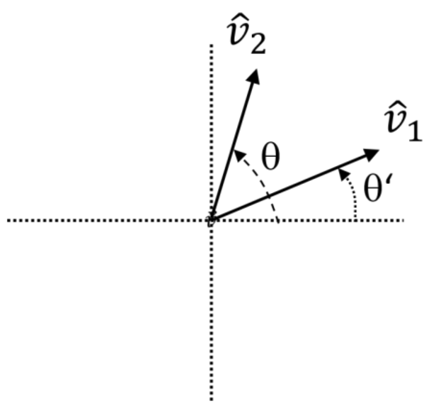
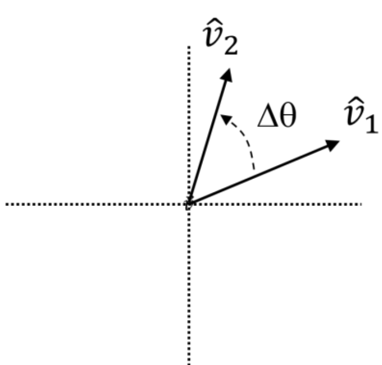
Inferring rotational motion from non-radially-aligned accelerometers

Review: rotational transformations between coordinate systems

Consider vectors $\hat{v}_1$ and $\hat{v}_2$ of equal magnitude, pointing in directions θ_1 and θ_2, respectively.



$\hat{v}_2$ can be obtained from $\hat{v}_1$ by a counterclockwise rotation of the latter by an amount $\Delta\theta$:



i.e. the two angles are related by

$$\theta_2 = \theta_1 + \Delta\theta$$

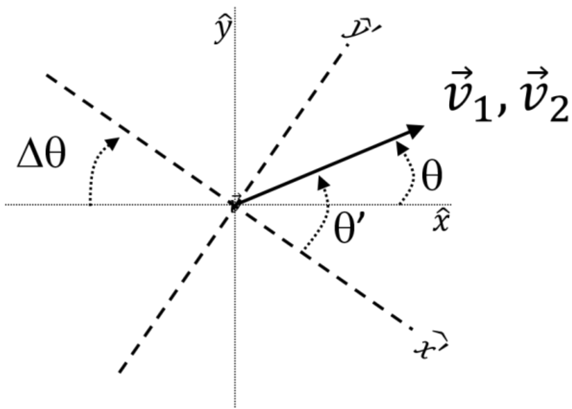
The rotation can be performed algebraically by multiplying $\hat{v}_1$ by a rotation matrix $R_{\Delta\theta}$:

$$\hat{v}_2 = R_{\Delta\theta}\hat{v}_1$$

where

$$R_\theta \equiv \begin{pmatrix} cos(\Delta\theta) & -sin(\Delta\theta) \\ sin(\Delta\theta) & cos(\Delta\theta) \end{pmatrix}$$

There are two equivalent ways of interpreting the effect of R. The first, as just described, is one in which we have a single coordinate system in which we obtain vector $\hat{v}_2$ by rotating $\hat{v}_1$ by an angle $\Delta\theta$. In the second view, $\hat{v}_1$ remains fixed in coordinate system S_1 while we rotate a 2nd coordinate system S_2 by the same angle $\Delta\theta$, but in the opposite (i.e. in the clockwise) direction. $\hat{v}_1$ and $\hat{v}_2$, in this case, represent the same vector pointing in the same direction, only expressed in terms of different coordinate systems.



Formulation of problem

In our problem, we have two coordinate systems at play. An accelerometer sensor is attached to a rotating rigid body. The $\hat{r}\hat{t}$ frame is formed from the radial and tangential directions defined at the sensor's attachment position $\vec{r}$. The sensor itself has its own internal $\hat{x}\hat{y}$ frame which is, generally speaking, not aligned with the $\hat{r}\hat{t}$ axes.

We begin with data from the sensor in the form of an acceleration vector $\vec{a}_{\hat{x}\hat{y}}$ expressed with respect to the $\hat{x}\hat{y}$ axes, i.e.

$$\vec{a}_{\hat{x}\hat{y}} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{bmatrix} a\ cos(\beta') \\ a\ sin(\beta') \end{bmatrix}$$

where

$$a \equiv \left|\vec{a}_{\hat{x}\hat{y}}\right|$$

and β' is the angle that the vector makes with the positive $\hat{x}$ axis, i.e.

$$\beta' \equiv tan^{-1}\left({}^{a_y}/_{a_x}\right)$$

Our problem is to determine $\vec{a}_{\hat{r}\hat{t}}$, i.e. the components of that same acceleration vector expressed as components of the local radial and tangential directions[1], i.e.

[1] associated with the (unknown) sensor position $\vec{r}$ on the rigid body.

$$\vec{a}_{\hat{r}\hat{t}} = \begin{bmatrix} a_r \\ a_t \end{bmatrix} = \begin{bmatrix} a\,cos(\beta) \\ a\,sin(\beta) \end{bmatrix}$$

where

$$\left|\vec{a}_{\hat{x}\hat{y}}\right| = \left|\vec{a}_{\hat{r}\hat{t}}\right| \equiv a$$

and β is the angle that the vector makes with the positive $\hat{r}$ axis, i.e.

$$\beta \equiv tan^{-1}\left({}^{a_t}/_{a_r}\right)$$

We interpret the non-alignment as a clockwise rotation of the $\hat{r}\hat{t}$ frame by an amount $\Delta\beta$ with respect to the $\hat{x}\hat{y}$ frame, i.e.

$$\beta = \beta' + \Delta\beta$$

in which the alignment condition $\Delta\beta = 0$ corresponds to

$$\begin{gathered} \hat{r} = \hat{x} \\ \hat{t} = \hat{y} \end{gathered}$$

Right-handedness

Given an established radial direction, there remain, in 2D, two choices for the positive tangential direction. We choose the tangential direction that produces an $\hat{r}\hat{t}$ coordinate system that is right-handed. This means, for example, that neither the tangential velocity nor the tangential acceleration are necessarily in the positive direction.

Implementation overview

We have previously established a cost function that tells us the degree to which pairs of temporally-adjacent acceleration vectors do not conform to circular-motion kinematics. This pair of vectors must be in $\hat{r}\hat{t}$ coordinates, and are denoted $\vec{a}_{\hat{r}\hat{t}\,(i)}$ and $\vec{a}_{\hat{r}\hat{t}\,(i+1)}$. We produce a trial vector pair by rotating $\vec{a}_{\hat{x}\hat{y}\,(i)}$ and $\vec{a}_{\hat{x}\hat{y}\,(i+1)}$ by an angle $\Delta\beta_{trial}$. Standard minimization procedures will iteratively evaluate the associated cost function and guide us towards a better choice of angle until we obtain the true angle corresponding to the real orientation of the $\hat{r}\hat{t}$ axes.

The remaining work to be done is to consider restrictions on the range of $\Delta\beta_{trial}$ values, which we resolve by extending the definition of our cost function.

Boundary conditions

Given

$$a_r = a\,cos(\beta)$$

the requirement

$$a_r \geq 0$$

implies a condition of

$$-90 \leq \beta \leq 90$$

for the allowable values of β .

For completeness, note that this implies a condition of

$$-90 - \beta' \leq \Delta\beta \leq 90 - \beta'$$

for the allowable values of rotation $\Delta\beta$.

Boundary values for the cost function

The minimization procedure could lead to the evaluation of the cost function for trial $\Delta\beta$ values that are outside the allowed domain just mentioned. We need to provide a computable continuation of the cost function for this domain. Minimization procedures require that such a continuation be continuous and differentiable for all values of $\Delta\beta$.

Recall the cost function:

$$c \equiv \dot{a_r} - \frac{\left(a_{t(i)}\right)^2}{r}\Delta t - 2a_{t(i)}\sqrt{\frac{a_{r(i)}}{r}}$$

Writing out the derivative as an explicit difference, we have

$$c = \frac{a_{r(i+1)} - a_{r(i)}}{\Delta t} - \frac{\left(a_{t(i)}\right)^2}{r}\Delta t - 2a_{t(i)}\sqrt{\frac{a_{r(i)}}{r}}$$

There is only one problematic square-root operation which imposes the condition[2]

$$a_{r(i)} \geq 0$$

The boundary value of the cost function, then, is

$$c_{boundary} = \frac{a_{r(i+1)}}{\Delta t} - \frac{\left(a_{t(i)}\right)^2}{r}\Delta t$$

If the algorithm generates a trial $\Delta\beta$ value outside the allowed range, then we return a cost given by

$$c_{outside} = c_{boundary} + penaltyFactor * (-\Delta\beta + 90)$$

This ensures that the cost function remains continuous across the boundary. We also want its 1st derivative to remain continuous: Noting that

$$\frac{\partial c_{outside}}{\partial(\Delta\beta)} = -penaltyFactor$$

the additional requirement is therefore

---

[2] Regarding the sign of $a_{r(i+1)}$, note that negative values, although unphysical, nevertheless leave the cost function computable and continuous.

$$penaltyFactor = -\left[\frac{\partial c}{\partial(\Delta\beta)}\right]_{\Delta\beta=90}$$

Student exercise:

Determine the full explicit expression for $c_{outside}$ in terms of $\Delta\beta$.

Using the 2 cases $\beta = 10deg$ (radially-dominant) and $\beta = 80deg$ (tangentially-dominant), plot the cost function over the range of $\Delta\beta$ values that correspond to

$$-90 - \beta_{outside} \leq \beta \leq 90 + \beta_{outside}$$

where

$$\beta_{outside} = 30deg$$