2D-DOA Estimation in Uniform Circular Arrays in the Presence of Mutual Coupling

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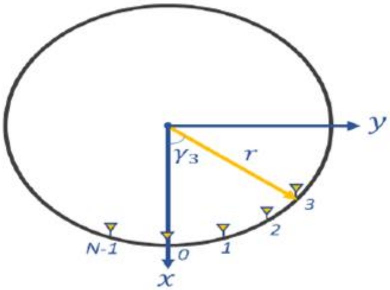
Outline

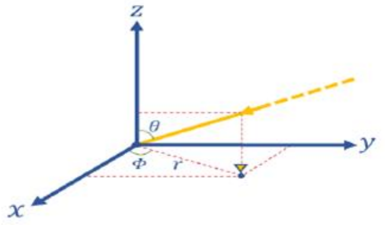
Importance and Applications of DOA Estimation

- **Fundamental Role:** DOA estimation is key in sensor array processing and is central to many modern signal processing systems.
- **Wide Applications:**
  - Wireless communications and radar.
  - Sonar, acoustics, and audio signal processing.
  - Seismology and medical imaging.
  - Autonomous vehicles and smart surveillance.
- **Impact:** Accurate DOA estimation enhances system performance in localization, tracking, and beamforming.

Motivation and Background

- **DOA Estimation:** In the presence of mutual coupling.





- **Challenges:**
  - Mutual coupling degrades performance.
  - Off-grid effects due to discretization of the continuous parameter space.
- **Approach:** An iterative auto-calibration scheme that jointly estimates:
  - Two-dimensional (2-D) DOAs.
  - Mutual Coupling Matrix (MCM).

Paper Contributions

- Use of an **integrated wideband dictionary** [1] to mitigate off-grid effects for 2-D DOA estimation.
- Reduction in computational complexity via **Sparse SVD-based representation** [2].
- Iterative procedure that jointly refines DOA and MCM estimates.
  - No need for a priori knowledge of the number of nonzero coupling coefficients.
- A simple formula for approximation of 2-D integration.

[1] M. Butsenko, J. Swärd, and A. Jakobsson, "Estimating sparse signals using integrated wideband dictionaries," *IEEE Transactions on Signal Processing*, vol. 66, no. 16, pp. 4170–4181, 2018.

[2] D. Malioutov, M. Çetin, and A. S. Willsky, "A sparse signal reconstruction perspective for source localization with sensor arrays," *IEEE Transactions on Signal Processing*, vol. 53, no. 8, pp. 3010–3022, 2005.

Signal Model Overview

- Consider K narrowband far-field sources impinging on a Uniform Circular Array (UCA) with N sensors.

- The array output is modeled as:

$$\mathbf{x}(t) = \mathbf{A}(\phi, \theta)\mathbf{s}(t) + \mathbf{e}(t), \quad t = t_1, t_2, \ldots, t_T,$$

- $\mathbf{A}(\phi, \theta)$ is the steering (manifold) matrix.

- signal vector $\mathbf{s}(t)$ and the noise vector $\mathbf{e}(t)$ are assumed to be zero-mean, circularly symmetric, and statistically independent random processes.

Array Manifold $A(\phi, \theta)$: Shape & Parameters

- **Dimensions:** $A(\phi, \theta) \in \mathbb{C}^{N \times K}$, where N is the number of sensors and K is the number of sources.
- **Steering Vector:** Each column $\mathbf{a}(\phi_i, \theta_i)$ corresponds to a source at azimuth ϕ_i and elevation θ_i.
- **UCA Example:**

$$a_n(\phi, \theta) = \exp\left(jk_0 r \sin\theta \, \cos\left(\phi - \frac{2\pi n}{N}\right)\right), \quad n = 0, 1, \ldots, N-1$$

- **Wave Number:** $k_0 = \frac{2\pi}{\lambda}$, with λ as the wavelength.
- **Parameter Ranges:** Counterclock-wise $\phi \in [0^\circ, 360^\circ)$ and $\theta \in [0^\circ, 90^\circ]$.

Mutual Coupling Effect

- In the presence of mutual coupling, the model becomes:

$$\mathbf{X} = \mathbf{C}\,\mathbf{A}(\phi, \theta)\,\mathbf{S} + \mathbf{E}$$

- $\mathbf{C}$ is the Mutual Coupling Matrix (MCM) with a complex symmetric circular Toeplitz structure.

- The coupling coefficients satisfy:

$$0 < |c_L| < \ldots < |c_2| < |c_1|$$

Calibration Approaches: Offline vs. Online

Offline Calibration

- Uses calibration sources with known locations.
- Requires dedicated calibration sessions.
- Direct measurement of the array manifold.
- Generally high accuracy in controlled settings.

Online (Auto-) Calibration

- No need for external calibration sources.
- Calibration is performed during regular operation.
- Automatically compensates for effects like mutual coupling.
- More practical in dynamic, real-world environments.

Overview of the Proposed Method

- **Step 1: Sparse SVD-based Representation**
  - Project data onto the dominant signal subspace.
  - Reduce data dimension and computational cost.
- **Step 2: DOA Estimation**
  - Use an integrated wideband dictionary over azimuth and elevation.
  - Apply a zooming procedure (coarse-to-fine grid) to mitigate off-grid effects.
  - Solve the penalized regression (LASSO) problem.
- **Step 3: MCM Estimation**
  - Exploit the circular Toeplitz structure.
  - Estimate coupling coefficients by minimizing a cost function based on the noise subspace.
- **Iteration:** Alternate between DOA and MCM estimation until convergence.

Sparse SVD-based Representation

- Perform SVD on the measured data:

$$\mathbf{X} = \mathbf{U}_s \Sigma_s \mathbf{V}_s^H + \mathbf{U}_c \Sigma_c \mathbf{V}_c^H$$

- Retain only the K dominant singular vectors:

$$\mathbf{X}_{\mathrm{SV}} = \mathbf{X}\mathbf{V}_s$$

- Preserves key signal information while reducing computational complexity.

Gridding Issue vs. Continuous Parameter Space

Gridding Issue

- Discretizes a continuous space.
- Causes off-grid mismatches.
- Finer grids increase complexity.

Continuous Parameter Space

- Handles parameters continuously.
- Avoids grid mismatch errors.
- Enhances resolution with balanced complexity.

DOA Estimation with Integrated Dictionary

- Construct a dictionary over the 2-D angular domain (ϕ and θ).
- Use an **integrated wideband element**:

$$\mathbf{a}_{n_r}(\phi_i, \theta_i) = \int_{\phi_i}^{\phi_{i+1}} \int_{\theta_i}^{\theta_{i+1}} \exp\left(jk_0 r \sin\theta \cos\left(\phi - \frac{2\pi n_r}{N} \right) \right) \, d\theta \, d\phi.$$

- **Zooming Procedure:**
  1. Start with a coarse grid.
  2. Identify active regions.
  3. Refine the dictionary over these regions.
- Formulate the sparse recovery problem using LASSO [3].
- We have obtained a closed form formula for this integration (see the appendix)!

[3] R. Tibshirani, "Regression shrinkage and selection via the lasso," *Journal of the Royal Statistical Society Series B: Statistical Methodology*, vol. 58, no. 1, pp. 267–288, 1996.

DOA Estimation with Integrated Dictionary- Zooming Process

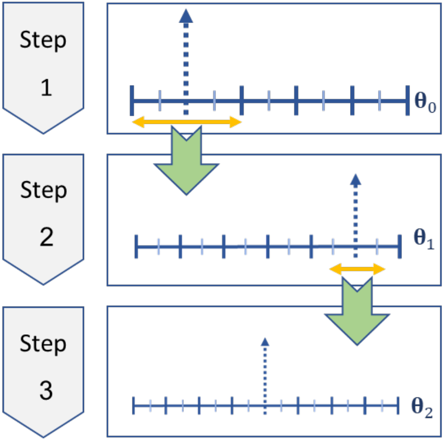


Figure: Illustration of the zooming process over intervals/bands of angles.

Computation of the 2-D Integral

The integrated wideband dictionary elements involve the computation of the following integral:

$$\mathbf{a}_{n_r}(\phi_i, \theta_i) = \int_{\phi_i}^{\phi_{i+1}} \int_{\theta_i}^{\theta_{i+1}} e^{jk_0 r \sin\theta \cos(\phi - \frac{2\pi n_r}{N})} \, d\theta \, d\phi$$
$$= \dots$$
$$= \sum_{n=0}^{\infty} \frac{(k_0 r)^n}{n!} \left(\int_{\phi_i - \frac{2\pi n_r}{N}}^{\phi_{i+1} - \frac{2\pi n_r}{N}} \cos^n \phi \, d\phi \right) \left(\int_{\theta_i}^{\theta_{i+1}} \sin^n \theta \, d\theta \right)$$

The above integrals can be simplified recursively as:

$$\int \sin^n \theta \, d\theta = -\frac{1}{n} \sin^{n-1}\theta \cos\theta + \frac{n-1}{n} \int \sin^{n-2} \theta \, d\theta,$$
$$\int \cos^n \phi \, d\phi = \frac{1}{n} \cos^{n-1}\phi \sin\phi + \frac{n-1}{n} \int \cos^{n-2} \phi \, d\phi.$$

For ~ 10 terms it gives the same precision as the built-in integral2 in Matlab

LASSO Formulation for DOA Estimation

$$\mathbf{s}_d = \arg\min_{\hat{\mathbf{s}}} \left\{ \underbrace{\|\mathbf{x} - \mathbf{D}\hat{\mathbf{s}}\|_2^2}_{\text{Data fitting term}} + \underbrace{\gamma\|\hat{\mathbf{s}}\|_1}_{\text{Sparsity-promoting term}} \right\}$$

Interpretation of Terms:

- **Data fitting term:** Ensures the reconstructed signal fits the observed data.
- **Sparsity-promoting term:** Encourages solutions with fewer active sources (sparse solutions).

Selecting Regularization Parameter (γ):

$$\gamma = \alpha \cdot \max_i |\mathbf{a}_i^H \mathbf{x}|, \quad 0 < \alpha \leq 1$$

- α controls the sparsity level.
- Optimal α chosen based on simulations (typically $\alpha \leq 0.4$).

LASSO in Signal Processing and Machine Learning

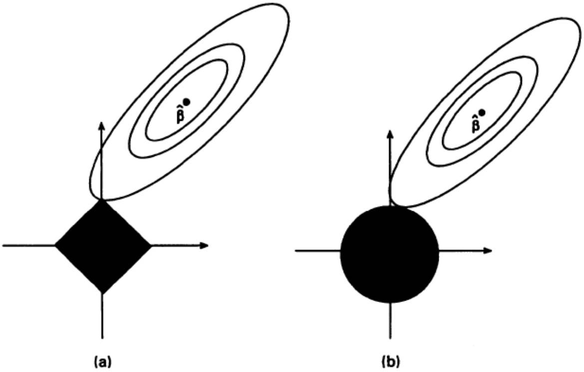


Figure: Figures [4] a) LASSO b) Ridge Regression. .

[4] R. Tibshirani, “Regression shrinkage and selection via the lasso,” *Journal of the Royal Statistical Society Series B: Statistical Methodology*, vol. 58, no. 1, pp. 267–288, 1996.

Mutual Coupling Matrix Estimation

- Given the estimated DOAs, determine the MCM by minimizing:

$$J = \left\| \mathbf{E}_n^H \mathbf{C} \mathbf{A}(\phi, \theta) \right\|_F^2 .$$

- Leverage the symmetric circular Toeplitz structure to rewrite the cost function in terms of the mutual coupling vector $\mathbf{c}$:

$$J = \mathbf{c}^H \mathbf{U}(\phi, \theta) \mathbf{c}.$$

- With the constraint $c_1 = 1$, $\mathbf{c}$ is obtained as the eigenvector corresponding to the smallest eigenvalue.

Algorithm Outline:

1. Initialize the MCM: $\mathbf{C}_0 = \mathbf{I}$.
2. Select zooming steps and dictionary bands for azimuth and elevation.
3. Compute the SVD of the data.
4. Form the integrated wideband dictionary.
5. Estimate DOAs using LASSO.
6. Estimate the MCM using the noise subspace.
7. Update the dictionary based on active bands.
8. Repeat steps 5–7 until convergence.

Simulation Setup and Metrics

- **UCA Configuration:** $N = 15$ sensors; radius $r = \lambda$.
- **Sources:** Three far-field sources with angles of arrival $(18.3^\circ, 243.4^\circ)$, $(83.6^\circ, 60^\circ)$, and $(73.9^\circ, 357.8^\circ)$
- **Params:** $M = 500$; $T = 200$; noise modeled as white Gaussian.
- Mutual coupling coefficients:
  $c_1 = 1,\ c_2 = 0.79 + j\,0.432,\ c_3 = 0.35 + j\,0.16$, and $c_k = 0$ for $k \geq 4$.
- **Metrics:**
  - RMSE for DOA estimation.
  - RMSE for coupling coefficient estimation.

Choosing α and Zooming Steps (Cont'd)

- The integrated wideband dictionary divides the 2-D angular domain into bands for both elevation and azimuth.
- **Elevation bands:**
  - Stage 1: $B_1^{\theta} = 30$
  - Stage 2: $B_2^{\theta} = 10$ (grey curve) **or** $B_2^{\theta} = 5$ (brown curve)
  - Stage 3: $B_3^{\theta} = 3$
- **Azimuth bands:**
  - Stage 1: $B_1^{\phi} = 120$
  - Stage 2: $B_2^{\phi} = 10$ (grey curve) **or** $B_2^{\phi} = 5$ (brown curve)
  - Stage 3: $B_3^{\phi} = 3$

Choosing α and Zooming Steps

- The parameter α (see Eq. (11)) controls the sparsity of the solution.
- As shown in the figure, for the grey curve configuration, using $\alpha \leq 0.4$ yields a high probability of correctly estimating the number of DOAs.

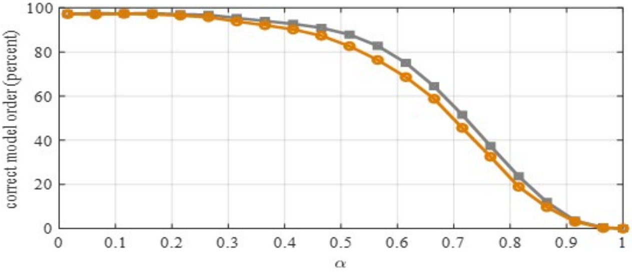


Figure: Probability of correct DOA estimation vs. α.

Key Simulation Results

$$\text{RMSE} = \sqrt{\frac{1}{KM}\sum_{k=1}^{K}\sum_{m=1}^{M}\left[(\theta_k^m - \theta_k)^2 + (\phi_k^m - \phi_k)^2\right]},$$

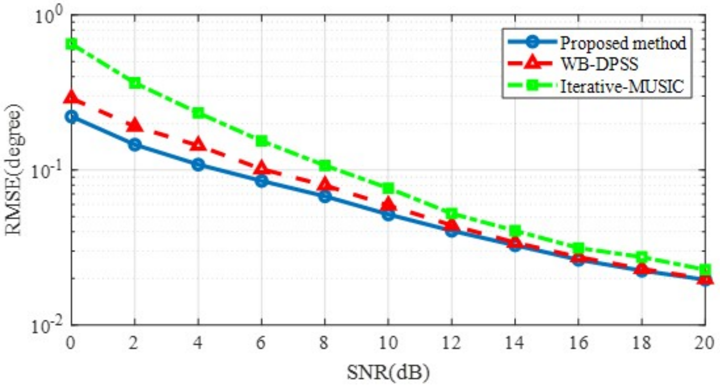


Figure: The RMSE of angle estimation as a function of the SNR.

Key Simulation Results

$$\text{RMSE}_{\text{coupling coeff.}} = \frac{\sqrt{\sum_{m=1}^{M} \|\hat{\mathbf{c}}^m - \mathbf{c}\|^2}}{\|\mathbf{c}\|} \times 100\%,$$

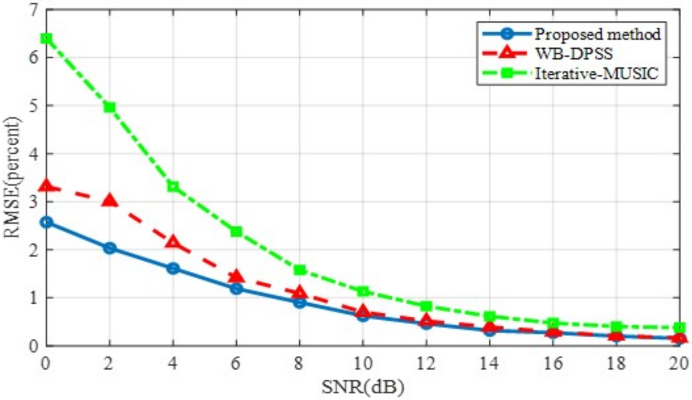


Figure: The RMSE of mutual coupling coefficients estimation as a function of the SNR.

Conclusions and Future Work

- **Summary:**
  - Introduced an iterative auto-calibration algorithm for joint 2-D DOA and MCM estimation.
  - Utilized a sparse representation framework with an integrated wideband dictionary.
  - Exploited the circular symmetry of the mutual coupling matrix.
- **Advantages:**
  - Reduced computational complexity.
  - High accuracy without prior knowledge of coupling sparsity.
- **Future Directions:**
  - Extension to nonuniform arrays.
  - Applications to arbitrary geometries.

Codes and Slides

Email: Zavarehbozorgasl@u.boisestate.edu

github.com/zavareh1/ClippedQuantFL

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Q&A

Thank You!

Questions?