Properties of the cost function

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General case

Recall our expression for the cost function, where we have dropped the indices for readability:

$$c = \dot{a_r} - \frac{(a_t)^2}{r}\Delta t - 2a_t\sqrt{\frac{a_r}{r}}$$

this equation is of the form

$$c = A - {}^{B}\!/_{r} - {}^{C}\!/_{\sqrt{r}}$$

where A and C can be independently of either sign and B is non-negative. Are there any conditions for which minimization of (the square of) might fail?

Changing coordinates to

$$z = \frac{1}{\sqrt{r}}$$

we have

$$c = A + Dz + Ez^2$$

where D =-C and E = -B

The determinant is

$$D^2 - 4AE$$

$$= 4{a_t}^2\frac{a_r}{r} - 4\dot{a_r}\frac{(a_t)^2}{r}\Delta t$$

The first term is always non-negative. The determinant itself is non-negative when

$$4{a_t}^2\frac{a_r}{r} > 4\dot{a_r}\frac{(a_t)^2}{r}\Delta t$$

which reduces to

$$a_r > \dot{a_r}\Delta t$$

or

$$a_r > \Delta a_r$$

So there is a real solution iff the change in a_r over the time interval is smaller than a_r itself. This does not seem to be a very intuitive result.

Limiting behaviours, simplifications

We are interested in visualizing how our cost function varies with the value of the radius *r*. This will allow us to visualize how the minimization algorithm progresses toward the optimal solution. What values should be set for the other quantities in the equation, in order to then plot the dependence on *r*? Can we identify limiting conditions or regimes of behaviour that help to categorize the types of scenarios we can expect?

To begin, note first that the value of ardot is not independent: it is uniquely determined by ar and at:

$$\dot{a_r} = \frac{(a_t)^2}{r}\Delta t + 2a_t\sqrt{\frac{a_r}{r}}$$

This leaves us with just two parameters to consider, a_r and a_t.

Vanishing value for a_t

When a_t is zero, $\dot{a_r}$ is zero, with no dependence on *r*, therefore affords no solution. We only get a signal, so to speak, when the system changes angular velocity.

Two simplification cases

Let`s rewrite a_r by factoring out a_t:

$$\dot{a_r} = a_t\left[\frac{a_t}{r}\Delta t + 2\sqrt{\frac{a_r}{r}}\right]$$

The two cases to consider are when the first or the second term dominates. Let us compare them using the latter case:

$$\frac{a_t}{r}\Delta t \ll 2\sqrt{\frac{a_r}{r}}$$

which we can reshuffle in various ways:

$$a_t \Delta t \ll 2\sqrt{r a_r}$$

$$\Delta v \ll 2\sqrt{r \omega^2 r}$$

$$\Delta v \ll 2\omega r$$

$$\frac{\Delta r \omega}{\omega} \ll 2r$$

$$\frac{\Delta \omega}{\omega} \ll 2$$

Since we are dealing with a limit, we can drop the numerical factor yielding the condition

$$\frac{\Delta \omega}{\omega} \ll 1$$

or

$$\frac{\Delta v}{v} \ll 1$$

i.e. as the fractional change in velocity vanishes. Can also rewrite as

$$\Delta v \ll v$$

We can refer to our two limiting situations, then, as

1. "small velocity" (as in small compared to the change in velocity over that time interval)
   a. Could also refer to this as the "kickstart" regime
2. "small change in velocity" (over the time interval, relative to the velocity itself).
   a. Could also refer to this as the "quasi-uniform" regime

Small-velocity limit

$$\dot{a_r} = a_t \frac{a_t}{r} \Delta t$$

$$c(r) = \dot{a_r} - \frac{(a_t)^2}{r} \Delta t$$

i.e. of the form

$$c(r) = A - \frac{B}{r}$$

where B is positive and A can be positive (speeding up) or negative (slowing down). Keep in mind we need to produce a cost function that is non-negative by squaring this one:

$$\acute{c}(r) = c^2(r)$$

c'(r) (and c(r)) has a minimal value of zero when

$$A = \frac{B}{r}$$

which admits a solution

$$r = \frac{B}{A}$$

Since B is always positive, r is positive only if A is negative. If A is positive, c(r) takes on a minimal value of A at r->infinity. This is an unphysical result: we can infer *r* when the system is speeding up, but not when it is slowing down. Where did we go wrong?

Small-change-in-velocity limit

$$\dot{a_r} = 2a_t\sqrt{\frac{a_r}{r}}$$

$$c(r) = \dot{a_r} - 2a_t\sqrt{\frac{a_r}{r}}$$

$$c(r) = A - \frac{B}{\sqrt{r}}$$

where both A and B can take on either sign independently of each other. Here we have the same unphysical result when A and B are of the same sign.

Another attempt

Once again:

$$c = \dot{a_r} - \frac{(a_t)^2}{r}\Delta t - 2a_t\sqrt{\frac{a_r}{r}}$$

where

$$\dot{a_r} = \frac{a_{r(i+1)} - a_{r(i)}}{\Delta t}$$

make easier to read using notation

T==at ; R1 == ar(i) ; R2 == ar(i+1), D = deltaT

$$c = \frac{R2 - R1}{D} - \frac{T^2}{r}D - 2T\sqrt{\frac{R1}{r}}$$

and another attempt

starting with

$$a_{r(i+1)} = r\omega_{(i+1)}{}^2 = r[\omega_i{}^2 + \alpha^2(\Delta t)^2 + 2\omega_i\alpha\Delta t]$$

replace alpha and omega with expressions involving at and ar:

$$a_{r(i+1)} = r\left[\frac{a_r}{r} + \frac{{a_t}^2}{r^2}\,(\Delta t)^2 + 2\sqrt{\frac{a_r}{r}}\frac{a_t}{r}\Delta t\right]$$

$$a_{r(i+1)} = a_r + \frac{{a_t}^2}{r}\,(\Delta t)^2 + 2\sqrt{\frac{a_r}{r}}\,a_t\Delta t$$

$$\dot{a_r} = \frac{{a_t}^2}{r}\,\Delta t + 2\sqrt{\frac{a_r}{r}}\,a_t$$

and another attempt #4

Starting with

$$\dot{a_r} = a_t\left[\frac{a_t}{r}\Delta t + 2\sqrt{\frac{a_r}{r}}\right]$$

and replace $\{a_r \quad a_t\}$ with $\{v \quad \Delta v\}$ using

$$a_r = \frac{v^2}{r}$$
$$a_t = \frac{\Delta v}{\Delta t}$$

yields a number of alternate expressions, all of which have a simple inverse dependence on *r* :

$$= a_t\left[\frac{a_t}{r}\Delta t + 2\frac{v}{r}\right]$$

$$= \frac{a_t}{r}\left[a_t\Delta t + 2v\right]$$

$$= \frac{a_t}{r}\left[\Delta v + 2v\right]$$

$$= \frac{1}{r}\frac{\Delta v}{\Delta t}\left[\Delta v + 2v\right]$$