Practising LaTex to present Ramanujan's Master Theorem

Zaw Myat

November 2023

1 Introduction

In mathematics, Ramanujan's Master Theorem, named after Srinivasa Ramanujan [2], is a technique that provides an analytic expression for the Mellin transform of an analytic function.

$$f(x) = \sum_{k=0}^{\infty} \frac{\varphi(k)}{k!}(-x)^k$$

then the Mellin transform of $f(x)$ is given by

$$\int_0^{\infty} x^{s-1} f(x) dx = \Gamma(s)\varphi(-s)$$

where $\gamma(s)$ is the gamma function.

It was widely used by Ramanujan to calculate definite integrals and infinite series.

Higher-dimensional versions of this theorem also appear in quantum physics (through Feynman diagrams).[7]

A similar result was also obtained by Glaisher [4].

2 Alternative formalism

An alternative formulation of Ramanujan's Master Theorem is as follows:

$$\int_0^{\infty} x^{s-1} \left(\lambda(0) - x\lambda(1) + x^2\lambda(2) - \cdots\right) dx = \frac{\pi}{\sin(\pi s)}\lambda(-s)$$

which gets converted to the above from after substituting $\lambda(n) \equiv \frac{\varphi(n)}{\gamma(1+n)}$ and using the function equation for the gamma function.

The integral above is convergent for $0 < \mathcal{R}e(s) < 1$ subject to growth conditions on φ.[1]

3 Proof

A proof subject to "natural" assumptions (though not the weakest necessary conditions) to Ramanujan's Master theorem was provided by G. H. Hardy[8] (chapter XI) employing the residue theorem and the well-known Mellin inversion theorem.

4 Application to Bernoulli polynomials

The generating function of the Bernoulli polynomials $B_k(x)$ is given by:

$$\frac{ze^{xz}}{e^z-1}=\sum_{k=0}^{\infty}B_k(x)\frac{z^k}{k!}$$

These polynomials are given in terms of the Hurwitz zeta function:

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}$$

by $\zeta(1-n,a)=-\frac{B_n(a)}{n}$ for $n\geq 1$. Using the Ramanujan master theorem and the generating function of Bernoulli polynomials one has the following integral representation.[3]

$$\int_0^{\infty}x^{s-1}\left(\frac{e^{-ax}}{1-e^{-x}}-\frac{1}{x}\right)dx=\gamma(s)\zeta(s,a)$$

which is valid for $0<\mathcal{R}e(s)<1$.

5 Application to the gamma function

Weiserstrass's definition of the gamma function

$$\Gamma(x)=\frac{e^{-\gamma x}}{x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)^{-1}e^{\frac{x}{n}}$$

is equivalent to expression

$$log\Gamma(1+x)=-\gamma x+\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}(-x)^k$$

where $\zeta(k)$ is the Riemann zeta function.
Then applying Ramanujan master theorem we have:

$$\int_0^{\infty}x^{s-1}\frac{\gamma x+log\Gamma(1+x)}{x^2}dx=\frac{\pi}{\sin(\pi s)}\frac{\zeta(2-s)}{2-s}$$

valid for $0<\mathcal{R}e(s)<1$.

Special cased of $s = \frac{1}{2}$ and $s = \frac{3}{4}$ are

$$\int_0^\infty \frac{\gamma x + log\Gamma(1+x)}{x^{\frac{5}{2}}} dx = \frac{2\pi}{3}\zeta\left(\frac{3}{2}\right)$$

$$\int_0^\infty \frac{\gamma x + log\Gamma(1+x)}{x^{\frac{9}{4}}} dx = \sqrt{2}\frac{4\pi}{5}\zeta\left(\frac{5}{4}\right)$$

6 Application to Bessel functions

The Bessel function of the first kind has the power series

$$J_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+v+1)k!}\left(\frac{z}{2}\right)^{2k+v}$$

By Ramanujan's Master Theorem, together with some identities for the gamma function and rearranging, we can evaluate the integral

$$\frac{2^{v-2s}\pi}{\sin(\pi(s-v))}\int_0^\infty z^{s-1-\frac{v}{2}} J_v(\sqrt{z})dz = \Gamma(s)\Gamma(s-v)$$

valid for $0 < 2\mathcal{R}(s) < \mathcal{R}(v) + \frac{3}{2}$.

Equivalently, if the spherical Bessel function $j_v(z)$ is preferred, the formula becomes

$$\frac{2^{v-2s}\sqrt{\pi}(1-2s+2v)}{cos(\pi(s-v))}\int_0^\infty z^{s-1-\frac{v}{2}} j_v(\sqrt{z})dz = \Gamma(s)\Gamma\left(\frac{1}{2}+s-v\right)$$

valid for $0 < 2\mathcal{R}e(s) < \mathcal{R}e(v) + 2$.

The solution is remarkable in that it is able to interpolate across the major identities for the gamma function. In particular, the choice of $J_0(\sqrt{z})$ gives the square of the gamma function, $j_0(\sqrt{z})$ gives the duplication formula, $z^{\frac{-1}{2}}J_1(\sqrt{z})$ gives the reflection formula, and fixing to the evaluable $s = \frac{1}{2}$ or $s = 1$ gives the gamma function by itself, up to reflection and scaling.

7 Bracket integration method

The bracket integration method (method of brackets) applies Ramanujan's Master Theorem to a broad range of integrals. The bracket integration method generates an integral of a series expansion, introduces simplifying notations, solves linear equations, and completes the integration using formulas arising from Ramanujan's Master Theorem.[5]

7.1 Generate an integral of a series expansion

This method transforms the integral to an integral of a series expansion involving M variables, $x_1, \cdots x_M$, and S summation parameters, $n_1, \cdots n_S$. A multivariate integral may assume this form.[7]

$$\int_0^\infty \cdots \int_0^\infty \sum_{n_1,\cdots,n_s=0}^{\infty} \varphi(n_1 \cdots n_S) \prod_{j=1}^{S} \left(\frac{(-1)_j^n}{n_j!} \right) \prod_{j=1}^{M} (x_j)^{(-c_j + a_{j1}.n_1 + \cdots + a_{jS}.n_S - 1)} \tag{1}$$

7.2 Apply special notations

- The bracket ($\langle \cdots \rangle$), indicator (ϕ), and monomial power notations replace terms in the series expansion.[7]

$$\int_0^\infty x^{c+bn-1} dx \rightarrow < c + b \cdot n > \tag{2}$$

$$\frac{(-1)^n}{n!} \rightarrow \phi_n \tag{3}$$

$$\prod_{j=1}^{S} \left(\frac{(-1)_j^n}{n_j!} \right) \rightarrow \phi_n 1, \cdots, n_s \tag{4}$$

$$\left(\sum_{k=1}^{P} u_k \right)^{\mp d} \rightarrow \sum_{n_1,\cdots,n_p=0}^{\infty} \varphi_{n_1,\cdots,n_P} \prod_{k=1}^{P} u_k^{n_k} \frac{\langle \pm d + \sum_{j=1}^{P} n_j \rangle}{\Gamma(\pm d)} \tag{5}$$

- Application of these notations transforms the integral to a bracket series containing B brackets.[6]

$$\sum_{n_1,\cdots,n_s=0}^{\infty} \varphi(n_1 \cdots n_s) \phi(n_1 \cdots n_s) \prod_{j=1}^{B} \left\langle -c_j + \sum_{k=1}^{S} a_{jk} \cdot n_k \right\rangle \tag{6}$$

- Each bracket series has an index defined as index = number of sums - number of brackets.

- Among all bracket series representations of an integral, the representation with a minimal index is preferred.

8 Solve linear equations

- The array of coefficients a_{jk} must have maximum rank, linearly independent leading columns to solve the following set of linear equations.
- If the index is non-negative, solve this equation set for each $n_j^\star$. The terms $n_j^\star$ may be linear functions of $n_{B+1} \cdots n_S$.

$$-c_j + \sum_{k=1}^{B} a_{jk} \cdot n_k^\star + \sum_{k=B+1}^{S} a_{jk} \cdot n_k = 0 \tag{7}$$

- If the index is zero, equation 7 simplifies to solving this equation set for each $n_j^\star$

$$-c_j + \sum_{k=1}^{B} a_{jk} \cdot n_k^\star = 0 \tag{8}$$

- If the index is negative, the integral cannot be determined.

9 Apply formulas

- If the index is non-negative, the formula for the integral is this form.[6]

$$\sum_{n_{B+1} \cdots n_S = 0}^{\infty} \frac{\varphi(\varphi_1^\star \cdots n_B^\star, n_{B+1} \cdots n_S) \cdot \prod_{j-1}^{B} \Gamma(-n_j^\star)}{det|A|} \tag{9}$$

- These rules apply.[5]
  - A series is generated for each choice of free summation parameters, $\{n_B + 1, \cdots Ns\}$
  - Series converging in a common region are added.
  - If a choice generates a divergent series or null series (a series with zero valued terms), the series is rejected.
  - A bracket series of negative index is assigned no value.
  - If all series are rejected, then the method cannot be applied.
  - If the index is zero, the formula 9 simplifies to this formula and no sum occurs.

$$\frac{\varphi(n_1^\star \cdots n_S^\star) \cdot \prod_{j=1}^{S} \Gamma(-n_j^\star))}{det|A|} \tag{10}$$

10 Mathematical basis

- Apply this variable transformation to the general integral form equation 1[1]

$$y_k = x_1^{a_{1k}} . \cdots . x_M^{a_{Mk}} \tag{11}$$

- This is the transformed integral (12) and the result from applying Ramanujan's Master Theorem (13).

$$(det|A|^{-1}) \cdot \int_0^\infty \cdots \int_0^\infty \sum_{n_1, \cdots, n_S = 0}^{\infty} \varphi(n_1 \cdots n_S) \prod_{j=1}^{S} \left(\frac{(-1)^{n_j}}{n_j!} \right) \prod_{j=1}^{M} (y_j)^{-n_j^\star + n_j - 1} dy_1 \cdots dy_M \tag{12}$$

$$= \sum_{n_{M+1} \cdots n_S = 0}^{\infty} \frac{\varphi(n_1^\star \cdots n_M^\star, n_{M+1} \cdots n_s) \cdot \prod_{j=1}^{M} \Gamma(-n_j^\star)}{det|A|} \tag{13}$$

- The number of brackets (B) equals the number of integrals (M) 2. In addition to generating the algorithm's formulas (9, 10), the variable transformation also generates the algorithm's linear equations (7,8).[6]

11 Example

- The bracket integration method is applied to this integral.

$$\int_0^\infty x^{\frac{3}{2}} \cdot e^{\frac{-x^3}{2}} dx$$

- Generate the integral of a series expansion (1)

$$\int_0^\infty \sum_{n=0}^{\infty} 2^{-n} \cdot \frac{(-1)^n}{n!} \cdot x^{(3 \cdot n + \frac{5}{2}) - 1} dx$$

- Apply special notations (equation 2, 3).

$$\sum_{n=0}^{\infty} 2^{-n} \cdot \phi(n) \cdot \langle 3 \cdot n + \frac{5}{2} \rangle$$

- Solve the linear equation (equation 8).

$$3 \cdot n^\star + \frac{5}{2} = 0, n^\star = \frac{-5}{6}$$

- Apply the formula (equation 9)

$$\frac{2^{\frac{5}{6}} \cdot \Gamma(\frac{5}{6})}{3}$$

References

[1] Ivan; Harrison Marshall; Moll Victor H.; Straub Armin Amdeberhan, Tewodros; Gonzalez. Ramanujan's master theorem. *The Ramanujan Journal*, (29):1–3, 2012.

[2] B. Berndt. *Ramanujan's Notebooks, Part I.* New York: Springer-Verlag, 1985.

[3] V Espinosa, O.; Moll. On some definite integrals involving the hurwitz zeta function. part 2. *The Ramanujan Journal*, 6(4):449–468, 2002.

[4] J.W.L. Glaisher. A new formula in definite integrals. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 48(315):53–55, 1874.

[5] Lin; Moll Victor H Gonzalez, Ivan; Jiu. An extension of the method of brackets. part 2. *Open Mathematics*, 18(1):983–995, 1 January 2020.

[6] Victor H Gonzalez, Ivan; Moll. Definite integrals by the method of brackets. part 1. *Advances in Applied Mathematics*, 45(1):50–73, July 2010.

[7] V.H.; Schmidt Iván González, Iván; Moll. A generalized ramanujan master theorem applied to the evaluation of feynman diagrams, 2011.

[8] G.H. Hardy. *Ramanujan: Twelve lectures on subjects suggested by his life and work*. New York: Springer-Verlag, New York, NY: Chelsea, 3rd edition, 1978.