Pulse function

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1 Derive the function

Define the function as a charging/discharging of capacitor. To take into account finite width of light pulse and clipping capacitor[1] we parametrize the pulse function by two time constants: rise time τ_1 and discharge time τ_2.

$$p(t) = (1 - e^{-t/\tau_1})e^{-t/\tau_2}, \quad t > 0 \tag{1}$$

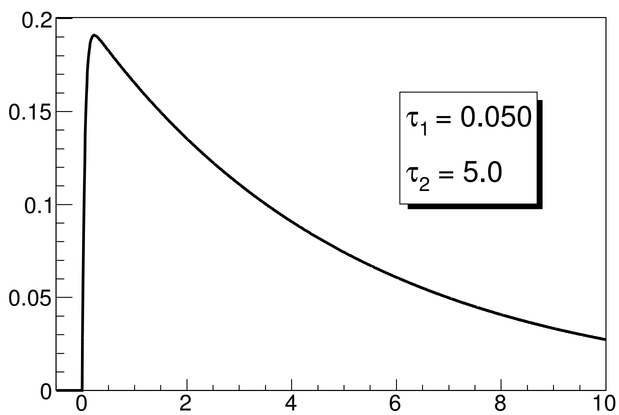

Figure 1: Pulse function.

The pulse function is defined on $(0, +\infty)$. Normalize it to 1.

$$\int_0^\infty (1 - e^{-x/\tau_1}) e^{-x/\tau_2} dx =$$
$$\int_0^\infty e^{-x/\tau_2} dx - \int_0^\infty e^{-x/\tau_1} e^{-x/\tau_2} dx =$$
$$\tau_2 + \frac{1}{\frac{1}{\tau_1} + \frac{1}{\tau_2}} = \frac{\tau_2^2}{\tau_1 + \tau_2}$$

Write pulse function normalized with factor A as

$$p(x) = A \frac{\tau_1 + \tau_2}{\tau_2^2} (1 - e^{-x/\tau_1}) e^{-x/\tau_2} \tag{2}$$

Fig.1 shows normalized pulse function for $A = 1$, $\tau_1 = 0.050$ and $\tau_2 = 1$.

Because $\int_0^\infty p(t)dt = A$

$[A] = QR$

$[p(x)] = [R]\frac{[Q]}{[t]} = [R][I] = [V] = \text{Volts}$

2 Convolution with scintillator decay

Let's assume that scintillator decay time is T and normalize scintillator decay function $s(t)$ to 1:

$$\int_0^\infty s(t)dt = 1, \quad \int_0^\infty e^{-t/T} dt = T$$

so

$$s(t) = \frac{1}{T} e^{-t/T} \tag{3}$$

Convolute pulse function with scintillator decay

$$P(x) = \int_0^x s(t) p(x - t) dt \tag{4}$$

In general case with integration limits a and b

$$P(x) = \int_a^b s(t)p(x-t)dt = \begin{vmatrix} z = x-t & dz = -dt \\ t = a & z = x-a \\ t = b & z = x-b \end{vmatrix} = -\int_{x-a}^{x-b} s(x-z)p(z)dz$$
$$= \int_{x-b}^{x-a} s(x-z)p(z)dz$$

Because in our case $a = 0$ and $b = x$

$$\boxed{P(x) = \int_0^x s(t-x)p(t)dt} \tag{5}$$

We will use Eq(5) for $P(x)$ for following calculations.

$$\begin{aligned} P(x) &= A\frac{\tau_1+\tau_2}{\tau_2^2}\frac{1}{T}\int_0^x e^{-\frac{x-t}{T}}(1-e^{-t/\tau_1})e^{-t/\tau_2}dt \\ &= A\frac{\tau_1+\tau_2}{\tau_2^2}\frac{1}{T}e^{-\frac{x}{T}}\Big[\int_0^x e^{-(\frac{1}{\tau_2}-\frac{1}{T})t}dt - \int_0^x e^{-(\frac{1}{\tau_1}+\frac{1}{\tau_2}-\frac{1}{T})t}dt\Big] \\ &= A\frac{\tau_1+\tau_2}{\tau_2^2}\frac{1}{T}e^{-\frac{x}{T}}\Big[\int_0^x e^{-(\frac{1}{\tau_2}-\frac{1}{T})t}dt - \int_0^x e^{-(\frac{1}{\tau_{12}}-\frac{1}{T})t}dt\Big] \end{aligned}$$

where

$$\boxed{\tau_{12} = \frac{\tau_1\tau_2}{\tau_1+\tau_2}} \tag{6}$$

$$\begin{aligned} P(x) &= A\frac{\tau_1+\tau_2}{\tau_2^2}\frac{1}{T}e^{-\frac{x}{T}}\Big[\frac{-1}{\frac{1}{\tau_2}-\frac{1}{T}}\left(e^{-(\frac{1}{\tau_2}-\frac{1}{T})x}-1\right) - \frac{-1}{\frac{1}{\tau_{12}}-\frac{1}{T}}\left(e^{-(\frac{1}{\tau_{12}}-\frac{1}{T})x}-1\right)\Big] \\ &= A\frac{\tau_1+\tau_2}{\tau_2^2}e^{-\frac{x}{T}}\Big[\frac{-\tau_2}{T-\tau_2}\left(e^{-(\frac{1}{\tau_2}-\frac{1}{T})x}-1\right) - \frac{-\tau_{12}}{T-\tau_{12}}\left(e^{-(\frac{1}{\tau_{12}}-\frac{1}{T})x}-1\right)\Big] \\ &= A\frac{\tau_1+\tau_2}{\tau_2^2}e^{-\frac{x}{T}}\Big[\frac{\tau_2}{T-\tau_2}\left(1-e^{-(\frac{1}{\tau_2}-\frac{1}{T})x}\right) - \frac{\tau_{12}}{T-\tau_{12}}\left(1-e^{-(\frac{1}{\tau_{12}}-\frac{1}{T})x}\right)\Big] \\ &= A\frac{\tau_1+\tau_2}{\tau_2^2}\Big[\frac{\tau_2}{T-\tau_2}\left(e^{-\frac{x}{T}}-e^{-\frac{x}{\tau_2}}\right) - \frac{\tau_{12}}{T-\tau_{12}}\left(e^{-\frac{x}{T}}-e^{-\frac{x}{\tau_{12}}}\right)\Big] \end{aligned}$$

$$P(x) = A\frac{\tau_1+\tau_2}{\tau_2^2}\Big(\tau_2\frac{e^{-x/T}-e^{-x/\tau_2}}{T-\tau_2} - \tau_{12}\frac{e^{-x/T}-e^{-x/\tau_{12}}}{T-\tau_{12}}\Big) \tag{7}$$

Define

$$\boxed{I_{T\tau} = \tau\frac{e^{-x/T}-e^{-x/\tau}}{T-\tau}} \tag{8}$$

then

$$\boxed{P(x) = A\frac{\tau_1+\tau_2}{\tau_2^2}\Big(I_{T\tau_2}(x) - I_{T\tau_{12}}(x)\Big)} \tag{9}$$

Fig.2 shows result of convolution function from Fig.1 with scintillator decay function with decay time $T = 40$. Note on shift of the maximum from about 0 to about 10. Using Eq(7) we can estimate shift of the maximum for the function plotted on Fig.2. Because $\tau_1 << \tau_2$ we can neglect the second term in parenthesis. If we equate to 0 derivative of the first term we will find

$$\begin{aligned}
\frac{1}{T}e^{-x/T} - \frac{1}{\tau_2}e^{-x/\tau_2} &= 0 \\
\frac{1}{T} - \frac{1}{\tau_2}e^{-\left(\frac{1}{\tau_2}-\frac{1}{T}\right)x} &= 0 \\
e^{-(\frac{1}{\tau_2}-\frac{1}{T})x} &= \frac{\tau_2}{T} \\
-\left(\frac{1}{\tau_2}-\frac{1}{T}\right)x &= ln\frac{\tau_2}{T} \\
&= \frac{T\tau_2}{T-\tau_2}ln\frac{T}{\tau_2} \\
\text{because of } T >> \tau_2 & \\
x &\approx \tau_2 ln\frac{T}{\tau_2}
\end{aligned}$$

In numbers

$$x \approx 5 \cdot ln\frac{40}{5} \approx 10$$

3 Special cases

NB: τ_2 is always finite while τ_1 and T can be 0. In function $I_{T\tau}$ τ can be τ_1 or τ_{12}; τ_{12} can be 0 when $\tau_1 = 0$.

Expression (8) will blow up during computing at $T \to \tau$. Except that it worth to consider cases $\tau \to 0$ and $T \to 0$.

Case 1 $\boxed{\tau \to 0}$ $I_{T\tau} \to 0$ $I_{T\tau} = 0$

Assume that $\tau > \epsilon$ now.

Case 2 $\boxed{T \to 0}$ $I_{T\tau} = e^{-x/\tau}$

Assume that both $\tau > \epsilon$ and $T > \epsilon$ now.

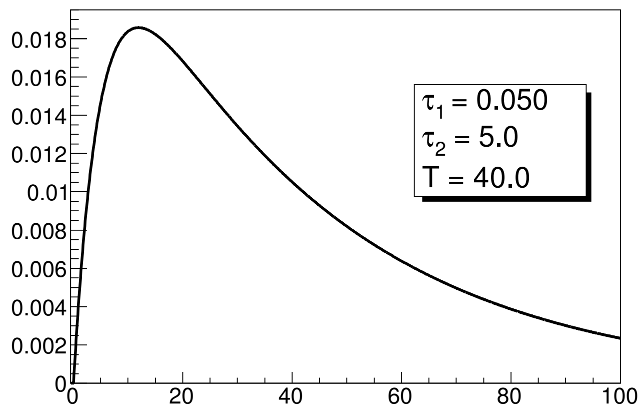

Figure 2: Convolution with scintillator decay function.

Case 3 $\boxed{\tau \to T}$

$$
\begin{aligned}
I_{T\tau} &= \tau \frac{e^{-x/T} - e^{-x/\tau}}{T-\tau} = \tau e^{-x/\tau} \frac{e^{(\frac{1}{\tau}-\frac{1}{T})x} - 1}{T-\tau} \\
&\underset{\tau\to T}{\longrightarrow} \quad \tau e^{-x/\tau} \frac{\frac{1}{\tau}-\frac{1}{T}}{T-\tau} x \quad \underset{\tau\to T}{\longrightarrow} \quad \cancel{\tau} e^{-x/\tau} \frac{x}{T\cancel{\tau}} = \frac{x}{T} e^{-x/\tau} \\
&= \frac{x}{\tau} e^{-x/\tau}
\end{aligned}
$$

4 Smearing with resolution function

To take into account signal jitter we convolute pulse function with resolution function: Gaussian with width σ

$$G(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} \tag{10}$$

Convolution integral runs from $-\infty$ to ∞.

$$\mathcal{P}_\sigma(x) = \int_{-\infty}^{\infty} dt P(t) G(t-x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} P(t) e^{-\frac{(t-x)^2}{2\sigma^2}} dt \tag{11}$$

Take into account that $P(t) = 0$ for $t < 0$

then

$$\mathcal{P}_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty P(t)e^{-\frac{(t-x)^2}{2\sigma^2}}dt \tag{12}$$

We can also use a step function

$$e(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

to extend integration to $-\infty$. The total area under smeared pulse function should be equal to area under unsmeared function:

$$\int_{-\infty}^{\infty}\mathcal{P}_\sigma(x)dx = \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}P(t)e(t)G(t-x)dt =$$
$$\int_{-\infty}^{\infty}dtP(t)e(t)\underbrace{\int_{-\infty}^{\infty}G(t-x)dx}_{=1} = \int_{-\infty}^{\infty}P(x)dx$$

The smearing does not introduce new special cases except $\sigma = 0$, therefore we will smear special cases separately.

In general case

$$\mathcal{P}_\sigma(x) = A\frac{\tau_1+\tau_2}{\tau_2^2}\Big(\frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty I_{T\tau_2}(t)e^{-\frac{(t-x)^2}{2\sigma^2}}dt - \frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty I_{T\tau_{12}}(t)e^{-\frac{(t-x)^2}{2\sigma^2}}dt\Big)$$

Denote

$$\boxed{I^\sigma_{T\tau} = \frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty I_{T\tau}(t)e^{-\frac{(t-x)^2}{2\sigma^2}}dt} \tag{13}$$

Here τ can serve as τ_1 or τ_{12}. Then

$$\boxed{\mathcal{P}_\sigma(x) = A\frac{\tau_1+\tau_2}{\tau_2^2}\Big(I^\sigma_{T\tau_2}(x) - I^\sigma_{T\tau_{12}}(x)\Big)} \tag{14}$$

$$\begin{aligned} I^\sigma_{T\tau}(x) &= \frac{1}{\sqrt{2\pi}\sigma}\frac{\tau}{T-\tau}\Big(\int_0^\infty e^{-t/T}e^{-\frac{(t-x)^2}{2\sigma^2}}dt - \int_0^\infty e^{-t/\tau}e^{-\frac{(t-x)^2}{2\sigma^2}}dt\Big) \\ &= \frac{\tau}{T-\tau}\Big(\frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty e^{-t/T}e^{-\frac{(t-x)^2}{2\sigma^2}}dt - \frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty e^{-t/\tau}e^{-\frac{(t-x)^2}{2\sigma^2}}dt\Big) \end{aligned} \tag{15}$$

If we denote

$$I^\sigma_T(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty e^{-t/T}e^{-\frac{(t-x)^2}{2\sigma^2}}dt \tag{16}$$

we can write Eq(15) as

$$I^\sigma_{T\tau}(x) = \frac{\tau}{T-\tau}(I^\sigma_T(x) - I^\sigma_\tau(x)) \tag{17}$$

To calculate integral $I_{\mathcal{T}}^{\sigma}(x)$ Eq(16) rewrite

$$e^{-\frac{t}{\mathcal{T}}}e^{-\frac{(t-x)^2}{2\sigma^2}}=e^{-\frac{x}{\mathcal{T}}}e^{\frac{\sigma^2}{2\mathcal{T}^2}}e^{-(\frac{t}{\sigma\sqrt{2}}-\frac{x-\sigma^2/\mathcal{T}}{\sigma\sqrt{2}})^2} \tag{18}$$

then

$$\begin{aligned}I_{\mathcal{T}}^{\sigma}(x)&=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x}{\mathcal{T}}}e^{\frac{\sigma^2}{2\mathcal{T}^2}}\int_0^{\infty}e^{-(\frac{t}{\sigma\sqrt{2}}-\frac{x-\sigma^2/\mathcal{T}}{\sigma\sqrt{2}})^2}dt\\&=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x}{\mathcal{T}}}e^{\frac{\sigma^2}{2\mathcal{T}^2}}\sigma\sqrt{2}\int_0^{\infty}e^{-(\frac{t}{\sigma\sqrt{2}}-\frac{x-\sigma^2/\mathcal{T}}{\sigma\sqrt{2}})^2}\frac{dt}{\sigma\sqrt{2}}\end{aligned}$$

Use substitution

$$\begin{aligned}z&=\frac{t}{\sigma\sqrt{2}} & u&=z-\frac{x-\sigma^2/\mathcal{T}}{\sigma\sqrt{2}}\\ z&=0 & u&=-\frac{x-\sigma^2/\mathcal{T}}{\sigma\sqrt{2}}\end{aligned}$$

then

$$\begin{aligned}I_{\mathcal{T}}^{\sigma}(x)&=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x}{\mathcal{T}}}e^{\frac{\sigma^2}{2\mathcal{T}^2}}\sigma\sqrt{2}\int_{-\frac{x-\sigma^2/\mathcal{T}}{\sigma\sqrt{2}}}^{\infty}e^{-u^2}du\\&=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x}{\mathcal{T}}}e^{\frac{\sigma^2}{2\mathcal{T}^2}}\sigma\sqrt{2}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{2}{\sqrt{\pi}}\int_{-\frac{x-\sigma^2/\mathcal{T}}{\sigma\sqrt{2}}}^{\infty}e^{-u^2}du\\&=\frac{1}{2}e^{-\frac{x}{\mathcal{T}}}e^{\frac{\sigma^2}{2\mathcal{T}^2}}\cdot\frac{2}{\sqrt{\pi}}\int_{-\frac{x-\sigma^2/\mathcal{T}}{\sigma\sqrt{2}}}^{\infty}e^{-u^2}du\end{aligned}$$

A complementary error function $erfc(x)$ is defined as $erfc(x)=1-erf(x)$, where $erf(x)$ is error function

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du \tag{19}$$

The complementary error function can be also written in form

$$erfc(x)=\frac{2}{\sqrt{\pi}}\int_x^{\infty}e^{-u^2}du \tag{20}$$

See plots on Fig. 3 and Fig. 4. Note that $erfc(x)$ is always positive and $erfc(-\infty)=2$.

In terms of $erfc(x)$

$$I_{\mathcal{T}}^{\sigma}(x)=\frac{1}{2}e^{-\frac{x}{\mathcal{T}}}e^{\frac{\sigma^2}{2\mathcal{T}^2}}erfc(-\frac{x-\sigma^2/\mathcal{T}}{\sigma\sqrt{2}}) \tag{21}$$

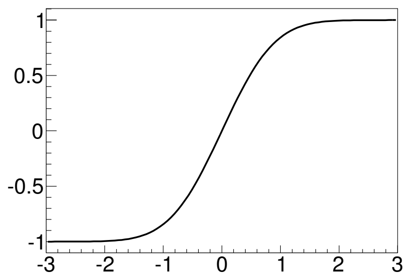

Figure 3: Error function $erf(x)$

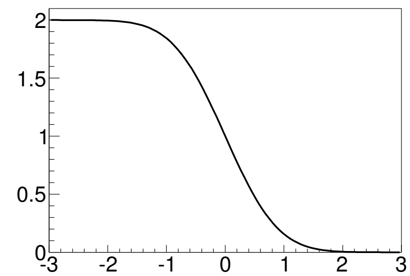

Figure 4: Complementary error function $erfc(x)$

Now the Eq(17) becomes

$$I^{\sigma}_{T\tau}(x) = \frac{\tau}{T-\tau}\left(\frac{1}{2}e^{-\frac{x}{T}}e^{\frac{\sigma^2}{2T^2}}erfc(-\frac{x-\sigma^2/T}{\sigma\sqrt{2}}) - \frac{1}{2}e^{-\frac{x}{\tau}}e^{\frac{\sigma^2}{2\tau^2}}erfc(-\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}})\right) \tag{22}$$

This expression is need to be plugged in Eq(14).

Fig.5 shows an example of pulse function convoluted with scintillator decay and resolution functions.

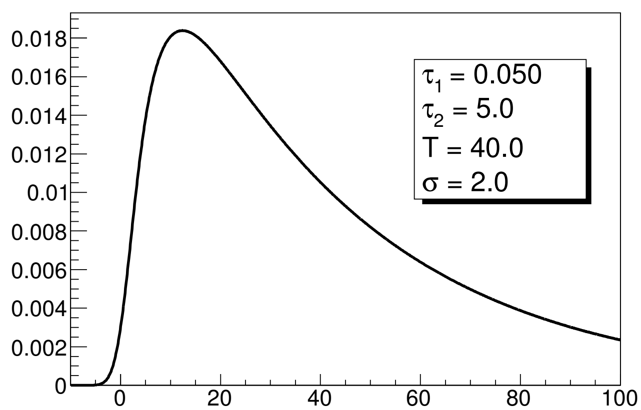

Figure 5: Convolution with scintillator decay and resolution functions.

4.1 Smearing in special cases

Case 1 $\boxed{\tau_1 \to 0}$ $I_{T\tau} \to 0 \Rightarrow I^{\sigma}_{T\tau} = 0$

Assume that $\tau > \epsilon$ now.

Case 2 $\boxed{T \to 0}$ $I_{T\tau} = e^{-x/\tau} \Rightarrow I^{\sigma}_{T\tau} = \frac{1}{2}e^{-\frac{x}{\tau}}e^{\frac{\sigma^2}{2\tau^2}}erfc(-\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}})$
For numerical computation we should to consider case when σ is small. We assume that $\sigma > 0$ and $x > 0$. Consider argument of the $erfc$

$$-\frac{x - \sigma^2/\tau}{\sigma\sqrt{2}}$$

If $\sigma \ll x - \sigma^2/\tau$ the $erfc = 2$ otherwise we can safely divide by σ.

Assume that both $\tau > \epsilon$ and $T > \epsilon$ now.

Case 3 $\boxed{\tau \to T}$ $I_{T\to\tau}(x) = \frac{x}{\tau}e^{-x/\tau} \Rightarrow I^{\sigma}_{T\to\tau} = \frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty \frac{t}{\tau}e^{-\frac{t}{\tau}}e^{-\frac{(t-x)^2}{2\sigma^2}}dt$
Calculate this integral.

$$\begin{aligned}
I^{\sigma}_{T\to\tau} &= \frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty \frac{t}{\tau}e^{-\frac{t}{\tau}}e^{-\frac{(t-x)^2}{2\sigma^2}}dt = \\
&= \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x}{\tau}}e^{\frac{\sigma^2}{2\tau^2}}\frac{1}{\tau}(\sigma\sqrt{2})^2\int_0^\infty \frac{t}{\sigma\sqrt{2}}\frac{dt}{\sigma\sqrt{2}}e^{-(\frac{t}{\sigma\sqrt{2}}-\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}})} \\
&= \left|\begin{array}{l}
z = \dfrac{t}{\sigma\sqrt{2}} - \dfrac{x-\sigma^2/\tau}{\sigma\sqrt{2}} \\
\dfrac{t}{\sigma\sqrt{2}} = z + \dfrac{x-\sigma^2/\tau}{\sigma\sqrt{2}} \\
dz = \dfrac{dt}{\sigma\sqrt{2}} \\
t = 0 \quad z = -\dfrac{x-\sigma^2/\tau}{\sigma\sqrt{2}} \\
t = \infty \quad z = \infty
\end{array}\right| \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{x}{\tau}}e^{\frac{\sigma^2}{2\tau^2}}\frac{2\sigma}{\tau}\int_{-\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}}}^\infty \Big(z + \frac{x-\sigma^2/\tau}{\sigma\sqrt{2}}\Big)dze^{-z^2} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{x}{\tau}}e^{\frac{\sigma^2}{2\tau^2}}\frac{2\sigma}{\tau}\Big[\int_{-\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}}}^\infty zdze^{-z^2} + \frac{x-\sigma^2/\tau}{\sigma\sqrt{2}}\int_{-\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}}}^\infty dze^{-z^2}\Big] \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{x}{\tau}}e^{\frac{\sigma^2}{2\tau^2}}\frac{2\sigma}{\tau}\Big[\frac{1}{2}\int_{-\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}}}^\infty dz^2e^{-z^2} + \frac{x-\sigma^2/\tau}{\sigma\sqrt{2}}\frac{\sqrt{\pi}}{2}\frac{2}{\sqrt{\pi}}\int_{-\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}}}^\infty dze^{-z^2}\Big] \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{x}{\tau}}e^{\frac{\sigma^2}{2\tau^2}}\frac{2\sigma}{\tau}\Big[\frac{1}{2}e^{-(\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}})^2} + \frac{x-\sigma^2/\tau}{\sigma\sqrt{2}}\frac{\sqrt{\pi}}{2}erfc(-\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}})\Big] \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{x}{\tau}}e^{\frac{\sigma^2}{2\tau^2}}\frac{\sigma}{\tau}\Big[e^{-(\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}})^2} + \sqrt{\pi}\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}}erfc(-\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}})\Big]
\end{aligned}$$

Finally

$$I^{\sigma}_{T\to\tau} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{\tau}} e^{\frac{\sigma^2}{2\tau^2}} \frac{\sigma}{\tau} \Big[e^{-(\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}})^2} + \sqrt{\pi}\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}} erfc(-\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}}) \Big] \quad (23)$$

Move σ in Eq.(23) into parentheses to simplify numerical calculations.

$$I^{\sigma}_{T\to\tau} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{\tau}} e^{\frac{\sigma^2}{2\tau^2}} \frac{1}{\tau} \Big[\sigma e^{-(\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}})^2} + \sqrt{\pi}\frac{x-\sigma^2/\tau}{\sqrt{2}} erfc(-\frac{x-\sigma^2/\tau}{\sigma\sqrt{2}}) \Big] \quad (24)$$

References

[1] Tests of timing properties of silicon photomultipliers, Nuclear Instruments and Methods in Physics Research Section A, Volume 616, Issue 1, 21 April 2010, Pages 38-44, A. Ronzhin, M. Albrow, K. Byrum, M. Demarteau, S. Los, E. May, E. Ramberg, J. Va'vra, A. Zatserklyaniy