KU Math Club
3/4/20

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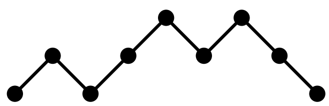
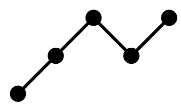
Experiment #1: Dyck Paths

A *Dyck path* is a sequence of up-steps $\nearrow$ and down-steps $\searrow$ obeying the following three rules:

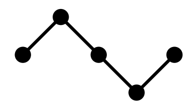
(1) All of the steps have the same length.
(2) The number of northeast steps is the same as the number of southeast steps.
(3) The path cannot ever go further south than its starting point.



Dyck path
with 8 steps



Not a Dyck path
(violates rule 2)



Not a Dyck path
(violates rule 3)

How many different Dyck paths are there with 2, 4, 6, or 8 steps?

Number of steps	Number of Dyck paths
2	
4	
6	
8	

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Experiment #2: Standard Tableaux

You have a set of tiles numbered 1, 2, . . . , N, where N is some positive even number. Your task is to arrange the tiles into a rectangle with two rows. Both rows need to be in increasing order left to right, and all columns need to be in increasing order top to bottom. The fancy term for a grid like this is a $2 \times n$ *standard tableau*.

1	**2**	**4**	**7**	**9**	**10**
3	**5**	**6**	**8**	**11**	**12**

Legal arrangement

1	**3**	**6**	**7**	**8**	**10**
2	**4**	**5**	**9**	**11**	**12**

Illegal arrangement
(third column out of order)

How many different $2 \times n$ standard tableaux are there with 2, 4, 6, or 8 tiles?

Number of tiles	Number of standard tableaux
2	
4	
6	
8	

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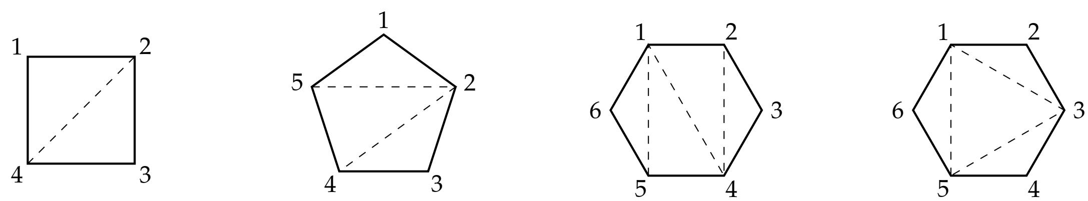
Experiment #3: Triangulating Polygons

Start with a regular polygon and number its vertices $1, 2, \ldots, N$. It can be cut into triangles ("triangulated") by making cuts that start and end at vertices of the original polygon. (The resulting triangles don't have to be congruent; in fact we don't care at all what they look like.) No two of the cuts should overlap.



Here are some useful facts. You can either take my word for it, or convince yourself that they're true (they're not hard):

- If you start with a triangle ($N = 3$), then you don't need to make any cuts at all.
- If you start with a square ($N = 4$), then you just need one cut and you get two triangles.
- If the original polygon has N sides, then the number of cuts you need is $N - 3$, no matter which cuts you make.
- The number of triangles you end up with will always be $N - 2$.

How many different ways are there to triangulate a regular N-sided polygon for $N = 3$, 4, 5, or 6?

Polygon	Number of possible triangulations
Triangle (3 sides)	
Square (4 sides)	
Pentagon (5 sides)	
Hexagon (6 sides)	

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Experiment #4: Arranging Tiles

You are given tiles numbered 1, 2, . . . , N, where N is some positive number. Your task is to arrange the tiles in order, left to right, *so that no three tiles are in order*. The rule applies to *every* set of three tiles (not just ones that appear consecutively).

3	**5**	**2**	**4**	**1**

Acceptable

3	**1**	**5**	**2**	**4**

Unacceptable
(1,2,4 appear in order)

How many different arrangements are there if you start with 1, 2, 3, or 4 tiles?

Number of tiles	Number of possible arrangements
1	
2	
3	
4	

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The Catalan Numbers

The sequence of *Catalan numbers* starts

$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862,\ 16796.$$

They have both a nice closed formula and a nice quadratic recurrence:

$$C_n = \frac{1}{n+1}\binom{2n}{n} \qquad\qquad C_0 = 1; \qquad C_n = C_{n-1} + \sum_{k=1}^{n-1} C_k C_{n-k-1} \quad (n>0)$$

These are one of the most frequently encountered sequences in combinatorics and are of huge importance. You have just seen four kinds of combinatorial objects that C_n counts:

- Dyck paths with $2n$ steps
- $2 \times n$ standard tableaux
- Triangulations of an $(n+3)$-sided polygon
- Permutations of $\{1,\ldots,n\}$ with no three numbers in order (a.k.a. "123-avoiding")

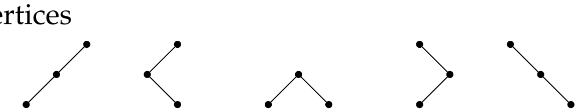
When two families of combinatorial objects have the same size, there must be a bijection between them (or so combinatorialists hope). Some bijections are not so hard (e.g., between Dyck paths and standard tableaux); some are much trickier. Enumerative combinatorics is all about finding bijections. See if you can find your own bijections between some of these families.

Here are just a few other things the Catalan numbers count:

- Parenthesizations of an $(n+1)$-term product

  $$((ab)c)d \quad (a(bc))d \quad (ab)(cd) \quad a((bc)d) \quad (a(b(cd))$$

- Binary trees with n vertices

- Nondecreasing sequences $(a_1,\ldots,a_n)$ of positive integers such that $a_i \le i$ for all i

  $$111 \quad 112 \quad 122 \quad 113 \quad 123$$

- Sequences $(1,a_1,\ldots,a_n,1)$ of positive integers such that each a_i is ≥ 2 and divides the sum of its two neighbors (one of my favorites)

  $$12531 \quad 12341 \quad 13231 \quad 14321 \quad 13521$$

Richard Stanley, for many years the leading figure in enumerative and algebraic combinatorics, maintains a list of known interpretations of the Catalan numbers. Exercise 6.19 in Volume 2 of Stanley's book *Enumerative Combinatorics* lists 66 known interpretations, challenging the reader to find $66 \times 65 = 4290$ bijections. EC2 was published in 1999; the list has since grown considerably and had reached 207 by 2013 when Stanley stopped keeping track. Stanley wrote a whole book about the Catalan numbers, published in 2015.