrix $\mathbf{A}$ and obtains an estimate of $\mathbf{w}$. Therefore,

$$\hat{\mathbf{v}} = \mathbf{A}\hat{\mathbf{w}} \Rightarrow \hat{\mathbf{w}} = \mathbf{A}^{-1}\hat{\mathbf{v}}.$$

Here the inverse of $\mathbf{A}$ is done in $\mathbb{Z}/p\mathbb{Z}$ and so $\mathbf{A}$ is required to be full rank in $\mathbb{Z}/p\mathbb{Z}$ for successful decoding. This decoding operation is the main reason we are interested in the structure of field.

In order to understand the inner workings of this system model a basic implementation has been done. The MATLAB code is enclosed in Annex (I).

## 5.3 Proposed Design Methodology

At this point we know that a constellation is a set of points $\mathcal{A}$ and we have done a first approach to design. However, we have not seen a methodology in order to build them. The reason is because there are no hard-and-fast rules for designing constellations.

In this work it is proposed a methodology to design constellations based on the following steps:

**Step 1:** Select a ring where we are going to design the constellation.

First we need to know which kind of points will form the constellation. In this work we will focus on design constellations in $\mathbb{Z}[i]$ and $\mathbb{Z}[w]$.

**Step 2:** Select a kind of prime.

This choice is necessary in order to define the constellation size in the Step 3. We have seen in Theorems 3.6 and 3.9 that each prime in $\mathbb{Z}^+$ can have one of the different types of factorization in both rings $\mathbb{Z}[i]$ and $\mathbb{Z}[w]$.

We are interested only in primes because the number of elements in a field is determined by a power of a prime and the structure of field allows us to recover the message at the receiver.

**Step 3:** Choose the size $M$ of the constellation.

It will be a power of the chosen prime in Step 2. This step can be summarized as selecting a power for the prime in the previous step.

**Step 4:** Determine a field with $M$ elements in $\mathbb{Z}$ and a field with $M$ elements in the selected ring.

In this step we are interested in finding fields with a finite number of elements using modular arithmetic.

**Step 5:** Define the constellation mapping and its inverse mapping

**5.1** We have to propose a constellation mapping using the defined fields in Step 4.

Theoretically always exists an isomorphism between two fields with the same number of elements however in practice we need to check that the proposed mapping is a bijection. If it is not, we will propose another mapping until a positive answer is obtained.

It is important to have in mind that if the proposed constellation mapping is not a bijection it will not have practical interest. We need to be aware that without this condition we cannot recover the message at the receiver and so we cannot complete the communication process.

**5.2** Finally, we need to actually define the inverse mapping, which existence is given by the fact that the constellation mapping is a bijection. This will allow us to recover the message at the receiver and finish the communication process.

Once we have completed this set of steps, we can build the constellation, find $\mathcal{A}$ using the constellation mapping and finally test it, doing the analysis and comparing the results.

In Figure 19 it is shown the flow diagram of the proposed methodology to design constellations.

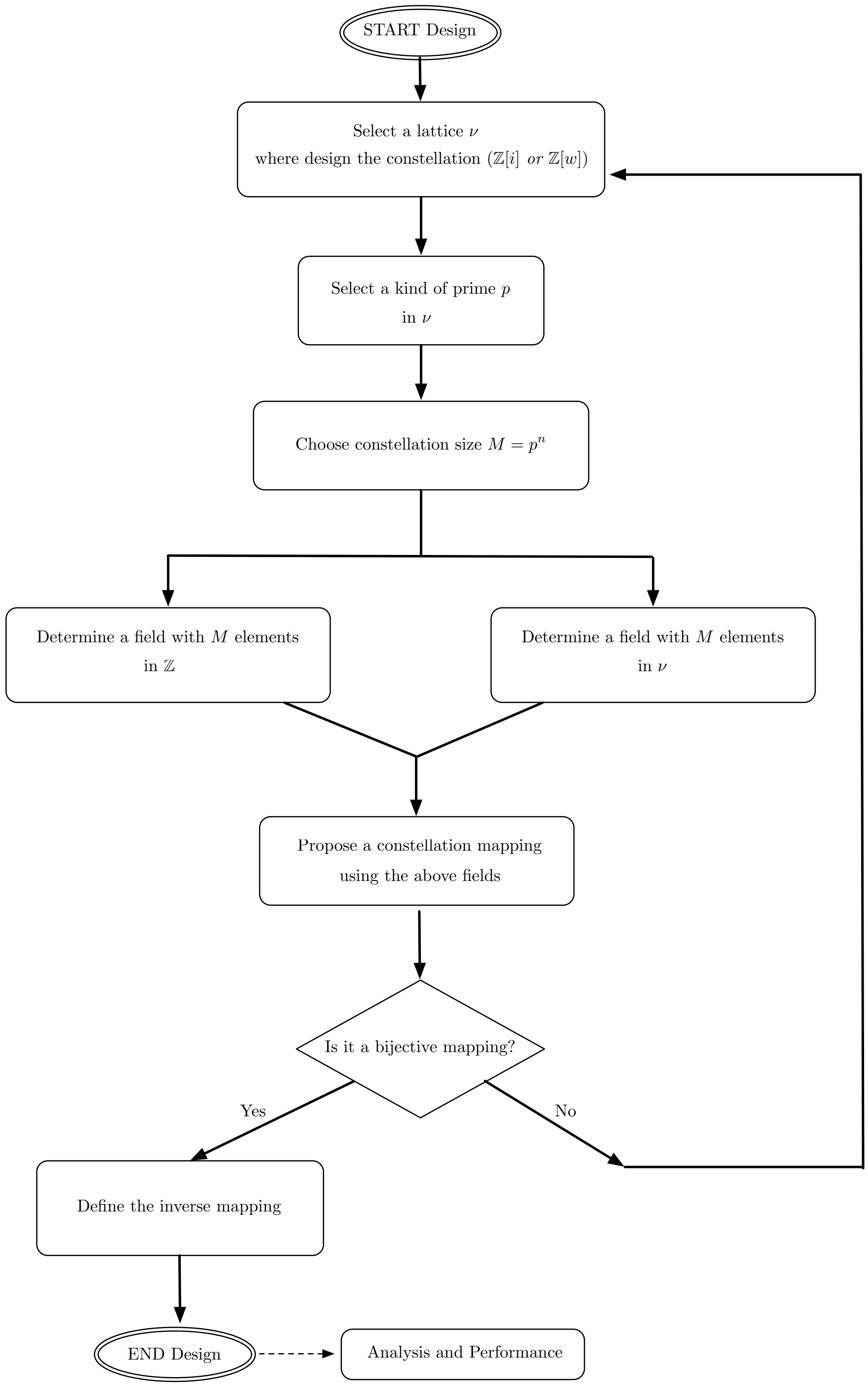



Figure 19: Flow Diagram Design Methodology

## 5.4 Design in $\mathbb{Z}[i]$

### 5.4.1 Constellation Design for Primes $p \equiv 1 \bmod 4$ in $\mathbb{Z}[i]$

**Step 1**: Following the steps of the proposed methodology we first choose the ring $\mathbb{Z}[i]$ where we are going to design the constellation. $\mathcal{A}$ will be a set of GAUSSIAN Integers points.

**Step 2**: We select primes $p$ in $\mathbb{Z}^+$ with type of factorization $p \equiv 1 \bmod 4$ in $\mathbb{Z}[i]$. We have seen in Theorem 3.6 that this type of primes in $\mathbb{Z}^+$ can be written as a product of two relative primes in $\mathbb{Z}[i]$, that is to say, $p = \pi\pi^*$.

**Step 3**: We choose the constellation size as $M = p$, or equivalently $n = 1$ using the same notation in Figure 19.

**Step 4**: In order to determine a field with $M = p$ elements in each ring $\mathbb{Z}$ and $\mathbb{Z}[i]$ we have used modular arithmetic:

- We know by Theorem 3.16 that if $p$ is prime in $\mathbb{Z}$, then $\mathbb{Z}/p\mathbb{Z}$ is a field and the number of elements of this field is determined by the absolute value of $p$. So we propose $\mathbb{Z}/p\mathbb{Z}$ as a field in $\mathbb{Z}$ with $M = p$ elements.
- On the other hand, $\pi$ is a prime in $\mathbb{Z}[i]$ and we know by Theorem 3.5 that $\mathbb{Z}[i]$ is a Principal Ideal Domain (PID) so we are in the hypothesis of Theorem 3.15 which allows us to conclude that $(\pi)$ is a prime ideal. Moreover, in a PID a prime ideal is a maximal ideal by Theorem 3.14 hence $(\pi)$ is a maximal ideal in the ring $\mathbb{Z}[i]$. Finally by Theorem 3.3 $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$ is a field.

  In this case the number of elements is determined by Theorem 3.13 using the norm of $\pi = a + bi$ . It is defined as $N(\pi) = a^2 + b^2 = (a+bi)(a-bi) = \pi\pi^* = p$, so $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$ is a field with $p$ elements.

Therefore the proposed fields in this step are $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$.

**Step 5:** In this case, $p$ decomposes in $\mathbb{Z}[i]$ as $p = \pi\pi^*$, with $\pi$ a prime number in $\mathbb{Z}[i]$.

Since $\#\left(\mathbb{Z}[i]/\pi\mathbb{Z}[i]\right) = N(\pi) = p$, the isomorphism we are looking for is $\mathbb{F}_p \cong \mathbb{Z}[i]/\pi\mathbb{Z}[i]$ and is obtained as follows:

**5.1** The constellation mapping between the fields defined above is defined using the modulo function (See reference [2]).

$$
\begin{aligned}
\mu:\ \mathbb{F}_p &\longrightarrow \mathbb{Z}[i]/\pi\mathbb{Z}[i] \\
x &\longmapsto \mu(x) = x - \left[\frac{x\pi^*}{\pi\pi^*}\right]\pi
\end{aligned}
$$

Figure 20: Constellation Mapping for Primes $p \equiv 1 \bmod 4$

We have studied it in detail in section (5.1).

**5.2** In section (5.1) it is proved that the modulo function defined as above is a bijective mapping which inverse is defined as:

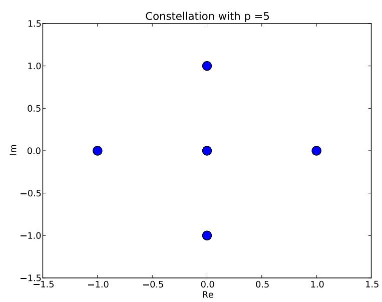
$$
\begin{array}{rcccl}
\mu^{-1}: & \mathbb{Z}[i]/\pi\mathbb{Z}[i] & \longrightarrow & \mathbb{F}_p & \\
 & a & \longmapsto & \mu^{-1}(a) = (a(v\pi^*) + a^*(u\pi^*)) \bmod p &
\end{array}
$$

Figure 21: Inverse Constellation Mapping for Primes $p \equiv 1 \bmod 4$

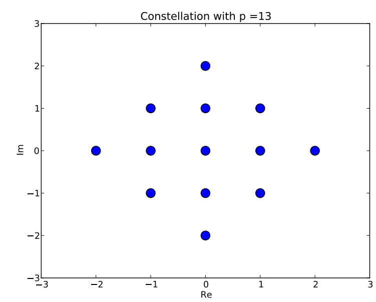
with $u\pi + v\pi^* = 1$.

This last step completes the design of $p \equiv 1 \bmod 4$ constellations in $\mathbb{Z}[i]$.

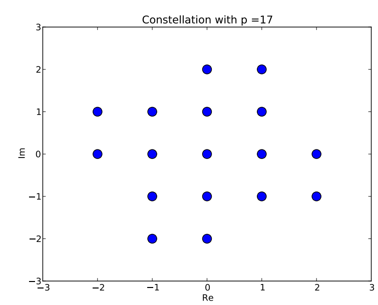
Now, we can see in Figure 22 the obtained constellations for different values of $p$:



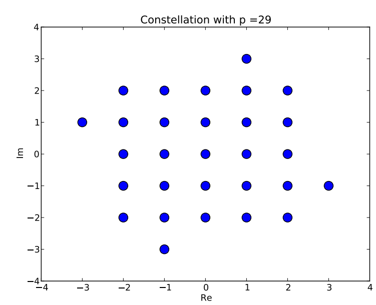

(a) $p \equiv 1 \bmod 4$ Constellation with $p = 5$




(b) $p \equiv 1 \bmod 4$ Constellation with $p = 13$




(c) $p \equiv 1 \bmod 4$ Constellation with $p = 17$




(d) $p \equiv 1 \bmod 4$ Constellation with $p = 29$

Figure 22: Four Examples of $p \equiv 1 \bmod 4$ Constellation Using Different Values of $p$

### 5.4.2 Constellation Design for Primes $p \equiv 3 \bmod 4$ in $\mathbb{Z}[i]$

Similarly to the previous case and following the steps of the proposed methodology we have proposed the next design. It is based on the extension for primes $p \equiv 3 \bmod 4$ proposed in [2] and it is extended along this section.

**Step 1**: We choose the ring $\mathbb{Z}[i]$. $\mathcal{A}$ will be a set of GAUSSIAN Integers points.

**Step 2**: We are going to work with primes $p$ in $\mathbb{Z}^+$ with type of factorization $p \equiv 3 \bmod 4$ in $\mathbb{Z}[i]$. We have seen in Theorem 3.6 that this type of primes are primes in $\mathbb{Z}^+$ which stay primes in $\mathbb{Z}[i]$.

**Step 3**: We choose the constellation size as $M = p^2$, or equivalently $n = 2$ using the same notation in Figure 19.

**Step 4**: In order to determine a field with $M = p^2$ elements in each ring $\mathbb{Z}$ and $\mathbb{Z}[i]$ we have done the following:

- Since $p \equiv 3 \bmod 4$ is prime in $\mathbb{Z}[i]$. By Theorem 3.15, $(p)$ is a prime ideal in $\mathbb{Z}[i]$, and so a maximal ideal because $\mathbb{Z}[i]$ is a PID (see Theorem 3.5). So, using Theorem 3.3 we obtain that $\mathbb{Z}[i]/p\mathbb{Z}[i]$ is a field.

  The number of elements is determined by Theorem 3.13 using the norm of $p$ . It is defined as $N(p) = pp^* = p^2$, so we propose $\mathbb{Z}[i]/p\mathbb{Z}[i]$ as a field in $\mathbb{Z}[i]$ with $M = p^2$ elements.

- In order to build a field with $M = p^2$ elements in $\mathbb{Z}$ we need to be aware that simply doing the quotient ring $\mathbb{Z}/p^2\mathbb{Z}$ does not guarantee the structure of field because $p^2$ is not a prime. However, we know that when $n > 1$ $\mathbb{F}_{p^n}$ can be represented as the field of equivalence classes of polynomials whose coefficients belong to $\mathbb{F}_p$ (any irreducible polynomial of degree $n$ yields the same field up to an isomorphism).

  We are interested in applying Theorem 3.17. We have the hypothesis that $p$ is prime, now our goal is to determine a monic irreducible $m(x)$ in $\mathbb{F}_p[X]$ of degree $n = 2$.

  We propose $x^2 + 1$ as a monic irreducible in $\mathbb{F}_p[X]$.

  It is easy to prove its irreducibility $x^2 + 1$; it has two roots $i$ and $-i$ but none in $\mathbb{F}_p$ so we can conclude that $x^2 + 1$ is a monic irreducible in $\mathbb{F}_p[X]$

  Finally using Theorem 3.17 we propose $\mathbb{F}_p[X]/(x^2 + 1)$ as a field with $M = p^2$ elements.

Therefore the proposed fields in this step are $\mathbb{Z}[i]/p\mathbb{Z}[i]$ and $\mathbb{F}_p[X]/(x^2 + 1)$.

**Step 5**: The isomorphism we are looking for is $\mathbb{Z}[i]/p\mathbb{Z}[i] \cong \mathbb{F}_p[X]/(x^2 + 1)$ with $X$ corresponding $i$ and is obtained as follows:

We are going to show that $\mathbb{Z}[i]/p\mathbb{Z}[i] \cong \mathbb{F}_p[X]/(x^2 + 1)$ with $X$ corresponding $i$ proving that each one is isomorphic to $\mathbb{Z}[X]/(p, x^2 + 1)$.

*Proof.* First we need to proof that $\mathbb{Z}[X]/(x^2 + 1) \cong \mathbb{Z}[i]$ with $X \mapsto i$

Consider the ring morphism

$$\begin{array}{rcl} \psi : \mathbb{Z}[X] & \longrightarrow & \mathbb{Z}[i] \\ P(X) & \longmapsto & P(i) \end{array}$$

Figure 23: Ring Morphism between $\mathbb{Z}[X]$ and $\mathbb{Z}[i]$

This is clearly surjective: every element $a + bi$ in $\mathbb{Z}[i]$ has a corresponding element $a + bX$ in $\mathbb{Z}[X]$ given by $\psi(a + bX) = a + bi$.

The kernel contains $(x^2 + 1)$. Moreover, if ones writes the euclidean division of $P(X)$ by $x^2 + 1$, one obtains a remainder of degree 1, $a + bX$, which is zero if and only if $a + bi = \psi(P(X))$ is zero, so the kernel is the ideal

$(x^2+1)$.

By the First Isomorphism Theorem we know that the image of $\psi$ is isomorphic to the quotient ring $\mathbb{Z}[X]/\mathrm{Kernel}(\psi)$. Finally, using that $\psi$ is surjective, $\mathrm{Image}(\psi) = \mathbb{Z}[i]$, we obtain $\mathbb{Z}[X]/(x^2+1) \cong \mathbb{Z}[i]$ with $X \mapsto i$.

$$\begin{tikzcd} \mathbb{Z}[X] \arrow[r, "\psi"] \arrow[d, "\pi"'] & \mathbb{Z}[i] = \mathrm{Image}(\psi) \\ \begin{array}{c}\mathbb{Z}[X]/\mathrm{Kernel}(\psi)\\ = \\ \mathbb{Z}[X]/(x^2+1)\end{array} \arrow[ur, "\cong", "\hat{\psi}"'] & \end{tikzcd}$$

Figure 24: First Isomorphism Theorem Diagram

where $\pi\colon \mathbb{Z}[X] \to \mathbb{Z}[X]/(x^2+1)$ is the canonical projection studied in detail in section (3.2.5).

Finally, let $p$ be a prime and $\mathbb{Z}[i]$ a PID (hence an integral domain) by Theorem 3.15 We know that $p\mathbb{Z}[i]$ is a prime ideal and the inverse of a prime ideal is a prime ideal too. Therefore we can assert that $\psi^{-1}(p\mathbb{Z}[i]) = (p, x^2+1)$.

Hence, $\mathbb{Z}[i]/p\mathbb{Z}[i] \cong \mathbb{Z}[X]/(p, x^2+1)$.

On the other hand, since $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ we have $\mathbb{F}_p[X]/(x^2+1) \cong \mathbb{Z}[X]/(p, x^2+1)$:

$$\begin{aligned} \mathbb{F}_p[X]/(x^2+1) &\cong (\mathbb{Z}/p\mathbb{Z})[X]/(x^2+1) \\ &\cong (\mathbb{Z}[X]/(p))/(x^2+1) \\ &\cong \mathbb{Z}[X]/(p, x^2+1) \end{aligned}$$

And that ends the proof.

$\square$

This proof allows us to define the constellations mapping as follows:

**5.1** The constellation mapping is defined as:

$$\begin{tikzcd} \gamma: \mathbb{F}_p[X]/(x^2+1) \arrow[r] & \mathbb{Z}[i]/p\mathbb{Z}[i] \\ x \arrow[r, mapsto] & i \end{tikzcd}$$

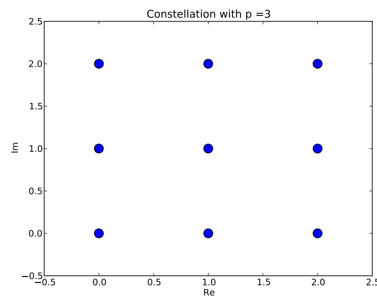
Figure 25: Constellation Mapping for Primes $p \equiv 3 \bmod 4$

**5.2** The inverse constellation mapping is defined as:

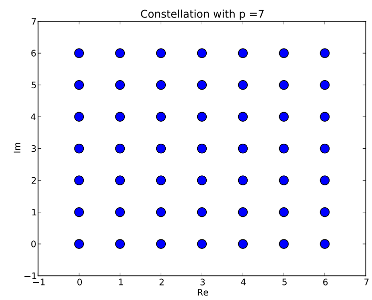
$$\begin{array}{rcl}\gamma^{-1}: \mathbb{Z}[i]/p\mathbb{Z}[i] & \longrightarrow & \mathbb{F}_p[X]/(x^2+1) \\ i & \longmapsto & x\end{array}$$

Figure 26: Inverse Constellation Mapping for Primes $p \equiv 3 \bmod 4$

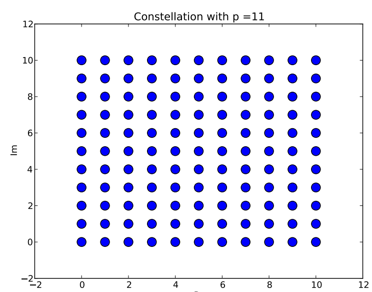
At this point we have finished the process of designing the constellation. Now in Figure 27 we can observe the resulting design of $p \equiv 3 \bmod 4$ constellations in $\mathbb{Z}[i]$ for different values of $p$.



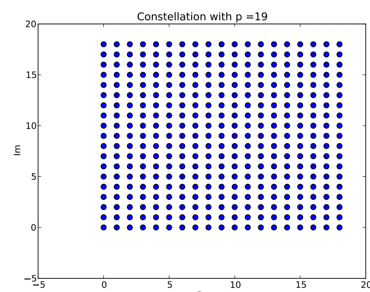

(a) $p \equiv 3$ modulo 4 Constellation with $p = 3$




(b) $p \equiv 3 \bmod 4$ Constellation with $p = 7$




(c) $p \equiv 3 \bmod 4$ Constellation with $p = 11$




(d) $p \equiv 3 \bmod 4$ Constellation with $p = 19$

Figure 27: Four Examples of $p \equiv 3 \bmod 4$ Constellation Using Different Values of $p$

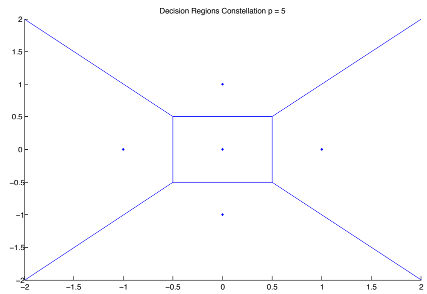
## 5.5 Best Performing Design(s) in $\mathbb{Z}[i]$

### 5.5.1 Decision Regions

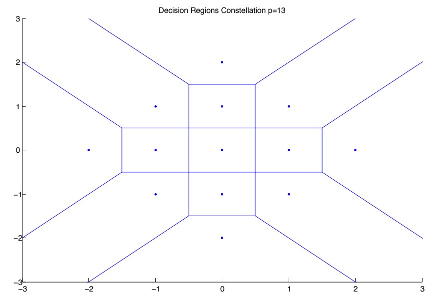
We have seen in section (4.1.1) that the decision region for a point $x_i$ in the constellation $\mathcal{A} = \{x_i\}_{i=0,\cdots,M}$, denoted $\mathcal{R}_{x_i}$, is the set of points of the complex plane that are closer to $x_i$ than to any other point of the signal constellation. Moreover, the decision region for a point $x_i$ is a polygon, often an irregular polygon, whose sides are the perpendicular bisectors of the lines between $x_i$ and the neighbours of $x_i$. This last is because in all the designs we are using Euclidean distance.

Now, we are interested in showing the resulting decision regions for the proposed constellations.

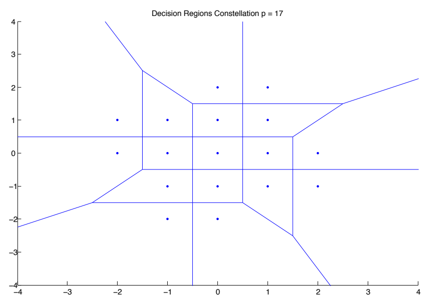
We have obtained the next decision regions for $1 \bmod 4$ constellations:



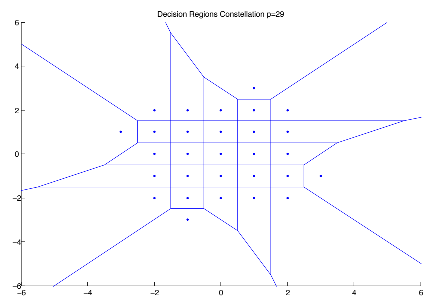

(a) $p \equiv 1 \bmod 4$ Constellation with $p = 5$




(b) $p \equiv 1 \bmod 4$ Constellation with $p = 13$



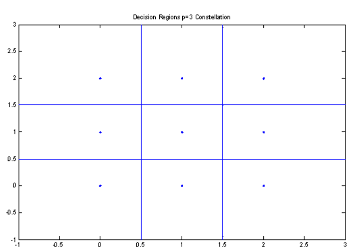

(c) $p \equiv 1 \bmod 4$ Constellation with $p = 17$



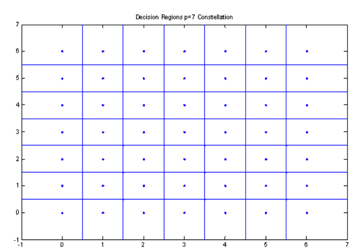

(d) $p \equiv 1 \bmod 4$ Constellation with $p = 29$

Figure 28: Decision Regions of $p \equiv 1 \bmod 4$ Constellation

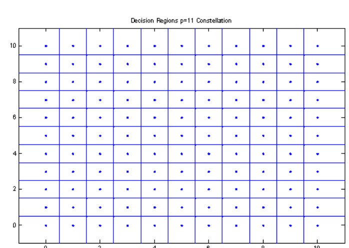
We enclose the resulting decision regions for constellations $3 \bmod 4$ in $\mathbb{Z}[i]$



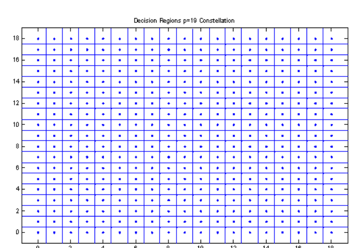

(a) $p \equiv 3 \bmod 4$ Constellation with $p = 3$




(b) $p \equiv 3 \bmod 4$ Constellation with $p = 7$




(c) $p \equiv 3 \bmod 4$ Constellation with $p = 11$




(d) $p \equiv 3 \bmod 4$ Constellation with $p = 19$

Figure 29: Decision Regions of $p \equiv 3 \bmod 4$ Constellation

Where we can see that $3 \bmod 4$ constellations have the same decision region's shape than common QAM.

### 5.5.2 Analysis of Error Probability

We implement the Nearest Neighbor Union Bound, explained in section (4.1.2.2)

$$P_e \leq N_e \cdot Q\left[\frac{d_{\min}}{2\sigma}\right]. \tag{29}$$

We start working with $1 \bmod 4$ constellations and proceed to do the analysis with $3 \bmod 4$ constellations.

Using the algorithms described in sections (4.1.3.1) and (4.1.3.2) we obtain $d_{\min}$ and $N_e$ parameters for $1 \bmod 4$ constellations.

| | |
|---|---|
| $\mathbb{Z}[i]$ Constellation $1 \bmod 4$ with $p = 5$ | $d_{\min} = 1$ |
| $\mathbb{Z}[i]$ Constellation $1 \bmod 4$ with $p = 13$ | $d_{\min} = 1$ |
| $\mathbb{Z}[i]$ Constellation $1 \bmod 4$ with $p = 17$ | $d_{\min} = 1$ |
| $\mathbb{Z}[i]$ Constellation $1 \bmod 4$ with $p = 29$ | $d_{\min} = 1$ |

Table 5: $d_{\min}$ Numerical Results for $\mathbb{Z}[i]$ Constellation $1 \bmod 4$

| | |
|---|---|
| $\mathbb{Z}[i]$ Constellation $1 \bmod 4$ with $p = 5$ | $N_e = 3.2$ |
| $\mathbb{Z}[i]$ Constellation $1 \bmod 4$ with $p = 13$ | $N_e = 3.6923$ |
| $\mathbb{Z}[i]$ Constellation $1 \bmod 4$ with $p = 17$ | $N_e = 3.7647$ |
| $\mathbb{Z}[i]$ Constellation $1 \bmod 4$ with $p = 29$ | $N_e = 4.1379$ |

Table 6: $N_e$ Numerical Results for $1 \bmod 4$ Constellations in $\mathbb{Z}[i]$

For constellations $p = 5$, $p = 13$, $p = 17$ and $p = 29$ in $\mathbb{Z}[i]$ we plot the Nearest Neighbor Union Bound in the next figure. We also plot $4$-QAM and $16$-QAM as a reference.

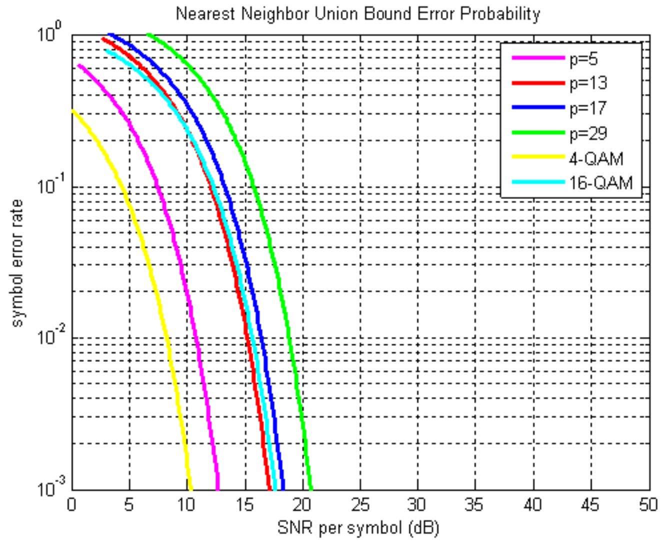



Figure 30: Nearest Neighbor Union Bound for $1 \bmod 4$ Constellations

We can see that constellations $1 \bmod 4$ are at most as good as QAM constellations but not better. For instance, $p = 17$ and 16-QAM present almost the same behavior, we can observe a 1dB difference an just one point difference. However, for $p = 5$ and 4-QAM, common QAM outperforms constellations $1 \bmod 4$, here we can see an average 3dB difference. Therefore, constellations $1 \bmod 4$ represent an alternative to common QAM but not imply improvement.

We consider now the constellations $3 \bmod 4$ in $\mathbb{Z}[i]$.

We compute $d_{\min}$

| | |
|---|---|
| $\mathbb{Z}[i]$ Constellation $3 \bmod 4$ with $p = 3$ | $d_{\min} = 1$ |
| $\mathbb{Z}[i]$ Constellation $3 \bmod 4$ with $p = 7$ | $d_{\min} = 1$ |
| $\mathbb{Z}[i]$ Constellation $3 \bmod 4$ with $p = 11$ | $d_{\min} = 1$ |
| $\mathbb{Z}[i]$ Constellation $3 \bmod 4$ with $p = 19$ | $d_{\min} = 1$ |

Table 7: $d_{\min}$ Numerical Results for $\mathbb{Z}[i]$ Constellation $3 \bmod 4$

We can see that the $d_{\min}$ value obtained for all the constellations in $\mathbb{Z}[i]$ remains the same with a value of $d_{\min} = 1$. This can be understood as a consequence of how constellations in $\mathbb{Z}[i]$ are generated using modulo function: the set of classes defining the points of a constellation $\mathcal{A}_1$ are included in the set of classes defining another constellation $\mathcal{A}_2$ with a larger dimension. Therefore if the first constellation achieves the minimum distance of the ring all the other constellations larger than it will have the same value of $d_{\min}$.

We also compute $N_e$ and obtain:

| | |
|---|---|
| $\mathbb{Z}[i]$ Constellation $3 \bmod 4$ with $p = 3$ | $N_e = 2.6667$ |
| $\mathbb{Z}[i]$ Constellation $3 \bmod 4$ with $p = 7$ | $N_e = 3.4286$ |
| $\mathbb{Z}[i]$ Constellation $3 \bmod 4$ with $p = 11$ | $N_e = 3.6364$ |
| $\mathbb{Z}[i]$ Constellation $3 \bmod 4$ with $p = 19$ | $N_e = 3.7895$ |

Table 8: $N_e$ Numerical Results for $3 \bmod 4$ Constellations in $\mathbb{Z}[i]$

And we plot the Nearest Neighbor Union Bound for $p = 3$, $p = 7$, $p = 11$ and $p = 19$ for $3 \bmod 4$ constellations in $\mathbb{Z}[i]$. We plot 16-QAM, 64-QAM and 256-QAM as a reference.

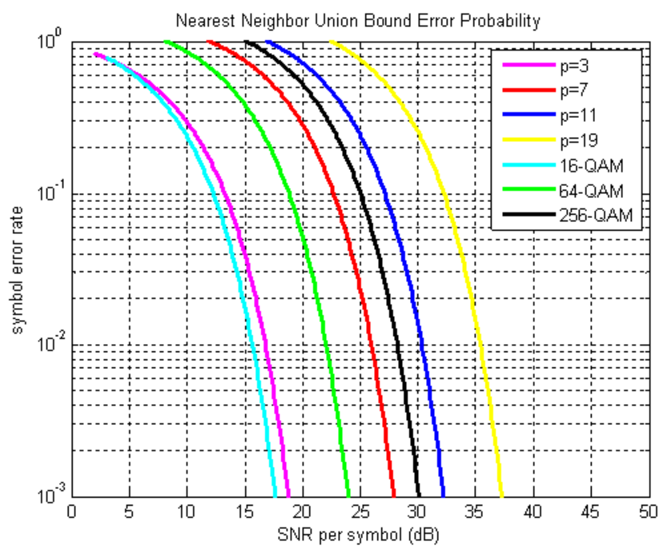



Figure 31: Nearest Neighbor Union Bound for $3 \bmod 4$ Constellations

We can see that constellations $3 \bmod 4$ in $\mathbb{Z}[i]$ show worse results than QAM constellations. For instance, we can see that common 16-QAM outperforms $p = 3$ (9 points constellation). Moreover, for $p = 7$ (49 points constellation) we can see there is an average 4dB difference with 64-QAM, which means that $3 \bmod 4$ is worse than QAM. Therefore, $3 \bmod 4$ constellations in $\mathbb{Z}[i]$ are much worse than common QAM.

## 5.6 Design in $\mathbb{Z}[w]$

### 5.6.1 Constellation Design for Primes $p \equiv 1 \bmod 6$ in $\mathbb{Z}[w]$

Now, we are going to design constellations in the ring of EISENSTEIN Integers, as in the previous sections we will follow the steps of the proposed methodology.

**Step 1**: First, we have chosen the ring $\mathbb{Z}[w]$. The proposed constellation $\mathcal{A}$ will be a set of EISENSTEIN Integers points.

**Step 2:** We select primes $p$ in $\mathbb{Z}^+$ with type of factorization $p \equiv 1 \bmod 6$ in $\mathbb{Z}[w]$. We have seen in Theorem 3.9 that this type of primes can be written as sum $p = a^2 + 3b^2$. Therefore, such primes $p$ in $\mathbb{Z}^+$ are the product of two conjugate primes in EISENSTEIN Integers:

$$p = a^2 + 3b^2 = \pi\pi^* \tag{30}$$

where $\pi = a + b + w2b$ and the conjugate of $\pi$ is $\pi^* = a + b + w^2 2b$.

**Step 3**: We choose the constellation size as $M = p$, or equivalently $n = 1$ using the same notation in Figure 19.

**Step 4:** In order to determine a field with $M = p$ elements in each ring $\mathbb{Z}$ and $\mathbb{Z}[i]$ we have used modular arithmetic:

- We know by Theorem 3.16 that if $p$ is prime in $\mathbb{Z}$, then $\mathbb{Z}/p\mathbb{Z}$ is a field and the number of elements of this field is determined by the absolute value of $p$. So we propose $\mathbb{Z}/p\mathbb{Z}$ as a field in $\mathbb{Z}$ with $M = p$ elements.
- On the other hand, $\pi$ is a prime in $\mathbb{Z}[w]$ and we know by Theorem 3.8 that $\mathbb{Z}[w]$ is a Principal Ideal Domain (PID) so we are in the hypothesis of Theorem 3.15 which allows us to conclude that $(\pi)$ is a prime ideal. Moreover, in a PID a prime ideal is a maximal ideal by Theorem 3.14 hence $(\pi)$ is a maximal ideal in the ring $\mathbb{Z}[w]$. Finally by Theorem 3.3 $\mathbb{Z}[w]/\pi\mathbb{Z}[w]$ is a field.

  In this case the number of elements is determined by Theorem 3.13 using the norm of $\pi = a + b + w2b$. It is defined as $N(\pi) = \pi\pi^* = p$, so $\mathbb{Z}[w]/\pi\mathbb{Z}[w]$ is a field with $p$ elements.

Therefore, the proposed fields in this step are $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}[w]/\pi\mathbb{Z}[w]$.

**Step 5**: In this case, $p$ decomposes in $\mathbb{Z}[w]$ as $p = \pi\pi^*$, with $\pi$ a prime number in $\mathbb{Z}[w]$.

Since $\#(\mathbb{Z}[w]/\pi\mathbb{Z}[w]) = N(\pi) = p$, the isomorphism we are looking for is $\mathbb{F}_p \cong \mathbb{Z}[w]/\pi\mathbb{Z}[w]$ and is obtained as follows:

**5.1** The constellation mapping between the fields defined above is defined using the modulo function (see reference [9]):

$$\begin{aligned} \tilde{\mu} :\ \mathbb{F}_p &\longrightarrow \mathbb{Z}[w]/\pi\mathbb{Z}[w] \\ x &\longmapsto \tilde{\mu}(x) = x - \left[\frac{x\pi^*}{\pi\pi^*}\right]\pi \end{aligned}$$

Figure 32: Constellation Mapping for Primes $p \equiv 1 \bmod 6$

We have studied it in detail in section (5.4) for GAUSSIAN Integers, all the study is the same using the analog theorems for EISENSTEIN Integers.

**5.2** Using the same result obtained in section (5.4) for GAUSSIAN Integers and adapt it for EISENSTEIN Integers, we can assert that the modulo function defined above is a bijective mapping which inverse is defined as:

$$\begin{aligned} \mu^{-1} :\ \mathbb{Z}[w]/\pi\mathbb{Z}[w] &\longrightarrow \mathbb{F}_p \\ a &\longmapsto \mu^{-1}(a) = (a(v\pi^*) + a^*(u\pi^*))\mathrm{mod} p \end{aligned}$$

Figure 33: Constellation Inverse Mapping for Primes $p \equiv 1 \bmod 6$

with

$$u\pi + v\pi^* = 1. \tag{31}$$

Testing is straightforward, using the above equation (31) we immediately get the inverse mapping $\tilde{\mu}^{-1}$:

$$x = \tilde{\mu}^{-1}(a) \equiv (a(v\pi^*) + a^*(u\pi)) \bmod p, \tag{32}$$

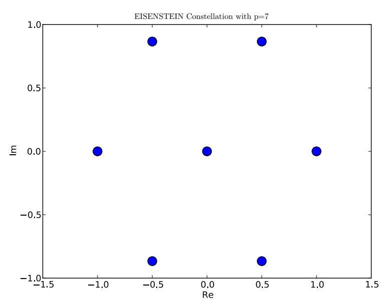
because if $x$ is an integer of $\mathbb{F}_p$ then $x = k\pi + a$ and $x = x^* = k^*\pi^* + a^*$, hence

$$a(v\pi^*) + a^*(u\pi) = (x - k\pi)(v\pi^*) + (x - k^*\pi^*)(u\pi) \equiv (x(v\pi^* + u\pi)) \bmod p \tag{33}$$

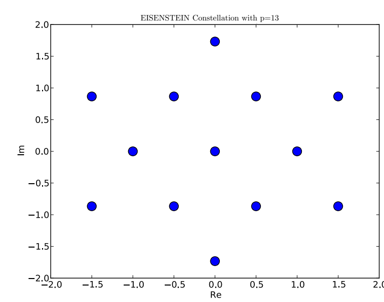
which equals $x$ by equation (31).

This last step completes the design of $p \equiv 1 \bmod 6$ constellations in $\mathbb{Z}[w]$.

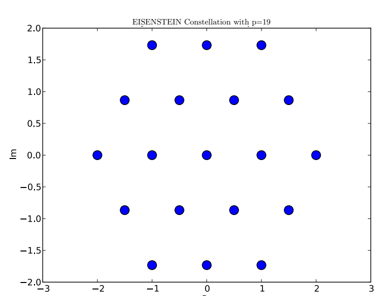
Now, we can see in Figure 34 the implemented constellations for different values of $p$:



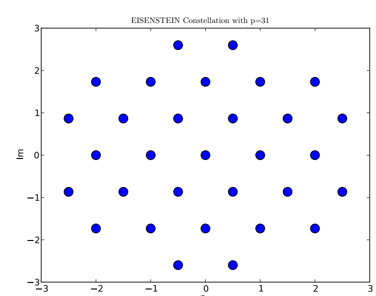

(a) $p \equiv 1 \bmod 6$ Constellation with $p = 7$




(b) $p \equiv 1 \bmod 6$ Constellation with $p = 13$




(c) $p \equiv 1 \bmod 6$ Constellation with $p = 19$




(d) $p \equiv 1 \bmod 6$ Constellation with $p = 31$

Figure 34: Four Examples of $p \equiv 1 \bmod 6$ Constellation Using Different Values of $p$

#### 5.6.2 Constellation Design for Primes $p \equiv 2 \bmod 3$ in $\mathbb{Z}[w]$

Similarly to the previous case and following the steps of the used methodology we have proposed the next design. It is based on the extension for primes $p \equiv 2 \bmod 3$ in reference [9] and it is extended along this section.

**Step 1**: We choose the ring $\mathbb{Z}[w]$. The constellation, $\mathcal{A}$, will be a set of EISENSTEIN Integers points.

**Step 2**: We are going to work with primes $p$ in $\mathbb{Z}^+$ with type of factorization $p \equiv 2 \bmod 3$ in $\mathbb{Z}[w]$. We have seen in Theorem 3.9 that this type of primes in $\mathbb{Z}^+$ which stay primes in $\mathbb{Z}[w]$.

**Step 3**: We choose the constellation size as $M = p^2$, or equivalently $n = 2$ using the same notation in Figure 19.

**Step 4**: In order to determine a field with $M = p^2$ elements in $\mathbb{Z}$ and $\mathbb{Z}[w]$ we have proceeded as follows:

- Since $p \equiv 2 \bmod 3$ is prime in $\mathbb{Z}[w]$ and it is a PID (Theorem 3.8) we have by Theorem 3.15 that $(p)$ is a prime ideal in $\mathbb{Z}[w]$. Moreover, in a PID a prime ideal is a maximal ideal by Theorem 3.14. Finally, using Theorem 3.3 we obtain that $\mathbb{Z}[w]/p\mathbb{Z}[w]$ is a field.

  The number of elements is determined by Theorem 3.13 using the norm of $p$. It is defined as $N(p) = pp^* = p^2$, so we propose $\mathbb{Z}[w]/p\mathbb{Z}[w]$ as a field in $\mathbb{Z}[w]$ with $M = p^2$.

- As in section (5.4.2), in order to build a field with $M = p^2$ elements in $\mathbb{Z}$ we need to be aware that simply doing the quotient ring $\mathbb{Z}/p^2\mathbb{Z}$ does not guarantee the structure of field because $p^2$ is not a prime. However, we know that when $n > 1$ $\mathbb{F}_{p^n}$ can be represented as the field of equivalence classes of polynomials whose coefficients belong to $\mathbb{F}_p$.

  We are interested in applying Theorem 3.17. We have the hypothesis that $p$ is prime, now our goal is to determine a monic irreducible $m(x)$ in $\mathbb{F}_p[X]$ of degree $n = 2$.

  We propose $x^2 + x + 1$ as a monic irreducible in $\mathbb{F}_p[X]$.

  It is easy to prove that $x^2 + x + 1$ is irreducible in $\mathbb{F}_p[X]$; $x^2 + x + 1$ has two roots $(-1 - \sqrt{-3}i)/2$ and $(-1 + \sqrt{-3}i)/2$ but none in $\mathbb{F}_p$ so $x^2 + x + 1$ as a monic irreducible in $\mathbb{F}_p[X]$.

  Finally using Theorem 3.17 we propose $\mathbb{F}_p[X]/(x^2 + x + 1)$ as a field with $M = p^2$ elements.

Therefore, the proposed fields in this step are $\mathbb{Z}[w]/p\mathbb{Z}[w]$ and $\mathbb{F}_p[X]/(x^2 + x + 1)$.

**Step 5**: The isomorphism we are looking for is $\mathbb{Z}[w]/p\mathbb{Z}[w] \cong \mathbb{F}_p[X]/(x^2 + x + 1)$ with $X$ corresponding $w$ and it is obtained step by step using the same proof as in section (5.4.2). Hence this proof allow us to define the constellation mappings as follows:

**5.1** The constellation mapping is defined as:

$$\begin{aligned} \tilde{\gamma}:\ \mathbb{F}_p[X]/(x^2+x+1) &\longrightarrow \mathbb{Z}[w]/p\mathbb{Z}[w] \\ x &\longmapsto w \end{aligned}$$

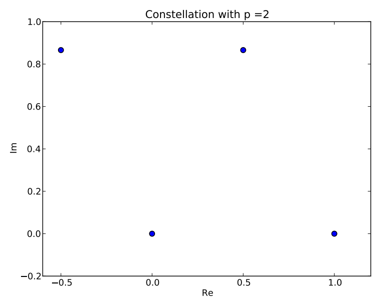
Figure 35: Constellation Mapping for Primes $p \equiv 2 \bmod 3$

**5.2** The inverse constellation mapping is defined as:

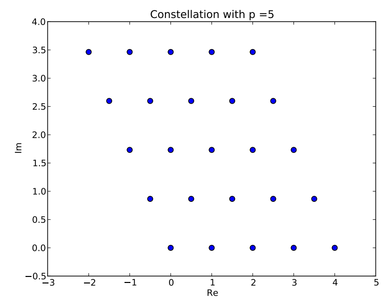
$$
\begin{aligned}
\tilde{\gamma}^{-1} : \mathbb{Z}[w]/p\mathbb{Z}[w] &\longrightarrow \mathbb{F}_p[X]/(x^2+x+1) \\
w &\longmapsto x
\end{aligned}
$$

Figure 36: Constellation Inverse Mapping for Primes $p \equiv 2 \bmod 3$

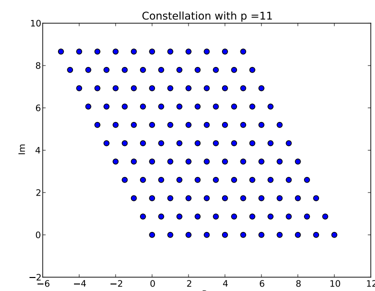
At this point we have finished the process of design. Now, in Figure 37 we can observe the resulting design of $p \equiv 2 \bmod 3$ constellations in $\mathbb{Z}[w]$ for different values of $p$:



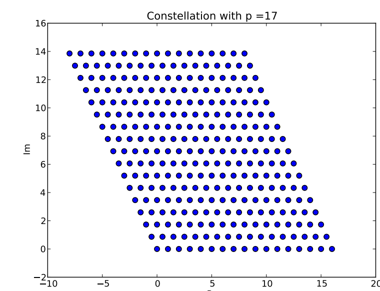

(a) $p \equiv 2 \bmod 3$ Constellation with $p = 2$




(b) $p \equiv 2 \bmod 3$ Constellation with $p = 5$




(c) $p \equiv 2 \bmod 3$ Constellation with $p = 11$



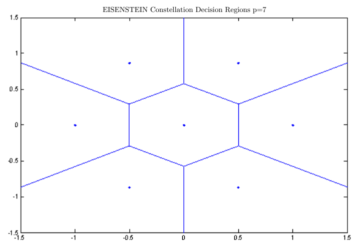

(d) $p \equiv 2 \bmod 3$ Constellation with $p = 17$

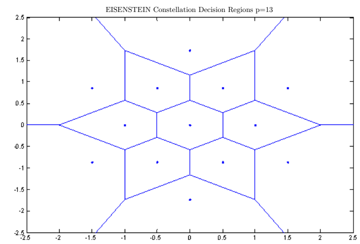
Figure 37: Four Examples of $p \equiv 2 \bmod 3$ Constellations Using Different Values of $p$

## 5.7 Best Performing Design(s) in $\mathbb{Z}[w]$

### 5.7.1 Decision Regions

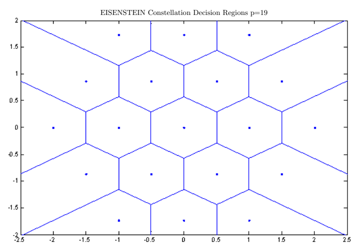
We plot the resulting decision regions for constellations $1 \bmod 6$ in $\mathbb{Z}[w]$:

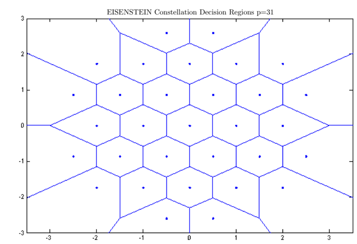



(a) EISENSTEIN Constellation $p \equiv 1 \bmod 6$ with $p = 7$

(b) EISENSTEIN Constellation $p \equiv 1 \bmod 6$ with $p = 13$



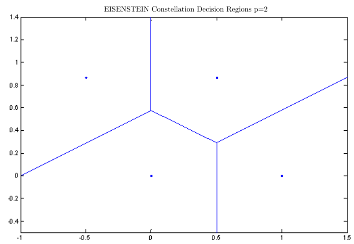




(c) EISENSTEIN Constellation $p \equiv 1 \bmod 6$ with $p = 19$

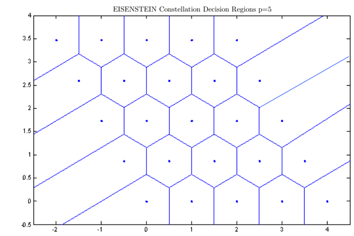
(d) EISENSTEIN Constellation $p \equiv 1 \bmod 6$ with $p = 31$

Figure 38: EISENSTEIN Constellation Decision Regions of $p \equiv 1 \bmod 6$

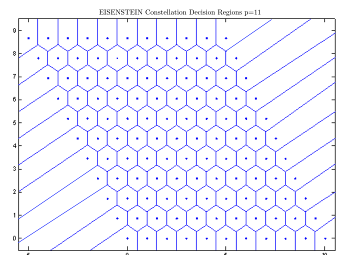
We plot the same results for constellations $2 \bmod 3$ in $\mathbb{Z}[w]$:



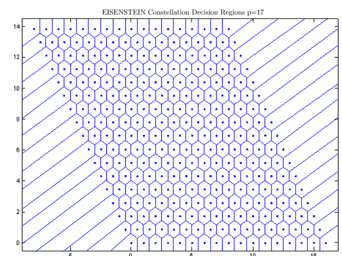

(a) EISENSTEIN Constellation $p \equiv 2 \bmod 3$ with $p = 2$




(b) EISENSTEIN Constellation $p \equiv 2 \bmod 3$ with $p = 5$




(c) EISENSTEIN Constellation $p \equiv 2 \bmod 3$ with $p = 11$




(d) EISENSTEIN Constellation $p \equiv 2 \bmod 3$ with $p = 17$

Figure 39: EISENSTEIN Constellation Decision Regions of $p \equiv 2 \bmod 3$

### 5.7.2 Analysis of Error Probability

We implement the Nearest Neighbor Union Bound, explained in section (4.1.2.2)

$$P_e \leq N_e \cdot Q\left[\frac{d_{\min}}{2\sigma}\right].$$

We start working with $1 \bmod 6$ constellations and proceed to do the analysis with $2 \bmod 3$.

Using the algorithms described in section (4.1.3.1) and (4.1.3.2) we obtain $d_{\min}$ and $N_e$ parameters for $1 \bmod 6$ constellations.

| | |
|---|---|
| $\mathbb{Z}[w]$ Constellation $1 \bmod 6$ with $p = 7$ | $d_{\min} = 1$ |
| $\mathbb{Z}[w]$ Constellation $1 \bmod 6$ with $p = 13$ | $d_{\min} = 1$ |
| $\mathbb{Z}[w]$ Constellation $1 \bmod 6$ with $p = 19$ | $d_{\min} = 1$ |
| $\mathbb{Z}[w]$ Constellation $1 \bmod 6$ with $p = 31$ | $d_{\min} = 1$ |

Table 9: $d_{\min}$ Numerical Results for $\mathbb{Z}[w]$ Constellation $1 \bmod 6$

| | |
|---|---|
| $\mathbb{Z}[w]$ Constellation $1 \bmod 6$ with $p = 7$ | $N_e = 3.4286$ |
| $\mathbb{Z}[w]$ Constellation $1 \bmod 6$ with $p = 13$ | $N_e = 4.6154$ |
| $\mathbb{Z}[w]$ Constellation $1 \bmod 6$ with $p = 19$ | $N_e = 4.4211$ |
| $\mathbb{Z}[w]$ Constellation $1 \bmod 6$ with $p = 31$ | $N_e = 5.0323$ |

Table 10: $N_e$ Numerical Results for $1 \bmod 3$ Constellations in $\mathbb{Z}[w]$

The results show consistency with the obtained decision regions for each constellation; constellations $1 \bmod 6$ in the ring $\mathbb{Z}[w]$ have a higher $N_e$ value in comparison with the constellations in $\mathbb{Z}[i]$. This is due to the hexagonal shape of their decision regions which allow them to have a bigger average number of neighbors.

We plot the Nearest Neighbor Union Bound for primes $1 \bmod 6$ with $p = 7$, $p = 13$, $p = 19$ and $p = 31$. We also plot 8-PSK and 16-QAM as a reference.

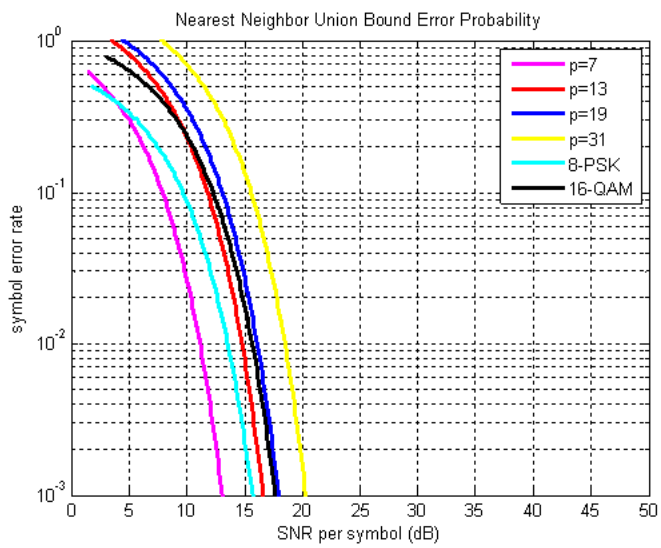



Figure 40: EISENSTEIN Constellations. Nearest Neighbor Union Bound for $1 \bmod 6$

We can see that constellations $1 \bmod 6$ in $\mathbb{Z}[w]$ show consistently better results than common QAM. In fact, here we can see something really interesting, for example, $p = 7$ constellation outperforms by far 8-PSK, we can see an average 3dB difference, which is an important improvement considering there is just one point difference. Moreover, $p = 19$ presents almost the same behavior as 16-QAM, which means that we are able to send 3 more points using the same SNR. Therefore, constellations $1 \bmod 6$ in $\mathbb{Z}[w]$ show impressive results when compared to common QAM and will definitely improve the performance of the overall system.

We consider now the constellations $2 \bmod 3$ in $\mathbb{Z}[w]$.

First we are going to compute $d_{\min}$

| | |
|---|---|
| $\mathbb{Z}[w]$ Constellation $2 \bmod 3$ with $p = 2$ | $d_{\min} = 1$ |
| $\mathbb{Z}[w]$ Constellation $2 \bmod 3$ with $p = 5$ | $d_{\min} = 1$ |
| $\mathbb{Z}[w]$ Constellation $2 \bmod 3$ with $p = 11$ | $d_{\min} = 1$ |
| $\mathbb{Z}[w]$ Constellation $2 \bmod 3$ with $p = 17$ | $d_{\min} = 1$ |

Table 11: $d_{\min}$ Numerical Results for $\mathbb{Z}[w]$ Constellation $2 \bmod 3$

We can see that the $d_{\min}$ value obtained for all the constellations in $\mathbb{Z}[w]$ remains the same with a value of $d_{\min} = 1$.

This can be understood as a consequence of how constellations in $\mathbb{Z}[w]$ are generated using modulo function: the set of classes defining the points of a constellation $\mathcal{A}_1$ are included in the set of classes defining another constellation $\mathcal{A}_2$ with a larger dimension. Therefore if the first constellation achieves the minimum distance of the ring all the other constellations larger than it will have the same value of $d_{\min}$.

| | |
|---|---|
| $\mathbb{Z}[w]$ Constellation $2 \bmod 3$ with $p = 2$ | $N_e = 2.5$ |
| $\mathbb{Z}[w]$ Constellation $2 \bmod 3$ with $p = 5$ | $N_e = 4.48$ |
| $\mathbb{Z}[w]$ Constellation $2 \bmod 3$ with $p = 11$ | $N_e = 5.2893$ |
| $\mathbb{Z}[w]$ Constellation $2 \bmod 3$ with $p = 17$ | $N_e = 5.5363$ |

Table 12: $N_e$ Numerical Results for $2 \bmod 3$ Constellations in $\mathbb{Z}[w]$

Finally, we plot the Nearest Neighbor Union Bound with primes $2 \bmod 3$ in $\mathbb{Z}[w]$ with $p = 2$, $p = 5$, $p = 11$ and $p = 17$. We also plot 4-QAM, 16-QAM and 256-QAM as a reference.

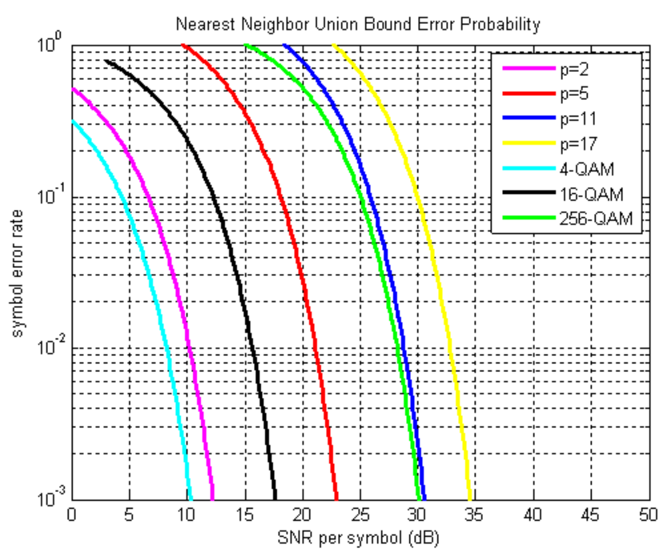



Figure 41: EISENSTEIN Constellations. Nearest Neighbor Union Bound for $2 \bmod 3$

We can see that constellations $2 \bmod 3$ in $\mathbb{Z}[w]$ are worse than common QAM. We can see for example that for $p = 2$ (4 points) the performance is worse than 4-QAM, where we can see an average 2dB difference. For $p = 11$ (121 points) we can see that there is just an average 0.5dB difference with 256-QAM. This implies that QAM is able to send $256$ points with the same overall performance as $p = 11$ ($121$ points) and therefore constellations $2 \bmod 3$ in $\mathbb{Z}[w]$ are not better than common QAM.

# 6 Conclusions and Further Work

We have studied in detail constellations over GAUSSIAN Integers and EISENSTEIN Integers. A theoretical introduction has been done as well as a description of the performance metrics used in the analysis. We have provided a basic system model implementation together with the necessary MATLAB codes to do the analysis.

As a conclusion of the proposed design, we have obtained good results using $1 \bmod 6$ constellation in the ring of $\mathbb{Z}[w]$, which is the best constellation design studied. Moreover, $1 \bmod 4$ in $\mathbb{Z}[i]$ appears as a good alternative to QAM, with similar performance results. However, it is also important to note that common QAM constellations have better performance than $3 \bmod 4$ in $\mathbb{Z}[i]$ and $2 \bmod 3$ in $\mathbb{Z}[w]$.

As a further work, constellation design over other types of integers could be proposed (for instance, HURWITZ Integers). Another important field of study is consider different distances, not only the Euclidean, and how this would affect to the error probability. Moreover, a more real implementation of the PNC scheme in Matlab would be of important use to do a more thorough analysis. Finally, the extension of this type of constellations to a more general system can be done.

# 7 Annex I

## Implementation

We are going to implement the system model with Matlab.

We use a RAYLEIGH faded channel model with coefficients rounded to the nearest GAUSSIAN Integer, which can be generated using a distribution gaussian both in the real and imaginary axis.

```
h=round((1/sqrt(2))*(randn(1,L)+j*randn(1,L)));
```

and circular symmetric complex GAUSSIAN noise $n \sim \mathcal{CN}(0, \sigma^2)$, where $\sigma^2$ is the noise power and can be calculated as:

$$SNR = \frac{\text{Average signal power}}{\text{noise power}} \Rightarrow \text{noise power} = \frac{\text{Average signal power}}{\text{SNR}} = \sigma^2$$

We can calculate the SNR measured in dB's as

$$SNR|_{dB} = 10 \cdot log(SNR_{Lineal}),$$
$$SNR_{Lineal} = 10^{\frac{SNR|_{dB}}{10}}.$$

The Average signal power of the constellation can be calculated as

$$\text{Average signal power} = \frac{1}{p} \sum_{c=1}^{p} x_c x_c^* \tag{34}$$

and therefore

$$\text{Noise power} = \frac{\frac{1}{p} \sum_{c=1}^{p} x_c x_c^*}{SNR_{Lineal}}. \tag{35}$$

Next, we generate the system model studied in the previous sections and we collect $L$ times the $\hat{v}$ values in order to estimate $w$.

A really important step in the implementation is computing the inverse matrix $A$ in modulo-$p$.

First, we need to know if the output matrix $A$ is invertible. An straightforward way is to compute its determinant and if the determinant is $0$ or has multiple factors with the modulo then the matrix is not invertible.

In the case $det \neq 0 \bmod p$ we need to follow the following steps in order to compute properly the inverse matrix.

We have to compute the inverse element modulo $p$ of the determinant in absolute value, using the extended euclidean algorithm, which can be done using the greatest common divisor function. Where we use the fact that if $det \neq 0 \bmod p$ the greatest common divisor between the determinant and $p$ is either 1 or $-1$. The procedure can be understood using $gcd = u \cdot det + v \cdot p$, then $gcd \mod p = u \cdot det \mod p$, where we can see that $u$ is the multiplicative inverse we are looking for (except for a unit factor).

Further, we need to calculate the adjoint matrix of $A$ and multiply it by the sign of the determinant. Finally, we multiply the inverse modulo p of the determinant with the adjoint matrix, and do the modulo $p$.

Once we have the inverse matrix A modulo $p$, we are able to calculate $\hat{w}$.

## Code

```
function [Perror,Perror_theory1, Perror_theory2]= system(p,pz,L)

wn=0:p-1;
iterations=500;
elements=[0:1:p-1];
residue_elements=elements-round(elements*conj(pz)/(p))*pz;
SNR_db = [0:1:50];
SNR = 10.^(SNR_db./10);
avg= sum(abs(residue_elements).^2)/p;
sigma2 = avg./SNR;

for m=1:length(SNR)
    derror=0;
    error2=0;
    for s=1:iterations

        %create index permutations
        v2 = randperm(size(wn,2));

        %permute w vector
        w2 = wn(v2);

        %choose L values
        w=w2(1:L);

        for r=1:L

            x=w-round(w*conj(pz)/(p))*pz;

            %E=sum(sqrt(x.*conj(x)))/L;

            h=round((1/sqrt(2))*(randn(1,L)+j*randn(1,L)));
            z = sqrt(sigma2(m)).*(1/sqrt(2))*(randn(1,1)+j*randn(1,1));
            y = sum(x.*h) + z;
```

```
            yml=round(y);
            phi = yml-round(yml*conj(pz)/(p))*pz;

            %Computing u and v
            [g,u,v]=CMPLX_GCD (pz, conj(pz));
            u = conj(g)*u;
            v = conj(g)*v;

            %Inverse
            invs = phi*(v*conj(pz))+conj(phi)*(u*pz);

            %modulo p
            muinvs(r) = mod(invs-round(invs/p)*p,p);

            %Coefficients Matrix A
            a = h-round(h*conj(pz)/(p))*pz;
            invsa = a*(v*conj(pz))+conj(a)*(u*pz);
            muinvsa = mod(invsa-round(invsa/p)*p,p);
            A(r,:)=muinvsa;

        end

        determinant=det(A);
        vdett=abs(determinant);

        if mod(round(vdett),p) ~= 0

            [g,u,v]=CMPLX_GCD (round(vdett),p);
            u=g*u;
            dettinvs=mod(round(u),p);
            adjunct=sign(determinant)*adj(A);
            D=dettinvs*adjunct;
            Ainvs=mod(round(D),p);

            finalw = mod(Ainvs * muinvs',p);

            if (length(find(finalw'~=w))>0)
                error2=error2+1;
            end
        else
            derror=derror+1;
        end
    end
    Perror(m)=(error2+derror)/(iterations*L);
end

t=0:1:length(SNR)-1;
P1=(1-(1/p));
for s=2:L
```

```
    P1=P1*(1-(1/p^s));
end
P11=1-P1;

PR=1-erf(1./(2.*sqrt(2.*sigma2)));
Perror_theory1=P11+L.*PR;
Perror_theory2=1-P1*(1-exp(-1./(8*(sigma2)))).^L;

end
```

# 8 Annex II

## $N_e$ Code

```
%constellation used
 x=[0,1,0,0,-1];
 y =[0,0,-1,1,0];

clear C
clear D
clear dimn
clear F

if length(x) == 4
    N_e=2;
else
[a,b]=voronoin([x(:) y(:)]);

for cc=1 : length(b)
    disp (b{cc});
end

C=[];
D=[];
for cc=1: length(b)
    B = cell2mat(b(i));
    if length(B)==2
        dimn(cc) = 1;
    else
        dimn(cc) = length(B);
    end

    for zz=2 : length(B)
        C=[C; [min(B(zz),B(zz-1)),max(B(zz),B(zz-1))]];
        if strmatch([min(B(zz),B(zz-1)),max(B(zz),B(zz-1))],D) < 1
            D = [D;[min(B(zz),B(zz-1)),max(B(zz),B(zz-1))]];
        end
    end
    if length(B)~=2
        C = [C; [min(B(1),B(end)),max(B(1),B(end))]];
        if strmatch([min(B(1),B(end)),max(B(1),B(end))],D) < 1
           D = [D;[min(B(1),B(end)),max(B(1),B(end))]];
        end
    end
end

F = zeros(length(x));

```

```
for k=1: length(D)
    v = strmatch(D(k,:),C);
    for zz=1:length(v)
        suma = 0;
        for hh =1:length(dimn)
            suma = suma + dimn(hh);
            if v(zz) <= suma
                t(zz) = hh;
                break
            end
        end
    end

    if(length(t)>2)
        for cc=2:length(t)
            d(cc-1)=sqrt((x(t(cc))-x(t(cc-1)))^2+(y(t(cc))-y(t(cc-1)))^2 );
        end
        d(length(t)) = sqrt((x(t(1))-x(t(end)))^2+(y(t(1))-y(t(end)))^2 );

        [BX,IX] = sort(d);
        if  IX(1)==length(t)
            t2=[t(end) t(1)];
        else
            t2=[t(IX(1)) t(IX(1)+1)];
        end

        if  IX(2)==length(t)
            t3=[t(end) t(1)];
        else
            t3=[t(IX(2)) t(IX(2)+1)];
        end

        for m=1:length(t2)
            for n =1:length(t2)
                if n~=m
                    F(t2(m),t2(n)) = 1;
                end
            end
        end
        for m=1:length(t3)
            for n =1:length(t3)
                if n~=m
                    F(t3(m),t3(n)) = 1;
                end
            end
        end
        clear t2
        clear t3
        clear d
```

```
    else
        for m=1:length(t)
            for n =1:length(t)
                if n~=m
                    F(t(m),t(n)) = 1;
                end
            end
        end
    end

    clear t
    clear v
end

%Ne result:
N_e = mean(sum(F,1))
end
```

## $d_{\min}$ Code

```
function [minim]=dmin(x,y)

minim = sqrt((x(1)-x(2))^2+(y(1)-y(2))^2);

for c=2:length(x)-1
    d = sqrt((x(c)-x(c+1))^2+(y(c)-y(c+1))^2);
    if d < minim
        minim = d;
    end
end
end
```

# 9 Annex III

## Constellations Codes

$1 \bmod 6$ **Constellation in** $\mathbb{Z}[w]$

```
def EISENSTEIN_constellation(a,b,p):

    real = []

    imaginary = []

    for n in range(0,p):

        axr1 = (float(n*a)/p)
        axr2 = (float(n*b)/p)

        a1 = math.ceil(axr1)
        a11 = math.floor(axr1)

        b1 = math.ceil(axr2)
        b11 = math.floor(axr2)

        alpha = n -(axr1 + axr2*cmath.exp(-2j*math.pi/3))*(a+b*cmath.exp(2j*math.pi/3))

        alpha1 = n -(a1 + b1*cmath.exp(-2j*math.pi/3))*(a+b*cmath.exp(2j*math.pi/3))

        alpha2 = n -(a11 + b1*cmath.exp(-2j*math.pi/3))*(a+b*cmath.exp(2j*math.pi/3))

        alpha3 = n -(a11 + b11*cmath.exp(-2j*math.pi/3))*(a+b*cmath.exp(2j*math.pi/3))

        alpha4 = n -(a1 + b11*cmath.exp(-2j*math.pi/3))*(a+b*cmath.exp(2j*math.pi/3))

        number2 = alpha - alpha1

        alpha_final = alpha1

        for cc in [alpha2,alpha3,alpha4]:

            number1 = alpha - cc

            c1 = number1*number1.conjugate()

            c2 =  number2*number2.conjugate()

```

```
            if c1.real < c2.real:
                number2 = number1
                alpha_final = cc
        real.append(alpha_final.real)
        imaginary.append(alpha_final.imag)
##    pylab.scatter(real, imaginary, s=150)
##
##
##    pylab.title('EISENSTEIN Constellation with p ='+str(p))
##    pylab.xlabel('Real')
##    pylab.ylabel('Imaginary')
##
##    pylab.show()
    return (real,imaginary)
```

$1 \bmod 4$ **Constellation in** $\mathbb{Z}[i]$

```
def modulo_p(n,p,pi):
        z = n*pi.conjugate()/p
        alpha = n - complex(round(z.real),round(z.imag)) *pi
        return alpha

def Z_constellation(p, pi):
    real = []
    imaginary = []
    for c in range(0,p):
        alpha = modulo_p(c,p,pi)
        real.append(alpha.real)
        imaginary.append(alpha.imag)
    print real
```

```

    print imaginary

##    pylab.scatter(real, imaginary, s=150)
##
##    pylab.title('Constellation with p ='+str(p))
##    pylab.xlabel('Real')
##    pylab.ylabel('Imaginary')
##
##
##    pylab.show()

    return (real,imaginary)
```

$3 \bmod 4$ **Constellation in** $\mathbb{Z}[i]$

```
def constellation_3mod4(p):

    real = []

    imaginary = []

    for z in range(0,p):

        for k in range(0,p):

            real.append(z)

            imaginary.append(k)

##    print real
##
##    print imaginary
##
##    pylab.scatter(real, imaginary, s=50)
##
##    pylab.title('Constellation with p ='+str(p))
##    pylab.xlabel('Real')
##    pylab.ylabel('Imaginary')
##
##
##    pylab.show()

    return (real,imaginary)
```

$2 \bmod 3$ **Constellation in** $\mathbb{Z}[w]$

```
def EISENSTEIN_2mod3(p):

    real = []

    imaginary = []

    for f in range(0,p):

        for k in range(0,p):

            n =k*cmath.exp(2j*math.pi/3)+f

            real.append(n.real)

            imaginary.append(n.imag)

##    print real
##
##    print imaginary
##
##    pylab.scatter(real, imaginary, s=50)
##
##    pylab.title('Constellation with p ='+str(p))
##    pylab.xlabel('Real')
##    pylab.ylabel('Imaginary')
##
##
##    pylab.show()

    return (real,imaginary)
```

**QAM Square Constellations**

```
def QAM(n):

    real = []

    imaginary = []

    k = []

    for c in range(0,2**(n/2)):

        k.append(c)

    for z in k:

```

```
            real.append(int(-2**(n/2.)+1+2*z))

            imaginary.append(int(-2**(n/2.)+1+2*z))

    r=[]
    imy=[]

    lista = [(x,y) for x in real for y in imaginary]

    for c in range(0,len(lista)):

            r.append(lista[c][0])

            imy.append(lista[c][1])

    for c in range(0,n+1,2):

            # M-QAM Square
            pylab.plot([-(-2**(c/2)+1),(-2**(c/2)+1),(-2**(c/2)+1),-(-2**(c/2)+1),-(-2**(c/2)+1)],[-(-2**(c/2)+1),-(-2**(c/2)+1),(-2**(c/2)+1),(-2**(c/2)+1),-(-2**(c/2)+1)], linewidth =1.7)


    pylab.scatter(r, imy, s=50)


##    pylab.title(str(n*n)+'-QAM Constellation')
    pylab.title('64-QAM Constellation')
    pylab.xlabel('Real')
    pylab.ylabel('Imaginary')

    pylab.show()

    return (r,imy)
```

**PSK Constellations**

```
def PSK(M):

        real =[]
        imaginary =[]

        for c in range(1,M+1):

                k = math.e**(2j*c*math.pi/M)

                real.append(k.real)
```

```
                imaginary.append(k.imag)


        pylab.scatter(real, imaginary, s=300)


        for c in range(0,M):

                pylab.plot([0,real[c]],[0,imaginary[c]], linewidth =1.7)


        pylab.title(str(M)+'-PSK Constellation')
        pylab.xlabel('Real')
        pylab.ylabel('Imaginary')

        pylab.show()

        return (real,imaginary)
```

# 10 Annex IV

The code of the Figure 5 has been extracted from http://rosettacode.org/wiki/Voronoi_diagram and modified in order to show the points and different distances. The final version of the code after these modifications is the next:

```
from PIL import Image
import random
import math
import ImageDraw

def generate_diagram_voronoi(width, height, num_cells):

        image = Image.new("RGB", (width, height))
        draw = ImageDraw.ImageDraw(image)

        putpixel = image.putpixel
        imgx, imgy = image.size
        nx = []
        ny = []
        nr = []
        ng = []
        nb = []

        for c in range(num_cells):
                nx.append(random.randrange(imgx))
                ny.append(random.randrange(imgy))
                nr.append(random.randrange(256))
                ng.append(random.randrange(256))
                nb.append(random.randrange(256))

        for y in range(imgy):
                for x in range(imgx):
                        dmin = math.hypot(imgx-1, imgy-1)
                        z = -1

                        for c in range(num_cells):
                                #d = math.hypot(nx[c]-x, ny[c]-y)
                                d = abs(nx[c]-x)+abs(ny[c]-y)

                                if d < dmin:
                                        dmin = d
                                        z = c

                                draw.polygon([(nx[c]+3, ny[c]+3), (nx[c]-3,ny[c]+3), (nx[c]-3,ny[c]-3), (nx[c]+3,ny[c]-3)], fill="red", outline="green")

                        putpixel((x, y), (nr[z], ng[z], nb[z]))
```

```

        #image.save("Diagram_Euclidean_Voronoi.png", "PNG")
        image.save("Diagram_Manhattan_Voronoi.png", "PNG")

#generate_diagram_voronoi(500, 500, 25)
generate_diagram_voronoi(500, 500, 25)
```

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**Resum:**

L'objectiu principal d'aquest treball és el disseny de constel·lacions per a un sistema de comunicacions basat en Physical Layer Network Coding. El disseny es durà a terme dins de dos anells commutatius: els GAUSSIAN Integers i els EISENSTEIN Integers.

Usant com a punt de partida aquest sistema concret, primer hem identificat la teoria necessària per al disseny, així com les 'performance metrics' d'utilitat per analitzar les constel·lacions proposades. Tot seguit hem presentat les constel·lacions més usades avui en dia per utilitzar-les com a referència i finalment hem proposat una metodologia de disseny, a partir de la qual s'han proposat quatre constel·lacions.

Per últim, s'ha dut a terme l'anàlisi dels resultats obtinguts així com la implementació del sistema estudiat de Physical Layer Network Coding.

**Resumen:**

El objetivo principal de este trabajo es el diseño de constelaciones para un sistema de comunicaciones basado en Physical Layer Network Coding. El diseño se llevará a cabo dentro de dos anillos commutativos: los GAUSSIAN Integers y los EISENSTEIN Integers.

Usando como punto de partida este sistema concreto, primero hemos identificado la teoría necesaria para el diseño, así como las 'performance metrics' de utilidad para analizar las constelaciones propuestas. A continuación hemos presentado las constelaciones más usadas hoy en día para utilizarlas como referencia y finalmente hemos propuesto una metodología de diseño, a partir de la cual se han propuesto cuatro constelaciones.

Por último, se ha realizado el análisis de los resultados obtenidos, así como la implementación del sistema estudiado de Physical Layer Network Coding.

**Summary:**

The main goal of this work is to design constellations in a system of Physical Layer Network Coding. The design is based on constellations over two commutative rings: GAUSSIAN Integers and EISENSTEIN Integers.

Using this particular system as a working base, first we have identified the needed theory in the design, as well as the needed performance metrics in order to analyse the proposed constellations. Then, we have presented the most used constellations nowadays in order to be used as a comparison. Finally, we have proposed a design methodology and four constellation designs.

Further, the obtained results have been analysed and the particular studied system of Physical Layer Network Coding has been implemented.

DE ZARZA I CUBERO Irene
Bellaterra, July 2013