Inferring the radial parameter in rotational acceleration data

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Formulation

Consider a rigid body undergoing circular motion in the horizontal place. An accelerometer is attached to the body at a radial distance *r* from the axis of rotation. The accelerometer will provide a time-series output

$\vec{a}_{(i)} = \{a_{r(i)} \quad a_{t(i)}\}$

at regular time intervals Δt. Our initial problem is to infer the value of *r* from this data sequence.

Expression involving parameter r

Consider the system rotating at angular velocity ω_i at time t_i, and undergoing uniform angular acceleration α during the time interval

$$\Delta t = t_{i+1} - t_i$$

These quantities relate to the components of the total acceleration vector via the following pair of relations:

$$a_{r(i)} = r{\omega_i}^2$$
$$a_{t(i)} = \alpha(t_{i+1} - t_i)$$

Elementary rotational kinematics determines the new angular velocity at time t_{i+1}:

$$\omega_{(i+1)} = \omega_i + \alpha\Delta t$$

and therefore the new radial acceleration will be

$$a_{r(i+1)} = r{\omega_{(i+1)}}^2 = r[{\omega_i}^2 + \alpha^2(\Delta t)^2 + 2\omega_i \alpha \Delta t]$$

in which we used

$${\omega_{(i+1)}}^2 = (\omega_i + \alpha\Delta t)^2 = {\omega_i}^2 + \alpha^2(\Delta t)^2 + 2\omega_i \alpha \Delta t$$

We can now derive an expression for the rate of change of the radial acceleration:

$$\begin{aligned} \dot{a_r} &= \frac{1}{\Delta t}\left[a_{r(i+1)} - a_{r(i)}\right] \\ &= \frac{r}{\Delta t}[\alpha^2(\Delta t)^2 + 2\omega_i \alpha \Delta t] \\ &= r\alpha[\alpha\Delta t + 2\omega_i] \\ &= a_{t(i)}\left[\frac{a_{t(i)}}{r}\Delta t \pm 2\sqrt{\frac{a_{r(i)}}{r}}\right] \end{aligned}$$

where the sign ambiguity arise from the square-root operation.

Interpreting the sign ambiguity

Starting with the expression

$$\dot{a_r} = r[\alpha^2 \Delta t + 2\omega_i \alpha]$$

In the calculus limit $\Delta t \to 0$, this becomes

$$\dot{a_r} \to 2r\omega_i \alpha$$

If ω_i and α are of the same sign, then the magnitude of ω must increase and $\dot{a_r}$ is positive. When we rewrite this in terms of the components of the acceleration vector, viz. as

$$\dot{a_r} = \pm 2a_{t(i)}\sqrt{\frac{a_{r(i)}}{r}}$$

it is clear that we can introduce a rule for determining the sign as follows:

$$\dot{a_r} = sign(\omega_i \alpha)\ 2a_{t(i)}\sqrt{\frac{a_{r(i)}}{r}}$$

For non-infinitesimal Δt values, our full expression is therefore

$$\dot{a_r} = sign(\omega_i \alpha)\ 2a_{t(i)}\sqrt{\frac{a_{r(i)}}{r}} + \frac{\left(a_{t(i)}\right)^2}{r}\Delta t$$

Properties of cost function revisited. and resolved!

Consider our new expression for the cost function:

$$c = \dot{a_r} - sign(\omega_i \alpha)\ 2a_{t(i)}\sqrt{\frac{a_{r(i)}}{r}} - \frac{\left(a_{t(i)}\right)^2}{r}\Delta t$$

This is of the form

$$c = A + Bz + Cz^2$$

where we changed coordinates to

$$z = \frac{1}{\sqrt{r}}$$

and we have coefficients

$$A \equiv \dot{a_r}$$

$$B \equiv -sign(\omega\alpha)\ 2a_t\sqrt{a_r}$$

$$C \equiv -\ (a_t)^2 \Delta t$$

where we have dropped the indices for readability in what follows

The determinant is

$$B^2 - 4AC$$

$$= 4(a_t)^2 a_r + 4\dot{a_r}\ (a_t)^2 \Delta t$$

The first term is non-negative, the second depends on the sign of $\dot{a_r}$. The determinant itself, then, is non-negative when

$$4(a_t)^2 a_r > 4|\dot{a_r}|\ (a_t)^2 \Delta t$$

or

$$a_r > |\dot{a_r}|\ \Delta t$$

$$a_r > |\Delta a_r|$$

Once again, not an intuitive result.

But no, it IS intuitive! The 2nd term of the determinant is only problematic if $\dot{a_r}$ is negative. In that case, our condition is only insisting that the new radial acceleration be non-negative, as it should be. Specifically,

$$a_{r(i)} > |\Delta a_r|$$

$$a_{r(i)} > \left|a_{r(i+1)} - a_{r(i)}\right|$$

$$a_{r(i)} > a_{r(i)} - a_{r(i+1)}$$

$$0 > -a_{r(i+1)}$$

$$0 < a_{r(i+1)}$$

Old notes: initial interpretation of the sign of the second term

How to interpret the sign of the second term? Which sign do we use for which circumstances?

Let's go back to the following expression:

$$\dot{a_r} = r[\alpha^2 \Delta t + 2\omega_i \alpha]$$

The sign of the quantity

$$\omega_i \alpha$$

depends on whether the two factors have the same sign or not. In the following relation

$$\omega_{(i+1)} = \omega_i + \alpha \Delta t$$

consider the situation in which ω_i is a small negative value, and α is a large positive value. This can result in

$$|\omega_{(i+1)}| > |\omega_i|$$

i.e.

$$\dot{a_r} > 0$$

How large must the disparity between α and ω_i be for this to occur? This can be determined in the expression

$$\dot{a_r} = r[\alpha^2 \Delta t + 2\omega_i \alpha]$$

In the situation under consideration, the two terms are of opposite sign. The condition is therefore

$$\alpha^2 \Delta t > 2\omega_i \alpha$$

or

$$\alpha \Delta t > 2\omega_i$$

We can clearly see that this phenomenon is specific to the numerical context, for in the calculus limit of small intervals

$$\Delta t \to 0$$

the condition only occurs when $\omega_i = 0$, and so effectively never occurs, as this condition, in the presence of a non-zero value of α, lasts for an infinitesimally short time.

Our condition can also be expressed as:

$${}^{1}\!/_{2}\, \alpha \Delta t^2 > \omega_i \Delta t$$

for which insight is perhaps gained by comparing this to

$$\Delta\theta = \omega_i \Delta t + {}^{1}\!/_{2}\, \alpha \Delta t^2$$

which, in the condition under consideration, the two terms are of opposite sign.