Pulse function

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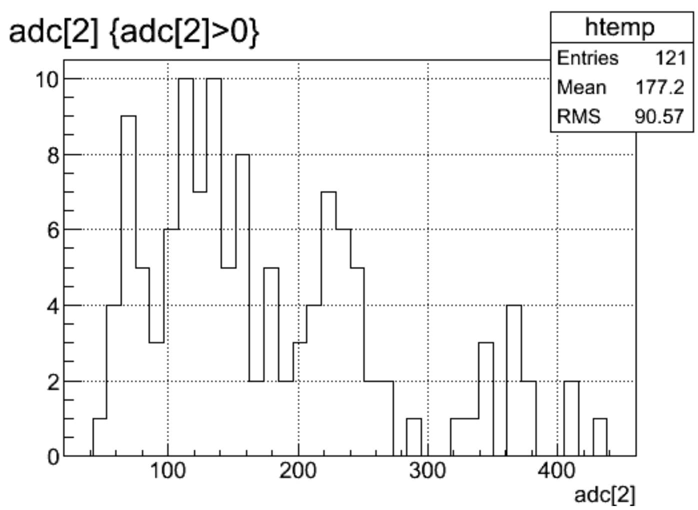
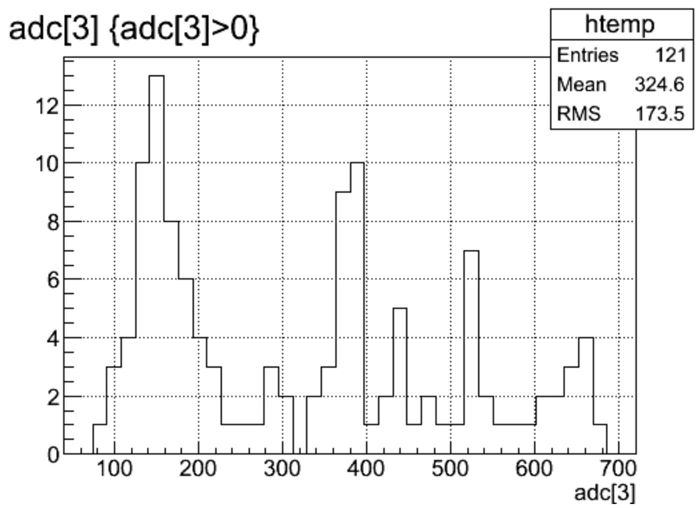
$y(x) = \frac{1}{x}$

$\int_o^\infty dx$



channel 3



1 Derive the function

Define the function as a charging/discharging of capacitor:

$p(t) = (1 - e^{-t/\tau})e^{-t/\tau}$, $t > 0$

Smear this function by convolution with Gaussian with sigma σ.

$y(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} p(t) e^{-\frac{(t-x)^2}{2\sigma^2}} dt$

because $p(t) = 0$ for $t < 0$

$y(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{0}^{\infty} p(t) e^{-\frac{(t-x)^2}{2\sigma^2}} dt$

or

$y(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{0}^{\infty} (1 - e^{-t/\tau}) e^{-t/\tau} e^{-\frac{(t-x)^2}{2\sigma^2}} dt$

$y(x) = \frac{1}{\sqrt{2\pi}\sigma} \left[\int_{0}^{\infty} e^{-t/\tau} e^{-\frac{(t-x)^2}{2\sigma^2}} dt - \int_{0}^{\infty} e^{-t/\tau} e^{-\frac{(t-x)^2}{2\sigma^2}} dt \right]$

Denote

$y(x) = \frac{1}{\sqrt{2\pi}\sigma} (I(x,\tau) - I(x,\tau/2))$

where

$I(x,\tau) = I = \int_{0}^{\infty} e^{-t/\tau} e^{-\frac{(t-x)^2}{2\sigma^2}} dt$

Rewriting and expressing the complete square we will have

$I = \sigma\sqrt{2} e^{-(\frac{x}{\tau} - \frac{\sigma^2}{2\tau^2})} \int_{0}^{\infty} \frac{dt}{\sigma\sqrt{2}} e^{-(\frac{t}{\sigma\sqrt{2}} - \frac{1}{\sigma\sqrt{2}}(x - \frac{\sigma^2}{\tau}))^2}$

$= \sigma\sqrt{2}e^{-(\frac{x}{\tau}-\frac{\sigma^2}{2\tau^2})}\int_{-\frac{1}{\sigma\sqrt{2}}(x-\frac{\sigma^2}{\tau})}^{\infty} e^{-z^2}dz$

$= \sigma\sqrt{2}e^{-(\frac{x}{\tau}-\frac{\sigma^2}{2\tau^2})}(\int_0^{\frac{1}{\sigma\sqrt{2}}(x-\frac{\sigma^2}{\tau})} e^{-z^2}dz + \int_0^{\infty} e^{-z^2}dz)$

$= \sigma\sqrt{2}e^{-(\frac{x}{\tau}-\frac{\sigma^2}{2\tau^2})}(\frac{\sqrt{\pi}}{2}\frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{\sigma\sqrt{2}}(x-\frac{\sigma^2}{\tau})} e^{-z^2}dz + \frac{\sqrt{\pi}}{2})$

$= \sigma\sqrt{2}e^{-(\frac{x}{\tau}-\frac{\sigma^2}{2\tau^2})}\frac{\sqrt{\pi}}{2}(1 + \frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{\sigma\sqrt{2}}(x-\frac{\sigma^2}{\tau})} e^{-z^2}dz)$

$= \sqrt{\frac{\pi}{2}}\sigma e^{-(\frac{x}{\tau}-\frac{\sigma^2}{2\tau^2})}(1 + erf\frac{x-\frac{\sigma^2}{\tau}}{\sigma\sqrt{2}})$

$= \sqrt{\frac{\pi}{2}}\sigma e^{-(\frac{x}{\tau}-\frac{\sigma^2}{2\tau^2})}(1 - erf\frac{\frac{\sigma^2}{\tau}-x}{\sigma\sqrt{2}})$

Finally,

$I(x,\tau) = \sqrt{\frac{\pi}{2}}\sigma e^{-(\frac{x}{\tau}-\frac{\sigma^2}{2\tau^2})}erfc\frac{\frac{\sigma^2}{\tau}-x}{\sigma\sqrt{2}}$

Correspondingly,

$I(x,\tau/2) = \sqrt{\frac{\pi}{2}}\sigma e^{-(\frac{2x}{\tau}-\frac{2\sigma^2}{\tau^2})}erfc\frac{\frac{2\sigma^2}{\tau}-x}{\sigma\sqrt{2}}$

Now consider different value for rise and discharge time, $\tau_1$and τ_2.

Now

$p(x) = (1 - e^{-\frac{t}{\tau_1}})e^{-\frac{t}{\tau_2}}$

and

$y(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_0^{\infty}(1 - e^{-t/\tau_1})e^{-t/\tau_2}e^{-\frac{(t-x)^2}{2\sigma^2}}dt$

$y(x) = \frac{1}{\sqrt{2\pi}\sigma}\left[\int_0^{\infty} e^{-t/\tau_2}e^{-\frac{(t-x)^2}{2\sigma^2}}dt - \int_0^{\infty} e^{-t(\frac{1}{\tau_1}+\frac{1}{\tau_2})}e^{-\frac{(t-x)^2}{2\sigma^2}}dt\right]$

Define $\tau_{12} = \frac{1}{\tau_1} + \frac{1}{\tau_2}$

Then

$y(x) = \frac{1}{\sqrt{2\pi}\sigma}\left[\int_0^{\infty} e^{-t/\tau_2}e^{-\frac{(t-x)^2}{2\sigma^2}}dt - \int_0^{\infty} e^{-\frac{t}{\tau_{12}}}e^{-\frac{(t-x)^2}{2\sigma^2}}dt\right]$

or

$y(x) = \frac{1}{\sqrt{2\pi}\sigma}(I(x,\tau_2) - I(x,\tau_{12}))$

After some optimization

$y(x) = \frac{1}{\sqrt{2\pi}\sigma}\sqrt{\pi/2}\sigma[I'(x,\tau_2) - I'(x,\tau_{12})]$

finally

$y(x) = \frac{1}{2}[I'(x,\tau_2) - I'(x,\tau_{12})]$

where

$I'(x,\tau) = e^{-(\frac{x}{\tau}-\frac{\sigma^2}{2\tau^2})} erfc \frac{\frac{\sigma^2}{\tau}-x}{\sigma\sqrt{2}}$

Example.

2 Convolution with scintillator decay

Lets represent scintillator decay function as

$s(t) = \frac{1}{T}e^{-t/T}$, $t \geq 0$

Pulse function $p(t) = A(1 - e^{-t/\tau_1})e^{-t/\tau_2}$

Convolution of the pulse function with scintillator decay.

Contribution to time moment x from pulse originated in t ($t \leq x$) with weight $s(t)$.

$p(x-t)s(t)$

and

$P(x) = \int_0^x p(x-t)s(t)dt$

$P(x) = \frac{A}{T}\int_0^x (1 - e^{-\frac{x-t}{\tau_1}})e^{-\frac{x-t}{\tau_2}} e^{-t/T}dt$, $x \geq t$

$= \frac{A}{T}(I_1^T - I_2^T)$

where

$I_1^T = \int_0^x e^{-\frac{x-t}{\tau_2}} e^{-t/T}dt$

$I_2^T = \int_0^x e^{-(\frac{1}{\tau_1}+\frac{1}{\tau_2})(x-t)} e^{-t/T}dt$

Define

$I^T(x,\tau,T) = \int_0^x e^{-x/\tau}e^{t/\tau}e^{-t/T}dt$

$= e^{-x/\tau}\int_0^x e^{(\frac{1}{\tau}-\frac{1}{T})t}dt$

$= e^{-x/\tau}\frac{1}{\frac{1}{\tau}-\frac{1}{T}}\int_0^{(\frac{1}{\tau}-\frac{1}{T})x} e^z dz$

$= e^{-x/\tau}\frac{1}{\frac{1}{\tau}-\frac{1}{T}}(e^{(\frac{1}{\tau}-\frac{1}{T})x} - 1)$

Note, that $\frac{1}{\frac{1}{\tau}-\frac{1}{T}}(e^{(\frac{1}{\tau}-\frac{1}{T})x} - 1) \to x$ as $\frac{1}{\tau} - \frac{1}{T} \to 0$

Then, in terms of $I^T(x,\tau,T)$

$I_1^T = I^T(x,\tau_2,T)$, NB: τ_2

$I_2^T = I^T(x,\tau_{12},T)$, where $\tau_{12} = \frac{1}{\tau_1} + \frac{1}{\tau_2}$

The End.