Physical-Layer Network Coding: Design of Constellations over Rings

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Outline

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Introduction

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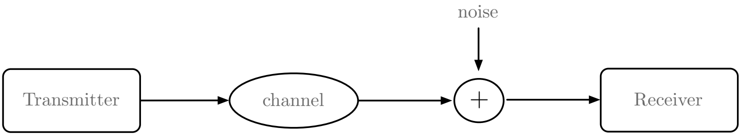

Note

Intermediate nodes can be added. These nodes would originally have the only function of forwarding the received messages.

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Physical-Layer Network Coding

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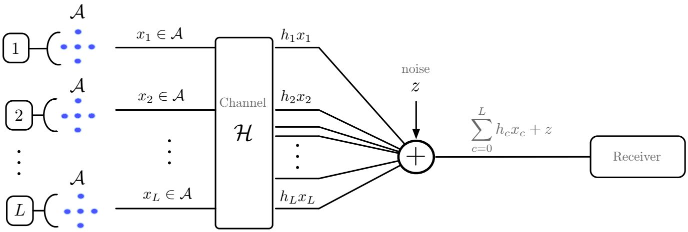
Network Coding

Allows intermediate nodes to combine messages before forwarding them.



Physical-Layer Network Coding

Exploits the network coding operation performed by nature.

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The Role of a Signal Constellation

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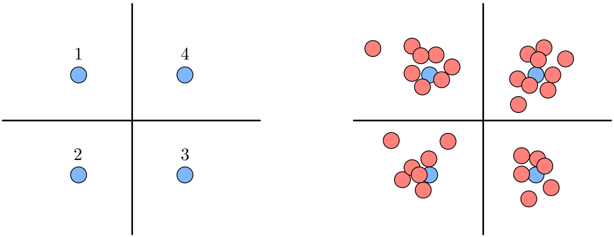
Definition

A signal constellation is a set of points in the complex plane used to describe all possible symbols used by a system to transmit data.



Transmission and reception points

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Objective

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General Objective

We tackle the design of new signal constellations for Physical-Layer Network Coding. Towards this aim, the appropriate algebraic tools need to be identified.

Design Objective

We aim at defining a design methodology and propose the best performing constellations. Performance will depend on the algebraically induced geometry.

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Introduction

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Commutative Rings

We are going to design constellations carved from the rings $\mathbb{Z}[i]$ and $\mathbb{Z}[w]$.

$$\{\text{Commutative Rings}\} \supset \{\text{PIDs}\} \supset \{\text{Euclidean Domains}\} \supset \{\text{Fields}\}$$

We Are Looking For

R/aR field, R PID and aR ideal.

The Ring $\mathbb{Z}[i]$

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$$\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$$

Definition

For $\alpha = a + ib \in \mathbb{Z}[i]$, its norm is defined as

$$N(\alpha) = \alpha\alpha^* = (a + bi)(a - bi) = a^2 + b^2.$$

Theorem: Norm Is Multiplicative.

For α and β in $\mathbb{Z}[*]$, $N(\alpha\beta) = N(\alpha)N(\beta)$.

$$N(\alpha\beta) = (\alpha\beta)(\alpha\beta)^* = \alpha\beta\alpha^*\beta^* = (\alpha\alpha^*)(\beta\beta^*) = N(\alpha)N(\beta).$$

The Ring $\mathbb{Z}[i]$

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Division Theorem

For $\alpha, \beta \in \mathbb{Z}[i]$ with $\beta \neq 0$, there are $\gamma, \rho \in \mathbb{Z}[i]$ such that

$$\alpha = \beta\gamma + \rho \text{ where } N(\rho) < N(\beta).$$

Proof.

- Let $\alpha, \beta \in \mathbb{Z}[i]$ with $\beta \neq 0$. Then $\alpha/\beta \in \mathbb{C} \Rightarrow \alpha/\beta = u + iv$ with $u, v \in \mathbb{R}$.
  - $a \in \mathbb{Z}$ close to $u \Rightarrow |u - a| \leq 1/2$.
  - $b \in \mathbb{Z}$ close to $v \Rightarrow |v - b| \leq 1/2$.

  Set $\gamma = a + ib \in \mathbb{Z}[i]$. Set $\rho = \alpha - \gamma\beta \in \mathbb{Z}[i]$.
- Remains to prove $N(\rho) < N(\beta)$. (Note $\beta \neq 0 \Rightarrow N(\beta) \neq 0$).
  - $N(\rho) = N((\rho/\beta)\beta) = N(\rho/\beta)N(\beta)$:
  - $N(\rho) < N(\beta) \Leftrightarrow N(\rho/\beta) < 1$
  - $\rho/\beta = (\alpha - \gamma\beta)/\beta = \alpha/\beta - \gamma = (u + iv) - (a + ib) = (u - a) + i(v - b)$.
  - $N(\rho/\beta) = (u - a)^2 + (v - b)^2 \leq 1/4 + 1/4 = 1/2 < 1$.

  Therefore $\alpha = \gamma\beta + \rho$ with $N(\rho) < N(\beta)$.

$\square$

The Ring $\mathbb{Z}[i]$

$\mathbb{Z}[i]$ as a PID

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Definition

An integral domain R is said to be an Euclidean domain if there is a function N from the set of nonzero elements of R to the set of non-negative integers such that

- (Division Theorem) given $a, b \in R$ with $b \neq 0$ there exist $q, r \in R$ such that $a = bq + r$ where $N(r) < N(b)$, and
- for all non-zero elements a and b of R we have $N(a) \leq N(ab)$.

Theorem

Euclidean domains are PIDs.

Proof.

Let C be any non-zero ideal of the Euclidean domain R, and $d \in C$ be a nonzero element of minimum norm.
We claim $(d) = C$. Certainly, $(d) \subseteq C$.
Let $a \in C$. By the Division Theorem, $a = qd + r$, with $r = 0$ or $N(r) < N(d)$.
Since $a - qd = r \in C$, by minimality of $N(d)$ we see $r = 0$ and $a = qd \in (d)$. □

The Ring $\mathbb{Z}[w]$

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$$\mathbb{Z}[w] = \{a + bw | a, b \in \mathbb{Z}\}$$

with w is a primitive cube root of 1:

$$w = e^{2\pi i/3} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \frac{1}{2}\left(-1 + i\sqrt{3}\right).$$

Definition

For $\alpha = a + wb \in \mathbb{Z}[w]$, its norm is defined as

$$N(\alpha) = \alpha\alpha^* = a^2 + b^2 - ab.$$

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Decision Regions

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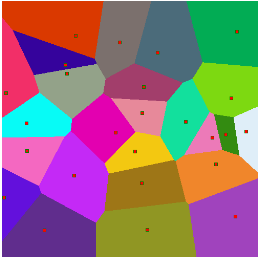
Definition

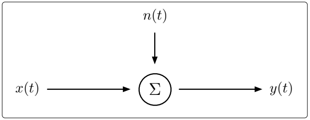
The decision region for a point x_c in the constellation $\mathcal{A} = \{x_c\}_{c=0,\cdots,M}$, denoted $\mathcal{R}_{x_c}$, is the set of points of the complex plane that are closer to x_c than to any other point of the signal constellation.



Probability of Error

Hypothesis

We assume an AWGN (Additive White Gaussian Noise) channel.



$$\equiv \quad y(t) = x(t) + n(t)$$

The noise $n(t)$ is a 1 dimensional random signal Gaussian with zero mean, variance σ^2 and probability distribution:

$$P_n(u) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}u^2}.$$

Assumption

The computation of P_e assumes the inputs x_c equally likely: $p_x(c) = \frac{1}{M} \forall c$.

Probability of Error

ML Detector Is the Optimum Detector

Which has decision rule of taking the point of the constellation the detected point is nearest to.

The Exact P_e

Corresponds to the sum of probabilities of having an error when transmitting a given symbol

$$P_e = \sum_{c=0}^{M-1} P_{e|c} \cdot P(c) = \frac{1}{M} \sum_{c=0}^{M-1} P_{e|c} = 1 - \frac{1}{M} \sum_{c=0}^{M-1} P_{r|c}.$$

Probability of Error

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Union Bound

The probability of error for the ML detector on the AWGN channel, with a M-point signal constellation with minimum distance d_{min} is bounded by

$$P_e \leq (M-1)Q\left[\frac{d_{\text{min}}}{2\sigma}\right].$$

The Nearest Neighbor Union Bound

The probability of error for the ML detector on the AWGN channel, with a M-point signal constellation with minimum distance d_{min} is bounded by

$$P_e \leq N_e Q\left[\frac{d_{\text{min}}}{2\sigma}\right].$$

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M-QAM Constellation

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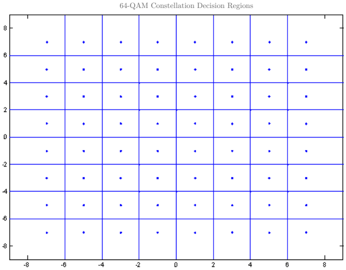
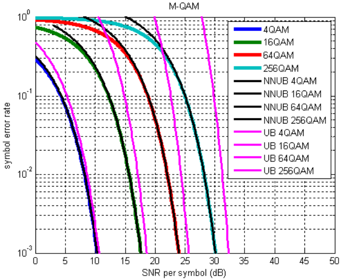
$$\mathcal{A} = \{a[n]\} = \{A(a_r[n] + i a_c[n])\},$$

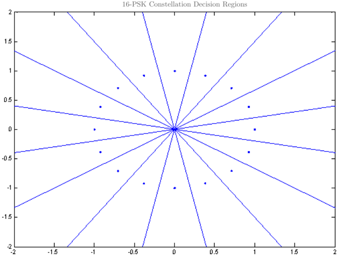
with $a_*[n]$ odd integers around zero.

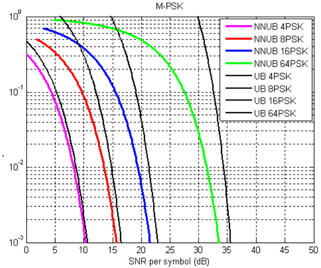




M-PSK Constellation

$$\mathcal{A} = \{Ae^{j2k\pi/M}\}, \quad k = 1, 2, \cdots, M.$$





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System Model

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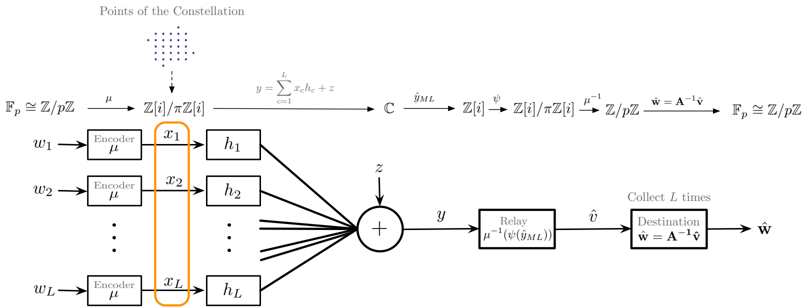

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Proposed Methodology

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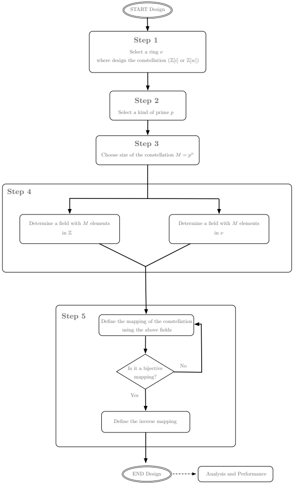
Methodology

- Step 1: select a ring ν.
- Step 2: select a type of prime p.
- Step 3: choose size of the constellation $M = p^n$.
- Step 4: determine a field in $\mathbb{Z}$ and ν.
- Step 5: define the mapping of the constellation and its inverse.



Outline

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$1 \bmod 4$ Constellation

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Design 1

- Step 1: ring $\mathbb{Z}[i]$.
- Step 2: primes $p \in \mathbb{Z}^+$, $p \equiv 1 \bmod 4$ in $\mathbb{Z}[i]$ $(p = \pi\pi^*)$.
- Step 3: $M = p$.
- Step 4: field with $M = p$ elements in $\mathbb{Z}$ and $\mathbb{Z}[i]$.
  - $\mathbb{Z}/p\mathbb{Z}$, $\#(\mathbb{Z}/p\mathbb{Z}) = |p| = p$.
  - $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$, $\#(\mathbb{Z}[i]/\pi\mathbb{Z}[i]) = N(\pi) = \pi\pi^* = p$.

Theorem

If R is a PID and $a \in R$ is irreducible then R/aR is a field.

$1 \bmod 4$ Constellation

Design 1

- Step 5: we are looking for $\mathbb{F}_p \cong \mathbb{Z}[i]/\pi\mathbb{Z}[i]$.

The first mapping from $\mathbb{F}_p$ to $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$ is defined as follows. We first state the division theorem in $\mathbb{Z}[i]$

$$x = \lambda\pi + \gamma,$$

$$\text{with } N(\gamma) < N(\pi),$$

$$\text{where } \lambda = \left[\frac{x}{\pi}\right] = \left[\frac{x\pi^*}{\pi\pi^*}\right].$$

If we solve for the residue

$$\gamma = x - \left[\frac{x\pi^*}{\pi\pi^*}\right]\pi.$$

$1 \bmod 4$ Constellation

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The mapping of the constellation is defined as:

$$\begin{aligned} \mu : \mathbb{F}_p &\longrightarrow \mathbb{Z}[i]/\pi\mathbb{Z}[i] \\ x &\longmapsto \mu(x) = x - \left[\frac{x\pi^*}{\pi\pi^*}\right]\pi \end{aligned}$$

The inverse mapping is defined as:

$$\begin{aligned} \mu^{-1} : \mathbb{Z}[i]/\pi\mathbb{Z}[i] &\longrightarrow \mathbb{F}_p \\ a &\longmapsto \mu^{-1}(a) = (a(v\pi^*) + a^*(u\pi^*)) \bmod p \end{aligned}$$

with $u\pi + v\pi^* = 1$.

$3 \bmod 4$ Constellation

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Design 2

- Step 1: ring $\mathbb{Z}[i]$.
- Step 2: primes $p \in \mathbb{Z}^+$, $p \equiv 3 \bmod 4$ in $\mathbb{Z}[i]$.
- Step 3: $M = p^2$.
- Step 4:
  - $\mathbb{Z}[i]/p\mathbb{Z}[i]$, $\#\left(\mathbb{Z}[i]/p\mathbb{Z}[i]\right) = N(p) = p^2$.
  - $\mathbb{F}_p[X]/(x^2+1)$, $\#\left(\mathbb{F}_p[X]/(x^2+1)\right) = p^2$.

Theorem

If R is a PID and $a \in R$ is irreducible then R/aR is a field.

$3 \bmod 4$ Constellation

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Design 2

- Step 5: we are looking for $\mathbb{F}_p[X]/(x^2+1) \cong \mathbb{Z}[i]/p\mathbb{Z}[i]$ (X corresponding i).

We are going to see that the two fields are isomorphic to $\mathbb{Z}[X]/(p, x^2+1)$.

- First, $\mathbb{Z}[X]/(x^2+1) \cong \mathbb{Z}[i]$ with $X \to i$.

$$\begin{aligned} \psi : \mathbb{Z}[X] &\longrightarrow \mathbb{Z}[i] \\ P(X) &\longmapsto P(i) \end{aligned}$$

Surjective with kernel $(1+x^2)$.

$3 \bmod 4$ Constellation

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By the NOETHER First Isomorphism Theorem:

$$\text{Image}(\psi) \cong \mathbb{Z}[X]/\text{Kernel}(\psi)$$

$$\begin{tikzcd}
\mathbb{Z}[X] \arrow[rr, "\psi"] \arrow[d, "\pi"'] & & \mathbb{Z}[i] = \text{Image}(\psi) \\
\begin{array}{c}\mathbb{Z}[X]/\text{Kernel}(\psi) \\ = \\ \mathbb{Z}[X]/(x^2+1)\end{array} \arrow[urr, "\cong", "\hat{\psi}"'] & &
\end{tikzcd}$$

We can assert $\psi^{-1}(p\mathbb{Z}[i]) = (p, x^2+1)$. Hence, $\mathbb{Z}[i]/p\mathbb{Z}[i] \cong \mathbb{Z}[X]/(p, x^2+1)$.

$3 \bmod 4$ Constellation

- Since $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ we have

$$\mathbb{F}_p[X]/(x^2+1) \cong (\mathbb{Z}/p\mathbb{Z})\,[X]/(x^2+1)$$

$$\cong (\mathbb{Z}[X]/(p))\,/(x^2+1) \cong \mathbb{Z}[X]/(p, x^2+1).$$

$3 \bmod 4$ Constellation

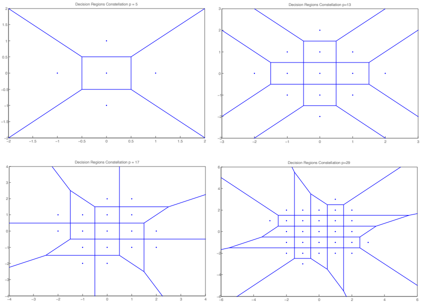
The mapping of the constellation is defined as:

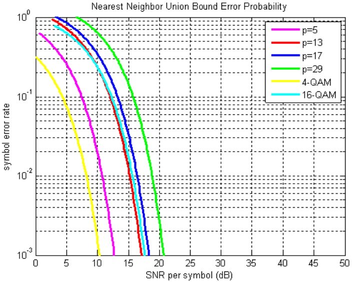
$$\begin{aligned} \gamma : \; \mathbb{F}_p[X]/(x^2+1) &\longrightarrow \mathbb{Z}[i]/p\mathbb{Z}[i] \\ x &\longmapsto i \end{aligned}$$

The inverse mapping is defined as:

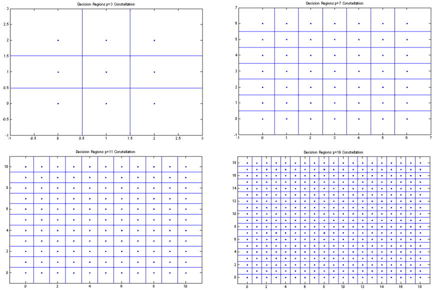
$$\begin{aligned} \gamma^{-1} : \mathbb{Z}[i]/p\mathbb{Z}[i] &\longrightarrow \mathbb{F}_p[X]/(x^2+1) \\ i &\longmapsto x \end{aligned}$$

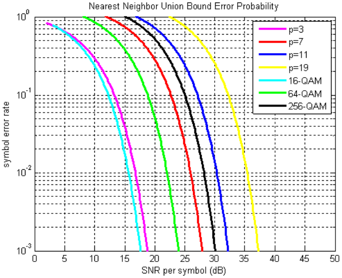
$1 \bmod 4$ Constellation





$3 \bmod 4$ Constellation





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$1 \bmod 6$ Constellation

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Design 3

- Step 1: ring $\mathbb{Z}[w]$.
- Step 2: primes $p \in \mathbb{Z}^+$, $p \equiv 1 \bmod 6$ in $Z[w]$ $(p = \pi\pi^*)$.
- Step 3: $M = p$.
- Step 4:
  - $\mathbb{Z}/p\mathbb{Z}$, $\#(\mathbb{Z}/p\mathbb{Z}) = |p| = p$.
  - $\mathbb{Z}[w]/\pi\mathbb{Z}[w]$, $\#(\mathbb{Z}[w]/\pi\mathbb{Z}[w]) = N(\pi) = \pi\pi^* = p$.

Theorem

If R is a PID and $a \in R$ is irreducible then R/aR is a field.

1 mod 6 Constellation

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Design 3

- Step 5: we are looking for $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}[w]/\pi\mathbb{Z}[w]$.

The mapping of the constellation is defined as:

$$\begin{aligned} \tilde{\mu} :\ \mathbb{F}_p &\longrightarrow \mathbb{Z}[w]/\pi\mathbb{Z}[w] \\ x &\longmapsto \tilde{\mu}(x) = x - \left[\frac{x\pi^*}{\pi\pi^*}\right]\pi \end{aligned}$$

The inverse mapping is defined as:

$$\begin{aligned} \mu^{-1} :\ \mathbb{Z}[w]/\pi\mathbb{Z}[w] &\longrightarrow \mathbb{F}_p \\ a &\longmapsto \mu^{-1}(a) = (a(v\pi^*) + a^*(u\pi^*))\mathrm{mod} p \end{aligned}$$

with $u\pi + v\pi^* = 1$.

$2 \bmod 3$ Constellation

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Design 4

- Step 1: ring $\mathbb{Z}[w]$.
- Step 2: primes $p \in \mathbb{Z}^+$, $p \equiv 2 \bmod 3$ in $Z[w]$.
- Step 3: $M = p^2$.
- Step 4:
  - $\mathbb{Z}[w]/p\mathbb{Z}[w]$, $\#\left(\mathbb{Z}[w]/p\mathbb{Z}[w]\right) = N(p) = p^2$.
  - $\mathbb{F}_p[X]/(x^2+x+1)$, $\#\left(\mathbb{F}_p[X]/(x^2+x+1)\right) = p^2$.

Theorem

If R is a PID and $a \in R$ is irreducible then R/aR is a field.

$2 \bmod 3$ Constellation

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Design 4

- Step 5: we are looking for $\mathbb{Z}[w]/p\mathbb{Z}[w] \cong \mathbb{F}_p[X]/(x^2+x+1)$ with X corresponding w.

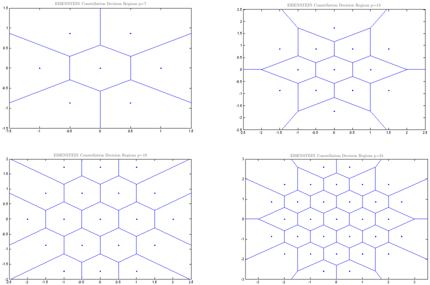
The mapping of the constellation is defined as:

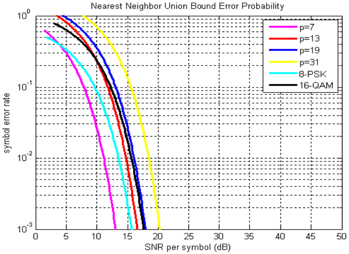
$$\begin{aligned} \tilde{\gamma}: \mathbb{F}_p[X]/(x^2+x+1) &\longrightarrow \mathbb{Z}[w]/p\mathbb{Z}[w] \\ x &\longmapsto w \end{aligned}$$

The inverse mapping is defined as:

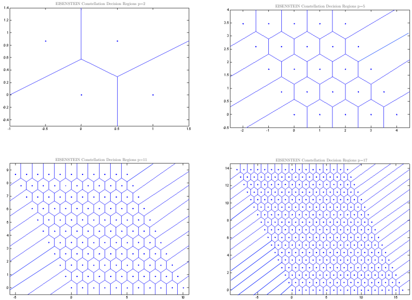
$$\begin{aligned} \tilde{\gamma}^{-1}: \mathbb{Z}[w]/p\mathbb{Z}[w] &\longrightarrow \mathbb{F}_p[X]/(x^2+x+1) \\ w &\longmapsto x \end{aligned}$$

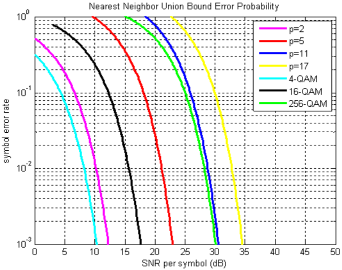
1 mod 6 Constellation





$2 \bmod 3$ Constellation





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Conclusions

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- Algebraic theory identification:
  - PID. $\surd$
  - Euclidean domain. $\surd$
  - Fields. $\surd$
- Performance metrics identification:
  - Decision regions. $\surd$
  - Nearest Neighbor Union Bound. $\surd$
- MATLAB parameters computation:
  - $d_{\min}$. $\surd$
  - N_e. $\surd$
- System model know how:
  - MATLAB implementation proposed. $\surd$

Conclusions

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- Design and performance of the constellations:
  - $\mathbb{Z}[i]$:
    - $1 \mod 4$. $\surd$
    - $3 \mod 4$. $\surd$
  - $\mathbb{Z}[w]$:
    - $1 \mod 6$. $\surd$
    - $2 \mod 3$. $\surd$

Conclusions

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And Finally the Best Constellations Are

- $1 \bmod 6$ constellation in $\mathbb{Z}[w]$ is the best performing constellation.
- $1 \bmod 4$ constellations in $\mathbb{Z}[i]$ appear as a good QAM alternative.
- QAM constellations have better performance than $3 \bmod 4$ in $\mathbb{Z}[i]$ and $2 \bmod 3$ in $\mathbb{Z}[w]$.

Thank You

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DE ZARZA I CUBERO Irene

Thank you