

Physical-Layer Network Coding: Design of Constellations over Rings

H. C. Zarza

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Outline

1 Introduction

2 Objectives

3 What Do We Need in Order to Design?

- Decision Regions
- Probability of Error
- M-QAM
- M-PSK

4 Proposed Designs

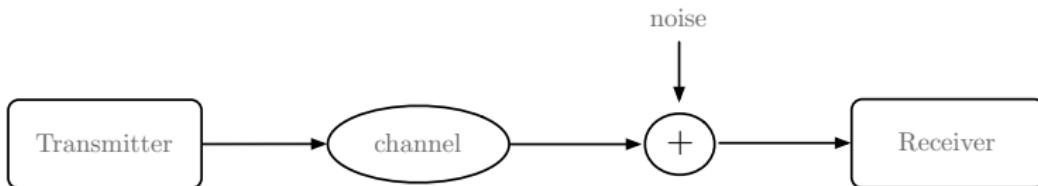
- 1 mod 4 Constellation
- 3 mod 4 Constellation
- Best Performance
- 1 mod 6 Constellation
- 2 mod 3 Constellation
- Best Performance

5 Conclusions

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Introduction



Note

Intermediate nodes can be added. These nodes would originally have the only function of forwarding the received messages.

Outline

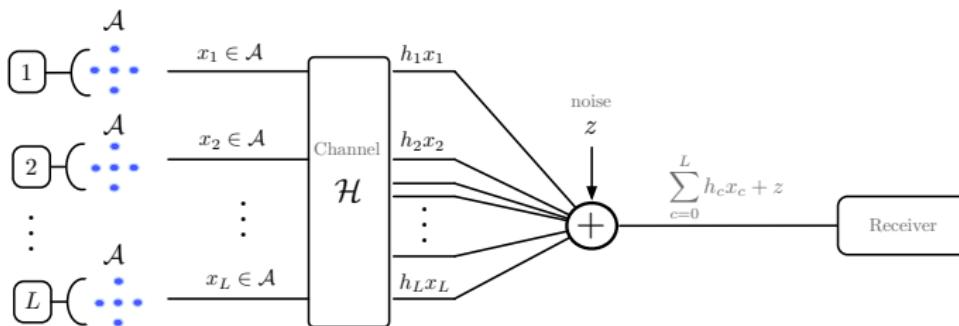
1 Introduction

- Physical-Layer Network Coding
- The Role of a Signal Constellation in the System

Physical-Layer Network Coding

Network Coding

Allows intermediate nodes to combine messages before forwarding them.



Physical-Layer Network Coding

Exploits the network coding operation performed by nature.

Outline

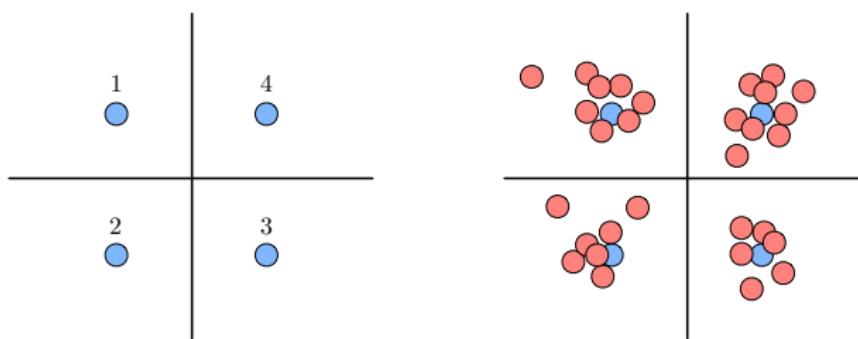
1 Introduction

- Physical-Layer Network Coding
- The Role of a Signal Constellation in the System

The Role of a Signal Constellation

Definition

A signal constellation is a set of points in the complex plane used to describe all possible symbols used by a system to transmit data.



Transmission and reception points

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Objective

General Objective

We tackle the design of new signal constellations for Physical-Layer Network Coding. Towards this aim, the appropriate algebraic tools need to be identified.

Design Objective

We aim at defining a design methodology and propose the best performing constellations. Performance will depend on the algebraically induced geometry.

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Outline

3 What Do We Need in Order to Design?

- Mathematical Theory
- Performance Metrics
- A Reference to Compare
- System Model
- Proposed Methodology

Introduction

Commutative Rings

We are going to design constellations carved from the rings $\mathbb{Z}[i]$ and $\mathbb{Z}[w]$.

$\{\text{Commutative Rings}\} \supset \{\text{PIDs}\} \supset \{\text{Euclidean Domains}\} \supset \{\text{Fields}\}$

We Are Looking For

R/aR field, R PID and aR ideal.

The Ring $\mathbb{Z}[i]$

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$$

Definition

For $\alpha = a + ib \in \mathbb{Z}[i]$, its norm is defined as

$$N(\alpha) = \alpha\alpha^* = (a + bi)(a - bi) = a^2 + b^2.$$

Theorem: Norm Is Multiplicative.

For α and β in $\mathbb{Z}[*]$, $N(\alpha\beta) = N(\alpha)N(\beta)$.

$$N(\alpha\beta) = (\alpha\beta)(\alpha\beta)^* = \alpha\beta\alpha^*\beta^* = (\alpha\alpha^*)(\beta\beta^*) = N(\alpha)N(\beta).$$

The Ring $\mathbb{Z}[i]$

Division Theorem

For $\alpha, \beta \in \mathbb{Z}[i]$ with $\beta \neq 0$, there are $\gamma, \rho \in \mathbb{Z}[i]$ such that

$$\alpha = \beta\gamma + \rho \text{ where } N(\rho) < N(\beta).$$

Proof.

- Let $\alpha, \beta \in \mathbb{Z}[i]$ with $\beta \neq 0$. Then $\alpha/\beta \in \mathbb{C} \Rightarrow \alpha/\beta = u + iv$ with $u, v \in \mathbb{R}$.
 - $a \in \mathbb{Z}$ close to $u \Rightarrow |u - a| \leq 1/2$.
 - $b \in \mathbb{Z}$ close to $v \Rightarrow |v - b| \leq 1/2$.
- Set $\gamma = a + ib \in \mathbb{Z}[i]$. Set $\rho = \alpha - \gamma\beta \in \mathbb{Z}[i]$.
- Remains to prove $N(\rho) < N(\beta)$. (Note $\beta \neq 0 \Rightarrow N(\beta) \neq 0$).
 - $N(\rho) = N((\rho/\beta)\beta) = N(\rho/\beta)N(\beta)$:
 - $N(\rho) < N(\beta) \Leftrightarrow N(\rho/\beta) < 1$
 - $\rho/\beta = (\alpha - \gamma\beta)/\beta = \alpha/\beta - \gamma = (u + iv) - (a + ib) = (u - a) + i(v - b)$.
 - $N(\rho/\beta) = (u - a)^2 + (v - b)^2 \leq 1/4 + 1/4 = 1/2 < 1$.

Therefore $\alpha = \gamma\beta + \rho$ with $N(\rho) < N(\beta)$.

The Ring $\mathbb{Z}[i]$

$\mathbb{Z}[i]$ as a PID

Definition

An integral domain R is said to be an Euclidean domain if there is a function N from the set of nonzero elements of R to the set of non-negative integers such that

- (Division Theorem) given $a, b \in R$ with $b \neq 0$ there exist $q, r \in R$ such that $a = bq + r$ where $N(r) < N(b)$, and
- for all non-zero elements a and b of R we have $N(a) \leq N(ab)$.

Theorem

Euclidean domains are PIDs.

Proof.

Let C be any non-zero ideal of the Euclidean domain R , and $d \in C$ be a nonzero element of minimum norm.

We claim $(d) = C$. Certainly, $(d) \subseteq C$.

Let $a \in C$. By the Division Theorem, $a = qd + r$, with $r = 0$ or $N(r) < N(d)$. Since $a - qd = r \in C$, by minimality of $N(d)$ we see $r = 0$ and $a = qd \in (d)$.



The Ring $\mathbb{Z}[w]$

$$\mathbb{Z}[w] = \{a + bw \mid a, b \in \mathbb{Z}\}$$

with w is a primitive cube root of 1:

$$w = e^{2\pi i/3} = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) = \frac{1}{2}(-1 + i\sqrt{3}).$$

Definition

For $\alpha = a + wb \in \mathbb{Z}[w]$, its norm is defined as

$$N(\alpha) = \alpha\alpha^* = a^2 + b^2 - ab.$$

Outline

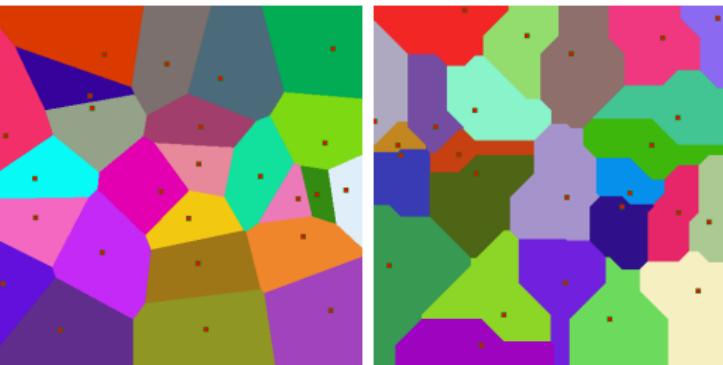
3 What Do We Need in Order to Design?

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Decision Regions

Definition

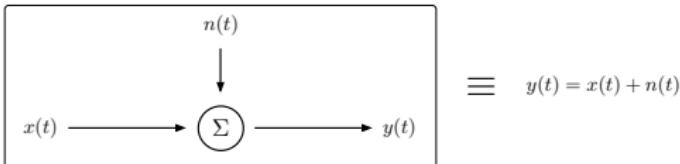
The decision region for a point x_c in the constellation $\mathcal{A} = \{x_c\}_{c=0, \dots, M}$, denoted \mathcal{R}_{x_c} , is the set of points of the complex plane that are closer to x_c than to any other point of the signal constellation.



Probability of Error

Hypothesis

We assume an AWGN (Additive White Gaussian Noise) channel.



The noise $n(t)$ is a 1 dimensional random signal Gaussian with zero mean, variance σ^2 and probability distribution:

$$P_n(u) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} u^2}.$$

Assumption

The computation of P_e assumes the inputs x_c equally likely: $p_x(c) = \frac{1}{M} \forall c$.

Probability of Error

ML Detector Is the Optimum Detector

Which has decision rule of taking the constellation point the detected point is nearest to.

The Exact P_e

Corresponds to the sum of probabilities of having an error when transmitting a given symbol

$$P_e = \sum_{c=0}^{M-1} P_{e|c} \cdot P(c) = \frac{1}{M} \sum_{c=0}^{M-1} P_{e|c} = 1 - \frac{1}{M} \sum_{c=0}^{M-1} P_{r|c}.$$

Probability of Error

Union Bound

The probability of error for the ML detector on the AWGN channel, with an M -point signal constellation with minimum distance d_{min} is bounded by

$$P_e \leq (M - 1)Q\left[\frac{d_{min}}{2\sigma}\right].$$

The Nearest Neighbor Union Bound

The probability of error for the ML detector on the AWGN channel, with an M -point constellation with minimum distance d_{min} is bounded by

$$P_e \leq N_e Q\left[\frac{d_{min}}{2\sigma}\right].$$

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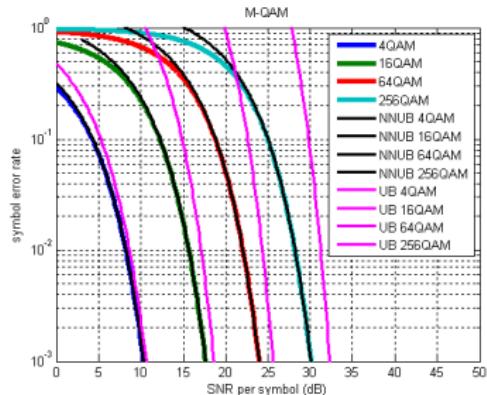
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M-QAM Constellation

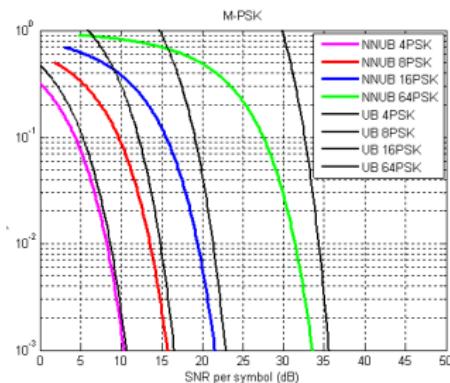
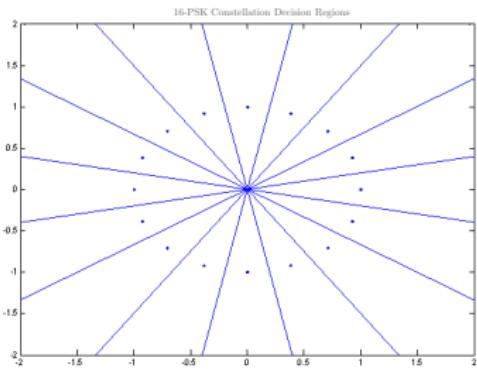
$$\mathcal{A} = \{a[n]\} = \{A(a_r[n] + ia_c[n])\},$$

with $a_*[n]$ odd integers around zero.



M-PSK Constellation

$$\mathcal{A} = \{Ae^{j2k\pi/M}\}, \quad k = 1, 2, \dots, M.$$



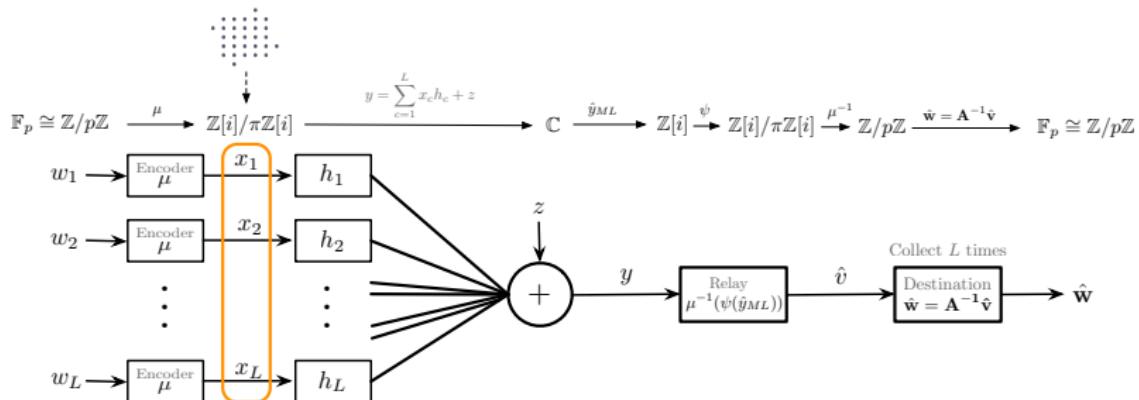
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System Model

Points of the Constellation



Outline

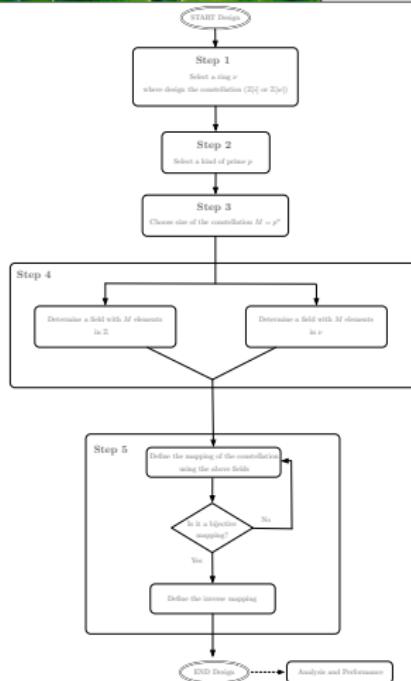
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Proposed Methodology

Methodology

- Step 1: Select a ring ν .
- Step 2: Select a type of prime p .
- Step 3: Choose size of the constellation $M = p^n$.
- Step 4: Determine a field in \mathbb{Z} and ν .
- Step 5: Define the mapping of the constellation and its inverse.



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4 Proposed Designs

- Designs in $\mathbb{Z}[i]$
 - 1 mod 4 Constellation
 - 3 mod 4 Constellation
 - Best Performance
- Designs in $\mathbb{Z}[w]$

1 mod 4 Constellation

Design 1

- Step 1: ring $\mathbb{Z}[i]$.
- Step 2: primes $p \in \mathbb{Z}^+$, $p \equiv 1 \pmod{4}$ in $\mathbb{Z}[i]$. ($p = \pi\pi^*$).
- Step 3: $M = p$.
- Step 4: field with $M = p$ elements in \mathbb{Z} and $\mathbb{Z}[i]$.
 - $\mathbb{Z}/p\mathbb{Z}$, $\#(\mathbb{Z}/p\mathbb{Z}) = |p| = p$.
 - $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$, $\#(\mathbb{Z}[i]/\pi\mathbb{Z}[i]) = N(\pi) = \pi\pi^* = p$.

Theorem

If R is a PID and $a \in R$ is irreducible, then R/aR is a field.

1 mod 4 Constellation

Design 1

- Step 5: We are looking for $\mathbb{F}_p \cong \mathbb{Z}[i]/\pi\mathbb{Z}[i]$.

The first mapping from \mathbb{F}_p to $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$ is defined as follows:
We first state the division theorem in $\mathbb{Z}[i]$

$$x = \lambda\pi + \gamma,$$

with $N(\gamma) < N(\pi)$,

$$\text{where } \lambda = \left[\frac{x}{\pi} \right] = \left[\frac{x\pi^*}{\pi\pi^*} \right].$$

If we solve for the residue

$$\gamma = x - \left[\frac{x\pi^*}{\pi\pi^*} \right] \pi.$$

1 mod 4 Constellation

The mapping of the constellation is defined as:

$$\begin{aligned}\mu : \mathbb{F}_p &\xrightarrow{\hspace{2cm}} \mathbb{Z}[i]/\pi\mathbb{Z}[i] \\ x &\longmapsto \mu(x) = x - \left[\frac{x\pi^*}{\pi\pi^*} \right] \pi\end{aligned}$$

The inverse mapping is defined as:

$$\begin{aligned}\mu^{-1} : \mathbb{Z}[i]/\pi\mathbb{Z}[i] &\xrightarrow{\hspace{2cm}} \mathbb{F}_p \\ a &\longmapsto \mu^{-1}(a) = (a(v\pi^*) + a^*(u\pi^*)) \bmod p\end{aligned}$$

with $u\pi + v\pi^* = 1$.

3 mod 4 Constellation

Design 2

- Step 1: ring $\mathbb{Z}[i]$.
- Step 2: primes $p \in \mathbb{Z}^+$, $p \equiv 3 \pmod{4}$ in $\mathbb{Z}[i]$.
- Step 3: $M = p^2$.
- Step 4:
 - $\mathbb{Z}[i]/p\mathbb{Z}[i]$, $\#(\mathbb{Z}[i]/p\mathbb{Z}[i]) = N(p) = p^2$.
 - $\mathbb{F}_p[X]/(x^2 + 1)$, $\#(\mathbb{F}_p[X]/(x^2 + 1)) = p^2$.

Theorem

If R is a PID and $a \in R$ is irreducible, then R/aR is a field.

3 mod 4 Constellation

Design 2

- Step 5: We are looking for $\mathbb{F}_p[X]/(x^2 + 1) \cong \mathbb{Z}[i]/p\mathbb{Z}[i]$ (X corresponding i).

We are going to see that the two fields are isomorphic to $\mathbb{Z}[X]/(p, x^2 + 1)$.

- First, $\mathbb{Z}[X]/(x^2 + 1) \cong \mathbb{Z}[i]$ with $X \rightarrow i$.

$$\psi : \mathbb{Z}[X] \longrightarrow \mathbb{Z}[i]$$

$$P(X) \longmapsto P(i)$$

Surjective with kernel $(1 + x^2)$.

3 mod 4 Constellation

By the NOETHER First Isomorphism Theorem:

$$\text{Image}(\psi) \cong \mathbb{Z}[X]/\text{Kernel}(\psi)$$

$$\begin{array}{ccc} \mathbb{Z}[X] & \xrightarrow{\psi} & \mathbb{Z}[i] = \text{Image}(\psi) \\ \pi \downarrow & & \swarrow \cong \hat{\psi} \\ \mathbb{Z}[X]/\text{Kernel}(\psi) & = & \mathbb{Z}[X]/(x^2 + 1) \end{array}$$

We can assert $\psi^{-1}(p\mathbb{Z}[i]) = (p, x^2 + 1)$. Hence,
 $\mathbb{Z}[i]/p\mathbb{Z}[i] \cong \mathbb{Z}[X]/(p, x^2 + 1)$.

3 mod 4 Constellation

- Since $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ we have

$$\begin{aligned}\mathbb{F}_p[X]/(x^2 + 1) &\cong (\mathbb{Z}/p\mathbb{Z})[X]/(x^2 + 1) \\ &\cong (\mathbb{Z}[X]/(p)) / (x^2 + 1) \cong \mathbb{Z}[X]/(p, x^2 + 1).\end{aligned}$$

3 mod 4 Constellation

the mapping of the constellation is defined as:

$$\gamma : \mathbb{F}_p[X]/(x^2 + 1) \longrightarrow \mathbb{Z}[i]/p\mathbb{Z}[i]$$

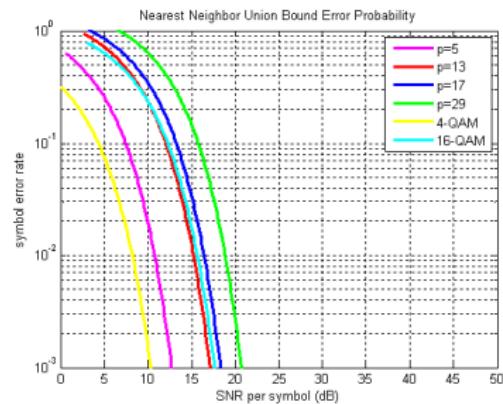
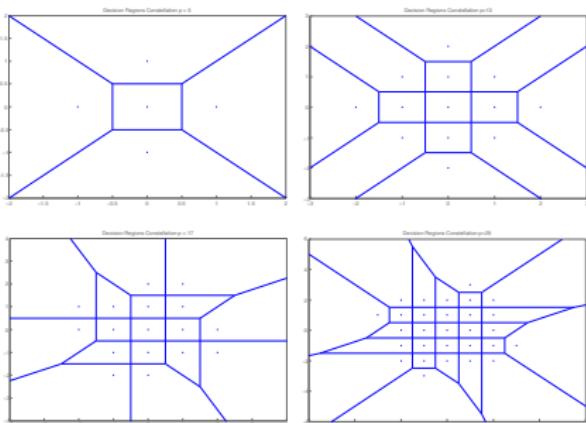
$$x \longmapsto i$$

The inverse mapping is defined as:

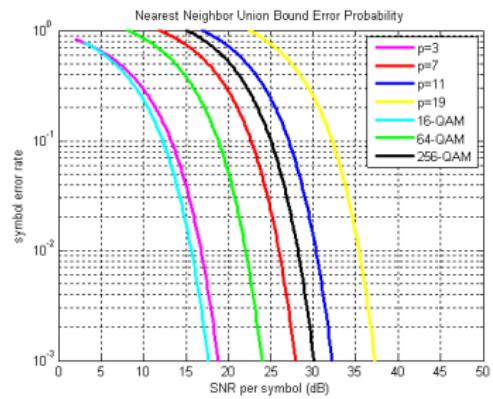
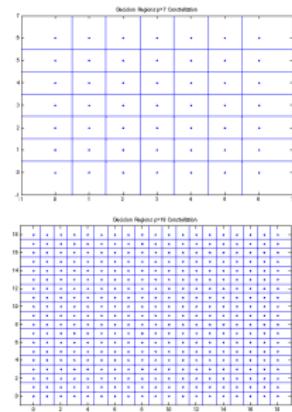
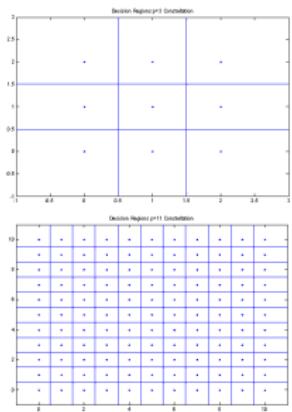
$$\gamma^{-1} : \mathbb{Z}[i]/p\mathbb{Z}[i] \longrightarrow \mathbb{F}_p[X]/(x^2 + 1)$$

$$i \longmapsto x$$

1 mod 4 Constellation



3 mod 4 Constellation



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4 Proposed Designs

- Designs in $\mathbb{Z}[i]$
- Designs in $\mathbb{Z}[w]$
 - 1 mod 6 Constellation
 - 2 mod 3 Constellation
 - Best Performance

1 mod 6 Constellation

Design 3

- Step 1: ring $\mathbb{Z}[w]$.
- Step 2: primes $p \in \mathbb{Z}^+$, $p \equiv 1 \pmod{6}$ in $\mathbb{Z}[w]$ ($p = \pi\pi^*$).
- Step 3: $M = p$.
- Step 4:
 - $\mathbb{Z}/p\mathbb{Z}$, $\#(\mathbb{Z}/p\mathbb{Z}) = |p| = p$.
 - $\mathbb{Z}[w]/\pi\mathbb{Z}[w]$, $\#(\mathbb{Z}[w]/\pi\mathbb{Z}[w]) = N(\pi) = \pi\pi^* = p$.

Theorem

If R is a PID and $a \in R$ is irreducible, then R/aR is a field.

1 mod 6 Constellation

Design 3

- Step 5: we are looking for $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}[w]/\pi\mathbb{Z}[w]$.

the mapping of the constellation is defined as:

$$\begin{aligned}\tilde{\mu} : \mathbb{F}_p &\longrightarrow \mathbb{Z}[w]/\pi\mathbb{Z}[w] \\ x &\longmapsto \tilde{\mu}(x) = x - \left[\frac{x\pi^*}{\pi\pi^*} \right] \pi\end{aligned}$$

The inverse mapping is defined as:

$$\begin{aligned}\mu^{-1} : \mathbb{Z}[w]/\pi\mathbb{Z}[w] &\longrightarrow \mathbb{F}_p \\ a &\longmapsto \mu^{-1}(a) = (a(v\pi^*) + a^*(u\pi^*)) \text{mod } p\end{aligned}$$

with $u\pi + v\pi^* = 1$.

2 mod 3 Constellation

Design 4

- Step 1: ring $\mathbb{Z}[w]$.
- Step 2: primes $p \in \mathbb{Z}^+$, $p \equiv 2 \pmod{3}$ in $\mathbb{Z}[w]$.
- Step 3: $M = p^2$.
- Step 4:
 - $\mathbb{Z}[w]/p\mathbb{Z}[w]$, $\#(\mathbb{Z}[w]/p\mathbb{Z}[w]) = N(p) = p^2$.
 - $\mathbb{F}_p[X]/(x^2 + x + 1)$, $\#(\mathbb{F}_p[X]/(x^2 + x + 1)) = p^2$.

Theorem

If R is a PID and $a \in R$ is irreducible, then R/aR is a field.

2 mod 3 Constellation

Design 4

- Step 5: We are looking for $\mathbb{Z}[w]/p\mathbb{Z}[w] \cong \mathbb{F}_p[X]/(x^2 + x + 1)$ with X corresponding w .

the mapping of the constellation is defined as:

$$\tilde{\gamma} : \mathbb{F}_p[X]/(x^2 + x + 1) \longrightarrow \mathbb{Z}[w]/p\mathbb{Z}[w]$$

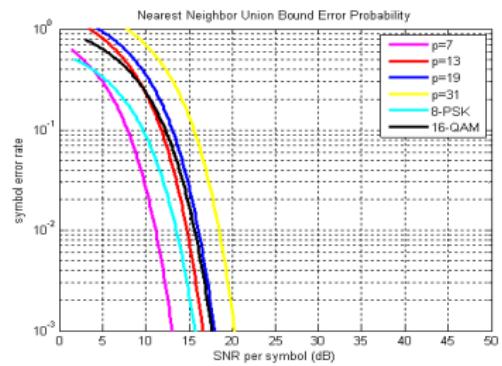
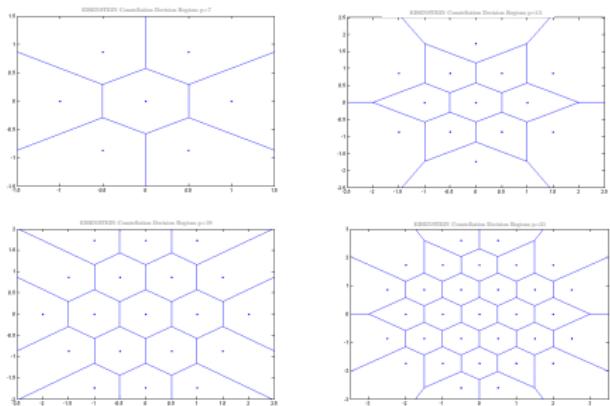
$$x \longmapsto w$$

The inverse mapping is defined as:

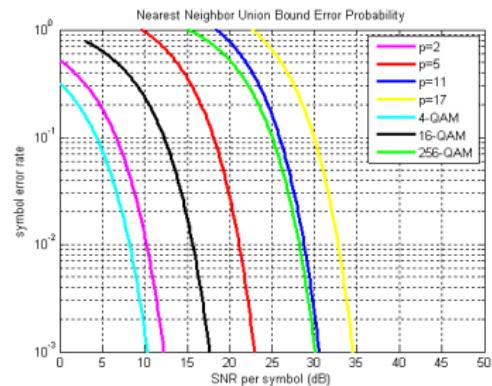
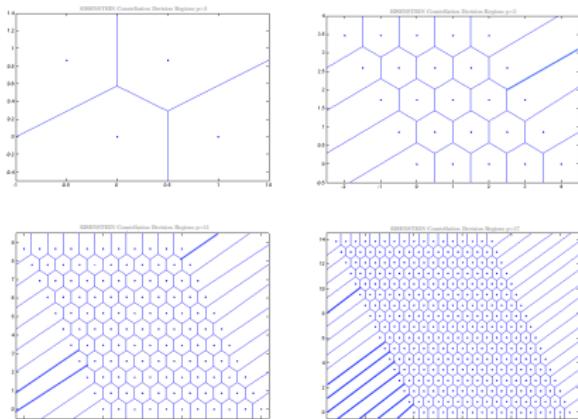
$$\tilde{\gamma}^{-1} : \mathbb{Z}[w]/p\mathbb{Z}[w] \longrightarrow \mathbb{F}_p[X]/(x^2 + x + 1)$$

$$w \longmapsto x$$

1 mod 6 Constellation



2 mod 3 Constellation



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Conclusions

- Algebraic theory identification:
 - PID ✓
 - Euclidean domain ✓
 - Fields ✓
- Performance metrics identification:
 - Decision regions ✓
 - Nearest Neighbor Union Bound ✓
- MATLAB parameters computation:
 - d_{\min} ✓
 - N_e ✓
- System model know how
 - MATLAB implementation proposed ✓

Conclusions

- Design and performance of the constellations
 - $\mathbb{Z}[i]$ ✓
 - $1 \pmod 4$
 - $3 \pmod 4$
 - $\mathbb{Z}[w]$ ✓
 - $1 \pmod 6$
 - $2 \pmod 3$

Conclusions

And Finally the Best Constellations Are

- 1 mod 6 constellation in $\mathbb{Z}[w]$ is the best performing constellation.
- 1 mod 4 constellations in $\mathbb{Z}[i]$ appear as a good QAM alternative.
- QAM constellations have better performance than 3 mod 4 in $\mathbb{Z}[i]$ and 2 mod 3 in $\mathbb{Z}[w]$.

Thank You

DE ZARZA I CUBERO Irene
Thank you