Numerical Differential Geometry/ Integration and Spectral Methods for 2D Surfaces

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Hecht Lab

HZDR, 2024

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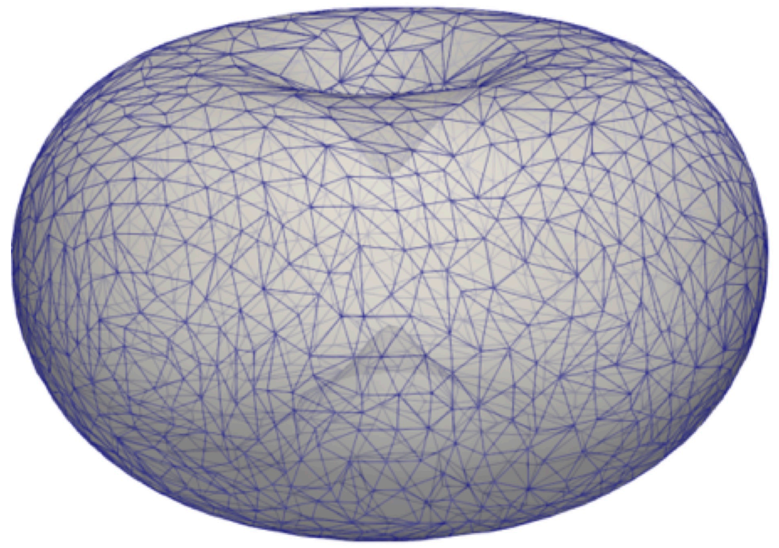
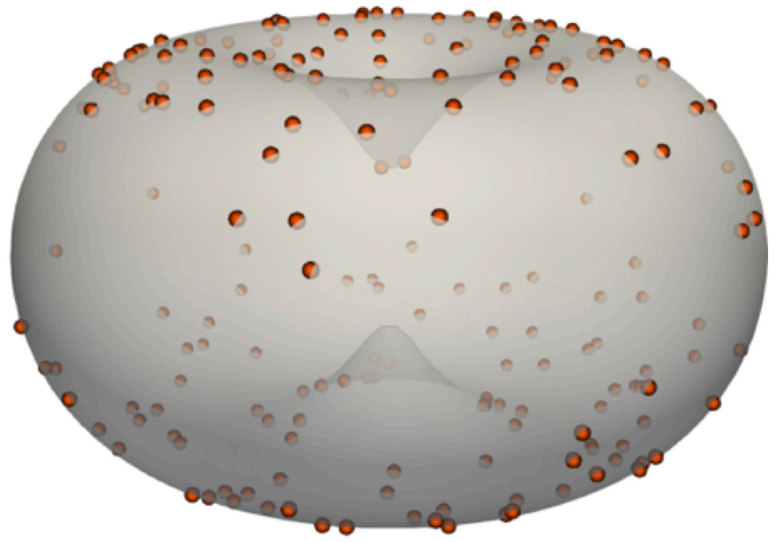
Numerical Differential Geometry

Classic mesh representation



No. of vertices = 1669

Global polynomial approach



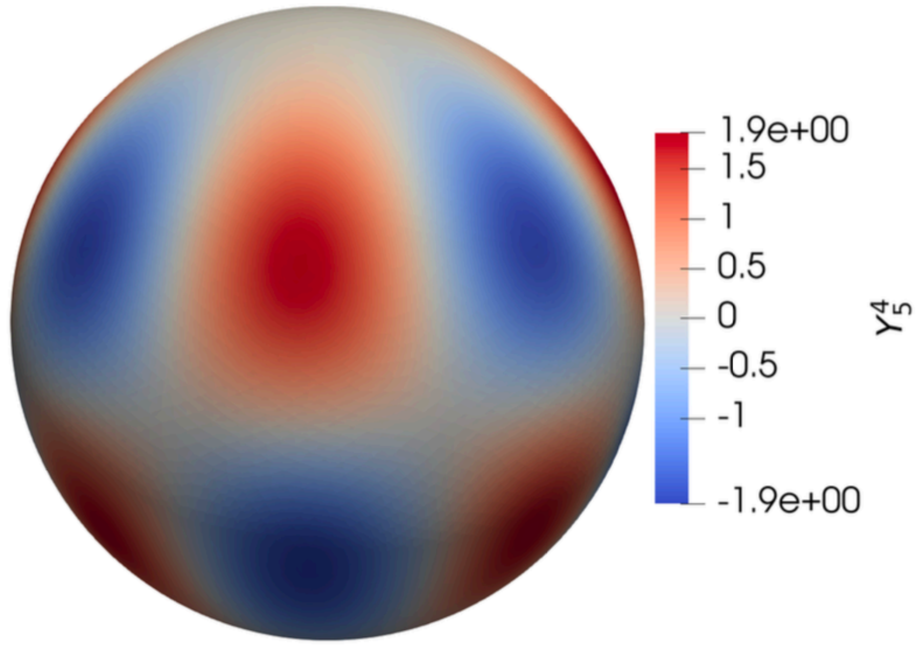
No. of points = 200

Order of approximation errors

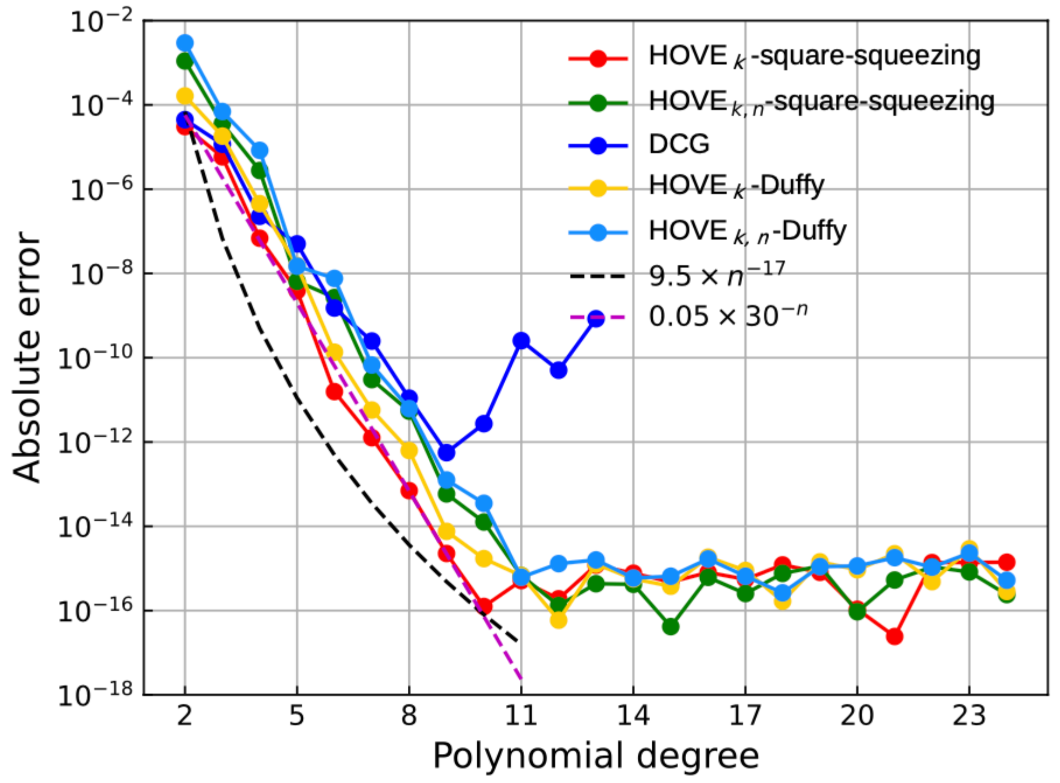
	Classic mesh	Global polynomial
Surface normal	$\sim 10^{-4}$	$\sim 10^{-12}$
Mean curvature	$\sim 10^{-2}$	$\sim 10^{-14}$
Gauss curvature	$\sim 10^{-2}$	$\sim 10^{-10}$

sur∫geopy

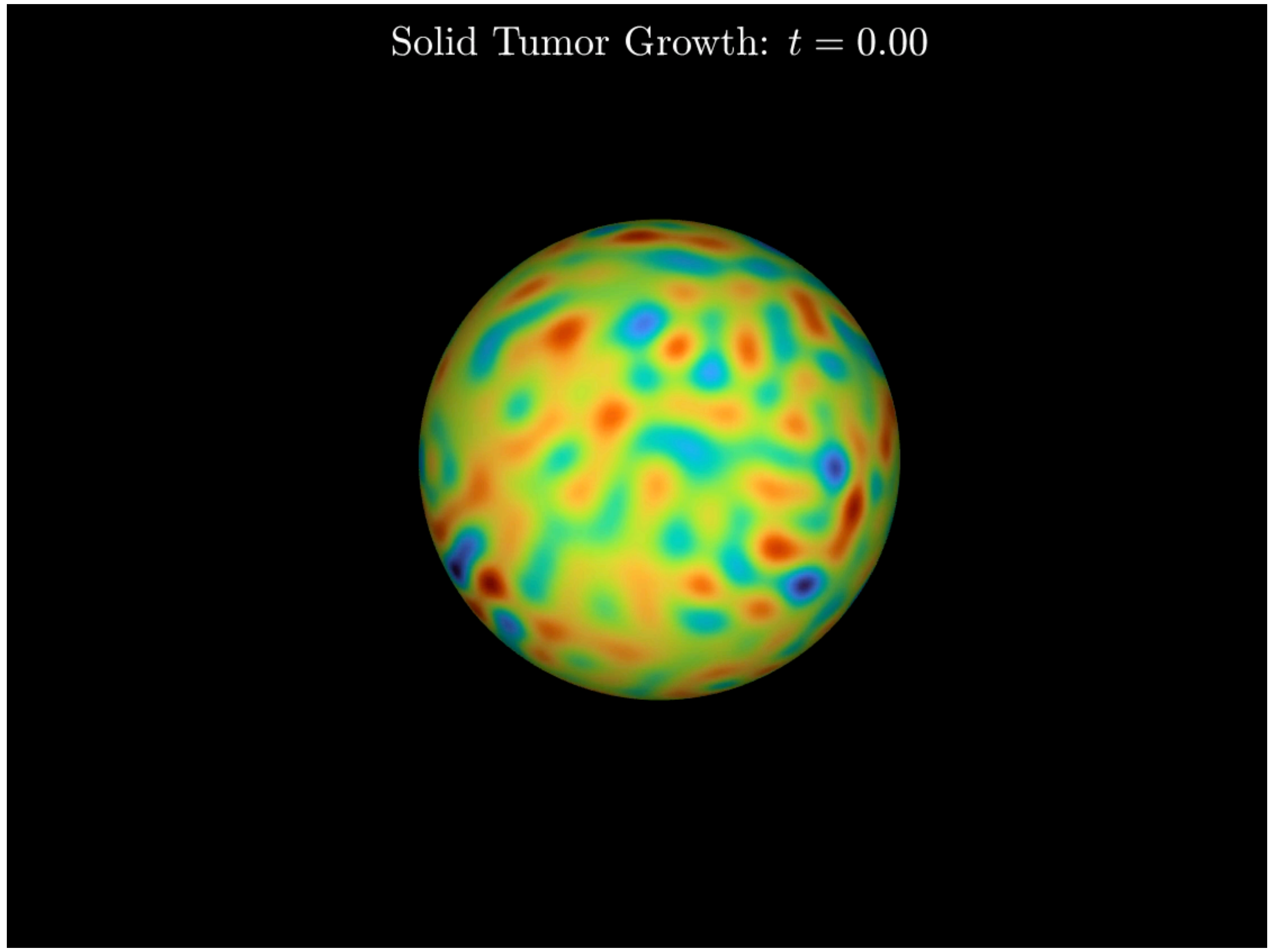
Numerical integration on surfaces



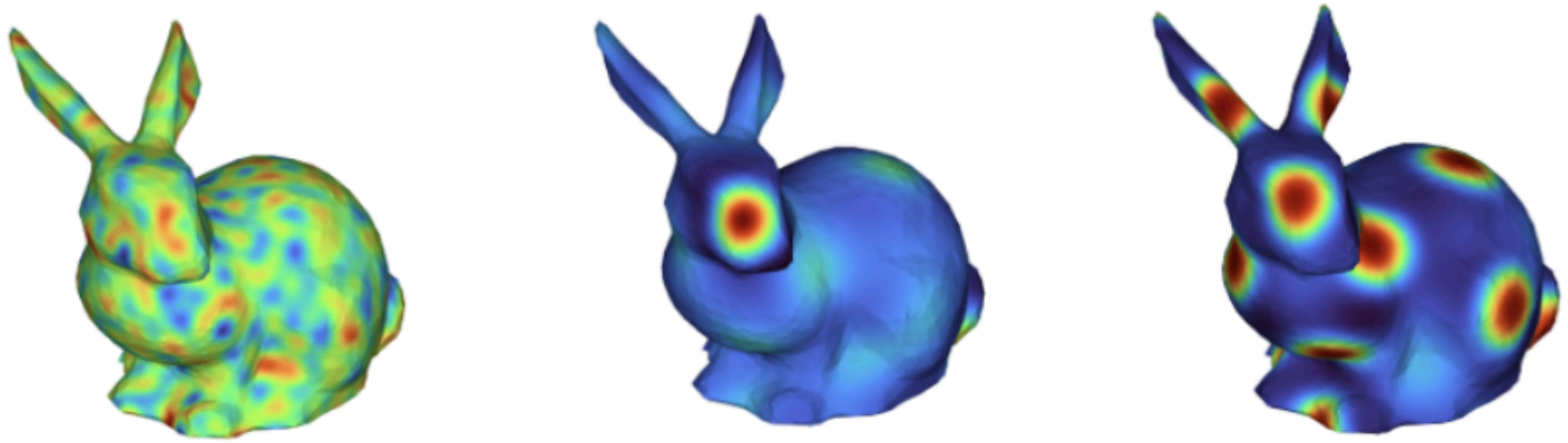
(a) Spherical harmonic Y_5^4



(b) Integration errors



Spectral Methods for 2D Surfaces



Numerical Differential Geometry

Home → SIAM Journal on Scientific Computing → Vol. 45, Iss. 4 (2023) → 10.1137/22M1536510

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Methods and Algorithms for Scientific Computing

Global Polynomial Level Sets for Numerical Differential Geometry of Smooth Closed Surfaces

Authors: Sachin Krishnan Thekke Veettil, Gentian Zavalani, Uwe Hernandez Acosta, Ivo F. Sbalzarini, and Michael Hecht

AUTHORS INFO & AFFILIATIONS

https://doi.org/10.1137/22M1536510

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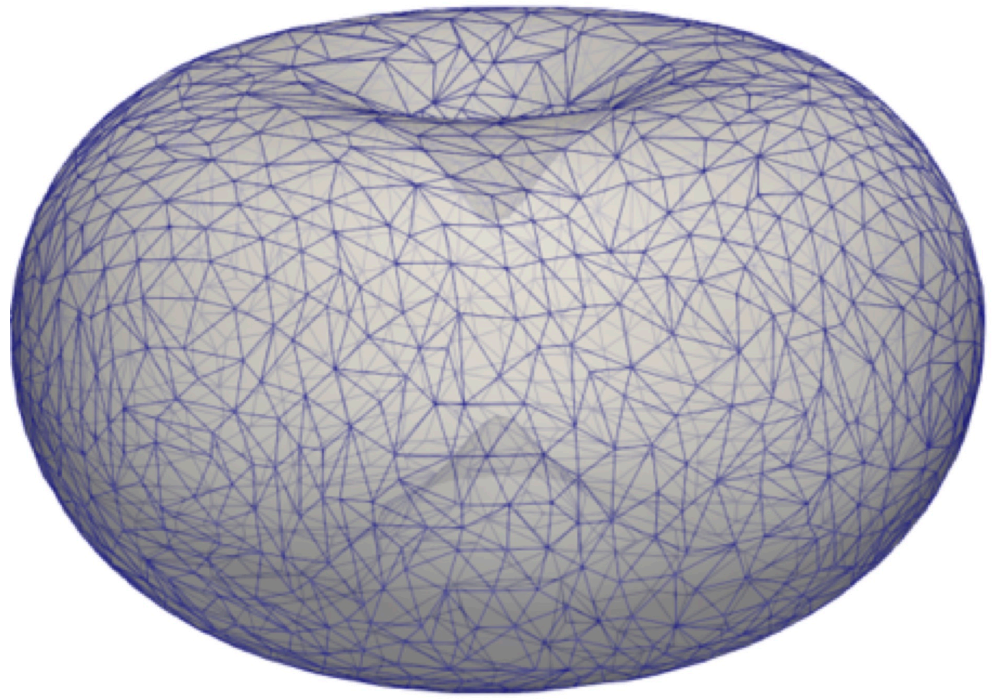
Numerical Differential Geometry

Classic mesh representation



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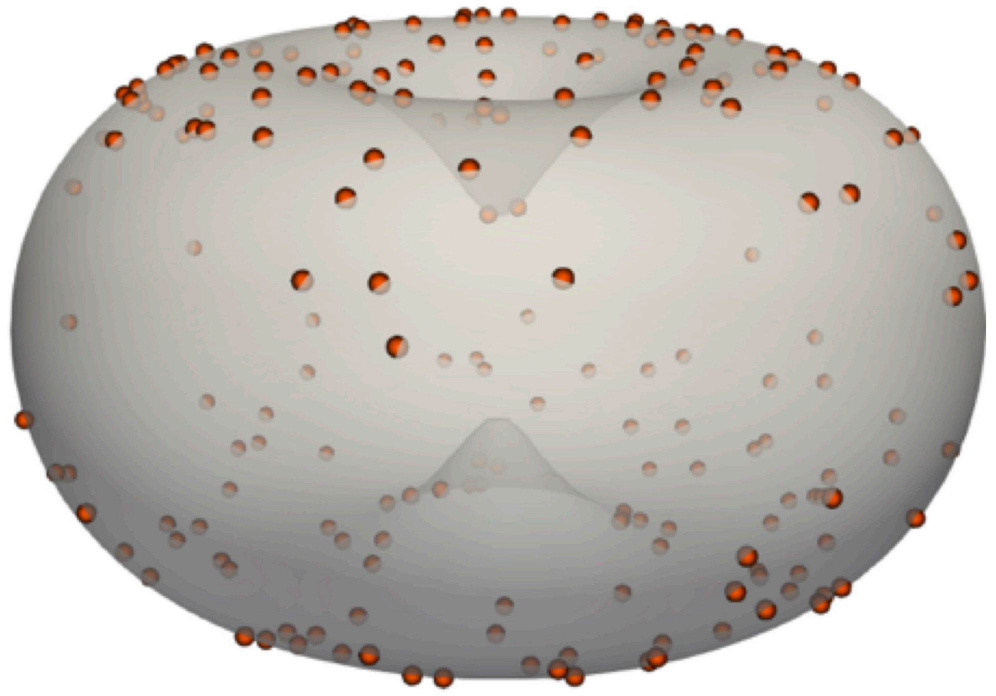

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Mean curvature flow using GPLS
Starting configuration : 200 points on biconcave disc

Global Polynomial Level Sets for Numerical
Differential Geometry of Smooth Curved Surfaces

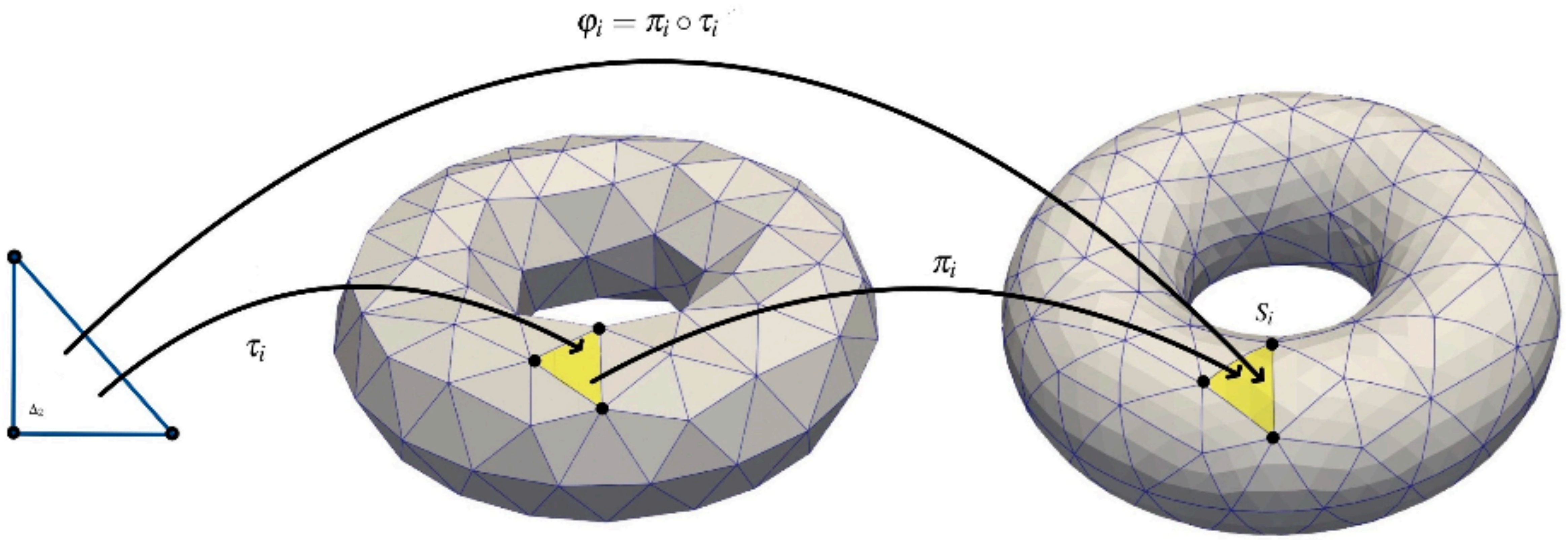
Sachin Krishnan Thekke Veettil, Gentian Zavalani, Uwe Hernandez Acosta,
Ivo F. Sbalzarini, Michael Hecht

High-order numerical integration on regular embedded surfaces

Our task

Let $S \subset \mathbb{R}^{d+1}$ be an oriented, connected, and smooth $d-$dimensional manifold, and $f : S \to \mathbb{R}$ an integrable function:

$$\int_S f(\mathrm{x})\, dS\,. \tag{1}$$



Given a triangulation of S, we can compute the integral of f by

$$\int_S f dS = \sum_{i=1}^{K} \int_{\triangle_2} f(\varphi_i(\mathrm{x}))\sqrt{\det(D\varphi_i^T(\mathrm{x})D\varphi_i(\mathrm{x}))}d\mathrm{x}\,. \tag{2}$$

Interpolation

Triangles

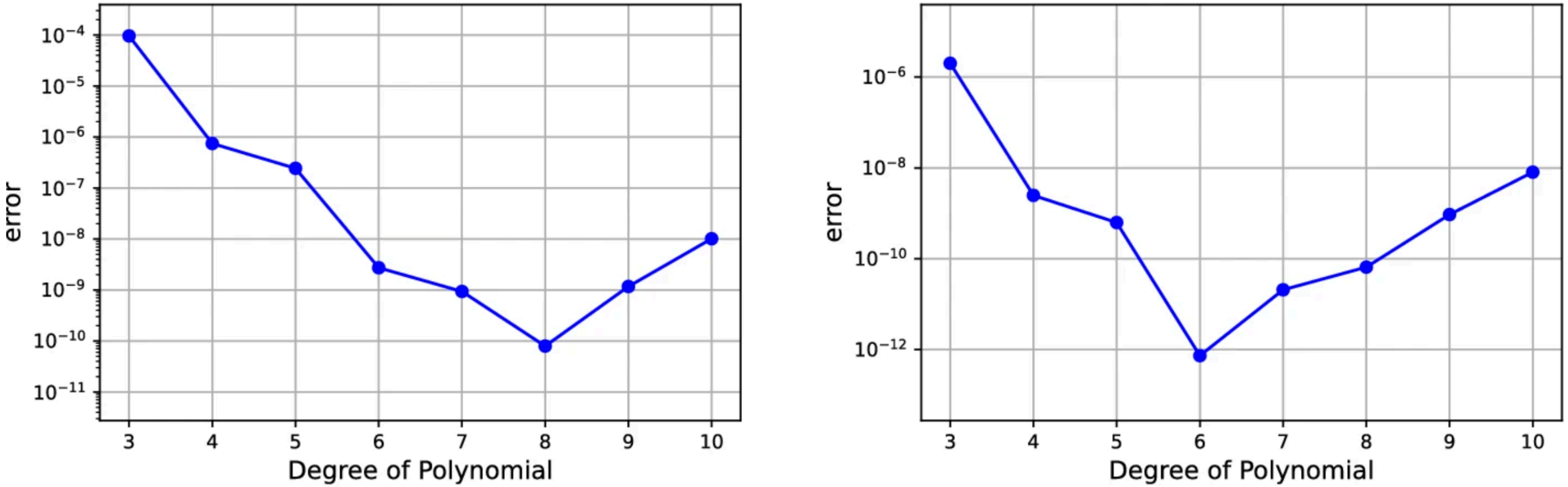
- Stable high-order polynomial interpolation on the triangle is hard to realize.
- Tri mesh (unstructured) are easy to generate.

Squares

- high-order polynomial interpolation on the square is easy.
- Suitable for higher-order schemes.
- The Quad mesh (structured) are hard to generate.

High-order numerical integration becomes unstable !

From: A note on the rate of convergence of integration schemes for closed surfaces



(a) torus with radii $R = 2, r = 1$

(b) ellipsoid with $a = b = 1, c = 0.6$.

Relative errors by integrating the Gaussian curvature over the torus and the ellipsoid using $N_\Delta = 2528,\ N_\Delta = 6152$ respectively

$$\int_S K_{\text{Gauss}}\, dS = 2\pi\chi(S),$$

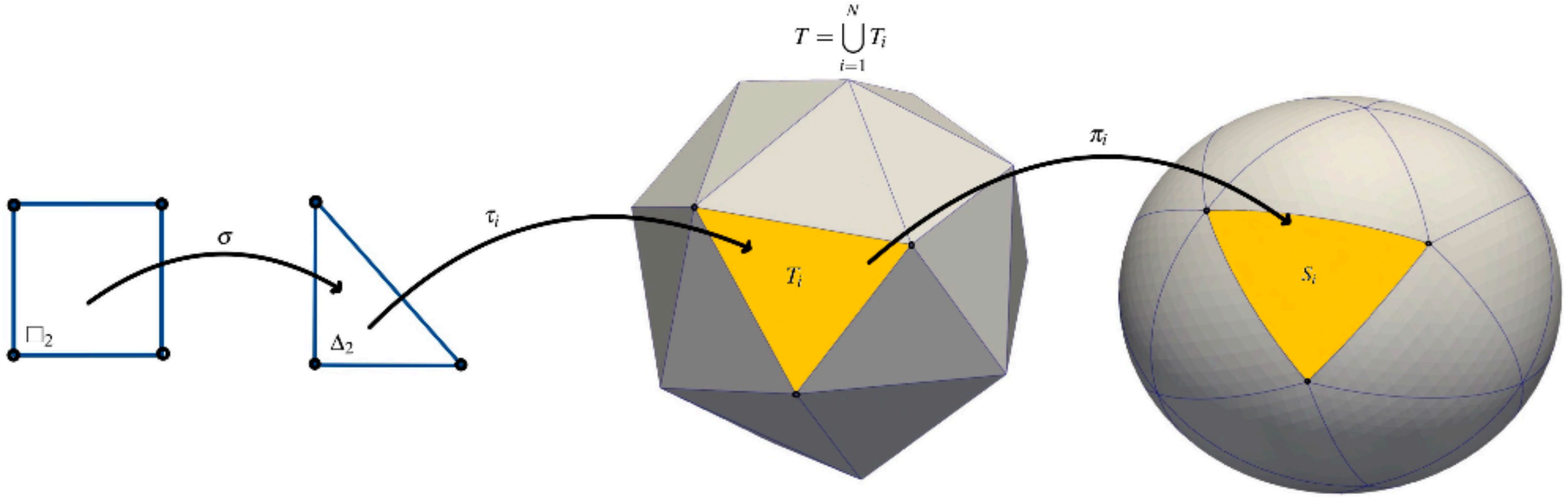
High-order numerical integration on regular embedded surfaces

Surface approximation using polynomial interpolation

- Setting

$$\varphi_i : \square_d \hookrightarrow S_i\,, \quad \varphi_i := \pi_i \circ \tau_i \circ \sigma\,, \quad \mathscr{S} = \{\varphi_i, \square_d, \varphi_i(\square_d)\}_{i=1,\dots,N}\,. \quad (6)$$

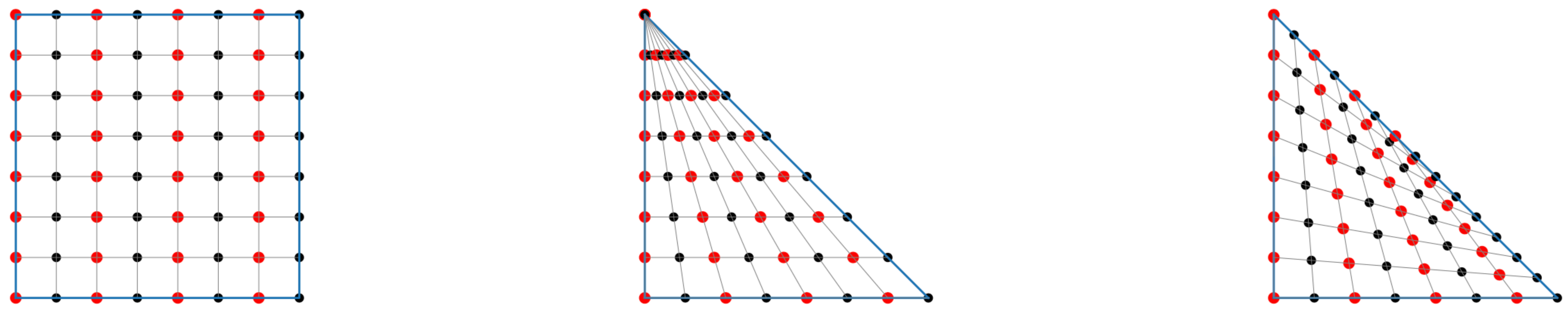
and get $\mathscr{S}$ a *r-regular, quadrilateral surface re-parametrization.*



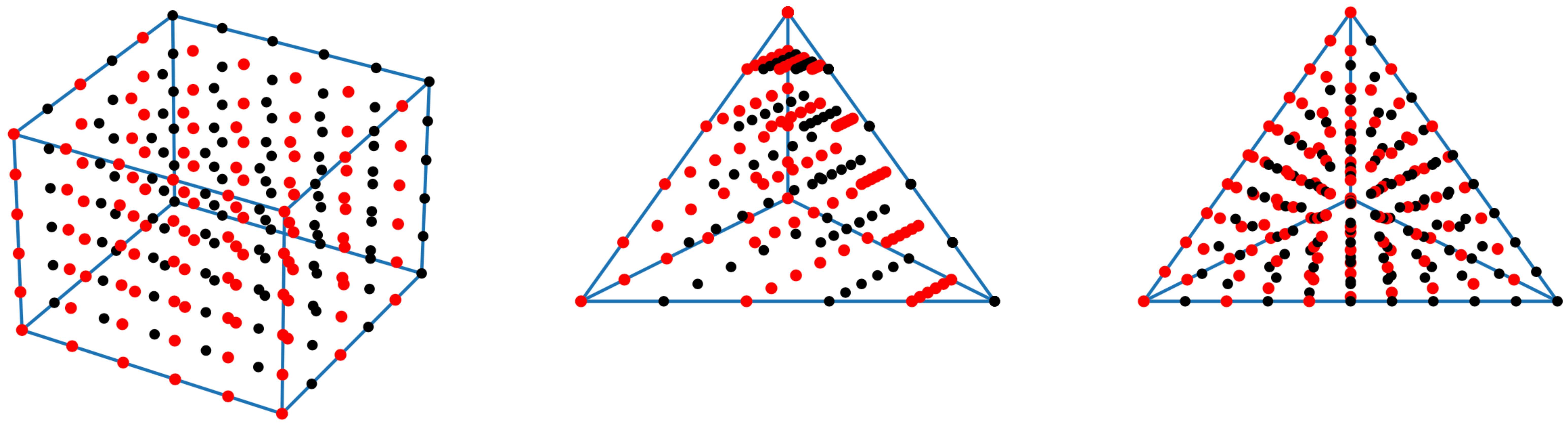
Given (6) we seek computing a n^{th}-order polynomial approximation:

$$Q_{d,n}\varphi_i = \sum_{\alpha \in A_{d,n}} \varphi_i(p_\alpha) L_\alpha\,.$$

Regular Quadriliteral Re-Parameterizations



(a) Standard square (b) Duffy’s transformation (c) Square-Squeezing



(a) standard hypercube (b) Duffy’s transformation (c) cube-squeezing

High-order numerical integration on regular embedded surfaces

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arXiv > math > arXiv:2311.13909

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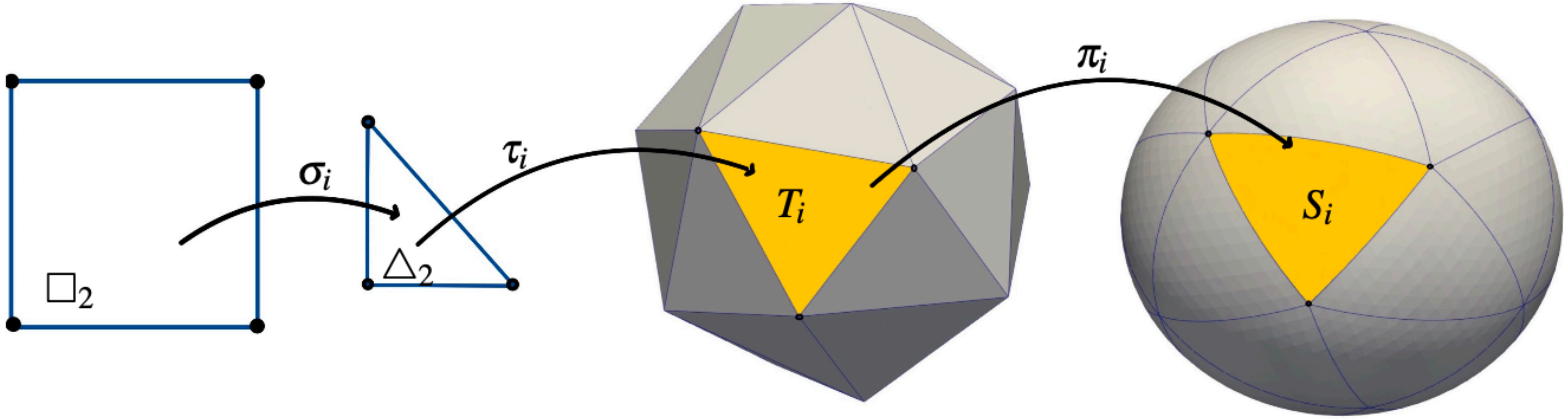
[Submitted on 23 Nov 2023]

High-Order Integration on regular triangulated manifolds reaches Super-Algebraic Approximation Rates through Cubical Re-parameterizations

Gentian Zavalani, Oliver Sander, Michael Hecht



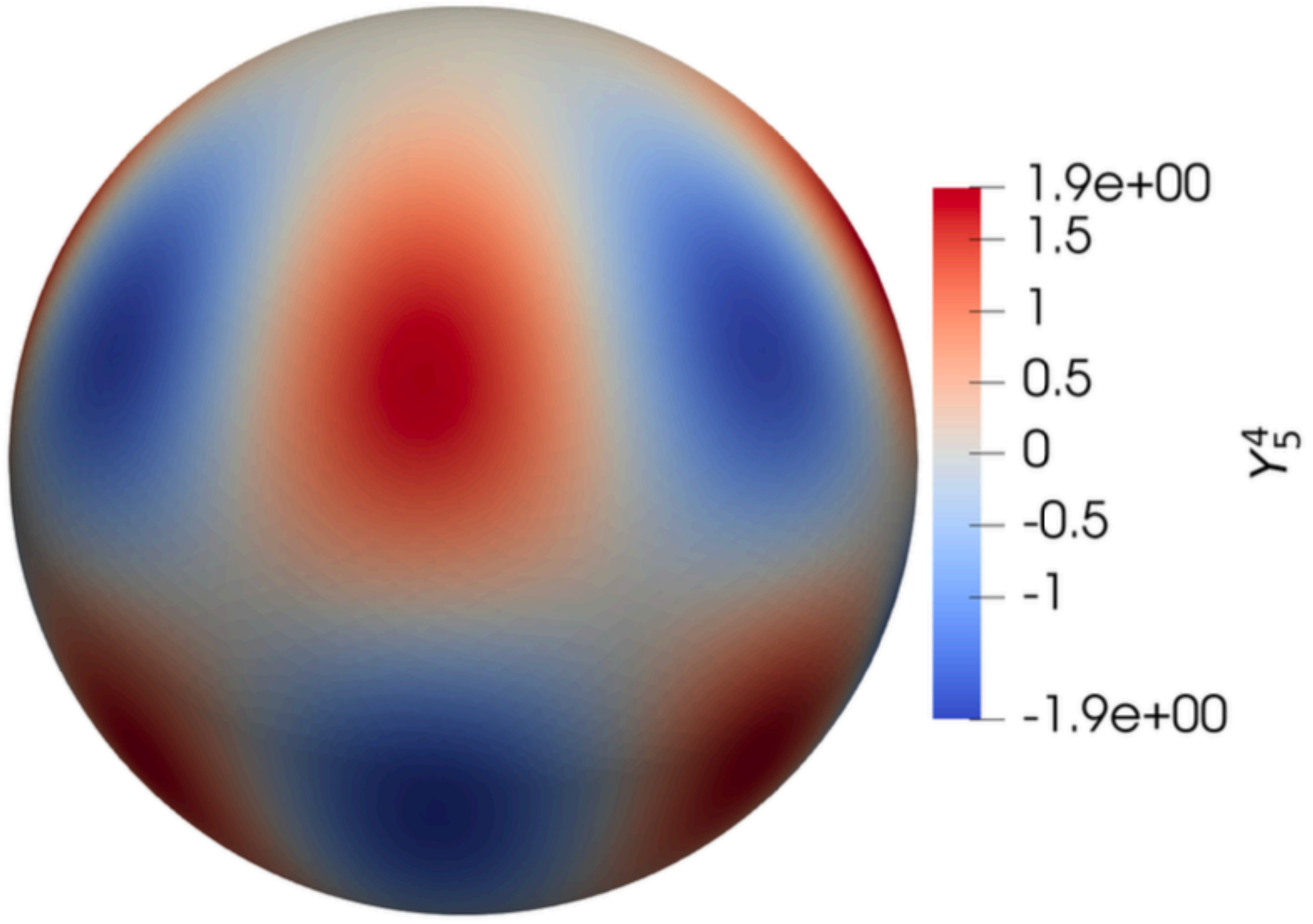
$$\varphi_i : \square_2 \hookrightarrow S_i\,, \quad \varphi_i := \pi_i \circ \tau_i \circ \sigma\,, \quad \mathcal{S} = \{\varphi_i, \square_2\,, \varphi_i(\square_2)\}_{i=1,\ldots,N}$$

$$\int_S f\,dS \approx \sum_{i=1}^{N} \int_{\square_2} Q_{k,f}(x)\sqrt{\det\left(DQ_{\varphi_i}^T DQ_{\varphi_i}\right)}\,dx$$

sur∫geopy

High-order numerical integration on regular embedded surfaces

$$\int_S Y_5^4 \, dS = 0 \,, \quad Y_5^4(x_1, x_2, x_3) = \frac{3\sqrt{385}(x_1^4 - 6x_2^2 x_1^2 + x_2^4)x_3}{16\sqrt{\pi}} \,,$$



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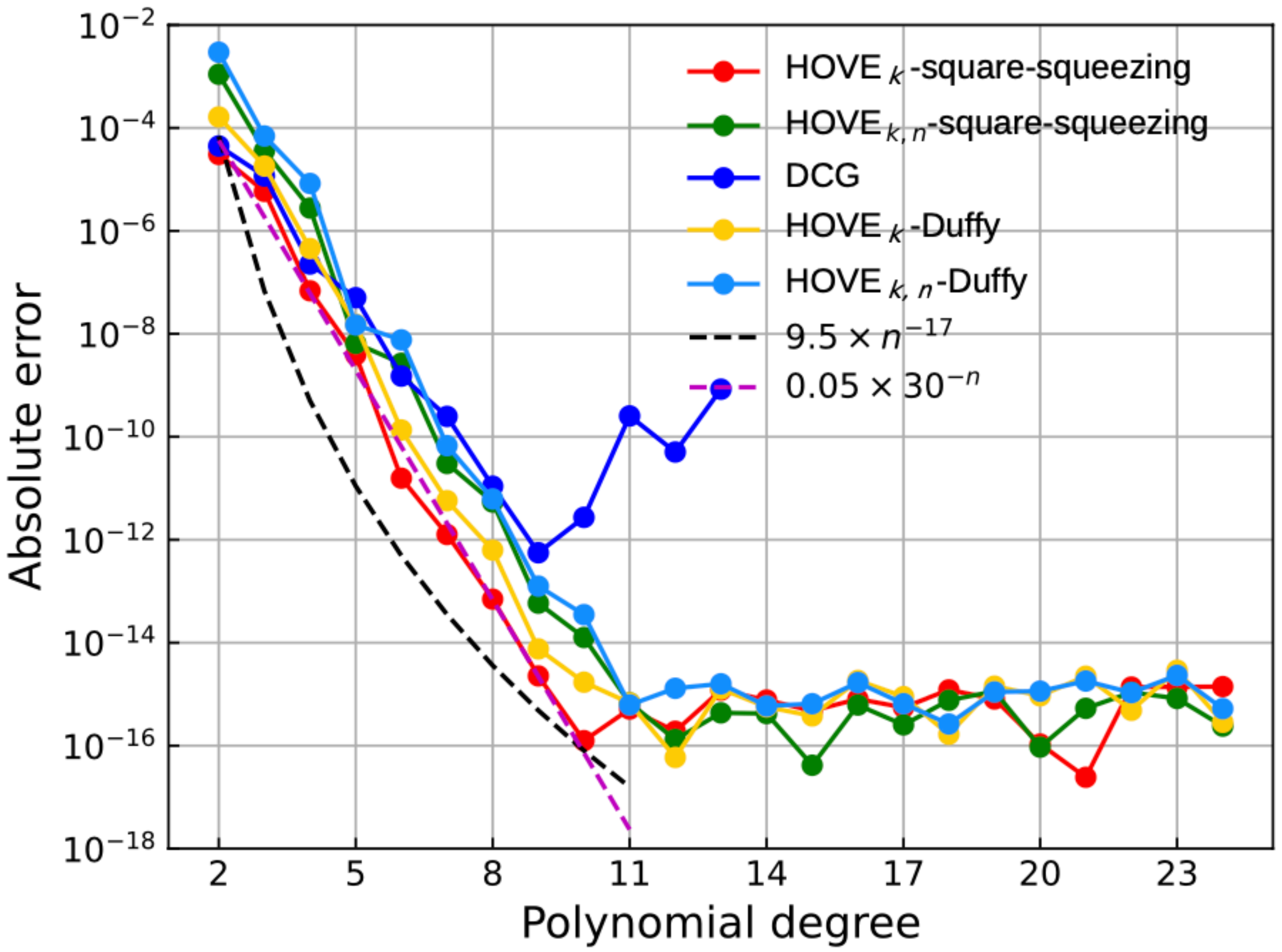
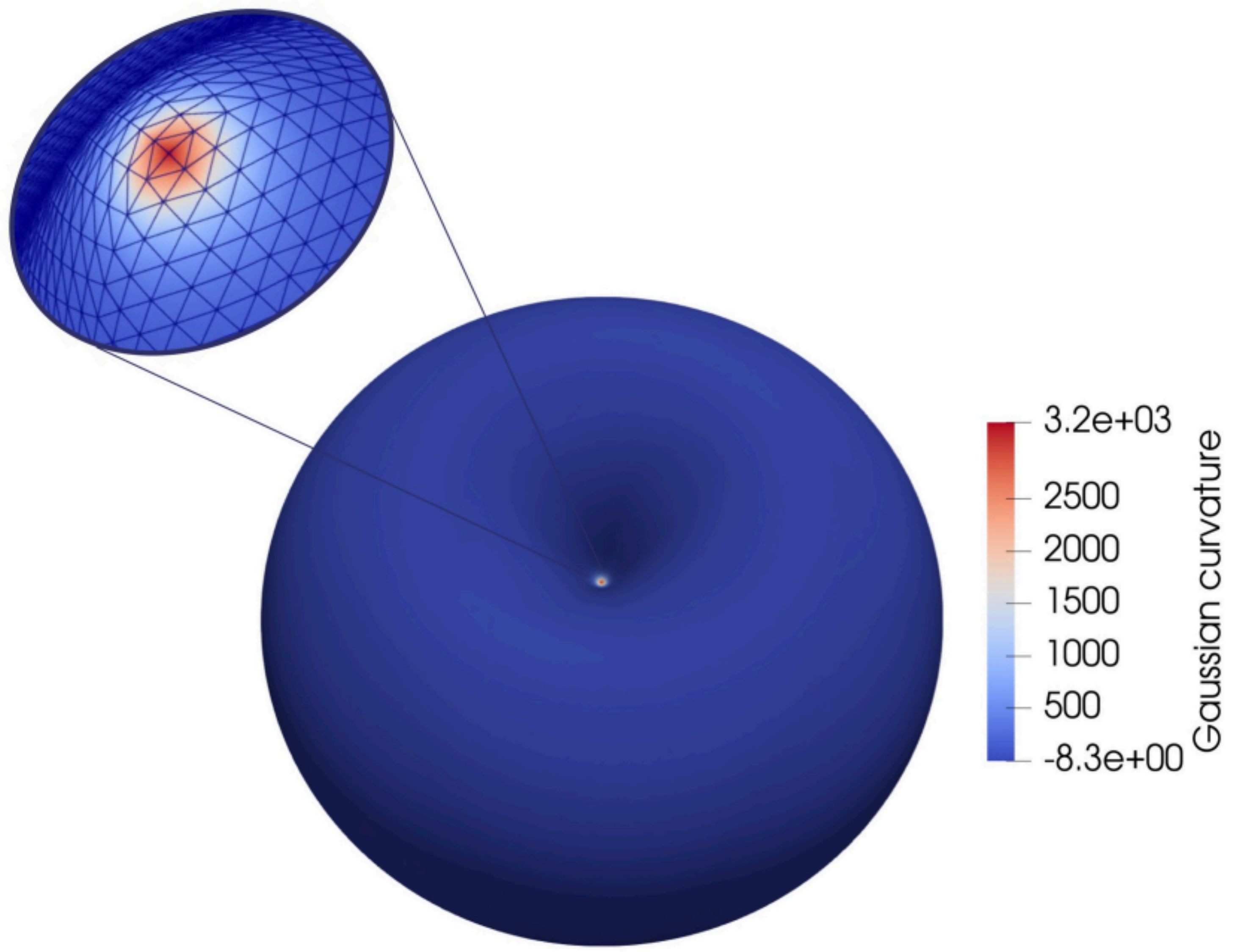


(b) Integration errors

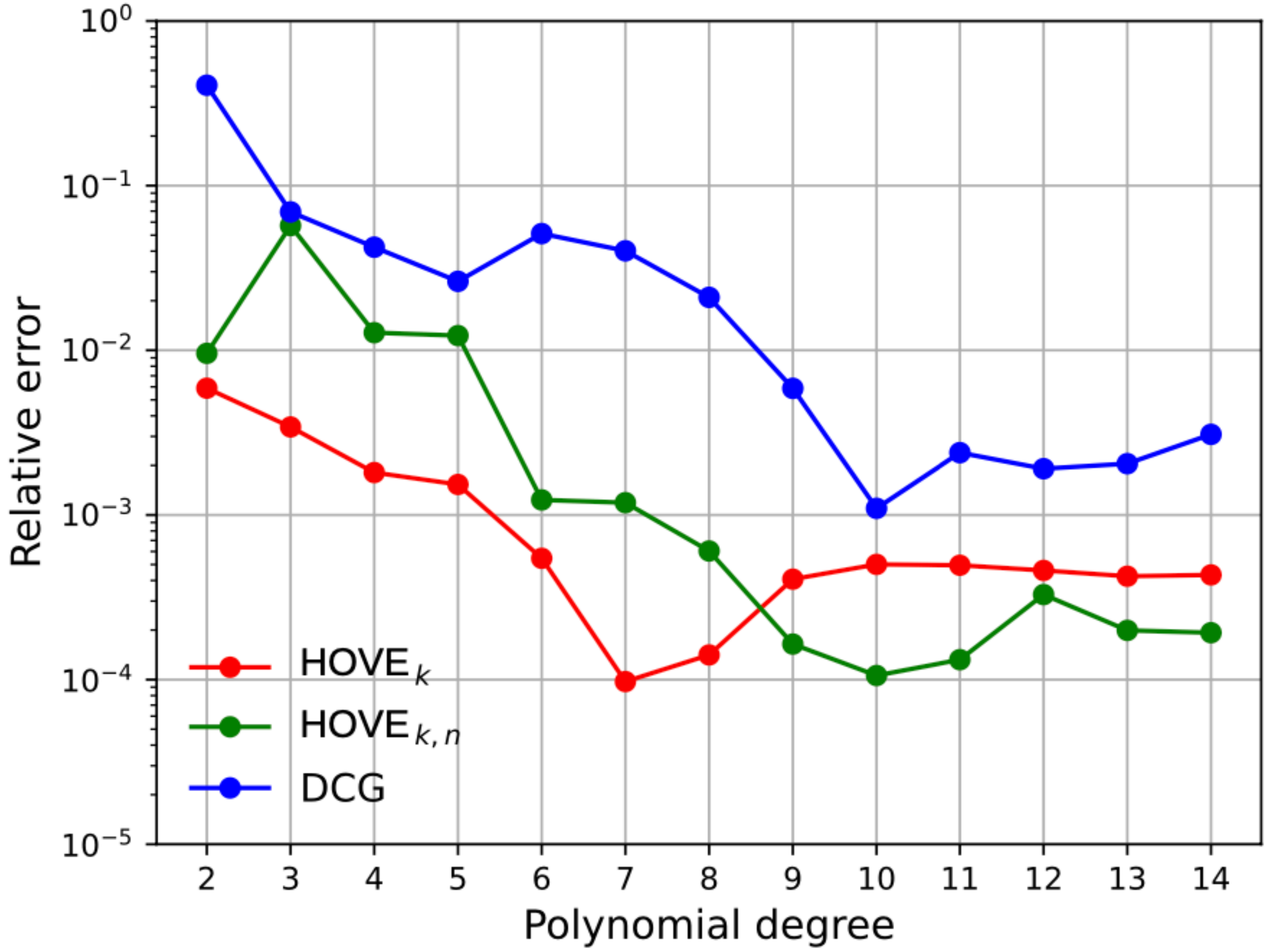
sur∫geopy

High-order numerical integration on regular embedded surfaces

$$\int_S K_{\text{Gauss}}\, dS = 2\pi\chi(S),$$



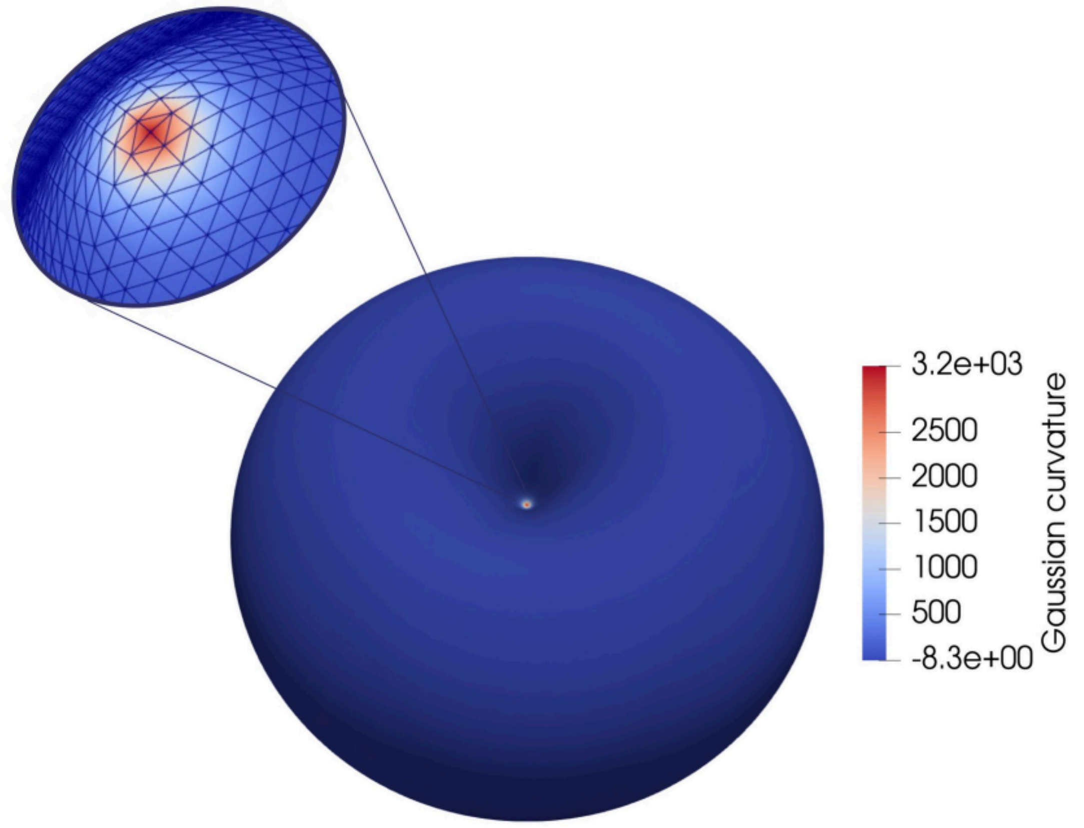
(b) Biconcave disc $c = 0.375$, $d = 0.5$ with 3144 triangles.



$$P_{\text{bicon}}(x, y, z) = (d^2 + x^2 + y^2 + z^2)^3 - 8d^2(y^2 + z^2) - c^4$$

sur∫geopy

High-order numerical integration on regular embedded surfaces
“h vs. p -refinement”



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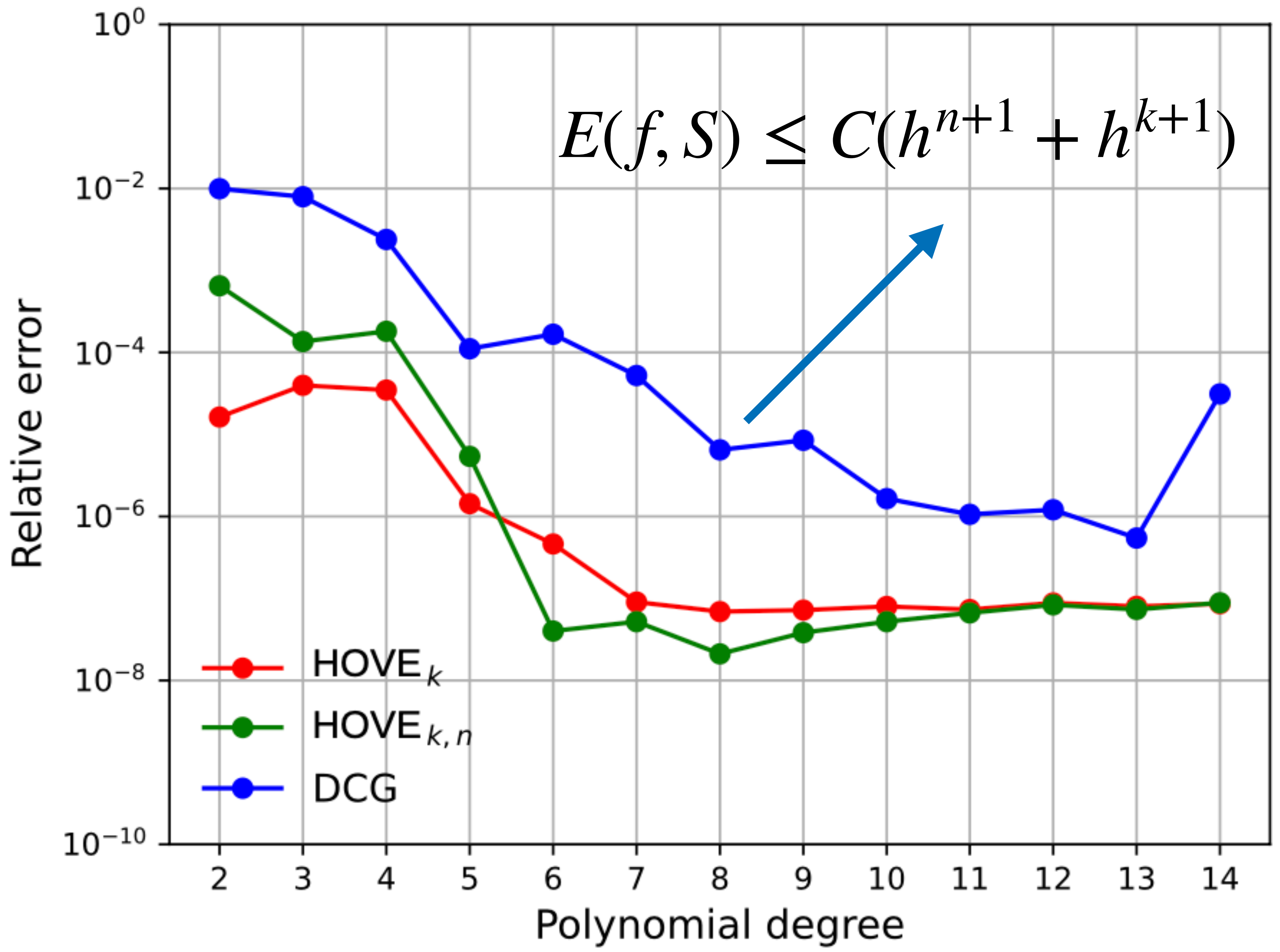
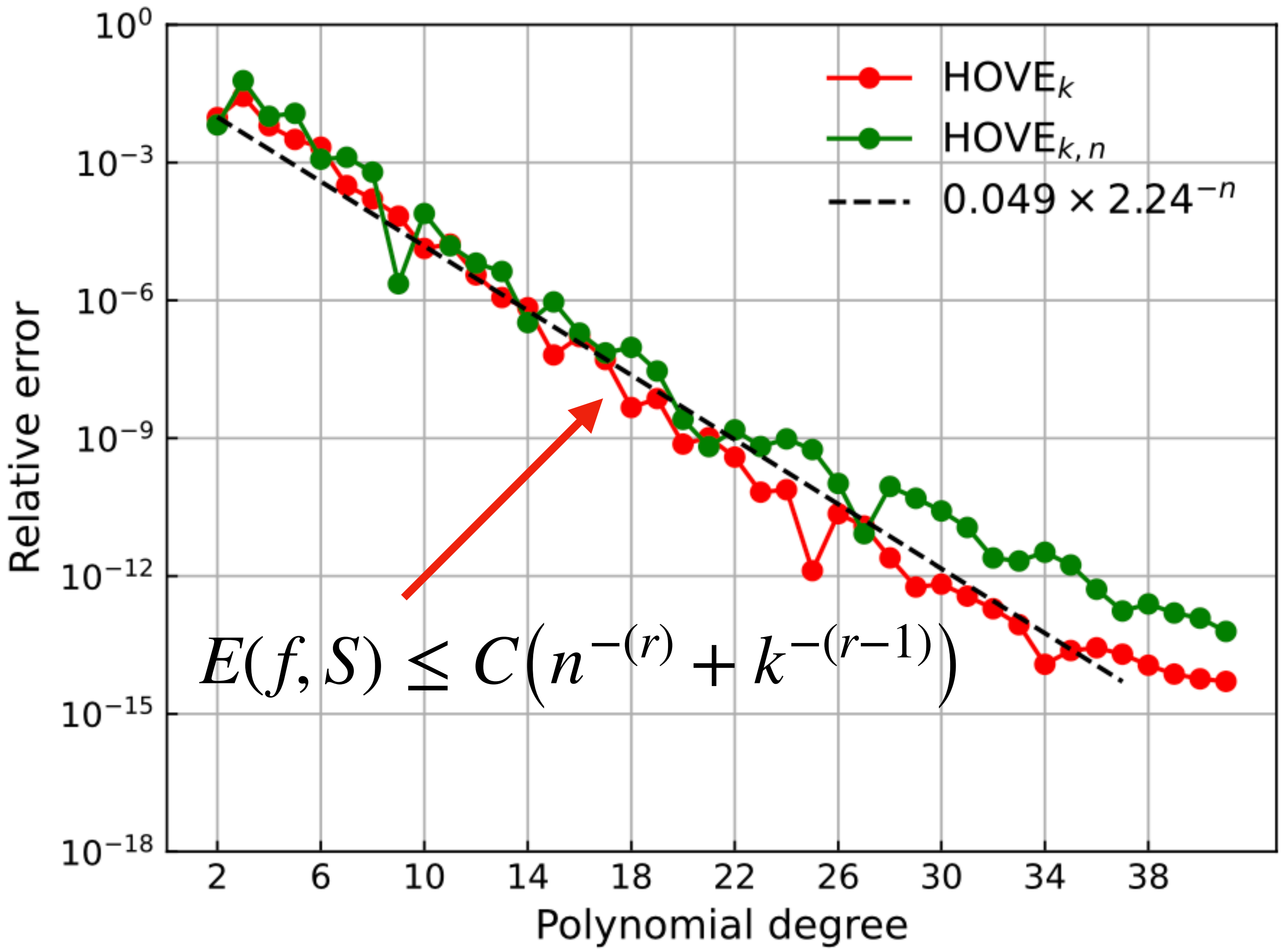


(a) $h-$refinement



(b) $p-$refinement

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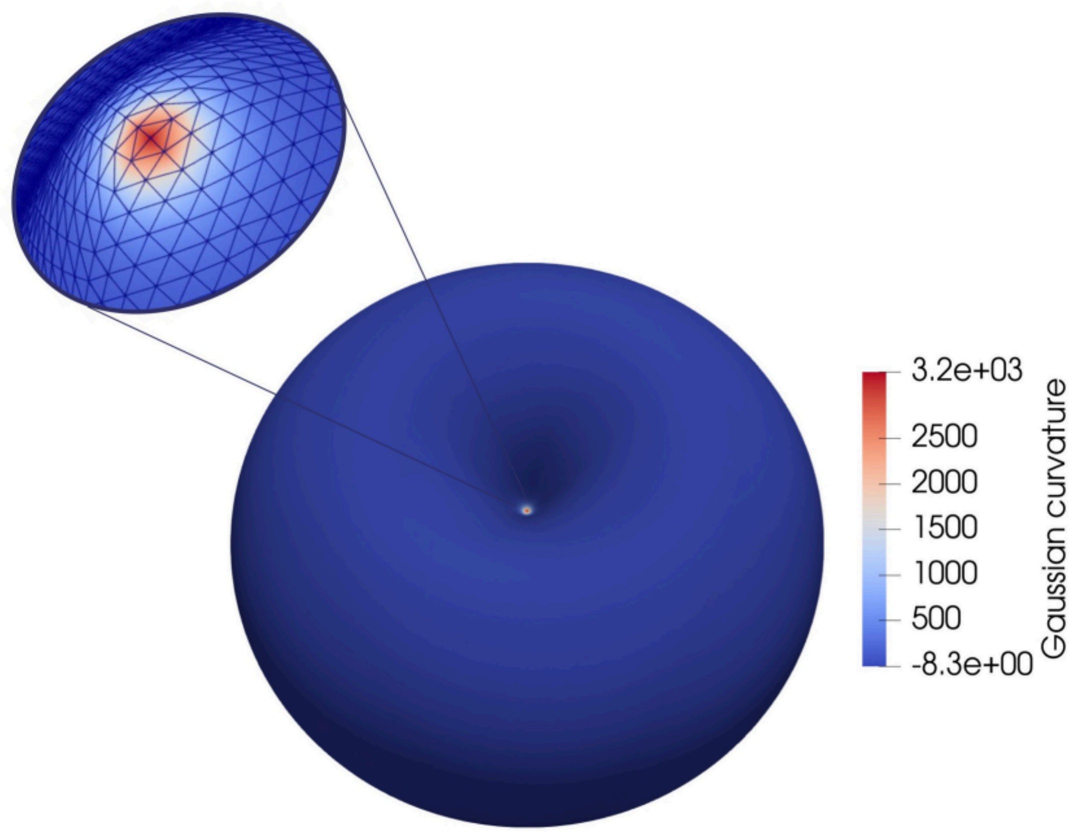
High-order numerical integration on regular embedded surfaces
"h vs. p -refinement"



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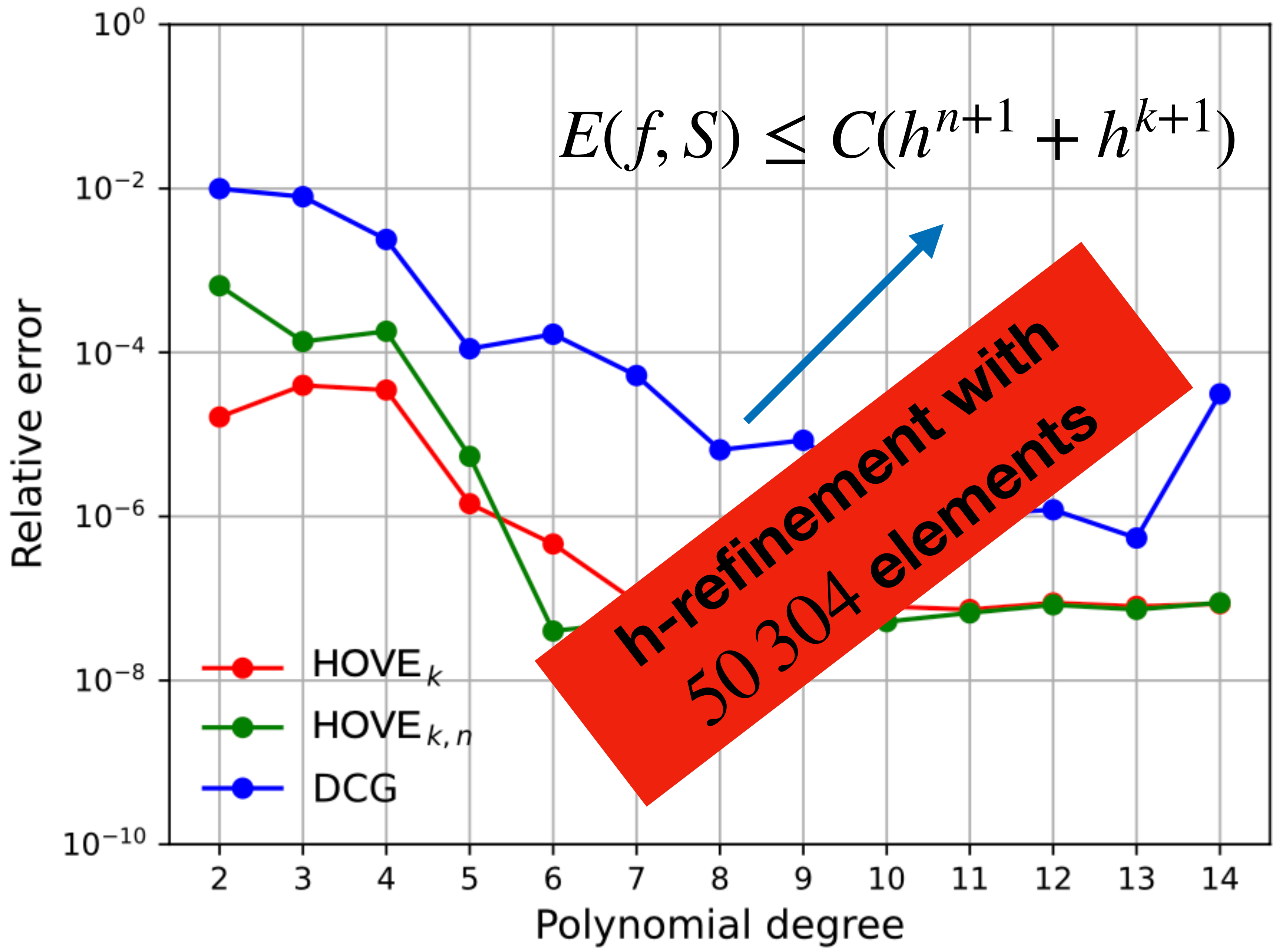

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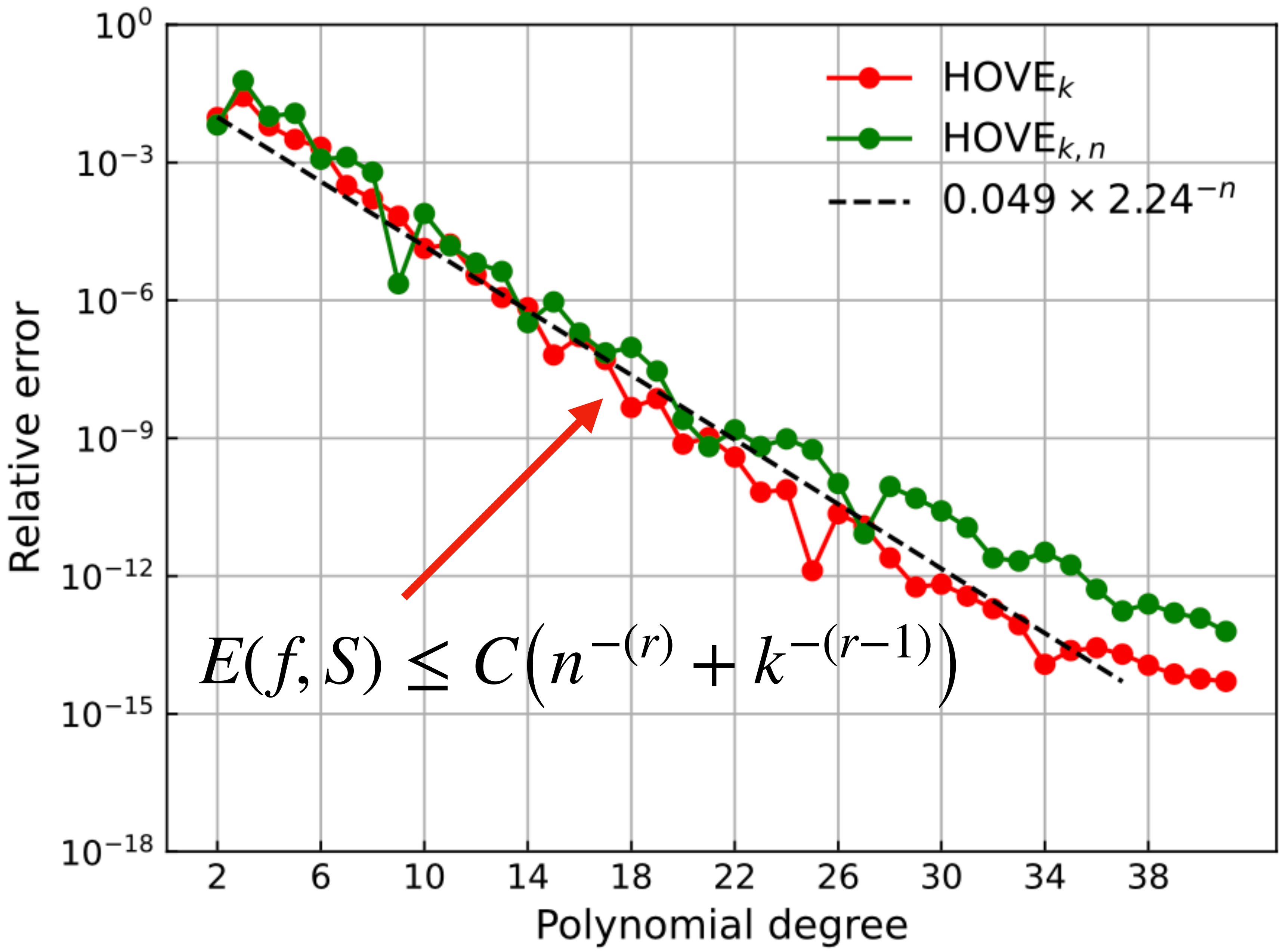

(b) $p-$refinement

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High-order numerical integration —speed up by spectral differentiation
ENUMATH—Lisbon, 2023

Speeding up using spectral differentiation

arXiv > math > arXiv:2403.09178

Mathematics > Numerical Analysis

[Submitted on 14 Mar 2024]

High-order numerical integration on regular embedded surfaces

Gentian Zavalani, Michael Hecht

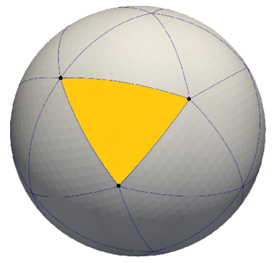
$$D_{\mathrm{x}} Q_{G_{2,k}} \varphi_i = D \otimes I, \quad D_{\mathrm{y}} Q_{G_{2,k}} \varphi_i = I \otimes D, \tag{11}$$

where $D_{\mathrm{x}} Q_{G_{2,k}} \varphi_i, D_{\mathrm{y}} Q_{G_{2,k}} \varphi_i \in \mathbb{C}^{(k+1)^2 \times (k+1)^2}$.

Representing the Jacobian as $DQ_{d,k}\varphi_i = [D_{\mathrm{x}} Q_{d,k}\varphi_i \,|\, D_{\mathrm{y}} Q_{d,k}\varphi_i]$, the ingredients above realise the HOSQ, computing the surface integral as:

$$\int_S f \, dS \approx \sum_{i=1}^{K} \int_{\square_2} f(\varphi(\mathrm{x})) \sqrt{\det((DQ_{d,k}\varphi_i(\mathrm{x}))^T DQ_{d,k}\varphi_i(\mathrm{x}))} \, d\mathrm{x} \tag{12}$$

$$\approx \sum_{i=1}^{K} \sum_{\mathrm{p} \in P} \omega_{\mathrm{p}} f(\varphi_i(\mathrm{p})) \sqrt{\det((DQ_{d,k}\varphi_i(p))^T DQ_{d,k}\varphi_i(p))}, \tag{13}$$



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ENUMATH—Lisbon, 2023

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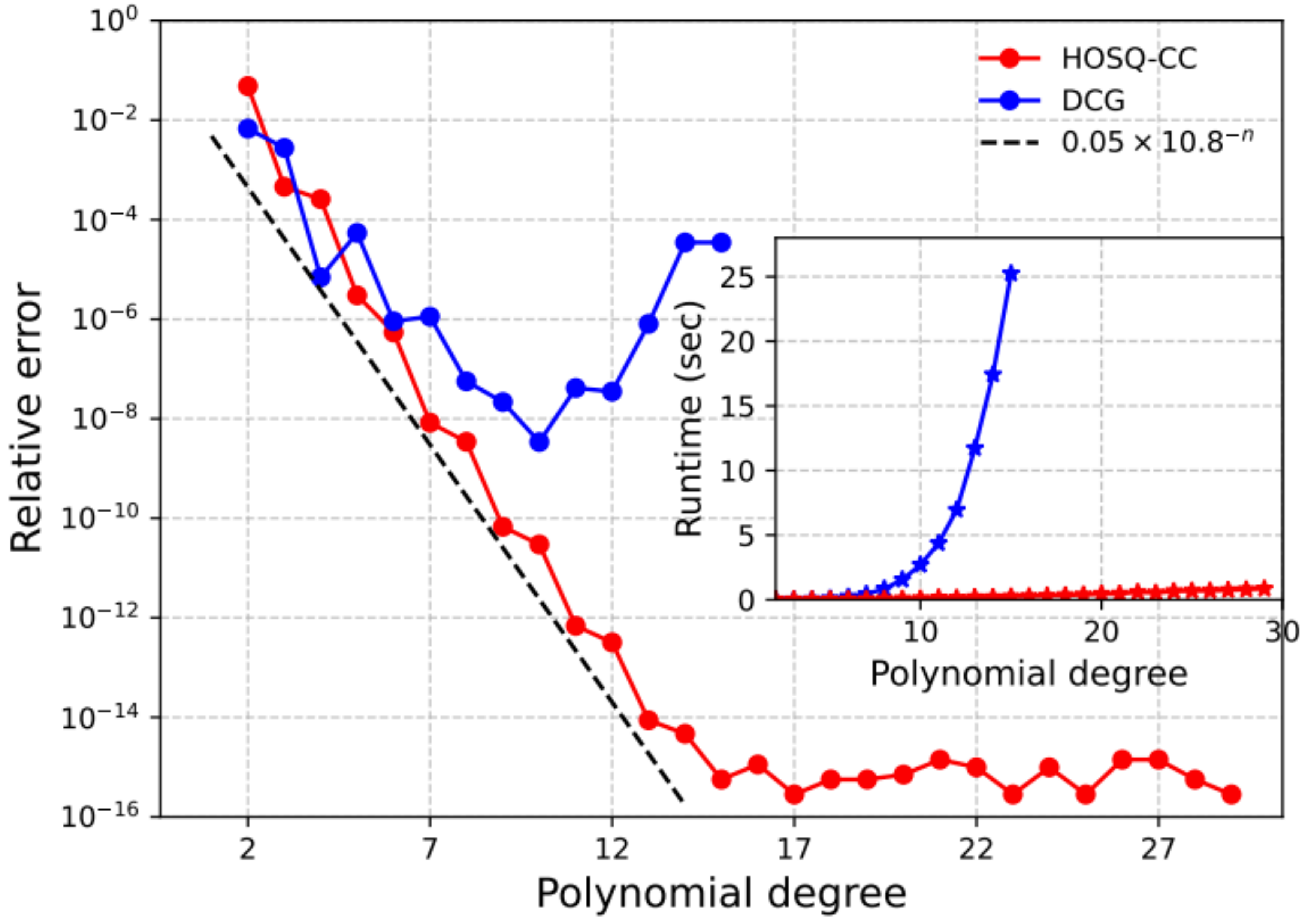
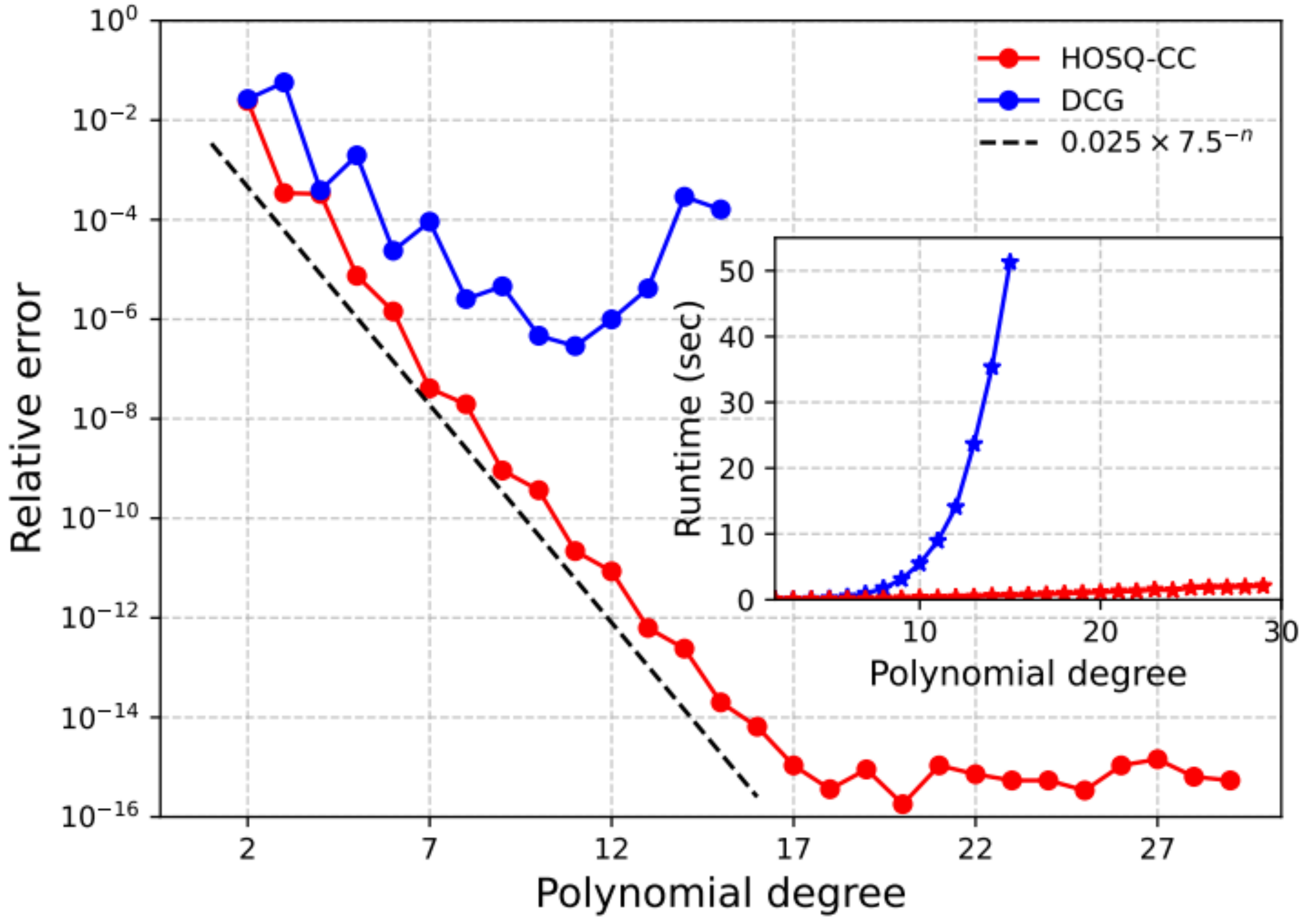


(a) Unit sphere



(b) Torus with radii $R = 2$ and $r = 1$

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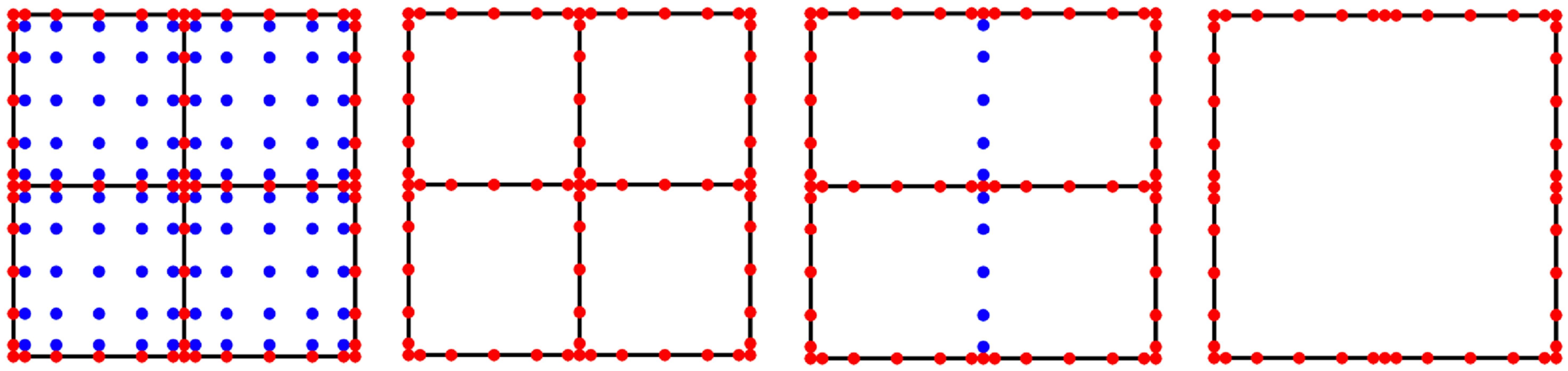
A high-order fast direct solver for surface PDEs

Hierarchical Poincaré–Steklov method



(a) Chebyshev mesh (b) boundary problem (c) merging solutions (d) hierachical reduction

1. Partition the geometry into a collection of small regions.
2. Discretize each small region with a spectral method.
3. ”Glue” regions together with approximate Dirichlet-to-Neumann (DtN) operators.

A high-order fast direct solver for PDEs on triangular meshes

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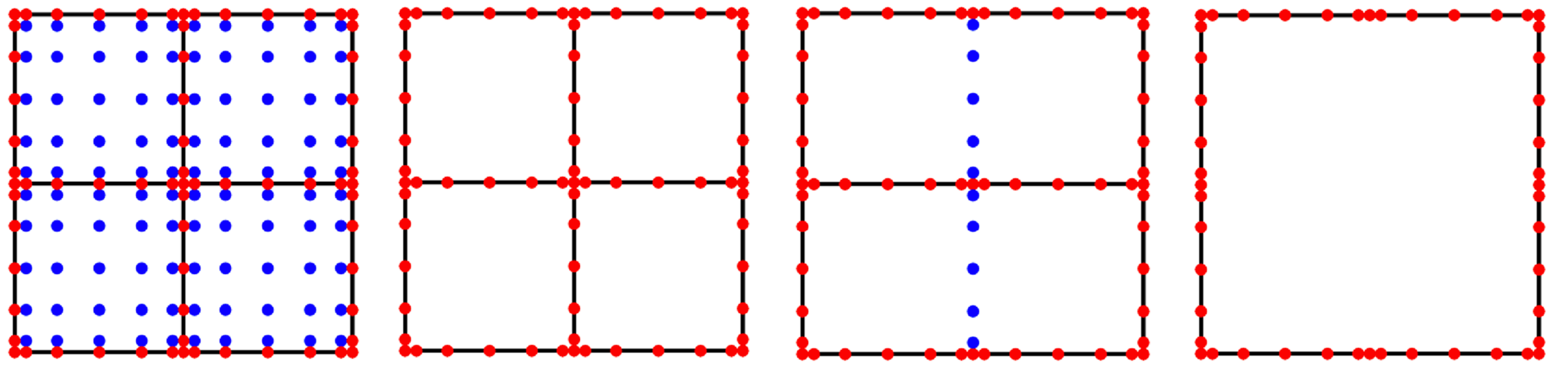
Hierarchical Poincaré–Steklov method



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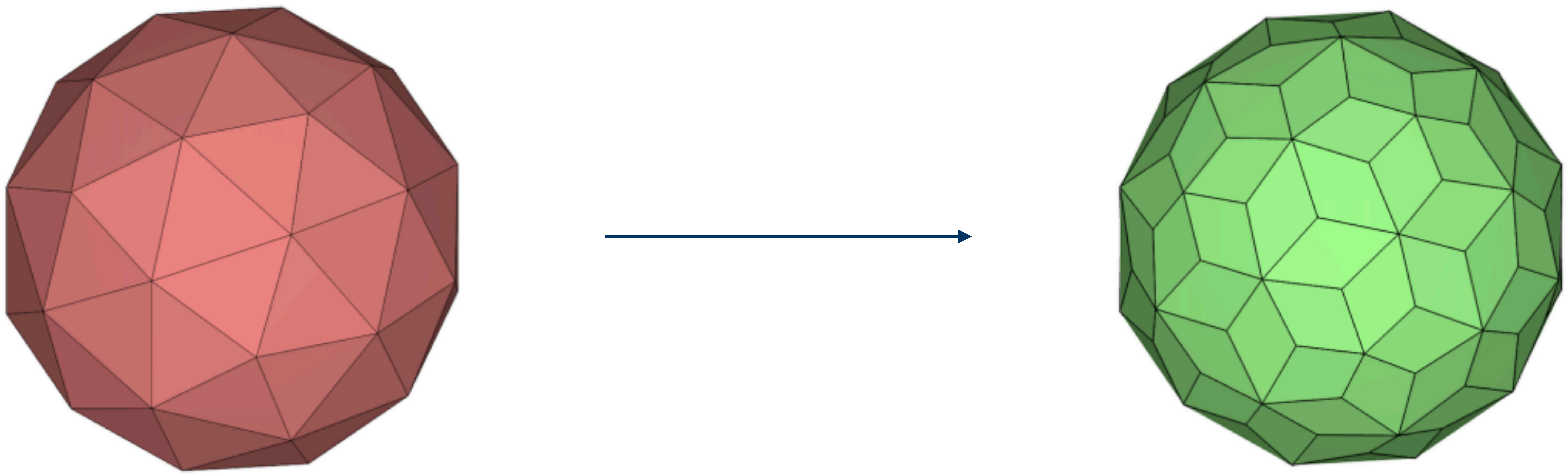
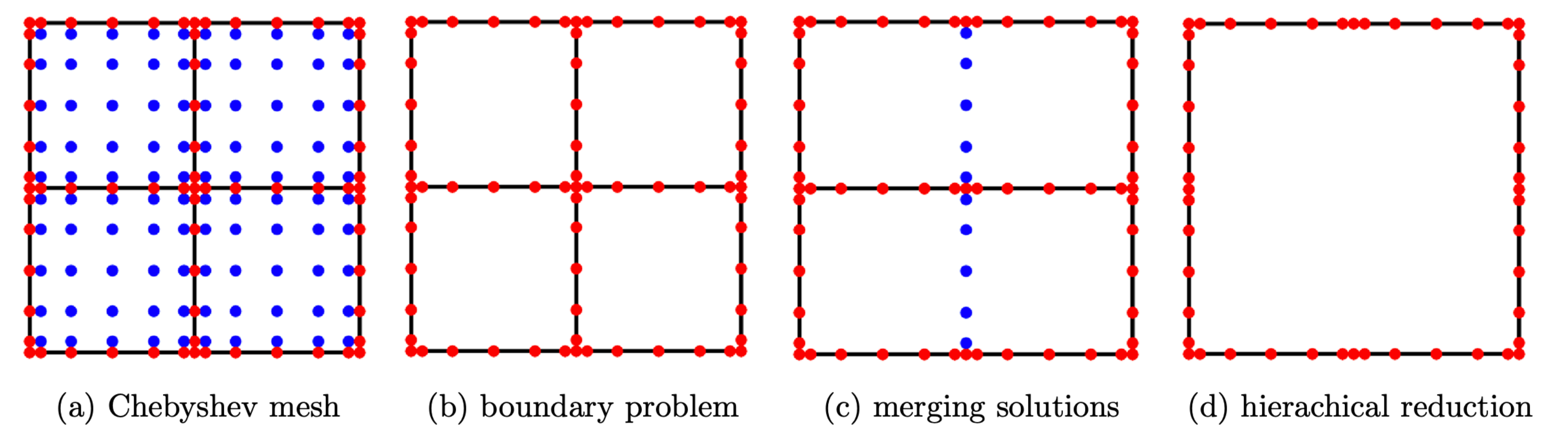
Extending the current approach to triangular meshes



A high-order fast direct solver for PDEs on triangular meshes

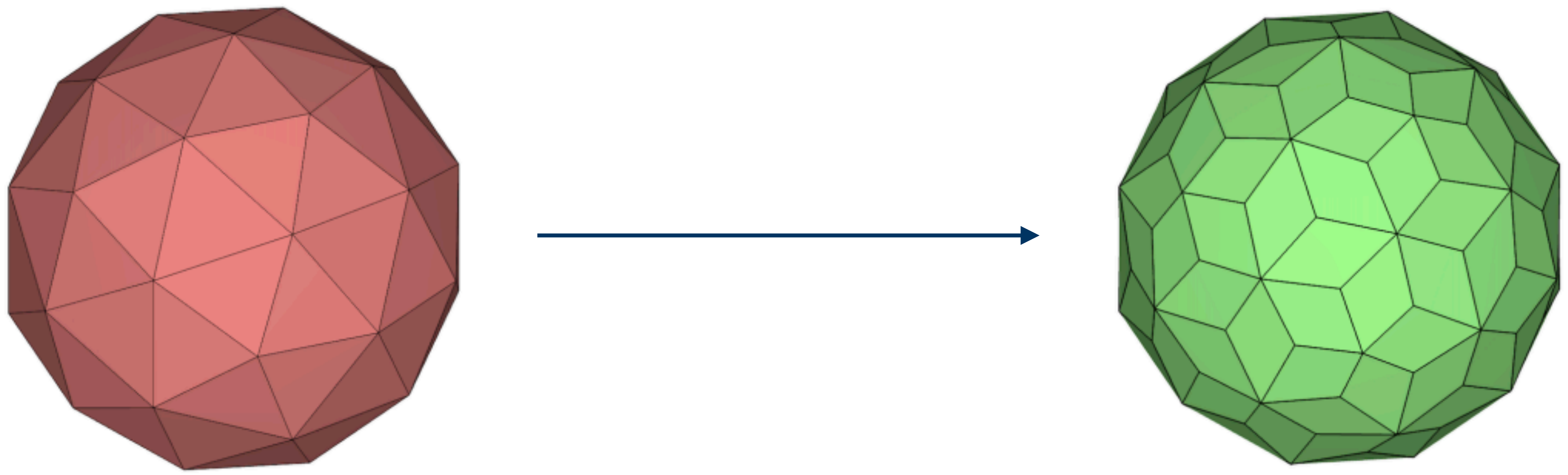


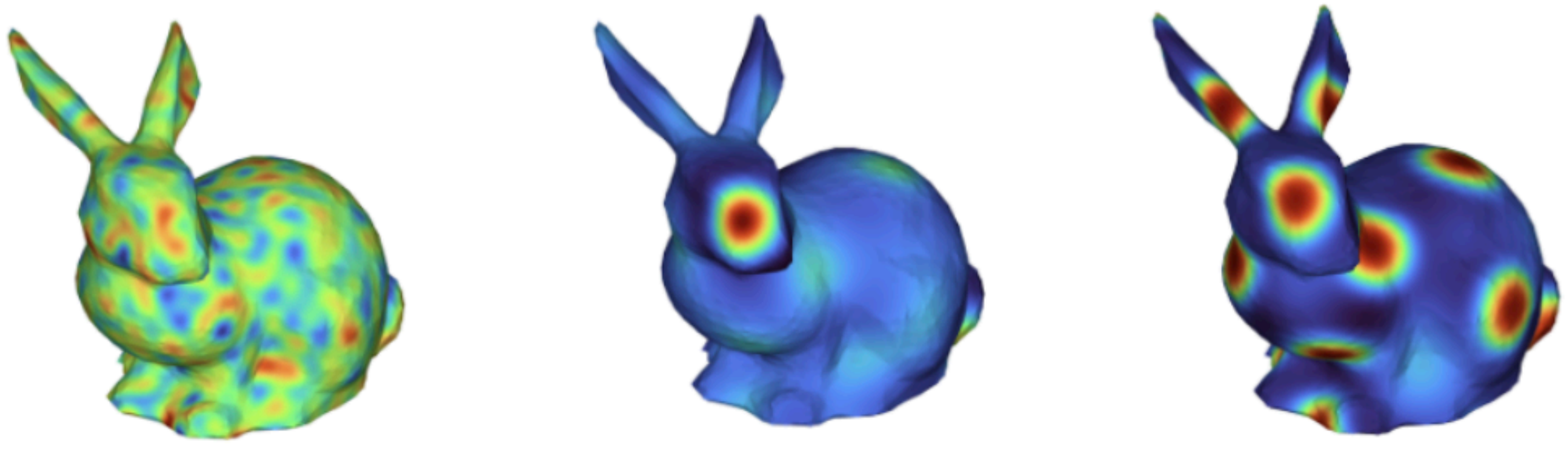
$$\frac{\partial u}{\partial t} = L_\Gamma u + N(u)$$

$$\frac{\partial u}{\partial t} = \delta_u \Delta_\Gamma u + \alpha u \left(1 - \tau_1 v^2\right) + v\left(1 - \tau_2 u\right),$$

$$\frac{\partial v}{\partial t} = \delta_v \Delta_\Gamma v + \beta v \left(1 + \frac{\alpha \tau_1}{\beta} u v\right) + u\left(\gamma + \tau_2 v\right)$$

Extending the current approach to triangular meshes





Turing model on a triangulated Standford Bunny shape, simulated at times $t = 0$, $t = 20$, and $t = 200$ using implicit–explicit backward differentiation (IMEX-BDF4) scheme

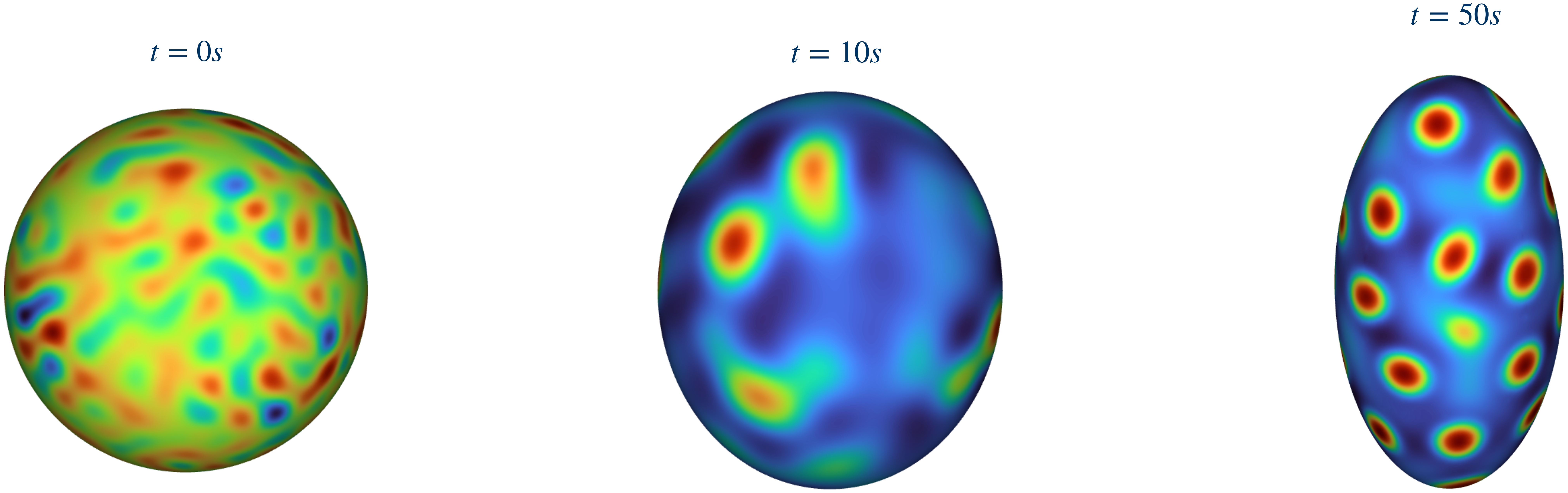
A high-order fast direct solver for PDEs on evolving surface

Time-dependent projection operator

$$\mathscr{P} : \mathscr{N}_\delta \to \mathscr{S}(t), \quad \mathscr{P}(x,t) = (T \circ \pi)(x,t)\,.$$

Anisotropic evolution of a sphere into an ellipsoid:
$T(x_1, x_2, x_3, t) = (x_1, x_2, x_3(1 + g_{rate}t))$ with growth rate $g_{rate} = 0.02$

$$\frac{\partial u}{\partial t} = L_\Gamma u + N(u)$$



A high-order fast direct solver for PDEs on evolving biological surfaces

Reaction Diffusion Systems on evolving surface

Turing system (Turing Patterns)

$$\frac{\partial u}{\partial t} = L_\Gamma u + N(u)$$

$$\frac{\partial u}{\partial t} = \delta_u \Delta_\Gamma u + \alpha u \left(1 - \tau_1 v^2\right) + v \left(1 - \tau_2 u\right),$$

$$\frac{\partial v}{\partial t} = \delta_v \Delta_\Gamma v + \beta v \left(1 + \frac{\alpha \tau_1}{\beta} uv\right) + u \left(\gamma + \tau_2 v\right)$$