Inferring the radial parameter in rotational acceleration data

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Formulation

Consider a rigid body undergoing circular motion in the horizontal place. An accelerometer is attached to the body at a radial distance *r* from the axis of rotation. The accelerometer will provide a time-series output

$\vec{a}_{(i)} = \{a_{r(i)} \quad a_{t(i)}\}$

at regular time intervals Δt. Our initial problem is to infer the value of *r* from this data. given only the accelerometer output.

Expression involving parameter r

Consider the systemssume angular velocity ω_i at time t_i, and uniform angular acceleration α during the time interval t_i to t_{i+1}.

The new angular velocity will be

$$\omega_{(i+1)} = \omega_i + \alpha\Delta t$$

$$\omega_{(i+1)}{}^2 = (\omega_i + \alpha\Delta t)^2 = \omega_i{}^2 + \alpha^2(\Delta t)^2 + 2\omega_i\alpha\Delta t$$

and the radial accelerations are therefore

$$a_{r(i)} = r\omega_i{}^2$$

$$a_{r(i+1)} = r\omega_{(i+1)}{}^2 = r[\omega_i{}^2 + \alpha^2(\Delta t)^2 + 2\omega_i\alpha\Delta t]$$

giving

$$\dot{a_r} = \frac{1}{\Delta t}\left[a_{r(i+1)} - a_{r(i)}\right]$$

$$= \frac{r}{\Delta t}[\alpha^2(\Delta t)^2 + 2\omega_i\alpha\Delta t] = r[\alpha^2\Delta t + 2\omega_i\alpha]$$

$$= r\left[\left(\frac{a_{t(i)}}{r}\right)^2 \Delta t + 2\frac{a_{t(i)}}{r}\sqrt{\frac{a_{r(i)}}{r}}\right]$$

$$= \frac{\left(a_{t(i)}\right)^2}{r}\Delta t + 2a_{t(i)}\sqrt{\frac{a_{r(i)}}{r}}$$

where we used

$\alpha = \frac{a_{t(i)}}{r}$ (1)

Solution via optimization methods

From the full expression, we can use minimization methods to find the optimal value for *r*, i.e. we can form a cost function

$$c \equiv \dot{a_r} - \frac{\left(a_{t(i)}\right)^2}{r}\Delta t - 2a_{t(i)}\sqrt{\frac{a_{r(i)}}{r}}$$

and minimize against r. We actually want the value of r that brings c closest to zero, so we will have to rectify it by an additional operation such as squaring it, eg we will minimize

$$c'(r) \equiv c^2(r)$$

(Ferenc: I pursue this line of thought in the next document; the rest can be skipped)

Student exercises

Direct algebraic solution

We can solve the above equation and obtain a closed-form solution for r.

r = ...

Are there any simplification opportunities? Any restrictions implied from the requirement that r must be non-negative? Can it be rewritten in a nicer form?

(Ferenc: I start exploring this a bit in the accompanying doc)

Solution under a simplification

By ignoring[1] the term in Δt, this simplifies to

$$\dot{a_r} = 2a_{t(i)}\sqrt{\frac{a_{r(i)}}{r}}$$

which can easily be solved for r.

$$r = 4{a_{t(i)}}^2 \frac{a_{r(i)}}{\dot{a_r}^2}$$

after which alpha is easily found.

Warm-up programming exercise for implementing a minimization problem

If alpha is known, eg. we generate data using a constant-angular-acceleration scenario, then we can implement a very simple cost function:

$$c = a_{t(i)} - \alpha r$$

We can minimize c by varying r, and obtain the optimal value for r. This will validate a simple minimization algorithm, into which we can inject increasingly sophisticated cost functions.

[1] The assumption is that $\alpha^2(\Delta t)^2 \ll 2\omega_i\alpha\Delta t$, which implies $\alpha^2\Delta t \ll 2\omega_i\alpha$; $\alpha\Delta t \ll 2\omega_i$; $\Delta\omega_i \ll \omega_i$
i.e. the change in omega is much smaller than its current value. This assumption would seem to fail near zero speed, with large accelerations. eg a “kickstart” scenario.