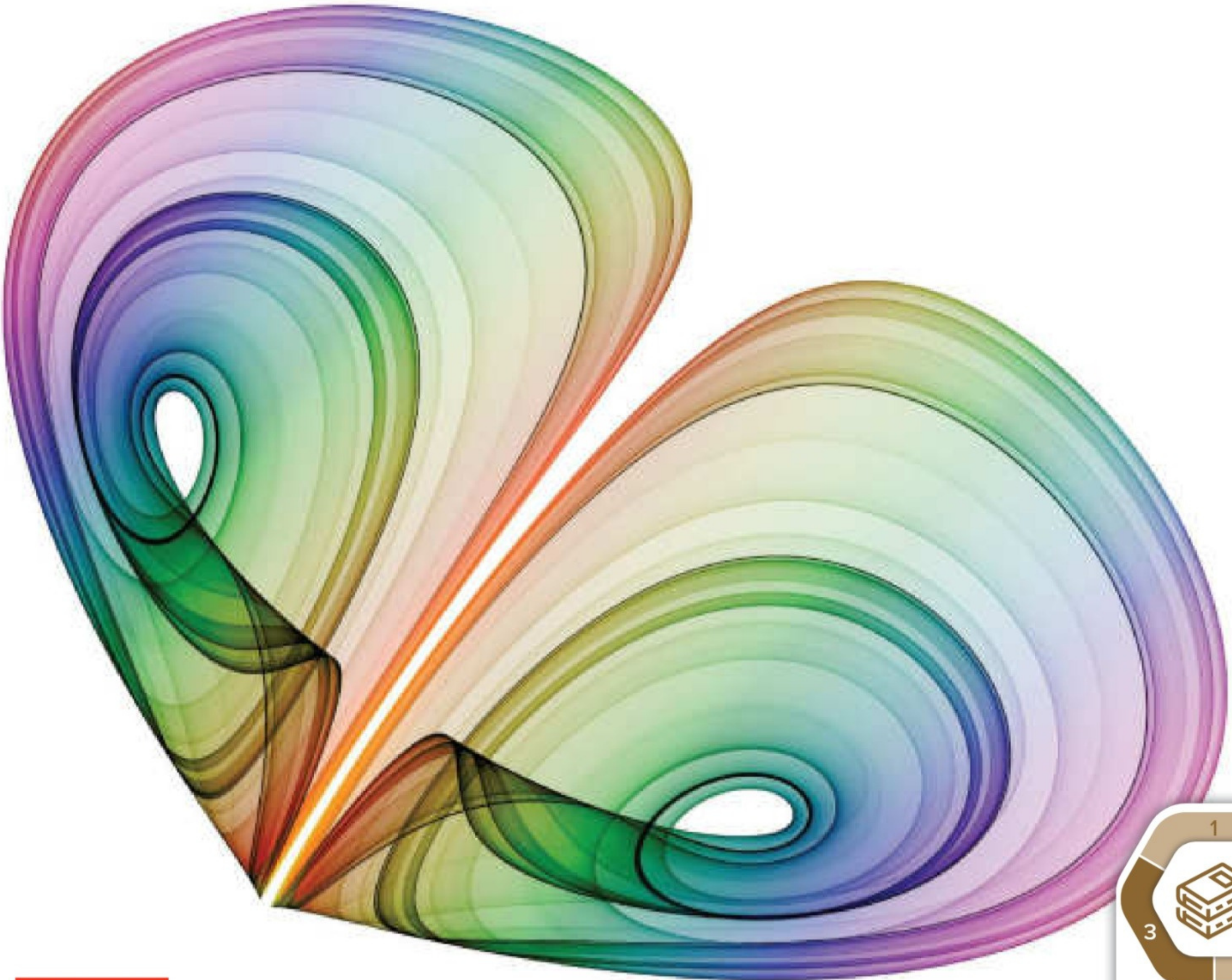
UNITED ARAB EMIRATES
MINISTRY OF EDUCATION

المدرسة الإماراتية

2021-2022

Mathematics

United Arab Emirates Edition



Mc Graw Hill

Grade 11 Advanced

McGraw-Hill Education

Mathematics

United Arab Emirates Edition

Advanced Stream

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Project: McGraw-Hill Education United Arab Emirates Edition Grade 11 Advanced Math T3

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mheducation.com/prek-12

McGraw Hill

Printed in the United Arab Emirates.

ISBN: 978-1-39-890811-6 *(Student Edition)*
MHID: 1-39-890811-8 *(Student Edition)*

1 2 3 4 5 6 7 8 9 XXX 22 21 20 19 18 17

ePub Edition

ISBN: 978-1-39-890815-4 *(Student Edition)*
MHID: 1-39-890815-0 *(Student Edition)*
ISBN: 978-1-39-890817-8 *(Teacher Edition)*
MHID: 1-39-890817-7 *(Teacher Edition)*

"Extensive knowledge and modern science must be acquired. The educational process we see today is in an ongoing and escalating challenge which requires hard work.
We succeeded in entering the third millennium, while we are more confident in ourselves."

H.H. Sheikh Khalifa Bin Zayed Al Nahyan
President of the United Arab Emirates

Contents in Brief

CHAPTER 9

Sequences and Series

CHAPTER 10 Statistics and Probability

CHAPTER 9

Sequences and Series

Then

You simplified and evaluated algebraic expressions.

Now

You will:

- Use arithmetic and geometric sequences and series.
- Use special sequences and iterate functions.
- Expand powers by using the Binomial Theorem.
- Prove statements by using mathematical induction.

Why?

CONSERVATION AND NATURE Mathematics occurs in aspects of nature in astonishing ways. The Fibonacci sequence manifests itself in seeds, flowers, pine cones, fruits, and vegetables. Sequences and series can further help us conserve our natural resources by making water filtration systems more efficient.

Get Ready for the Chapter

QuickCheck	**Quick**Review

Solve each equation.

1. $-6 = 7x + 78$
2. $768 = 3x^4$
3. $23 - 5x = 8$
4. $2x^3 + 4 = -50$
5. **PLANTS** Laila has 48 plants for her two gardens. She plants 12 in the small garden. In the other garden she wants 4 plants in each row. How many rows will she have?

Example 1

Solve $25 = 3x^3 + 400$.

$25 = 3x^3 + 400$	Original equation
$-375 = 3x^3$	Subtract 400 from each side.
$-125 = x^3$	Divide each side by 3.
$\sqrt[3]{-125} = \sqrt[3]{x^3}$	Take the cube root of each side.
$-5 = x$	Simplify.

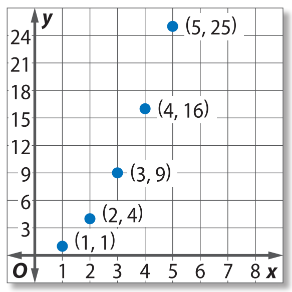
Graph each function.

6. $\{(1, 3), (2, 5), (3, 7), (4, 9), (5, 11)\}$
7. $\{(1, -15), (2, -12), (3, -9), (4, -6), (5, -3)\}$
8. $\left\{(1, 27), (2, 9), (3, 3), (4, 1), \left(5, \frac{1}{3}\right)\right\}$
9. $\left\{(1, 1), (2, 2), \left(3, \frac{5}{2}\right), \left(4, \frac{11}{4}\right), \left(5, \frac{23}{8}\right)\right\}$
10. **DAY CARE** A child care center has expenses of AED 450 per day. They charge AED 150 per child per day. The function $P(c) = 150c - 450$ gives the amount of money the center makes when there are c children there. How much will they make if there are 8 children?

Example 2

Graph the function $\{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$. State the domain and range.

The domain of a function is the set of all possible x-values. So, the domain of the function is $\{1, 2, 3, 4, 5\}$. The range of a function is the set of all possible y-values. So, the range of this function is $\{1, 4, 9, 16, 25\}$.



Evaluate each expression for the given value(s) of the variable(s).

11. $\frac{a}{3}(b + c)$ if $a = 9$, $b = -2$, and $c = -8$
12. $r + (n - 2)t$ if $r = 15$, $n = 5$, and $t = -1$
13. $x \cdot y^{z+1}$ if $x = -2$, $y = \frac{1}{3}$, and $z = 5$
14. $\frac{a(1 - bc)^2}{1 - b}$ if $a = -3$, $b = -4$, and $c = 1$

Example 3

Evaluate $2 \cdot 3^{x+y}$ if $x = -2$ and $y = -3$.

$2 \cdot 3^{x+y} = 2 \cdot 3^{-2+(-3)}$	Substitute.
$= 2 \cdot 3^{-5}$	Simplify.
$= \frac{2}{3^5}$	Rewrite with positive exponent.
$= \frac{2}{243}$	Evaluate the power.

Get Started on the Chapter

You will learn several new concepts, skills, and vocabulary terms as you study Chapter 9. To get ready, identify important terms and organize your resources.

FOLDABLES® **Study**Organizer

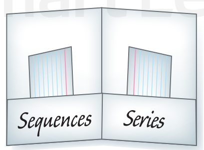
Sequences and Series Make this Foldable to help you organize your Chapter 9 notes about sequences and series. Begin with one $8\frac{1}{2}''$ by $11''$ sheet of paper.

1 **Fold** in half, matching the short sides.

2 **Unfold** and fold the long side up to form a pocket.

3 **Staple** or glue the outer edges to complete the pocket.

4 **Label** each side as shown. Use index cards to record notes and examples.



NewVocabulary

English

sequence
finite sequence
infinite sequence
arithmetic sequence
common difference
geometric sequence
common ratio
arithmetic means
series
arithmetic series
partial sum
geometric means
geometric series
convergent series
divergent series
recursive sequence
iteration
mathematical induction
induction hypothesis

ReviewVocabulary

coefficient the numerical factor of a monomial

coefficient

$15x^3$

formula a mathematical sentence that expresses the relationship between certain quantities

function a relation in which each element of the domain is paired with exactly one element in the range

LESSON 9-1

Sequences as Functions

Then

- You analyzed linear and exponential functions.

Now

1. Relate arithmetic sequences to linear functions.
2. Relate geometric sequences to exponential functions.

Why?

- During their routine, a high school marching band marches in rows. There is one performer in the first row, three performers in the next row, and five in the third row. This pattern continues for the rest of the rows.

NewVocabulary

sequence
term
finite sequence
infinite sequence
arithmetic sequence
common difference
geometric sequence
common ratio

Mathematical Practices

Reason abstractly and quantitatively.

Look for and make use of structure.

1 Arithmetic Sequences

A **sequence** is a set of numbers in a particular order or pattern. Each number in a sequence is called a **term**. A sequence may be a **finite sequence** containing a limited number of terms, such as {−2, 0, 2, 4, 6}, or an **infinite sequence** that continues without end, such as {0, 1, 2, 3, ...}. The first term of a sequence is denoted a_1, the second term is denoted a_2, and so on.

KeyConcept Sequences as Functions

Words A sequence is a function in which the domain consists of natural numbers, and the range consists of real numbers.

Symbols

Domain:	1	2	3	...	n	the position of a term
Range:	a_1	a_2	a_3	...	a_n	the terms of the sequence

Examples

Finite Sequence	Infinite Sequence
{3, 6, 9, 12, 15}	{3, 6, 9, 12, 15, ...}
Domain: {1, 2, 3, 4, 5}	Domain: {all natural numbers}
Range: {3, 6, 9, 12, 15}	Range: {y | y is a multiple of 3, $y \geq 3$}

In an **arithmetic sequence**, each term is determined by adding a constant value to the previous term. This constant value is called the **common difference**.

Consider the sequence 3, 6, 9, 12, 15. This sequence is arithmetic because the terms share a common difference. Each term is 3 more than the previous term.

3 6 9 12 15
+3 +3 +3 +3

Example 1 Identify Arithmetic Sequences

Determine whether each sequence is arithmetic.

a. 5, −6, −17, −28, ...

5 −6 −17 −28
−11 −11 −11

The common difference is −11.
The sequence is arithmetic.

b. −4, 12, 28, 42, ...

−4 12 28 42
+16 +16 +14

There is no common difference.
This is not an arithmetic sequence.

GuidedPractice

1A. 7, 12, 16, 20, ...

1B. −6, 3, 12, 21, ...

You can use the common difference to find terms of an arithmetic sequence.

Example 2 Graph an Arithmetic Sequence

Consider the arithmetic sequence 18, 14, 10, … .

a. Find the next four terms of the sequence.

Step 1 To determine the common difference, subtract any term from the term directly after it. The common difference is 10 − 14 or −4.

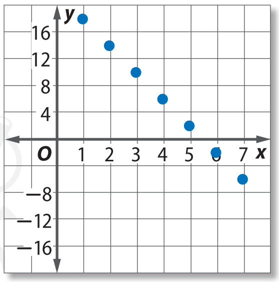
Step 2 To find the next term, add −4 to the last term.

Continue to add −4 to find the following terms.

$$10 \xrightarrow{+(-4)} 6 \xrightarrow{+(-4)} 2 \xrightarrow{+(-4)} -2 \xrightarrow{+(-4)} -6$$

The next four terms are 6, 2, −2, and −6.

b. Graph the first seven terms of the sequence.
The domain contains the terms {1, 2, 3, 4, 5, 6, 7} and the range contains the terms {18, 14, 10, 6, 2, −2, −6}. So, graph the corresponding ordered pairs.



Guided Practice

2. Find the next four terms of the arithmetic sequence 18, 11, 4, … . Then graph the first seven terms.

Notice that the graph of the terms of the arithmetic sequence lie on a line. An arithmetic sequence is a linear function in which the term number n is the independent variable, the term a_n is the dependent variable, and the common difference is the slope.

Real-World Link

Each year, about 100 bands compete in the Bands of America Grand National Championships.

Source: Bands of America

Real-World Example 3 Find a Term

MARCHING BANDS **Refer to the beginning of the lesson. Suppose the director wants to determine how many performers will be in the 14th row during the routine.**

Understand Because the difference between any two consecutive rows is 2, the common difference for the sequence is 2.

Plan Use point-slope form to write an equation for the sequence. Let $m = 2$ and $(x_1, y_1) = (3, 5)$. Then solve for $x = 14$.

Solve

$(y - y_1) = m(x - x_1)$	Point-slope form
$(y - 5) = 2(x - 3)$	$m = 2$ and $(x_1, y_1) = (3, 5)$
$y - 5 = 2x - 6$	Multiply.
$y = 2x - 1$	Add 5 to each side.
$y = 2(14) - 1$	Replace x with 14.
$y = 28 - 1$ or 27	Simplify.

Check You can find the terms of the sequence by adding 2, starting with row 1, until you reach row 14.

Guided Practice

3. MONEY Usama's employer offers him a pay rate of AED 33 per hour with a AED 0.50 raise every three months. How much will Usama earn per hour after 3 years?

2 Geometric Sequences

Another type of sequence is a geometric sequence. In a **geometric sequence**, each term is determined by multiplying a nonzero constant by the previous term. This constant value is called the **common ratio**.

Consider the sequence $\frac{1}{16}, \frac{1}{4}, 1, 4, 16$. This sequence is geometric because the terms share a common ratio. Each term is 4 times as much as the previous term.

$$\frac{1}{16} \xrightarrow{\times 4} \frac{1}{4} \xrightarrow{\times 4} 1 \xrightarrow{\times 4} 4 \xrightarrow{\times 4} 16$$

Example 4 Identify Geometric Sequences

Determine whether each sequence is geometric.

a. **$-2, 6, -18, 54, \ldots$**

Find the ratios of the consecutive terms.

$\frac{6}{-2} = -3 \qquad \frac{-18}{6} = -3 \qquad \frac{54}{-18} = -3$

The ratios are the same, so the sequence is geometric.

b. **$8, 16, 24, 32, \ldots$**

$\frac{16}{8} = 2 \qquad \frac{24}{16} = 1.5 \qquad \frac{32}{24} = 1.\overline{3}$

The ratios are not the same, so the sequence is not geometric.

WatchOut!

Ratios If you find the ratio of a term to the previous term, set up the remaining ratios the same way.

GuidedPractice

4A. $-8, 2, -0.5, 0.125, \ldots$

4B. $1, 3, 7, 15, \ldots$

When given a set of information, you can create a problem that relates a story.

Example 5 Graph a Geometric Sequence

Consider the geometric sequence 32, 8, 2, … .

a. **Find the next three terms of the sequence.**

Step 1 Find the value of the common ratio: $\frac{2}{8}$ or $\frac{1}{4}$.

Step 2 To find the next term, multiply the previous term by $\frac{1}{4}$.

Continue multiplying by $\frac{1}{4}$ to find the following terms.

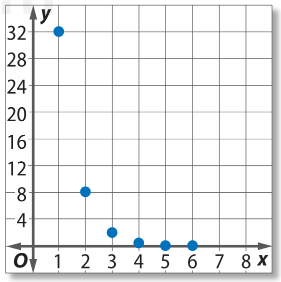
$$2 \xrightarrow{\times \frac{1}{4}} \frac{1}{2} \xrightarrow{\times \frac{1}{4}} \frac{1}{8} \xrightarrow{\times \frac{1}{4}} \frac{1}{32}$$

The next three terms are $\frac{1}{2}, \frac{1}{8}$, and $\frac{1}{32}$.

b. **Graph the first six terms of the sequence.**

Domain: $\{1, 2, 3, 4, 5, 6\}$

Range: $\left\{32, 8, 2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}\right\}$



GuidedPractice

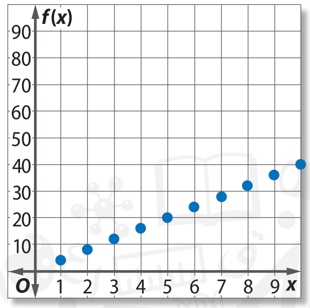
5. Find the next two terms of 7, 21, 63, … . Then graph the first five terms.

Review Vocabulary

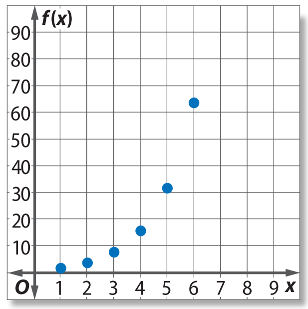
exponential function a function of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$

Examine the graph in Example 5. While the graph of an arithmetic sequence is linear, the graph of a geometric sequence is exponential and can be represented by $f(x) = r^x$, where r is the common ratio, $r > 0$, and $r \neq 1$.

Arithmetic



Geometric



x	1	2	3	4	5	6	7	8	9	10
$f(x)$	4	8	12	16	20	24	28	32	36	40

x	1	2	3	4	5	6
$f(x)$	2	4	8	16	32	64

Arithmetic and geometric sequences are functions in which the domain, defined by the term number n, contains the set of or subset of positive integers. The characteristics of arithmetic and geometric sequences can be used to classify sequences.

Example 6 Classify Sequences

Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

a. 16, 24, 36, 54, …

Check for a common difference.

$54 - 36 = 18 \qquad 36 - 24 = 12$ ✗

Check for a common ratio.

$\frac{54}{36} = \frac{3}{2} \qquad \frac{36}{24} = \frac{3}{2} \qquad \frac{24}{16} = \frac{3}{2}$ ✓

Because there is a common ratio, the sequence is geometric.

b. 1, 4, 9, 16, …

Check for a common difference.

$16 - 9 = 7 \qquad 9 - 4 = 5$ ✗

Check for a common ratio.

$\frac{16}{9} = 1.\overline{7} \qquad \frac{9}{4} = 2.25$ ✗

Because there is no common difference or ratio, the sequence is neither arithmetic nor geometric.

c. 23, 17, 11, 5, …

Check for a common difference.

$5 - 11 = -6 \qquad 11 - 17 = -6 \qquad 17 - 23 = -6$ ✓

Because there is a common difference, the sequence is arithmetic.

Guided Practice

6A. $\frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, \ldots$

6B. $2, -\frac{3}{2}, \frac{9}{8}, -\frac{27}{32}, \ldots$

6C. $-4, 4, 5, -5, \ldots$

Check Your Understanding

Example 1 **Determine whether each sequence is arithmetic. Write *yes* or *no*.**

1. 8, −2, −12, −22,

2. −19, −12, −5, 2, 9

3. 1, 2, 4, 8, 16

4. 0.6, 0.9, 1.2, 1.8, ...

Example 2 **Find the next four terms of each arithmetic sequence. Then graph the sequence.**

5. 6, 18, 30, ...

6. 15, 6, −3, ...

7. −19, −11, −3, ...

8. −26, −33, −40, ...

Example 3

9. **FINANCIAL LITERACY** Yasmin is saving her money to buy a car. She has AED 950, and she plans to save AED 320 per week from her job as a babysitter.

a. How much will Yasmin have saved after 8 weeks?

b. If the car costs AED 7,350, how long will it take her to save enough money at this rate?

Example 4 **Determine whether each sequence is geometric. Write *yes* or *no*.**

10. −8, −5, −1, 4, ...

11. 4, 12, 36, 108, ...

12. 27, 9, 3, 1, ...

13. 7, 14, 21, 28, ...

Example 5 **Find the next three terms of each geometric sequence. Then graph the sequence.**

14. 8, 12, 18, 27, ...

15. 8, 16, 32, 64, ...

16. 250, 50, 10, 2, ...

17. $9, -3, 1, -\frac{1}{3}, \ldots$

Example 6 **Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.**

18. 5, 1, 7, 3, 9, ...

19. 200, −100, 50, −25, ...

20. 12, 16, 20, 24, ...

Practice and Problem Solving

Example 1 **Determine whether each sequence is arithmetic. Write *yes* or *no*.**

21. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

22. −9, −3, 0, 3, 9

23. 14, −5, −19, ...

24. $\frac{2}{9}, \frac{5}{9}, \frac{8}{9}, \frac{11}{9}, \ldots$

Example 2 **Find the next four terms of each arithmetic sequence. Then graph the sequence.**

25. −4, −1, 2, 5, ...

26. 10, 2, −6, −14, ...

27. −5, −11, −17, −23, ...

28. −19, −2, 15, ...

29. $\frac{1}{5}, \frac{4}{5}, \frac{7}{5}, \ldots$

30. $\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3}$

Example 3

31. **THEATER** There are 28 seats in the front row of a theater. Each successive row contains two more seats than the previous row. If there are 24 rows, how many seats are in the last row of the theater?

32. **SENSE-MAKING** Ibrahim began an exercise program to get back in shape. He plans to row 5 minutes on his rowing machine the first day and increase his rowing time by one minute and thirty seconds each day.

a. How long will he row on the 18th day?

b. On what day will Ibrahim first row an hour or more?

c. Is it reasonable for this pattern to continue indefinitely? Explain.

Example 4 **Determine whether each sequence is geometric. Write *yes* or *no*.**

33. 21, 14, 7, ...
34. 124, 186, 248, ...
35. $-27, 18, -12, \ldots$
36. 162, 108, 72, ...
37. $\frac{1}{2}, -\frac{1}{4}, 1, -\frac{1}{2}, \ldots$
38. $-4, -2, 0, 2, \ldots$

Example 5 **Find the next three terms of the sequence. Then graph the sequence.**

39. $0.125, -0.5, 2, \ldots$
40. 18, 12, 8, ...
41. 64, 48, 36, ...
42. 81, 108, 144, ...
43. $\frac{1}{3}, 1, 3, 9, \ldots$
44. 1, 0.1, 0.01, 0.001, ...

Example 6 **Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.**

45. 3, 12, 27, 48, ...
46. $1, -2, -5, -8, \ldots$
47. 12, 36, 108, 324, ...
48. $-\frac{2}{5}, -\frac{2}{25}, -\frac{2}{125}, -\frac{2}{625}, \ldots$
49. $\frac{5}{2}, 3, \frac{7}{2}, 4, \ldots$
50. 6, 9, 14, 21, ...

51. READING Asma took an 800-page book on vacation. If she was already on page 112 and is going to be on vacation for 8 days, what is the minimum number of pages she needs to read per day to finish the book by the end of her vacation?

52. DEPRECIATION Amna's car is expected to depreciate at a rate of 15% per year. If her car is currently valued at AED 88,200, to the nearest dirham, how much will it be worth in 6 years?

53. REGULARITY When a piece of paper is folded onto itself, it doubles in thickness. If a piece of paper that is 0.1 mm thick could be folded 37 times, how thick would it be?

H.O.T. Problems Use Higher-Order Thinking Skills

54. REASONING Explain why the sequence 8, 10, 13, 17, 22 is not arithmetic.

55. OPEN ENDED Describe a real-life situation that can be represented by an arithmetic sequence with a common difference of 8.

56. CHALLENGE The sum of three consecutive terms of an arithmetic sequence is 6. The product of the terms is -42. Find the terms.

57. ERROR ANALYSIS Badr and Salem are determining whether the sequence 8, 8, 8, ... is *arithmetic*, *geometric*, *neither*, or *both*. Is either of them correct? Explain your reasoning.

Badr
The sequence has a common difference of 0. The sequence is arithmetic.

Salem
The sequence has a common ratio of 1. The sequence is geometric.

58. OPEN ENDED Find a geometric sequence, an arithmetic sequence, and a sequence that is neither geometric nor arithmetic that begins 3, 9, ...

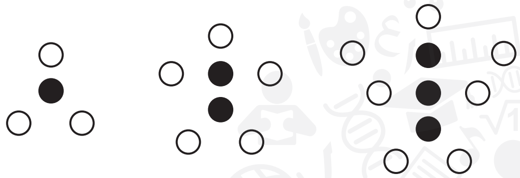
59. REASONING If a geometric sequence has a ratio r such that $|r| < 1$, what happens to the terms as n increases? What would happen to the terms if $|r| \geq 1$?

60. WRITING IN MATH Describe what happens to the terms of a geometric sequence when the common ratio is doubled. What happens when it is halved? Explain your reasoning.

Standardized Test Practice

61. SHORT RESPONSE Fawzia's rectangular bedroom measures 4.5 meters by 3.5 meters. She wants to purchase carpet for the bedroom that costs AED 108 per meter square, including tax. How much will it cost to carpet her bedroom?

62. The pattern of filled circles and white circles below can be described by a relationship between two variables.



Which rule relates w, the number of white circles, to f, the number of dark circles?

A $w = 3f$

B $f = \frac{1}{2}w - 1$

C $w = 2f + 1$

D $f = \frac{1}{3}w$

63. SAT/ACT Rana wanted to determine the average of her six test scores. She added the scores correctly to get T, but divided by 7 instead of 6. Her average was 12 less than the actual average. Which equation could be used to determine the value of T?

F $6T + 12 = 7T$

G $\frac{T}{7} = \frac{T-12}{6}$

H $\frac{T}{7} + 12 = \frac{T}{6}$

J $\frac{T}{6} = \frac{T-12}{7}$

K $\frac{T}{6} = 12 - \frac{T}{7}$

64. Find the next term in the geometric sequence $8, 6, \frac{9}{2}, \frac{27}{8}, \ldots$.

A $\frac{11}{8}$

B $\frac{27}{16}$

C $\frac{9}{4}$

D $\frac{81}{32}$

Spiral Review

Solve each system of equations.

65. $y = 5$
$y^2 = x^2 + 9$

66. $y - x = 1$
$x^2 + y^2 = 25$

67. $3x = 8y^2$
$8y^2 - 2x^2 = 16$

Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

68. $6x^2 + 6y^2 = 162$

69. $4y^2 - x^2 + 4 = 0$

70. $x^2 + y^2 + 6y + 13 = 40$

Graph each function.

71. $f(x) = \frac{6}{(x-2)(x+3)}$

72. $f(x) = \frac{-3}{(x-2)^2}$

73. $f(x) = \frac{x^2 - 36}{x + 6}$

74. HEALTH A certain medication is eliminated from the bloodstream at a steady rate. It decays according to the equation $y = ae^{-0.1625t}$, where t is in hours. Find the half-life of this substance.

Skills Review

Write an equation of each line.

75. passes through $(6, 4)$, $m = 0.5$

76. passes through $\left(2, \frac{1}{2}\right)$, $m = -\frac{3}{4}$

77. passes through $(0, -6)$, $m = 3$

78. passes through $(0, 4)$, $m = \frac{1}{4}$

79. passes through $(1, 3)$ and $\left(8, -\frac{1}{2}\right)$

80. passes through $(-5, 1)$ and $(5, 16)$

LESSON 9-2

Sequences, Series, and Sigma Notation

Then

- You used functions to generate ordered pairs and used graphs to analyze end behavior.

Now

1. Investigate several different types of sequences.
2. Use sigma notation to represent and calculate sums of series.

Why?

- Wafa developed a Web site where students at her high school can post their own social networking Web pages. A student at the high school is given a free page if he or she refers the Web site to five friends. The site starts with one page created by Wafa, who in turn, refers five friends that each create a page. Those five friends refer five more people each, all of whom develop pages, and so on.

NewVocabulary

sequence
term
finite sequence
infinite sequence
recursive sequence
explicit sequence
Fibonacci sequence
converge
diverge
series
finite series
*n*th partial sum
infinite series
sigma notation

1 Sequences

In mathematics, a **sequence** is an ordered list of numbers. Each number in the sequence is known as a **term**. A **finite sequence**, such as 1, 3, 5, 7, 9, 11, contains a finite number of terms. An **infinite sequence**, such as 1, 3, 5, 7, ..., contains an infinite number of terms.

Each term of a sequence is a function of its position. Therefore, an infinite sequence is a function whose domain is the set of natural numbers and can be written as $f(1) = a_1$, $f(2) = a_2$, $f(3) = a_3$, ..., $f(n) = a_n$, ..., where a_n denotes the nth term. If the domain of the function is only the first n natural numbers, the sequence is finite.

Infinitely many sequences exist with the same first few terms. To sufficiently define a *unique* sequence, a formula for the nth term or other information *must* be given. When defined *explicitly*, an **explicit formula** gives the nth term a_n as a function of n.

Example 1 Find Terms of Sequences

a. Find the next four terms of the sequence 2, 7, 12, 17, … .

The nth term of this sequence is not given. One possible pattern is that each term is 5 greater than the previous term. Therefore, a sample answer for the next four terms is 22, 27, 32, and 37.

b. Find the next four terms of the sequence 2, 5, 10, 17, … .

The nth term of this sequence is not given. If we subtract each term from the term that follows, we start to see a possible pattern.

$a_2 - a_1 = 5 - 2$ or 3 $\qquad a_3 - a_2 = 10 - 5$ or 5 $\qquad a_4 - a_3 = 17 - 10$ or 7

It appears that each term is generated by adding the next successive odd number. However, looking at the pattern, it may also be determined that each term is 1 more than each perfect square, or $a_n = n^2 + 1$. Using either pattern, a sample answer for the next four terms is 26, 37, 50, and 65.

c. Find the first four terms of the sequence given by $a_n = 2n(-1)^n$.

Use the explicit formula given to find a_n for $n = 1, 2, 3$, and 4.

$a_1 = 2 \cdot 1 \cdot (-1)^1$ or -2 $\quad n = 1$ $\qquad a_2 = 2 \cdot 2 \cdot (-1)^2$ or 4 $\quad n = 2$

$a_3 = 2 \cdot 3 \cdot (-1)^3$ or -6 $\quad n = 3$ $\qquad a_4 = 2 \cdot 4 \cdot (-1)^4$ or 8 $\quad n = 4$

The first four terms in the sequence are −2, 4, −6, and 8.

GuidedPractice

Find the next four terms of each sequence.

1A. 32, 16, 8, 4, …

1B. 1, 2, 4, 7, 11, 16, 22, …

1C. Find the first four terms of the sequenc e given by $a_n = n^3 - 10$.

Sequences can also be defined *recursively*. Recursively defined sequences give one or more of the first few terms and then define the terms that follow using those previous terms. The formula defining the nth term of the sequence is called a **recursive formula** or a *recurrence relation*.

StudyTip

Notation The term denoted a_n represents the nth term of a sequence. The term denoted a_{n-1} represents the term immediately before a_n. The term a_{n-2} represents the term two terms before a_n.

Example 2 Recursively Defined Sequences

Find the fifth term of the recursively defined sequence $a_1 = 1$, $a_n = a_{n-1} + 2n - 1$, where $n \geq 2$.

Since the sequence is defined recursively, all the terms before the fifth term must be found first. Use the given first term, $a_1 = 2$, and the recursive formula for a_n.

$$a_2 = a_{2-1} + 2(2) - 1 \qquad n = 2$$
$$= a_1 + 3 \qquad \text{Simplify.}$$
$$= 2 + 3 \text{ or } 5 \qquad a_1 = 2$$

$$a_3 = a_{3-1} + 2(3) - 1 \qquad n = 3$$
$$= a_2 + 5 \text{ or } 10 \qquad a_2 = 5$$

$$a_4 = a_{4-1} + 2(4) - 1 \qquad n = 4$$
$$= a_3 + 7 \text{ or } 17 \qquad a_3 = 10$$

$$a_5 = a_{5-1} + 2(5) - 1 \qquad n = 5$$
$$= a_4 + 9 \text{ or } 26 \qquad a_4 = 17$$

GuidedPractice

Find the sixth term of each sequence.

2A. $a_1 = 3$, $a_n = (-2)a_{n-1}$, $n \geq 2$

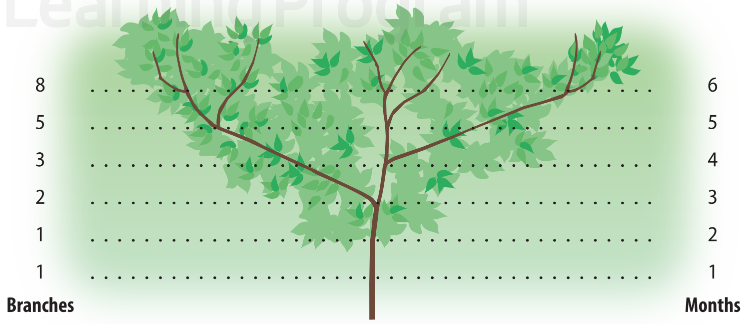
2B. $a_1 = 8$, $a_n = 2a_{n-1} - 7$, $n \geq 2$

The **Fibonacci sequence** describes many patterns found in nature. This sequence is often defined recursively.

Real-World Example 3 Fibonacci Sequence

NATURE **Suppose that when a plant first starts to grow, the stem has to grow for two months before it is strong enough to support branches. At the end of the second month, it sprouts a new branch and will continue to sprout one new branch each month. The new branches also each grow for two months and then start to sprout one new branch each month. If this pattern continues, how many branches will the plant have after 10 months?**

During the first two months, there will only be one branch, the stem. At the end of the second month, the stem will produce a new branch, making the total for the third month two branches. The new branch will grow and develop two months before producing a new branch of its own, but the original branch will now produce a new branch each month.



Real-WorldLink

Along with being found in flower petals, sea shells, and the bones in a human hand, Fibonacci sequences can also be found in pieces of art, music, poetry, and architecture.

Source: *Universal Principles of Design*

WatchOut!

Notation The first term of a sequence is occasionally denoted as a_0. When this occurs, the domain of the function describing the sequence is the set of whole numbers.

The following table shows the pattern.

Month	1	2	3	4	5	6	7	8	9	10
Branches	1	1	2	3	5	8	13	21	34	55

Each term is the sum of the previous two terms. This pattern can be written as the recursive formula $a_0 = 1$, $a_1 = 1$, $a_n = a_{n-2} + a_{n-1}$, where $n \geq 2$.

GuidedPractice

3. NATURE How many branches will a plant like the one described in Example 3 have after 15 months if no branches are removed?

Previously, you examined the end behavior of the graphs of functions. You learned that as the domains of some functions approach ∞, the ranges approach a unique number called a limit. As a function, an infinite sequence may also have a limit. If a sequence has a limit such that the terms approach a unique number, then it is said to **converge**. If not, the sequence is said to **diverge**.

TechnologyTip

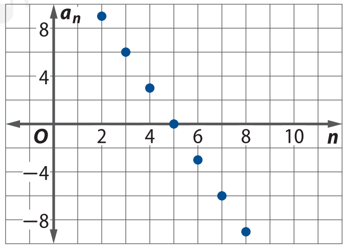
Convergent or Divergent Sequences If an explicit formula for a sequence is known, you can enter the formula in the Y= menu of a graphing calculator and graph the related function. Analyzing the end behavior of the graph can help you to determine whether the sequence is convergent or divergent.

Example 4 Convergent and Divergent Sequences

Determine whether each sequence is *convergent* or *divergent*.

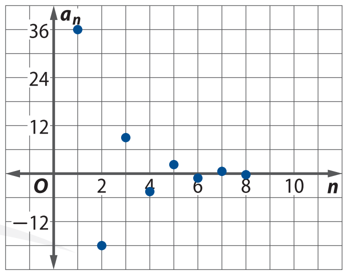
a. $a_n = -3n + 12$

The first eight terms of this sequence are 12, 9, 6, 3, 0, −3, −6, and −9. From the graph at the right, you can see that a_n does not approach a finite number. Therefore, this sequence is divergent.



b. $a_1 = 36$, $a_n = -\frac{1}{2}a_{n-1}$, $a \geq 2$

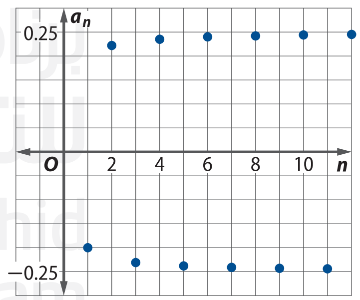
The first eight terms of this sequence are 36, −18, 9, −4.5, 2.25, −1.125, 0.5625, −0.28125, and 0.140625. From the graph at the right, you can see that a_n approaches 0 as n increases. This sequence has a limit and is therefore convergent.



c. $a_n = \dfrac{(-1)^n \cdot n}{4n + 1}$

The first twelve terms of this sequence are given or approximated below.

$a_1 = -0.2$ $\quad a_2 \approx 0.222$
$a_3 \approx -0.231$ $\quad a_4 \approx 0.235$
$a_5 \approx -0.238$ $\quad a_6 = 0.24$
$a_7 \approx -0.241$ $\quad a_8 \approx 0.242$
$a_9 \approx -0.243$ $\quad a_{10} \approx 0.244$
$a_{11} \approx -0.244$ $\quad a_{12} \approx 0.245$



It appears that when n is odd, a_n approaches $-\frac{1}{4}$, and when n is even, a_n approaches $\frac{1}{4}$. Since a_n does not approach one particular value, the sequence has no limit. Therefore, the sequence is divergent.

GuidedPractice

4A. $a_n = \dfrac{64}{2n}$ **4B.** $a_1 = 9$, $a_n = a_{n-1} + 4$ **4C.** $a_n = 3(-1)^n$

2 Series

A **series** is the indicated sum of all of the terms of a sequence. Like sequences, series can be finite or infinite. A **finite series** is the indicated sum of all the terms of a finite sequence, and an **infinite series** is the indicated sum of all the terms of an infinite sequence.

	Sequence	Series
Finite	1, 3, 5, 7, 9	$1 + 3 + 5 + 7 + 9$
Infinite	1, 3, 5, 7, 9, …	$1 + 3 + 5 + 7 + 9 + \ldots$

The sum of the first n terms of a series is called the ***n*th partial sum** and is denoted S_n. The nth partial sum of any series can be found by calculating each term up to the nth term and then finding the sum of those terms.

Example 5 The *n*th Partial Sum

a. Find the fourth partial sum of $a_n = (-2)^n + 3$.

Find the first four terms.

$a_1 = (-2)^1 + 3$ or 1 $\quad n = 1$

$a_2 = (-2)^2 + 3$ or 7 $\quad n = 2$

$a_3 = (-2)^3 + 3$ or -5 $\quad n = 3$

$a_4 = (-2)^4 + 3$ or 19 $\quad n = 4$

The fourth partial sum is $S_4 = 1 + 7 + (-5) + 19$ or 22.

b. Find S_3 of $a_n = \frac{4}{10^n}$.

Find the first three terms.

$a_1 = \frac{4}{10^1}$ or 0.4 $\quad n = 1$

$a_2 = \frac{4}{10^2}$ or 0.04 $\quad n = 2$

$a_3 = \frac{4}{10^3}$ or 0.004 $\quad n = 3$

The third partial sum is $S_3 = 0.4 + 0.04 + 0.004$ or 0.444.

GuidedPractice

5A. Find the sixth partial sum of $a_1 = 8$, $a_n = 0.5(a_{n-1})$, $n \geq 2$.

5B. Find the seventh partial sum of $a_n = 3\left(\frac{1}{10}\right)^n$.

StudyTip

Converging Infinite Sequences While it is necessary for an infinite sequence to converge to 0 in order for the corresponding infinite series to have a sum, it is not sufficient. Some infinite sequences converge to 0 and the corresponding infinite series still do not have sums.

Since an infinite series does not have a finite number of terms, you might assume that an infinite series has no sum S. This is true for the series below.

Infinite Sequence	Infinite Series	Sequence of First Four Partial Sums
1, 4, 7, 10, …	$1 + 4 + 7 + 10 + \ldots$	1, 5, 12, 22, …

However, some infinite series do have sums. For an infinite series to have a fixed sum S, the infinite sequence associated with this series must converge to 0. Notice the sequence of partial sums in the infinite series below appears to approach a sum of $0.\overline{1}$ or $\frac{1}{9}$.

Infinite Sequence	Infinite Series	Sequence of First Three Partial Sums
0.1, 0.01, 0.001, …	$0.1 + 0.01 + 0.001 + \ldots$	0.1, 0.11, 0.111, …

We will take a closer look at sums of infinite sequences in Lesson 9-3.

Series are often more conveniently notated using the uppercase Greek letter sigma Σ. A series written using this letter is said to be expressed using *summation notation* or **sigma notation**.

KeyConcept Sigma Notation

For any sequence $a_1, a_2, a_3, a_4, \ldots$, the sum of the first k terms is denoted

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \ldots + a_k,$$

where n is the index of summation, k is the upper bound of summation, and 1 is the lower bound of summation.

ReadingMath

Sigma Notation $\sum_{n=1}^{k} a_n$ is read *the summation from $n = 1$ to k of a sub n.*

In this notation, the lower bound indicates where to begin summing the terms of the sequence and the upper bound indicates where to end the sum. If the upper bound is given as ∞, the sigma notation represents an infinite series.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$$

Example 6 Sums in Sigma Notation

Find each sum.

a. $\sum_{n=1}^{5} (4n - 3)$

$$\sum_{n=1}^{5} (4n - 3) = [4(1) - 3] + [4(2) - 3] + [4(3) - 3] + [4(4) - 3] + [4(5) - 3]$$

$$= 1 + 5 + 9 + 13 + 17 \text{ or } 45$$

b. $\sum_{n=3}^{7} \frac{6n - 3}{2}$

$$\sum_{n=3}^{7} \frac{6n - 3}{2} = \frac{6(3) - 3}{2} + \frac{6(4) - 3}{2} + \frac{6(5) - 3}{2} + \frac{6(6) - 3}{2} + \frac{6(7) - 3}{2}$$

$$= 7.5 + 10.5 + 13.5 + 16.5 + 19.5 \text{ or } 67.5$$

c. $\sum_{n=1}^{\infty} \frac{7}{10^n}$

$$\sum_{n=1}^{\infty} \frac{7}{10^n} = \frac{7}{10^1} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \frac{7}{10^5} + \ldots$$

$$= 0.7 + 0.07 + 0.007 + 0.0007 + 0.00007 + \ldots$$

$$= 0.77777\ldots \text{ or } \frac{7}{9}$$

WatchOut!

Variations in Sigma Notation The index of summation does not have to be the letter n. It can be represented by any variable. For example, the summation in Example 6a could also be written as $\sum_{i=1}^{5} (4i - 3)$.

GuidedPractice

6A. $\sum_{n=1}^{5} \frac{n^2 - 1}{2}$

6B. $\sum_{n=7}^{13} (n^3 - n^2)$

6C. $\sum_{n=1}^{\infty} \frac{6}{10^n}$

Note that while the lower bound of a summation is often 1, a sum can start with any term p in a sequence as long as $p < k$. In Example 6b, the summation started with the 3rd term of the sequence and ended with the 7th term.

Exercises

Find the next four terms of each sequence. (Example 1)

1. 1, 8, 15, 22, …
2. 3, −6, 12, −24, …
3. 81, 27, 9, 3, …
4. 1, 3, 7, 13, …
5. −2, −15, −28, −41…
6. 1, 4, 10, 19, …

Find the first four terms of each sequence. (Example 1)

7. $a_n = n^2 - 1$
8. $a_n = -2^n + 7$
9. $a_n = \frac{n+7}{9-n}$
10. $a_n = (-1)^{n+1} + n$

11. **AUTOMOBILE LEASES** Lease agreements often contain clauses that limit the number of kilometers driven per year by charging a per-kilometer fee over that limit. For the car shown below, the lease requires that the number of kilometers driven each year must be no more than 15,000. (Example 2)

   a. Write the sequence describing the maximum number of allowed kilometers on the car at the end of every 12 months of the lease if the car has 1350 kilometers at the beginning of the lease.

   b. Write the first 4 terms of the sequence that gives the cumulative cost of the lease for a given month.

   c. Write an explicit formula to represent the sequence in part **b**.

   d. Determine the total amount of money paid by the end of the lease.

Find the specified term of each sequence. (Example 2)

12. 4th term, $a_1 = 5$, $a_n = -3a_{n-1} + 10$, $n \geq 2$
13. 7th term, $a_1 = 14$, $a_n = 0.5a_{n-1} + 3$, $n \geq 2$
14. 4th term, $a_1 = 0$, $a_n = 3^{a_{n-1}}$, $n \geq 2$
15. 3rd term, $a_1 = 3$, $a_n = (a_{n-1})^2 - 5a_{n-1} + 4$, $n \geq 2$

16. **WEB SITE** Wafa, the student from the beginning of the lesson, had great success expanding her Web site. Each student who received a referral developed a Web page and referred five more students to Wafa's site. (Example 3)

   a. List the first five terms of a sequence modeling the number of new Web pages created through Wafa's site.

   b. Suppose the school has 1576 students. After how many rounds of referrals did the entire student body have a Web page?

17. **BEES** Female honeybees come from fertilized eggs (male and female parent), while male honeybees come from unfertilized eggs (one female parent). (Example 3)

   a. Draw a family tree showing the 3 previous generations of a male honeybee (parents only).

   b. Determine the number of parent bees in the 11th previous generation of a male honeybee.

Determine whether each sequence is *convergent* or *divergent*. (Example 4)

18. $a_1 = 4, 1.5a_{n-1}, n \geq 2$
19. $a_n = \frac{5}{10^n}$
20. $a_n = -n^2 - 8n + 106$
21. $a_1 = -64, \frac{3}{4}a_{n-1}, n \geq 2$
22. $a_1 = 1, a_n = 4 - a_{n-1}, n \geq 2$
23. $a_n = n^2 - 3n + 1$
24. $a_n = \frac{n^2+4}{3+n}$
25. $a_1 = 9, a_n = \frac{a_{n-1}+3}{2}, n \geq 2$
26. $a_n = \frac{5n+6}{n}$
27. $a_n = \frac{5n}{5^n} + 1$

Find the indicated sum for each sequence. (Example 5)

28. 5th partial sum of $a_n = n(n-4)(n-3)$
29. 6th partial sum of $a_n = \frac{-5n+3}{n}$
30. S_8 of $a_1 = 1, a_n = a_{n-1} + (18 - n), n \geq 2$
31. S_4 of $a_1 = 64, a_n = -\frac{3}{4}a_{n-1}, n \geq 2$
32. 11th partial sum of $a_1 = 4, a_n = (-1)^{n-1}\left(|a_{n-1}| + 3\right), n \geq 2$
33. S_9 of $a_1 = -35, a_n = a_{n-1} + 8, n \geq 2$
34. 4th partial sum of $a_1 = 3, a_n = (a_{n-1} - 2)^3, n \geq 2$
35. S_4 of $a_n = \frac{(-3)^n}{10}$

Find each sum. (Example 6)

36. $\sum_{n=1}^{8} (6n - 11)$
37. $\sum_{n=4}^{11} (30 - 4n)$
38. $\sum_{n=1}^{7} [n^2(n-5)]$
39. $\sum_{n=2}^{7} (n^2 - 6n + 1)$
40. $\sum_{n=8}^{15} \left(\frac{n}{4} - 7\right)$
41. $\sum_{n=1}^{10} [(n-4)^2(n-5)]$
42. $\sum_{n=0}^{6} [(-2)^n - 9]$
43. $\sum_{n=1}^{3} 7\left(\frac{1}{10}\right)^{2n}$
44. $\sum_{n=1}^{\infty} 5\left(\frac{1}{10^n}\right)$
45. $\sum_{n=1}^{\infty} \frac{8}{10^n}$

46. **FINANCIAL LITERACY** Mazen's bank account had an initial deposit of AED 380, earning 3.5% interest per year compounded annually.

   a. Find the balance each year for the first five years.

   b. Write a recursive and an explicit formula defining his account balance.

   c. For very large values of n, which formula gives a more accurate balance? Explain.

47. **INVESTING** Hiyam invests AED 200 every 3 months. The investment pays an annual percentage rate of 8%, and the interest is compounded quarterly. If Hiyam makes each payment at the beginning of the quarter and the interest is posted at the end of the quarter, what will the total value of the investment be after 2 years?

48. **RIDES** The table shows the number of riders of the Mean Streak each year from 1998 to 2007. This ridership data can be approximated by $a_n = -\frac{1}{20}n + 1.3$, where $n = 1$ represents 1998, $n = 2$ represents 1999, and so on.

Mean Streak Roller Coaster

Year	Number of Riders (millions)	Year	Number of Riders (millions)
1998	1.31	2003	0.99
1999	1.15	2004	0.95
2000	1.14	2005	0.89
2001	1.09	2006	0.81
2002	1.05	2007	0.82

Source: *Cedar Fair Entertainment Company*

a. Sketch a graph of the number of riders from 1998 to 2007. Then determine whether the sequence appears to be *convergent* or *divergent*. Does this make sense in the context of the situation? Explain your reasoning.

b. Use the table to find the total number of riders from 1998 to 2005. Then use the explicit sequence to find the 8th partial sum of a_n. Compare the results.

c. If the sequence continues, find a_{14}. What does this number represent?

Copy and complete the table.

	Recursive Formula	Sequence	Explicit Formula
49.		6, 8, 10, 12, ...	
50.	$a_1 = 15, a_n = a_{n-1} - 1, n \geq 2$		
51.		7, 21, 63, 189, ...	
52.			$a_n = 10(-2)^n$
53.			$a_n = 8n - 3$
54.	$a_1 = 2, a_n = 4a_{n-1}, n \geq 2$		
55.	$a_1 = 3, a_n = a_{n-1} + 2n - 1, n \geq 2$		
56.			$a_n = n^2 + 1$

Write each series in sigma notation. The lower bound is given.

57. $-2 - 1 + 0 + 1 + 2 + 3 + 4 + 5;\ n = 1$

58. $\frac{1}{20} + \frac{1}{25} + \frac{1}{30} + \frac{1}{35} + \frac{1}{40} + \frac{1}{45};\ n = 4$

59. $8 + 27 + 64 + \cdots + 1000;\ n = 2$

60. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{128};\ n = 1$

61. $-8 + 16 - 32 + 64 - 128 + 256 - 512;\ n = 3$

62. $8\left(-\frac{1}{3}\right) + 8\left(\frac{1}{9}\right) + 8\left(-\frac{1}{27}\right) + \cdots + 8\left(-\frac{1}{243}\right);\ n = 1$

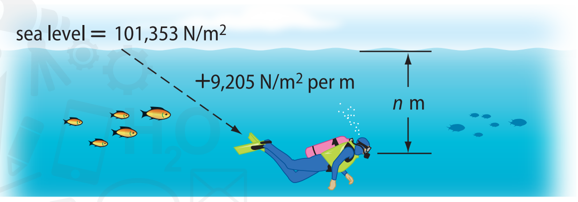
Determine whether each sequence is *convergent* or *divergent*. Then find the fifth partial sum of the sequence.

63. $a_n = \sin \frac{n\pi}{2}$

64. $a_n = n \cos \pi$

65. $a_n = e^{-\frac{n}{2}} \cos \pi n$

66. **WATER PRESSURE** The pressure exerted on the human body at sea level is 101,353 newton per square meter (N/m^2). For each additional meter below sea level, the pressure is about 9,205 N/m^2 greater, as shown.



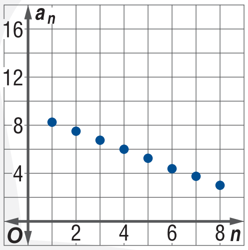
a. Write a recursive formula to represent a_n, the pressure at n meters below sea level. (*Hint*: Let $a_0 = 14.7$.)

b. Write the first three terms of the sequence and describe what they represent.

c. Scuba divers cannot safely dive deeper than 100 meters. Write an explicit formula to represent a_n. Then use the formula to find the water pressure at 100 meters below sea level.

Match each sequence with its graph.

67.



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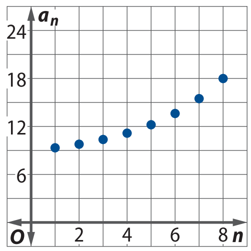

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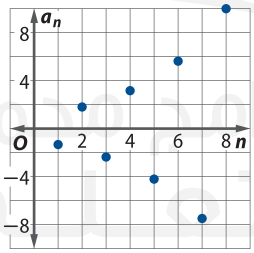

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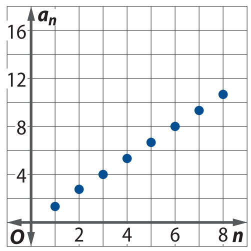

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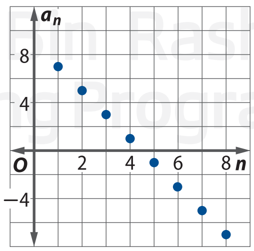

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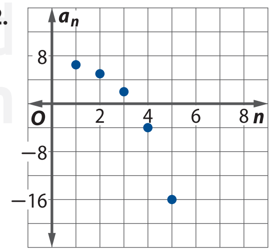


a. $a_n = \frac{4}{3}n$

b. $a_n = -\frac{3}{4}n + 9$

c. $a_n = \left(-\frac{4}{3}\right)^n$

d. $a_n = 8 - \frac{3}{4}(2^n)$

e. $a_n = 9 - 2n$

f. $a_n = \left(\frac{4}{3}\right)^n + 8$

73. **GOLDEN RATIO** Consider the Fibonacci sequence 1, 1, 2, 3, …, $a_{n-2} + a_{n-1}$.

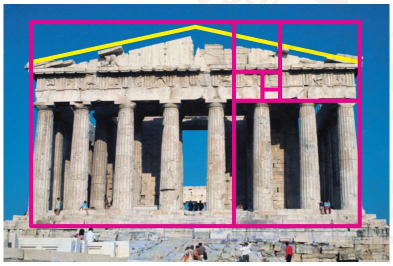
a. Find $\frac{a_n}{a_{n-1}}$ for the second through eleventh terms of the Fibonacci sequence.

b. Sketch a graph of the terms found in part **a.** Let $n - 1$ be the x-coordinate and $\frac{a_n}{a_{n-1}}$ be the y-coordinate.

c. Based on the graph found in part **b,** does this sequence appear to be convergent? If so, describe the limit to three decimal places. If not, explain why not.

d. In a *golden rectangle*, the ratio of the length to the width is about 1.61803399. This is called the *golden ratio*. How does the limit of the sequence $\frac{a_n}{a_{n-1}}$ compare to the golden ratio?

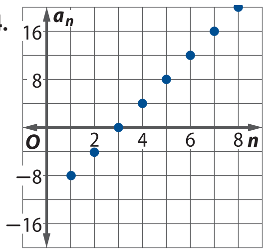
e. Golden rectangles are common in art and architecture. The Parthenon, in Greece, is an example of how golden rectangles are used in architecture.



Research golden rectangles and find two more examples of golden rectangles in art or architecture.

Determine whether each sequence is *convergent* or *divergent*.

74.



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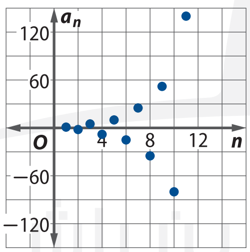

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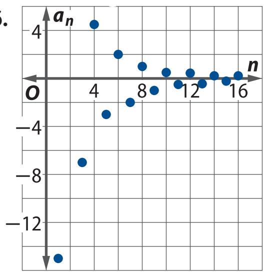

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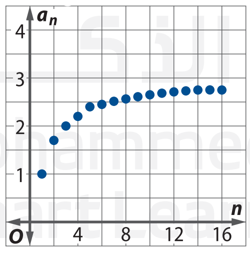


Write an explicit formula for each recursively defined sequence.

78. $a_1 = 10$; $a_n = a_{n-1} + 5$

79. $a_1 = 1.25$; $a_n = a_{n-1} - 0.5$

80. $a_1 = 128$; $a_n = 0.5a_{n-1}$

81. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate sums of infinite series.

a. **NUMERICAL** Calculate the first five terms of the infinite sequence $a_n = \frac{4}{10^n}$.

b. **GRAPHICAL** Use a graphing calculator to sketch $y = \frac{4}{10^x}$.

c. **VERBAL** Describe what is happening to the terms of the sequence as $n \to \infty$.

d. **NUMERICAL** Find the sum of the first 5 terms, 7 terms, and 9 terms of the series.

e. **VERBAL** Describe what is happening to the partial sums S_n as n increases.

f. **VERBAL** Predict the sum of the first n terms of the series. Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

82. **CHALLENGE** Consider the recursive sequence below.

$$a_n = a_{n - a_{n-1}} + a_{n - a_{n-2}} \text{ for } a_1 = 1,\ a_2 = 1,\ n \geq 3$$

a. Find the first eight terms of the sequence.

b. Describe the similarities and differences between this sequence and the other recursive sequences in this lesson.

83. **OPEN ENDED** Write a sequence either recursively or explicitly that has the following characteristics.

a. converges to 0

b. converges to 3

c. diverges

84. **WRITING IN MATH** Describe why an infinite sequence must not only converge, but converge to 0, in order for there to be a sum.

REASONING **Determine whether each statement is *true* or *false*. Explain your reasoning.**

85. $\sum_{n=1}^{5} (n^2 + 3n) = \sum_{n=1}^{5} n^2 + 3\sum_{n=1}^{5} n$

86. $\sum_{n=1}^{5} 3^n = \sum_{n=3}^{7} 3^{n-2}$

87. **CHALLENGE** Find the sum of the first 60 terms of the sequence below. Explain how you determined your answer.

$$15, 17, 2, -15, -17, \ldots,$$
$$\text{where } a_n = a_{n-1} - a_{n-2} \text{ for } n \geq 3$$

88. **WRITING IN MATH** Make an outline that could be used to describe the steps involved in finding the 300th partial sum of the infinite sequence $a_n = 2n - 3$. Then explain how to express the same sum using sigma notation.

Spiral Review

Graph each complex number on a polar grid. Then express it in rectangular form.

89. $2\left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right)$

90. $2.5\,(\cos 1 + i \sin 1)$

91. $5\,(\cos 0 + i \sin 0)$

Determine the eccentricity, type of conic, and equation of the directrix given by each polar equation.

92. $r = \dfrac{3}{2 - 0.5 \cos \theta}$

93. $r = \dfrac{6}{1.2 \sin \theta + 0.3}$

94. $r = \dfrac{1}{0.2 - 0.2 \sin \theta}$

Determine whether the points are collinear. Write *yes* or *no*.

95. $(-3, -1, 4), (3, 8, 1), (5, 12, 0)$

96. $(4, 8, 6), (0, 6, 12), (8, 10, 0)$

97. $(0, -4, 3), (8, -10, 5), (12, -13, 2)$

98. $(-7, 2, -1), (-9, 3, -4), (-5, 1, 2)$

Find the length and the midpoint of the segment with the given endpoints.

99. $(2, -15, 12), (1, -11, 15)$

100. $(-4, 2, 8), (9, 6, 0)$

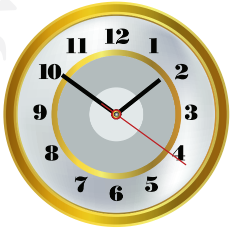
101. $(7, 1, 5), (-2, -5, -11)$

102. TIMING The path traced by the tip of the hour-hand of a clock can be modeled by a circle with parametric equations $x = 6 \sin t$ and $y = 6 \cos t$.

a. Find an interval for t in radians that can be used to describe the motion of the tip as it moves from 12 o'clock noon to 12 o'clock noon the next day.

b. Simulate the motion described in part **a** by graphing the equation in parametric mode on a graphing calculator.

c. Write an equation in rectangular form that models the motion of the hour-hand. Find the radius of the circle traced out by the hour-hand if x and y are given in centimeters.



Find the exact value of each expression.

103. $\tan \frac{\pi}{12}$

104. $\sin 75^\circ$

105. $\cos 165^\circ$

Find the partial fraction decomposition of each rational expression.

106. $\dfrac{10x^2 - 11x + 4}{2x^2 - 3x + 1}$

107. $\dfrac{1}{2x^2 + x}$

108. $\dfrac{x + 1}{x^3 + x}$

Skills Review for Standardized Tests

109. SAT/ACT The first term in a sequence is -5, and each subsequent term is 6 more than the term that immediately precedes it. What is the value of the 104th term?

A 607

B 613

C 618

D 619

E 615

110. REVIEW Find the exact value of $\cos 2\theta$ if $\sin \theta = -\frac{\sqrt{5}}{3}$ and $180^\circ < \theta < 270^\circ$.

F $-\frac{\sqrt{6}}{6}$

G $-\frac{4\sqrt{5}}{9}$

H $-\frac{\sqrt{30}}{6}$

J $-\frac{1}{9}$

111. The first four terms of a sequence are 144, 72, 36, and 18. What is the tenth term in the sequence?

A 0

B $\frac{9}{64}$

C $\frac{9}{32}$

D $\frac{9}{16}$

112. REVIEW How many 5-centimeter cubes can be stacked inside a box that is 10 centimeters long, 15 centimeters wide, and 5 centimeters tall?

F 5

G 6

H 15

J 20

LESSON

9-3 Arithmetic Sequences and Series

Then

- You determined whether a sequence was arithmetic.

Now

1. Find the nth term and arithmetic means for arithmetic sequences.
2. Find sums of arithmetic series.

Why?

- In the 18th century, a teacher asked his class of elementary students to find the sum of the counting numbers 1 through 100. A pupil named Karl Gauss correctly answered within seconds, astonishing the teacher. Gauss went on to become a great mathematician.

  He solved this problem by using an arithmetic series.

NewVocabulary
arithmetic means
series
arithmetic series
partial sum
sigma notation

Mathematical Practices
Look for and express regularity in repeated reasoning.

1 Arithmetic Sequences

In Lesson 9-1, you used the point-slope form to find a specific term of an arithmetic sequence. It is possible to develop an equation for any term of an arithmetic sequence using the same process.

Consider the arithmetic sequence $a_1, a_2, a_3, \ldots, a_n$ in which the common difference is d.

$(y - y_1) = m(x - x_1)$ Point-slope form

$(a_n - a_1) = d(n - 1)$ $(x, y) = (n, a_n)$, $(x_1, y_1) = (1, a_1)$, and $m = d$

$a_n = a_1 + d(n - 1)$ Add a_1 to each side.

You can use this equation to find any term in an arithmetic sequence when you know the first term and the common difference.

KeyConcept ***n*th Term of an Arithmetic Sequence**

The nth term a_n of an arithmetic sequence in which the first term is a_1 and the common difference is d is given by the following formula, where n is any natural number.

$$a_n = a_1 + (n - 1)d$$

You will prove this formula in Exercise 80.

Example 1 **Find the *n*th Term**

Find the 12th term of the arithmetic sequence 9, 16, 23, 30, … .

Step 1 Find the common difference.

$16 - 9 = 7 \qquad 23 - 16 = 7 \qquad 30 - 23 = 7$

So, $d = 7$.

Step 2 Find the 12th term.

$a_n = a_1 + (n - 1)d$ nth term of an arithmetic sequence

$a_{12} = 9 + (12 - 1)(7)$ $a_1 = 9$, $d = 7$, and $n = 12$

$= 9 + 77$ or 86 Simplify.

GuidedPractice

Find the indicated term of each arithmetic sequence.

1A. $a_1 = -4$, $d = 6$, $n = 9$

1B. a_{20} for $a_1 = 15$, $d = -8$

If you are given some terms of an arithmetic sequence, you can write an equation for the nth term of the sequence.

Example 2 Write Equations for the nth Term

Write an equation for the nth term of each arithmetic sequence.

a. $5, -13, -31, \ldots$

$d = -13 - 5$ or -18; 5 is the first term.

$a_n = a_1 + (n - 1)d$	nth term of an arithmetic sequence
$a_n = 5 + (n - 1)(-18)$	$a_1 = 5$ and $d = -18$
$a_n = 5 + (-18n + 18)$	Distributive Property
$a_n = -18n + 23$	Simplify.

b. $a_5 = 19, d = 6$

First, find a_1.

$a_n = a_1 + (n - 1)d$	nth term of an arithmetic sequence
$19 = a_1 + (5 - 1)(6)$	$a_5 = 19$, $n = 5$, and $d = 6$
$19 = a_1 + 24$	Multiply.
$-5 = a_1$	Subtract 24 from each side.

Then write the equation.

$a_n = a_1 + (n - 1)d$	nth term of an arithmetic sequence
$a_n = -5 + (n - 1)(6)$	$a_1 = -5$ and $d = 6$
$a_n = -5 + (6n - 6)$	Distributive Property
$a_n = 6n - 11$	Simplify.

Guided Practice

2A. $12, 3, -6, \ldots$

2B. $a_6 = 12, d = 8$

> **Study Tip**
>
> **Checking Solutions** Check your solution by using it to determine the first three terms of the sequence.

Sometimes you are given two terms of a sequence, but they are not consecutive terms of the sequence. The terms between any two nonconsecutive terms of an arithmetic sequence, called **arithmetic means**, can be used to find missing terms of a sequence.

> **Reading Math**
>
> **arithmetic mean** the average of two or more numbers
>
> **arithmetic means** the terms between any two nonconsecutive terms of an arithmetic sequence

Example 3 Find Arithmetic Means

Find the arithmetic means in the sequence -8, __?__, __?__, __?__, __?__, 22, … .

Step 1 Since there are four terms between the first and last terms given, there are $4 + 2$ or 6 total terms, so $n = 6$.

Step 2 Find d.

$a_n = a_1 + (n - 1)d$	nth term of an arithmetic sequence
$22 = -8 + (6 - 1)d$	$a_1 = -8$, $a_6 = 22$, and $n = 6$
$30 = 5d$	Distributive Property
$6 = d$	Divide each side by 5.

Step 3 Use d to find the four arithmetic means.

$$-8 \xrightarrow{+6} -2 \xrightarrow{+6} 4 \xrightarrow{+6} 10 \xrightarrow{+6} 16 \xrightarrow{+6} 22$$

The arithmetic means are -2, 4, 10, and 16.

Guided Practice

3. Find the five arithmetic means between -18 and 36.

2 Arithmetic Series

A **series** is formed when the terms of a sequence are added. An **arithmetic series** is the sum of the terms of an arithmetic sequence. The sum of the first n terms is called the **partial sum** and is denoted S_n.

KeyConcept Partial Sum of an Arithmetic Series

Formula	Given	The sum S_n of the first n terms is:
General	a_1 and a_n	$S_n = n\left(\frac{a_1 + a_n}{2}\right)$
Alternate	a_1 and d	$S_n = \frac{n}{2}[2a_1 + (n-1)d]$

Sometimes a_1, a_n, or n must be determined before the sum of an arithmetic series can be found. When this occurs, use the formula for the nth term.

Example 4 Use the Sum Formulas

Find the sum of 12 + 19 + 26 + ... + 180.

Step 1 $a_1 = 12$, $a_n = 180$, and $d = 19 - 12$ or 7.

We need to find n before we can use one of the formulas.

$a_n = a_1 + (n-1)d$	nth term of an arithmetic sequence
$180 = 12 + (n-1)(7)$	$a_n = 180$, $a_1 = 12$, and $d = 7$
$168 = 7n - 7$	Simplify.
$25 = n$	Solve for n.

Step 2 Use either formula to find S_n.

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$	Sum formula
$S_{25} = \frac{25}{2}[2(12) + (25-1)(7)]$	$n = 25$, $a_1 = 12$, and $d = 7$
$S_{25} = 12.5(192)$ or 2400	Simplify.

GuidedPractice

Find the sum of each arithmetic series.

4A. $2 + 4 + 6 + ... + 100$

4B. $n = 16$, $a_n = 240$, and $d = 8$.

You can use a sum formula to find terms of a series.

> **WatchOut!**
> **Common Difference** Don't confuse the sign of the common difference in an arithmetic sequence. Check that the rule actually produces the terms of a sequence.

Example 5 Find the First Three Terms

Find the first three terms of the arithmetic series in which $a_1 = 7$, $a_n = 79$, and $S_n = 430$.

Step 1 Find n.

$S_n = n\left(\frac{a_1 + a_n}{2}\right)$	Sum formula
$430 = n\left(\frac{7 + 79}{2}\right)$	$S_n = 430$, $a_1 = 7$, and $a_n = 79$
$430 = n(43)$	Simplify.
$10 = n$	Divide each side by 43.

Step 2 Find d.

$$a_n = a_1 + (n-1)d \qquad \text{nth term of an arithmetic sequence}$$
$$79 = 7 + (10-1)d \qquad a_n = 79,\ a_1 = 7, \text{ and } n = 10$$
$$72 = 9d \qquad \text{Subtract 7 from each side.}$$
$$8 = d \qquad \text{Divide each side by 9.}$$

Step 3 Use d to determine a_2 and a_3.

$a_2 = 7 + 8$ or 15 $\qquad a_3 = 15 + 8$ or 23

The first three terms are 7, 15, and 23.

GuidedPractice

Find the first three terms of each arithmetic series.

5A. $S_n = 120,\ n = 8,\ a_n = 36$ $\qquad$ **5B.** $a_1 = -24,\ a_n = 288,\ S_n = 5280$

The sum of a series can be written in shorthand by using **sigma notation**.

ReadingMath

Sigma Notation The name comes from the Greek letter sigma, which is used in the notation.

KeyConcept Sigma Notation

Symbols

last value of k

first value of k

$$\sum_{k=1}^{n} f(k)$$

formula for the terms of the series

Example

$$\sum_{k=1}^{12}(4k+2) = [4(1)+2] + [4(2)+2] + [4(3)+2] + \cdots + [4(12)+2]$$
$$= 6 + 10 + 14 + \cdots + 50$$

Standardized Test Example 6 Use Sigma Notation

Find $\sum_{k=4}^{18}(6k-1)$.

A 846 $\qquad$ **B** 910 $\qquad$ **C** 975 $\qquad$ **D** 1008

Read the Test Item

You need to find the sum of the series. Find a_1, a_n, and n.

Solve the Test Item

There are $18 - 4 + 1$ or 15 terms, so $n = 15$.

$a_1 = 6(4) - 1$ or 23 $\qquad a_n = 6(18) - 1$ or 107

Find the sum.

$$S_n = n\left(\frac{a_1 + a_n}{2}\right) \qquad \text{Sum formula}$$
$$S_{15} = 15\left(\frac{23 + 107}{2}\right) \qquad n = 15,\ a_1 = 23, \text{ and } a_n = 107$$
$$S_{15} = 15(65) \text{ or } 975 \qquad \text{The correct answer is C.}$$

Test-TakingTip

Perseverance

Sometimes it is necessary to break a problem into parts, solve each part, then combine the solutions of the parts.

GuidedPractice

6. Find $\sum_{m=9}^{21}(5m+6)$.

F 972 $\qquad$ **G** 1053 $\qquad$ **H** 1281 $\qquad$ **J** 1701

Check Your Understanding

Example 1 **Find the indicated term of each arithmetic sequence.**

1. $a_1 = 14$, $d = 9$, $n = 11$

2. a_{18} for 12, 25, 38, …

Example 2 **Write an equation for the *n*th term of each arithmetic sequence.**

3. 13, 19, 25, …

4. $a_5 = -12$, $d = -4$

Example 3 **Find the arithmetic means in each sequence.**

5. 6, $\underline{?}$, $\underline{?}$, $\underline{?}$, 42

6. −4, $\underline{?}$, $\underline{?}$, $\underline{?}$, 8

Example 4 **Find the sum of each arithmetic series.**

7. the first 50 natural numbers

8. $4 + 8 + 12 + \cdots + 200$

9. $a_1 = 12$, $a_n = 188$, $d = 4$

10. $a_n = 145$, $d = 5$, $n = 21$

Example 5 **Find the first three terms of each arithmetic series.**

11. $a_1 = 8$, $a_n = 100$, $S_n = 1296$

12. $n = 18$, $a_n = 112$, $S_n = 1098$

Example 6 **13. MULTIPLE CHOICE** Find $\sum_{k=1}^{12} (3k + 9)$.

A 45

B 78

C 342

D 410

Practice and Problem Solving

Example 1 **Find the indicated term of each arithmetic sequence.**

14. $a_1 = -18$, $d = 12$, $n = 16$

15. $a_1 = -12$, $n = 66$, $d = 4$

16. $a_1 = 9$, $n = 24$, $d = -6$

17. a_{15} for −5, −12, −19, …

18. a_{10} for −1, 1, 3, …

19. a_{24} for 8.25, 8.5, 8.75, …

Example 2 **Write an equation for the *n*th term of each arithmetic sequence.**

20. 24, 35, 46, …

21. 31, 17, 3, …

22. $a_9 = 45$, $d = -3$

23. $a_7 = 21$, $d = 5$

24. $a_4 = 12$, $d = 0.25$

25. $a_5 = 1.5$, $d = 4.5$

26. 9, 2, −5, …

27. $a_6 = 22$, $d = 9$

28. $a_8 = -8$, $d = -2$

29. $a_{15} = 7$, $d = \frac{2}{3}$

30. −12, −17, −22, …

31. $a_3 = -\frac{4}{5}$, $d = \frac{1}{2}$

32. STRUCTURE Jamal averaged 123 total pins per game in his bowling league this season. He is taking bowling lessons and hopes to bring his average up by 8 pins each new season.

a. Write an equation to represent the *n*th term of the sequence.

b. If the pattern continues, during what season will Jamal average 187 per game?

c. Is it reasonable for this pattern to continue indefinitely? Explain.

Example 3 **Find the arithmetic means in each sequence.**

33. 24, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, −1

34. −6, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, 49

35. −28, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, 7

36. 84, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, 39

37. −12, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, −66

38. 182, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, 104

Example 4 **Find the sum of each arithmetic series.**

39. the first 100 even natural numbers

40. the first 200 odd natural numbers

41. the first 100 odd natural numbers

42. the first 300 even natural numbers

43. $-18 + (-15) + (-12) + \cdots + 66$

44. $-24 + (-18) + (-12) + \cdots + 72$

45. $a_1 = -16$, $d = 6$, $n = 24$

46. $n = 19$, $a_n = 154$, $d = 8$

47. **CONTESTS** The prizes in a weekly radio contest began at AED 150 and increased by AED 50 for each week that the contest lasted. If the contest lasted for eleven weeks, how much was awarded in total?

Example 5 **Find the first three terms of each arithmetic series.**

48. $n = 32$, $a_n = -86$, $S_n = 224$

49. $a_1 = 48$, $a_n = 180$, $S_n = 1368$

50. $a_1 = 3$, $a_n = 66$, $S_n = 759$

51. $n = 28$, $a_n = 228$, $S_n = 2982$

52. $a_1 = -72$, $a_n = 453$, $S_n = 6858$

53. $n = 30$, $a_n = 362$, $S_n = 4770$

54. $a_1 = 19$, $n = 44$, $S_n = 9350$

55. $a_1 = -33$, $n = 36$, $S_n = 6372$

56. **PRIZES** A radio station is offering a total of AED 8,500 in prizes over ten hours. Each hour, the prize will increase by AED 100. Find the amounts of the first and last prize.

Example 6 **Find the sum of each arithmetic series.**

57. $\sum_{k=1}^{16} (4k - 2)$

58. $\sum_{k=4}^{13} (4k + 1)$

59. $\sum_{k=5}^{16} (2k + 6)$

60. $\sum_{k=0}^{12} (-3k + 2)$

61. **FINANCIAL LITERACY** Najla borrowed some money from her parents. She agreed to pay AED 50 at the end of the first month and AED 25 more each additional month for 12 months. How much does she pay in total after the 12 months?

62. **GRAVITY** When an object is in free fall and air resistance is ignored, it falls 16 meters in the first second, an additional 48 meters during the next second, and 80 meters during the third second. How many total meters will the object fall in 10 seconds?

Use the given information to write an equation that represents the *n*th term in each arithmetic sequence

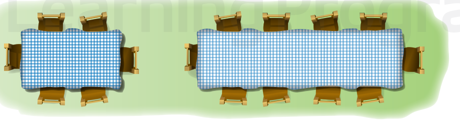
63. The 100th term of the sequence is 245. The common difference is 13.

64. The eleventh term of the sequence is 78. The common difference is -9.

65. The sixth term of the sequence is -34. The 23rd term is 119.

66. The 25th term of the sequence is 121. The 80th term is 506.

67. **MODELING** The rectangular tables in a reception hall are often placed end-to-end to form one long table. The diagrams below show the number of people who can sit at each of the table arrangements.



a. Make drawings to find the next three numbers as tables are added one at a time to the arrangement.

b. Write an equation representing the *n*th number in this pattern.

c. Is it possible to have seating for exactly 100 people with such an arrangement? Explain.

68. **PERFORMANCE** A certain company pays its employees according to their performance. Badria is paid a flat rate of AED 800 per week plus AED 96 for every unit she completes. If she earned AED 2,048 in one week, how many units did she complete?

69. **SALARY** Tarek currently earns AED 112,000 per year. If Tarek expects a AED 16,000 increase in salary every year, after how many years will he have a salary of AED 400,000 per year?

70. **SPORTS** While training for cross country, Sindiyya plans to run 3 kilometers per day for the first week, and then increase the distance by a half kilometer each of the following weeks.

a. Write an equation to represent the nth term of the sequence.

b. If the pattern continues, during which week will she be running 10 kilometers per day?

c. Is it reasonable for this pattern to continue indefinitely? Explain.

71. **MULTIPLE REPRESENTATIONS** Consider $\sum_{k=1}^{x}(2k+2)$.

a. **Tabular** Make a table of the partial sums of the series for $1 \leq k \leq 10$.

b. **Graphical** Graph (k, partial sum).

c. **Graphical** Graph $f(x) = x^2 + 3x$ on the same grid.

d. **Verbal** What do you notice about the two graphs?

e. **Analytical** What conclusions can you make about the relationship between quadratic functions and the sum of arithmetic series?

f. **Algebraic** Find the arithmetic series that relates to $g(x) = x^2 + 8x$.

Find the value of x.

72. $\sum_{k=3}^{x}(6k-5) = 928$

73. $\sum_{k=5}^{x}(8k+2) = 1032$

H.O.T. Problems Use Higher-Order Thinking Skills

74. **CRITIQUE** Eissa and Jassim are determining the formula for the nth term for the sequence $-11, -2, 7, 16, \ldots$. Is either of them correct? Explain your reasoning.

Eissa

$d = 16 - 7$ or 9, $a_1 = -11$

$a_n = -11 + (n-1)9$

$= 9n - 20$

Jassim

$d = 16 - 7$ or 9, $a_1 = -11$

$a_n = 9n - 11$

75. **REASONING** If a is the third term in an arithmetic sequence, b is the fifth term, and c is the eleventh term, express c in terms of a and b.

76. **CHALLENGE** There are three arithmetic means between a and b in an arithmetic sequence. The average of the arithmetic means is 16. What is the average of a and b?

77. **CHALLENGE** Find S_n for $(x + y) + (x + 2y) + (x + 3y) + \ldots$.

78. **OPEN ENDED** Write an arithmetic series with 8 terms and a sum of 324.

79. **WRITING IN MATH** Compare and contrast arithmetic sequences and series.

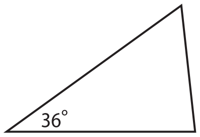
80. **PROOF** Prove the formula for the nth term of an arithmetic sequence.

81. **PROOF** Derive a sum formula that does not include a_1.

82. **PROOF** Derive the Alternate Sum Formula using the General Sum Formula.

Standardized Test Practice

83. SAT/ACT The measures of the angles of a triangle form an arithmetic sequence. If the measure of the smallest angle is 36°, what is the measure of the largest angle?



A 54°
B 75°
C 84°
D 90°
E 97°

84. The area of a triangle is $\frac{1}{2}q^2 - 8$ and the height is $q + 4$. Which expression best describes the triangle's base?

F $(q + 1)$ **H** $(q - 3)$
G $(q + 2)$ **J** $(q - 4)$.

85. The expression $1 + \sqrt{2} + \sqrt[3]{3}$ is equivalent to

A $\sum_{k=1}^{3} k^{\frac{1}{k}}$ **C** $\sum_{k=1}^{3} k^{-k}$

B $\sum_{k=1}^{3} k^{k}$ **D** $\sum_{k=1}^{3} \sqrt{k}$

86. SHORT RESPONSE Ahmed can type a 200-word essay in 6 hours. Husam can type the same essay in $4\frac{1}{2}$ hours. If they work together, how many hours will it take them to type the essay?

Spiral Review

Determine whether each sequence is arithmetic. Write *yes* or *no*. (Lesson 9-1)

87. $-6, 4, 14, 24, \ldots$ **88.** $2, \frac{7}{5}, \frac{4}{5}, \frac{1}{5}, \ldots$ **89.** $10, 8, 5, 1, \ldots$

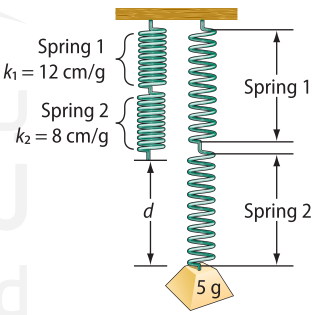
Solve each system of inequalities by graphing.

90. $x + 2y > 1$
$x^2 + y^2 \leq 25$

91. $x + y \leq 2$
$4x^2 - y^2 \geq 4$

92. $x^2 + y^2 \geq 4$
$4y^2 + 9x^2 \leq 36$

93. PHYSICS The distance a spring stretches is related to the mass attached to the spring. This is represented by $d = km$, where d is the distance, m is the mass, and k is the spring constant. When two springs with spring constants k_1 and k_2 are attached in a series, the resulting spring constant k is found by the equation $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$.



a. If one spring with constant of 12 centimeters per gram is attached in a series with another spring with constant of 8 centimeters per gram, find the resultant spring constant.

b. If a 5-gram object is hung from the series of springs, how far will the springs stretch? Is this answer reasonable in this context?

Graph each function. State the domain and range.

94. $f(x) = \frac{2}{3}(2^x)$ **95.** $f(x) = 4^x + 3$ **96.** $f(x) = 2\left(\frac{1}{3}\right)^x - 1$

Skills Review

Solve each equation. Round to the nearest ten-thousandth.

97. $5^x = 52$ **98.** $4^{3p} = 10$ **99.** $3^{n+2} = 14.5$ **100.** $16^{d-4} = 3^{3-d}$

LESSON 9-4

Geometric Sequences and Series

Then

- You determined whether a sequence was geometric.

Now

1. Find the nth term and geometric means for geometric sequences.
2. Find sums of geometric series.

Why?

- Hasan sees a new book in a bookshop. He e-mails a link for the author's Web site to five of his friends. They each forward the link to five of their friends. The link is forwarded again following the same pattern. How many people will receive the link on the eighth round of e-mails?

NewVocabulary
geometric means
geometric series

Mathematical Practices
Look for and express regularity in repeated reasoning.

1 Geometric Sequences

As with arithmetic sequences, there is a formula for the nth term of a geometric sequence. This formula can be used to determine any term of the sequence.

KeyConcept *n*th Term of a Geometric Sequence

The nth term a_n of a geometric sequence in which the first term is a_1 and the common ratio is r is given by the following formula, where n is any natural number.

$$a_n = a_1 r^{n-1}$$

You will prove this formula in Exercise 68.

Real-World Example 1 Find the *n*th Term

MUSIC **If the pattern continues, how many e-mails will be sent in the eighth round?**

Understand We need to determine the number of forwarded e-mails on the eighth round. Five e-mails were sent on the first round. Each of the five recipients sent five e-mails on the second round, and so on.

Plan This is a geometric sequence, and the common ratio is 5. Use the formula for the nth term of a geometric sequence.

Solve

$a_n = a_1 r^{n-1}$	nth term of a geometric sequence
$a_8 = 5(5)^{8-1}$	$a_1 = 5$, $r = 5$, and $n = 8$
$a_8 = 5(78{,}125)$ or $390{,}625$	$5^7 = 78{,}125$

Check Write out the first eight terms by multiplying by the common ratio.

5, 25, 125, 625, 3125, 15,625, 78,125, 390,625

There will be 390,625 e-mails sent on the 8th round.

GuidedPractice

1. **E-MAILS** Sumayya receives a joke in an e-mail that asks her to forward it to four of her friends. She forwards it, then each of her friends forwards it to four of their friends, and so on. If the pattern continues, how many people will receive the e-mail on the ninth round of forwarding?

If you are given some of the terms of a geometric sequence, you can determine an equation for finding the nth term of the sequence.

Example 2 Write an Equation for the nth Term

Write an equation for the nth term of each geometric sequence.

a. 0.5, 2, 8, 32, ...

$r = 8 \div 2$ or 4; 0.5 is the first term.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$a_n = 0.5(4)^{n-1}$ — $a_1 = 0.5$ and $r = 4$

b. $a_4 = 5$ and $r = 6$

Step 1 Find a_1.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$5 = a_1(6^{4-1})$ — $a_n = 5$, $r = 6$, and $n = 4$

$5 = a_1(216)$ — Evaluate the power.

$\frac{5}{216} = a_1$ — Divide each side by 216.

Step 2 Write the equation.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$a_n = \frac{5}{216}(6)^{n-1}$ — $a_1 = \frac{5}{216}$ and $r = 6$

GuidedPractice

Write an equation for the nth term of each geometric sequence.

2A. $-0.25, 2, -16, 128, \ldots$

2B. $a_3 = 16$, $r = 4$

Math HistoryLink

Archytas (428–347 B.C.) Geometric sequences, or geometric progressions, were first studied by the Greek mathematician Archytas. His studies of these sequences came from his interest in music and octaves.

Like arithmetic means, **geometric means** are the terms between two nonconsecutive terms of a geometric sequence. The common ratio r can be used to find the geometric means.

Example 3 Find Geometric Means

Find three geometric means between 2 and 1250.

Step 1 Since there are three terms between the first and last term, there are $3 + 2$ or 5 total terms, so $n = 5$.

Step 2 Find r.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$1250 = 2r^{5-1}$ — $a_n = 1250$, $a_1 = 2$, and $n = 5$

$625 = r^4$ — Divide each side by 2.

$\pm 5 = r$ — Take the 4th root of each side.

Step 3 Use r to find the four arithmetic means.

$2 \xrightarrow{\times 5} 10 \xrightarrow{\times 5} 50 \xrightarrow{\times 5} 250 \xrightarrow{\times 5} 1250$ or $2 \xrightarrow{\times -5} -10 \xrightarrow{\times -5} 50 \xrightarrow{\times -5} -250 \xrightarrow{\times -5} 1250$

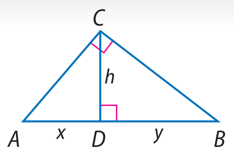
The geometric means are 10, 50, and 250 or -10, 50, and -250.

GuidedPractice

3. Find four geometric means between 0.5 and 512.

ReadingMath

Geometric Means A geometric mean can also be represented geometrically. In the figure below, h is the geometric mean between x and y.



2 Geometric Series

A **geometric series** is the sum of the terms of a geometric sequence. The sum of the first n terms of a series is denoted S_n. You can use either of the following formulas to find the partial sum S_n of the first n terms of a geometric series.

KeyConcept Partial Sum of a Geometric Series

Given	The sum S_n of the first n terms is:
a_1 and n	$S_n = \frac{a_1 - a_1 r^n}{1 - r}, r \neq 1$
a_1 and a_n	$S_n = \frac{a_1 - a_n r}{1 - r}, r \neq 1$

Real-World Example 4 Find the Sum of a Geometric Series

MUSIC Refer to the beginning of the lesson. If the pattern continues, what is the total number of e-mails sent in the eight rounds?

Five e-mails are sent in the first round and there are 8 rounds of e-mails. So, $a_1 = 5$, $r = 5$ and $n = 8$.

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ Sum formula

$S_8 = \frac{5 - 5 \cdot 5^8}{1 - 5}$ $a_1 = 5, r = 5$, and $n = 8$

$S_8 = \frac{-1{,}953{,}120}{-4}$ Simplify the numerator and denominator.

$S_8 = 488{,}280$ Divide.

There will be 488,280 e-mails sent after 8 rounds.

GuidedPractice

Find the sum of each geometric series.

4A. $a_1 = 2, n = 10, r = 3$

4B. $a_1 = 2000, a_n = 125, r = \frac{1}{2}$

As with arithmetic series, sigma notation can also be used to represent geometric series.

WatchOut!

Sigma Notation Notice in Example 5 that you are being asked to evaluate the sum from the 3rd term to the 10th term.

Example 5 Sum in Sigma Notation

Find $\sum_{k=3}^{10} 4(2)^{k-1}$.

Find a_1, r, and n. In the first term, $k = 3$ and $a_1 = 4 \cdot 2^{3-1}$ or 16. The base of the exponential function is r, so $r = 2$. There are $10 - 3 + 1$ or 8 terms, so $n = 8$.

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ Sum formula

$= \frac{16 - 16(2)^8}{1 - 2}$ $a_1 = 16, r = 2$, and $n = 8$

$= 4080$ Use a calculator.

GuidedPractice

Find each sum.

5A. $\sum_{k=4}^{12} \frac{1}{4} \cdot 3^{k-1}$

5B. $\sum_{k=2}^{9} \frac{2}{3} \cdot 4^{k-1}$

You can use the formula for the sum of a geometric series to help find a particular term of the series.

Example 6 Find the First Term of a Series

Find a_1 in a geometric series for which $S_n = 13{,}116$, $n = 7$, and $r = 3$.

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} \quad \text{Sum formula}$$

$$13{,}116 = \frac{a_1 - a_1(3^7)}{1 - 3} \quad S_n = 13{,}116,\ r = 3, \text{ and } n = 7$$

$$13{,}116 = \frac{a_1(1 - 3^7)}{1 - 3} \quad \text{Distributive Property}$$

$$13{,}116 = \frac{-2186a_1}{-2} \quad \text{Subtract.}$$

$$13{,}116 = 1093a_1 \quad \text{Simplify.}$$

$$12 = a_1 \quad \text{Divide each side by 1093.}$$

GuidedPractice

6. Find a_1 in a geometric series for which $S_n = -26{,}240$, $n = 8$, and $r = -3$.

Check Your Understanding

Example 1

1. REGULARITY Ismail is making a family tree for his grandfather. He was able to trace many generations. If Ismail could trace his family back 10 generations, starting with his parents how many ancestors would there be?

Example 2

Write an equation for the nth term of each geometric sequence.

2. 2, 4, 8, ...

3. 18, 6, 2, ...

4. $-4, 16, -64, \ldots$

5. $a_2 = 4$, $r = 3$

6. $a_6 = \frac{1}{8}$, $r = \frac{3}{4}$

7. $a_2 = -96$, $r = -8$

Example 3

Find the geometric means of each sequence.

8. 0.25, _?_, _?_, _?_, 64

9. 0.20, _?_, _?_, _?_, 125

Example 4

10. GAMES Muna arranges some rows of dominoes so that after she knocks over the first one, each domino knocks over two more dominoes when it falls. If there are ten rows, how many dominoes does Muna use?

Example 5

Find the sum of each geometric series.

11. $\sum_{k=1}^{6} 3(4)^{k-1}$

12. $\sum_{k=1}^{8} 4\left(\frac{1}{2}\right)^{k-1}$

Example 6

Find a_1 for each geometric series described.

13. $S_n = 85\frac{5}{16}$, $r = 4$, $n = 6$

14. $S_n = 91\frac{1}{12}$, $r = 3$, $n = 7$

15. $S_n = 1020$, $a_n = 4$, $r = \frac{1}{2}$

16. $S_n = 121\frac{1}{3}$, $a_n = \frac{1}{3}$, $r = \frac{1}{3}$

Practice and Problem Solving

Example 1

17. **WEATHER** Heavy rain in Bilal's town caused the river to rise. The river rose three centimeters the first day, and each day after rose twice as much as the previous day. How much did the river rise in five days?

Find a_n for each geometric sequence.

18. $a_1 = 2400, r = \frac{1}{4}, n = 7$

19. $a_1 = 800, r = \frac{1}{2}, n = 6$

20. $a_1 = \frac{2}{9}, r = 3, n = 7$

21. $a_1 = -4, r = -2, n = 8$

22. **BIOLOGY** A certain bacteria grows at a rate of 3 cells every 2 minutes. If there were 260 cells initially, how many are there after 21 minutes?

Example 2

Write an equation for the nth term of each geometric sequence.

23. $-3, 6, -12, \ldots$

24. $288, -96, 32, \ldots$

25. $-1, 1, -1, \ldots$

26. $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \ldots$

27. $8, 2, \frac{1}{2}, \ldots$

28. $12, -16, \frac{64}{3}, \ldots$

29. $a_3 = 28, r = 2$

30. $a_4 = -8, r = 0.5$

31. $a_6 = 0.5, r = 6$

32. $a_3 = 8, r = \frac{1}{2}$

33. $a_4 = 24, r = \frac{1}{3}$

34. $a_4 = 80, r = 4$

Example 3

Find the geometric means of each sequence.

35. 810, ?, ?, ?, 10

36. 640, ?, ?, ?, 2.5

37. $\frac{7}{2}$, ?, ?, ?, $\frac{56}{81}$

38. $\frac{729}{64}$, ?, ?, ?, $\frac{324}{9}$

39. Find two geometric means between 3 and 375.

40. Find two geometric means between 16 and -2.

Example 4

41. **PERSEVERANCE** A certain water filtration system can remove 70% of the contaminants each time a sample of water is passed through it. If the same water is passed through the system four times, what percent of the original contaminants will be removed from the water sample?

Find the sum of each geometric series.

42. $a_1 = 36, r = \frac{1}{3}, n = 8$

43. $a_1 = 16, r = \frac{1}{2}, n = 9$

44. $a_1 = 240, r = \frac{3}{4}, n = 7$

45. $a_1 = 360, r = \frac{4}{3}, n = 8$

46. **VACUUMS** A vacuum claims to pick up 80% of the dirt every time it is run over the carpet. Assuming this is true, what percent of the original amount of dirt is picked up after the seventh time the vacuum is run over the carpet?

Example 5

Find the sum of each geometric series.

47. $\sum_{k=1}^{7} 4(-3)^{k-1}$

48. $\sum_{k=1}^{8} (-3)(-2)^{k-1}$

49. $\sum_{k=1}^{9} (-1)(4)^{k-1}$

50. $\sum_{k=1}^{10} 5(-1)^{k-1}$

Example 6

Find a_1 for each geometric series described.

51. $S_n = -2912, r = 3, n = 6$

52. $S_n = -10{,}922, r = 4, n = 7$

53. $S_n = 1330, a_n = 486, r = \frac{3}{2}$

54. $S_n = 4118, a_n = 128, r = \frac{2}{3}$

55. $a_n = 1024, r = 8, n = 5$

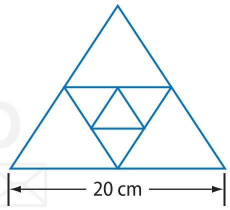
56. $a_n = 1875, r = 5, n = 7$

57. **SCIENCE** One minute after it is released, a gas-filled balloon has risen 100 meters. In each succeeding minute, the balloon rises only 50% as far as it rose in the previous minute. How far will it rise in 5 minutes?

58. **CHEMISTRY** Radon has a half-life of about 4 days. This means that about every 4 days, half of the mass of radon decays into another element. How many grams of radon remain from an initial 60 grams after 4 weeks?

59. **REASONING** A virus goes through a computer, infecting the files. If one file was infected initially and the total number of files infected doubles every minute, how many files will be infected in 20 minutes?

60. **GEOMETRY** In the figure, the sides of each equilateral triangle are twice the size of the sides of its inscribed triangle. If the pattern continues, find the sum of the perimeters of the first eight triangles.



61. **PENDULUMS** The first swing of a pendulum travels 30 centimeters. If each subsequent swing travels 95% as far as the previous swing, find the total distance traveled by the pendulum after the 30th swing.

62. **PHONE CHAINS** A school established a phone chain in which every staff member calls two other staff members to notify them when the school closes due to weather. The first round of calls begins with the superintendent calling both principals. If there are 94 total staff members and employees at the school, how many rounds of calls are there?

63. **TELEVISIONS** High Tech Electronics advertises a weekly installment plan for the purchase of a popular brand of high definition television. The buyer pays AED 15 at the end of the first week, AED 16.50 at the end of the second week, AED 18.15 at the end of the third week, and so on for one year. (Assume that 1 year = 52 weeks.)

   a. What will the payments be at the end of the 10th, 20th, and 40th weeks?

   b. Find the total cost of the TV.

   c. Why is the cost found in part **b** not entirely accurate?

H.O.T. Problems Use Higher-Order Thinking Skills

64. **PROOF** Derive the General Sum Formula using the Alternate Sum Formula.

65. **PROOF** Derive a sum formula that does not include a_1.

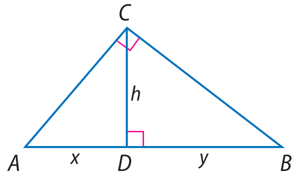
66. **OPEN ENDED** Write a geometric series for which $r = \frac{3}{4}$ and $n = 6$.

67. **REASONING** Explain how $\sum_{k=1}^{10} 3(2)^{k-1}$ needs to be altered to refer to the same series if $k = 1$ changes to $k = 0$. Explain your reasoning.

68. **PROOF** Prove the formula for the nth term of a geometric sequence.

69. **CHALLENGE** The fifth term of a geometric sequence is $\frac{1}{27}$th of the eighth term. If the ninth term is 702, what is the eighth term?

70. **CHALLENGE** Use the fact that h is the geometric mean between x and y in the figure at the right to find h^4 in terms of x and y.



71. **OPEN ENDED** Write a geometric series with 6 terms and a sum of 252.

72. **WRITING IN MATH** How can you classify a sequence? Explain your reasoning.

Standardized Test Practice

73. Which of the following is closest to $\sqrt[3]{7.32}$?

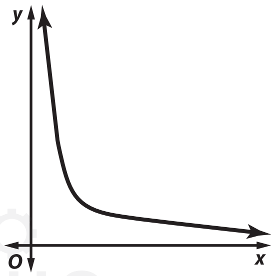
A 1.8
B 1.9
C 2.0
D 2.1

74. The first term of a geometric series is 5, and the common ratio is −2. How many terms are in the series if its sum is −6825?

F 5
G 9
H 10
J 12

75. SHORT RESPONSE Ayesha has a savings account. She withdraws half of the contents every year. After 4 years, she has AED 2,000 left. How much did she have in the savings account originally?

76. SAT/ ACT The curve below could be part of the graph of which function?



A $y = \sqrt{x}$
B $y = x^2 - 5x + 4$
C $y = -x + 20$
D $y = \log x$
E $xy = 4$

Spiral Review

77. MONEY Manal bought a high-definition LCD television at the electronics store. She paid AED 800 immediately and AED 300 each month for a year and a half. How much did Manal pay in total for the TV? (Lesson 9-2)

Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning. (Lesson 9-1)

78. $\frac{1}{10}, \frac{3}{5}, \frac{7}{20}, \frac{17}{20}, \ldots$

79. $-\frac{7}{25}, -\frac{13}{50}, -\frac{6}{25}, -\frac{11}{50}, \ldots$

80. $-\frac{22}{3}, -\frac{68}{9}, -\frac{208}{27}, -\frac{632}{81}, \ldots$

Find the center and radius of each circle. Then graph the circle.

81. $(x - 3)^2 + (y - 1)^2 = 25$

82. $(x + 3)^2 + (y + 7)^2 = 81$

83. $(x - 3)^2 + (y + 7)^2 = 50$

84. Suppose y varies jointly as x and z. Find y when $x = 9$ and $z = -5$, if $y = -90$ when $z = 15$ and $x = -6$.

85. SHOPPING A certain store found that the number of customers who will attend a sale can be modeled by $N = 125\sqrt[3]{100Pt}$, where N is the number of customers expected, P is the percent of the sale discount, and t is the number of hours the sale will last. Find the number of customers the store should expect for a sale that is 50% off and will last four hours.

Skills Review

Evaluate each expression if $a = -2$, $b = \frac{1}{3}$, and $c = -12$.

86. $\frac{3ab}{c}$

87. $\frac{a - c}{a + c}$

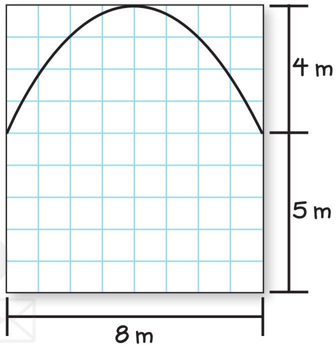
88. $\frac{a^3 - c}{b^2}$

89. $\frac{c + 3}{ab}$

EXPLORE 9-5

Algebra Lab
Area Under a Curve

A soccer stadium is being redesigned so that there is an archway above the main entrance. A scale drawing of the archway is created in which each line on the grid paper represents one meter of the actual archway. The designer modeled the shape of the top with the quadratic equation $y = -0.25x^2 + 3x$.



Activity

Find the area of the opening under the archway.

Method 1

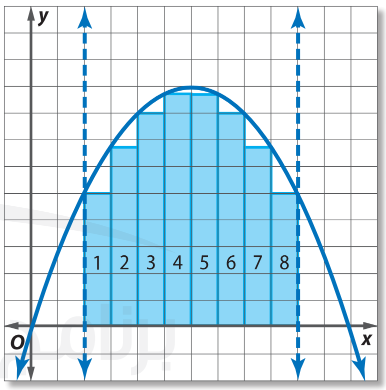
Step 1 Make a table of values for $y = -0.25x^2 + 3x$. Then graph the equation.

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	2.75	5	6.75	8	8.75	9	8.75	8	6.75	5	2.75	0

Step 2 Divide the figure into regions.

To estimate the area inside the archway, you can divide the archway into rectangles as shown in blue.

Because the left and right sides of the archway are 5 meters high and $y = 5$ when $x = 2$ and when $x = 10$, the opening of the entrance extends from $x = 2$ to $x = 10$.



Step 3 Find the area of the regions.

Rectangle	1	2	3	4	5	6	7	8
Width (m)	1	1	1	1	1	1	1	1
Height (m)	5	6.75	8	8.75	8.75	8	6.75	5
Area (m^2)	5	6.75	8	8.75	8.75	8	6.75	5

The approximate area of the archway is the sum of the areas of the rectangles.

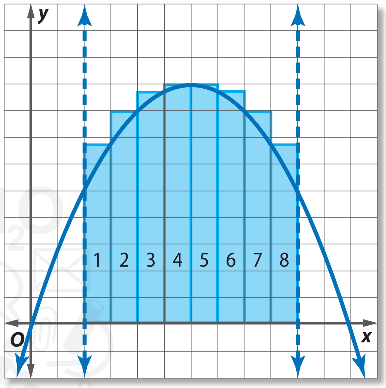
$5 + 6.75 + 8 + 8.75 + 8.75 + 8 + 6.75 + 5 = 57$ m^2

(continued on the next page)

Area Under a Curve *Continued*

Method 2

Step 1 Draw a second graph of the equation and divide into regions. Divide the archway into rectangles as shown in blue.



Step 2 Find the area of the regions.

Rectangle	1	2	3	4	5	6	7	8
Width (m)	1	1	1	1	1	1	1	1
Height (m)	6.75	8	8.75	9	9	8.75	8	6.75
Area (m^2)	6.75	8	8.75	9	9	8.75	8	6.75

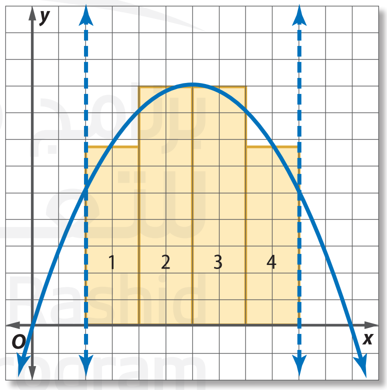
The approximate area of the archway is the sum of the areas of the rectangles.

$6.75 + 8 + 8.75 + 9 + 9 + 8.75 + 8 + 6.75 = 65 \text{ m}^2$

Both Method 1 and Method 2 illustrate how to approximate the area under a curve over a specified interval.

Analyze the Results

1. Is the area of the regions calculated using Method 1 greater than or less than the actual area of the archway? Explain your reasoning.
2. Is the area of the regions calculated using Method 2 greater than or less than the actual area of the archway? Explain your reasoning.
3. Compare the area estimates for both methods. How could you find the best estimate for the area inside the archway? Explain your reasoning.
4. The diagram shows a third method for finding an estimate of the area of the archway. Is this estimate for the area greater than or less than the actual area? How does this estimate compare to the other two estimates of the area?



Exercises

Estimate the area described by any method. Make a table of values, draw graphs with rectangles, and make a table for the areas of the rectangles. Compare each estimate to the actual area.

5. the area under the curve for $y = -x^2 + 4$, from $x = -2$ to $x = 2$, and above the x–axis
6. the area under the curve for $y = x^3$, from $x = 0$ to $x = 4$, and above the x–axis
7. the area under the curve for $y = x^2$, from $x = -3$ to $x = 3$, and above the x–axis

LESSON 9-5

Infinite Geometric Series

Then

- You found sums of finite geometric series.

Now

1. Find sums of infinite geometric series.
2. Write repeating decimals as fractions.

Why?

- In a game of American Football, with their opponent on the 10-yard line, the defense is penalized half the distance to the goal, placing the ball on the 5-yard line. If they continue to be penalized in this way, where will the ball eventually be placed? Will they ever reach the goal line? How many total penalty yards will the defense have incurred? These questions can be answered by looking at infinite geometric series.

NewVocabulary

infinite geometric series
convergent series
divergent series
infinity

Mathematical Practices

Attend to precision.
Look for and express regularity in repeated reasoning.

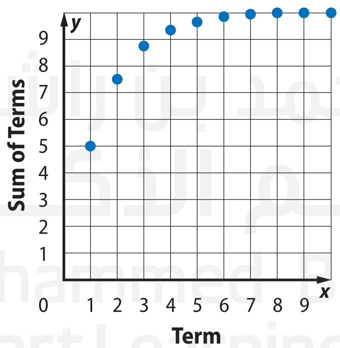
1 Infinite Geometric Series

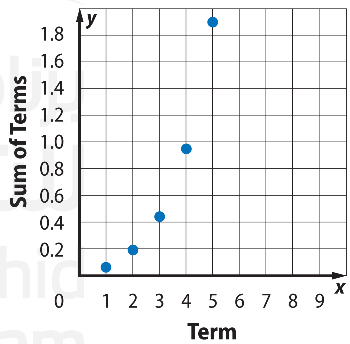
An **infinite geometric series** has an infinite number of terms. A series that has a sum is a **convergent series**, because its sum converges to a specific value. A series that does not have a sum is a **divergent series**.

When you evaluated the sum S_n of an infinite geometric series for the first n terms, you were finding the partial sum of the series. It is also possible to find the sum of an entire series. In the application above, it seems that the ball will eventually reach the goal line, and the defense will be penalized a total of 10 yards. This value is the actual sum of the infinite series $5 + 2.5 + 1.25 + \ldots$. The graph of S_n for $1 \leq n \leq 10$ is shown on the left below. As n increases, S_n approaches 10.

KeyConcept Convergent and Divergent Series

Convergent Series	Divergent Series
Words The sum approaches a finite value.	**Words** The sum does not approach a finite value.
Ratio $\lvert r \rvert < 1$	**Ratio** $\lvert r \rvert \geq 1$
Example $5 + 2.5 + 1.25 + \ldots$	**Example** $\frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \ldots$





Example 1 Convergent and Divergent Series

Determine whether each infinite geometric series is *convergent* or *divergent*.

a. $54 + 36 + 24 + \ldots$

Find the value of r.

$r = \frac{36}{54}$ or $\frac{2}{3}$; since $-1 < \frac{2}{3} < 1$, the series is convergent.

b. $8 + 12 + 18 + \ldots$

$r = \frac{12}{8}$ or 1.5; since $1.5 > 1$, the series is divergent.

GuidedPractice

1A. $2 + 3 + 4.5 + \ldots$

1B. $100 + 50 + 25 + \ldots$

StudyTip

Absolute Value Recall that $|r| < 1$ means $-1 < r < 1$.

When $|r| < 1$, the value of r^n will approach 0 as n increases. Therefore, the partial sums of the infinite geometric series will approach $\frac{a_1 - a_1(0)}{1 - r}$ or $\frac{a_1}{1 - r}$.

KeyConcept Sum of an Infinite Geometric Series

The sum S of an infinite geometric series with $|r| < 1$ is given by

$$S = \frac{a_1}{1 - r}.$$

If $|r| \geq 1$, the series has no sum.

When an infinite geometric series is divergent, $|r| \geq 1$ and the series has no sum because the absolute value of r^n will increase infinitely as n increases.

The table at the right shows the partial sums for the divergent series $4 + 16 + 64 + \ldots$. As n increases, S_n increases rapidly without limit.

n	S_n
5	1364
10	1,398,100
15	1,431,655,764

Example 2 Sum of an Infinite Series

Find the sum of each infinite series, if it exists.

Determine whether each infinite geometric series is *convergent* or *divergent*.

a. $\frac{2}{3} + \frac{6}{15} + \frac{18}{75} + \ldots$

Step 1 Find the value of r to determine if the sum exists.

$r = \frac{6}{15} \div \frac{2}{3}$ or $\frac{3}{5}$ Divide consecutive terms.

Since $\left|\frac{3}{5}\right| < 1$, the sum exists.

Step 2 Use the formula to find the sum.

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{\frac{2}{3}}{1 - \frac{3}{5}}$ $a_1 = \frac{2}{3}$ and $r = \frac{3}{5}$

$= \frac{2}{3} \div \frac{2}{5}$ or $\frac{5}{3}$ Simplify.

b. $6 + 9 + 13.5 + 20.25 + \ldots$

$r = \frac{9}{6}$ or 1.5; since $|1.5| \geq 1$, the series diverges and the sum does not exist.

GuidedPractice

2A. $4 - 2 + 1 - 0.5 + \ldots$

2B. $16 + 20 + 25 + \ldots$

StudyTip

Convergence and Divergence A series converges when the absolute value of a term is smaller than the absolute value of the previous term. An infinite arithmetic series will always be divergent.

Sigma notation can be used to represent infinite series. If a sequence goes to **infinity**, it continues without end. The infinity symbol ∞ is placed above the $\sum$ to indicate that a series is infinite.

Example 3 Infinite Series in Sigma Notation

Find $\sum_{k=1}^{\infty} 18\left(\frac{4}{5}\right)^{k-1}$.

$S = \dfrac{a_1}{1-r}$ — Sum formula

$= \dfrac{18}{1-\frac{4}{5}}$ — $a_1 = 18$ and $r = \frac{4}{5}$

$= \dfrac{18}{\frac{1}{5}}$ or 90 — Simplify.

GuidedPractice

3. Find $\sum_{k=1}^{\infty} 12\left(\frac{3}{4}\right)^{k-1}$.

2 Repeating Decimals

A repeating decimal is the sum of an infinite geometric series. For instance, $0.\overline{45} = 0.454545...$ or $0.45 + 0.0045 + 0.000045 + ...$. The formula for the sum of an infinite series can be used to convert the decimal to a fraction.

Problem-SolvingTip

Sense-Making In many cases, it is possible to solve a problem in more than one way. Use the method with which you are most comfortable.

Example 4 Write a Repeating Decimal as a Fraction

Write $0.\overline{63}$ as a fraction.

Method 1 Use the sum of an infinite series.

$$0.\overline{63} = 0.63 + 0.0063 + ...$$

$$= \frac{63}{100} + \frac{63}{10{,}000} + ...$$

$S = \dfrac{a_1}{1-r}$ — Sum formula

$= \dfrac{\frac{63}{100}}{1-\frac{1}{100}}$ — $a_1 = \frac{63}{100}$ and $r = \frac{1}{100}$

$= \dfrac{63}{99}$ or $\dfrac{7}{11}$ — Simplify.

Method 2 Use algebraic properties.

$x = 0.\overline{63}$ — Let $x = 0.\overline{63}$.

$x = 0.636363...$ — Write as a repeating decimal.

$100x = 63.636363...$ — Multiply each side by 100.

$99x = 63$ — Subtract x from $100x$ and $0.\overline{63}$ from $63.\overline{63}$.

$x = \dfrac{63}{99}$ or $\dfrac{7}{11}$ — Divide each side by 99.

StudyTip

Repeating Decimals Every repeating decimal is a rational number and can be written as a fraction.

GuidedPractice

4. Write $0.\overline{21}$ as a fraction.

Check Your Understanding

Example 1 **Determine whether each infinite geometric series is *convergent* or *divergent*.**

1. $16 - 8 + 4 - \ldots$

2. $32 - 48 + 72 - \ldots$

3. $0.5 + 0.7 + 0.98 + \ldots$

4. $1 + 1 + 1 + \ldots$

Example 2 **Find the sum of each infinite series, if it exists.**

5. $440 + 220 + 110 + \ldots$

6. $520 + 130 + 32.5 + \ldots$

7. $\frac{1}{4} + \frac{3}{8} + \frac{9}{16} + \ldots$

8. $\frac{32}{9} + \frac{16}{3} + 8 + \ldots$

9. **SENSE-MAKING** A certain medicine has a half-life of 8 hours after it is administered to a patient. What percent of the medicine is still in the patient's system after 24 hours?

Example 3 **Find the sum of each infinite series, if it exists.**

10. $\sum_{k=1}^{\infty} 5 \cdot 4^{k-1}$

11. $\sum_{k=1}^{\infty} (-2) \cdot (0.5)^{k-1}$

12. $\sum_{k=1}^{\infty} 3 \cdot \left(\frac{4}{5}\right)^{k-1}$

13. $\sum_{k=1}^{\infty} \frac{1}{2} \cdot \left(\frac{3}{4}\right)^{k-1}$

Example 4 **Write each repeating decimal as a fraction.**

14. $0.\overline{35}$

15. $0.\overline{642}$

Practice and Problem Solving

Example 1 **Determine whether each infinite geometric series is *convergent* or *divergent*.**

16. $21 + 63 + 189 + \ldots$

17. $480 + 360 + 270 + \ldots$

18. $\frac{3}{4} + \frac{9}{8} + \frac{27}{16} + \ldots$

19. $\frac{5}{6} + \frac{10}{9} + \frac{40}{27} + \ldots$

20. $0.1 + 0.01 + 0.001 + \ldots$

21. $0.008 + 0.08 + 0.8 + \ldots$

Example 2 **Find the sum of each infinite series, if it exists.**

22. $18 + 21.6 + 25.92 + \ldots$

23. $-3 - 4.2 - 5.88 - \ldots$

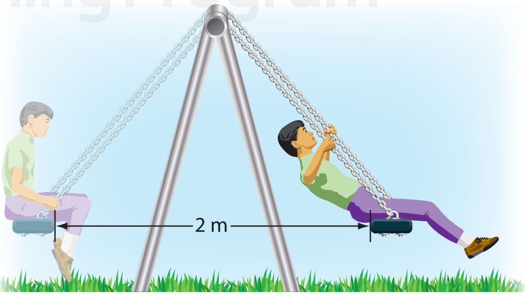
24. $\frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \ldots$

25. $\frac{12}{5} + \frac{6}{5} + \frac{3}{5} + \ldots$

26. $21 + 14 + \frac{28}{3} + \ldots$

27. $32 + 40 + 50 + \ldots$

28. **SWINGS** If Hassan does not push any harder after his initial swing, the distance traveled per swing will decrease by 10% with each swing. If his initial swing traveled 2 meters, find the total distance traveled when he comes to rest.



Example 3 **Find the sum of each infinite series, if it exists.**

29. $\sum_{k=1}^{\infty} \frac{4}{3} \cdot \left(\frac{5}{4}\right)^{k-1}$

30. $\sum_{k=1}^{\infty} \frac{1}{4} \cdot 3^{k-1}$

31. $\sum_{k=1}^{\infty} \frac{5}{3} \cdot \left(\frac{3}{7}\right)^{k-1}$

32. $\sum_{k=1}^{\infty} \frac{2}{3} \cdot \left(\frac{4}{3}\right)^{k-1}$

33. $\sum_{k=1}^{\infty} \frac{8}{3} \cdot \left(\frac{5}{6}\right)^{k-1}$

34. $\sum_{k=1}^{\infty} \frac{1}{8} \cdot \left(\frac{1}{12}\right)^{k-1}$

Example 4 **Write each repeating decimal as a fraction.**

35. $0.3\overline{21}$

36. $0.1\overline{45}$

37. $2.\overline{18}$

38. $4.\overline{96}$

39. $0.12\overline{14}$

40. $0.43\overline{36}$

41. **FANS** A fan is running at 10 revolutions per second. After it is turned off, its speed decreases at a rate of 75% per second. Determine the number of revolutions completed by the fan after it is turned off.

42. **PRECISION** Sally deposited AED 5,000 into an account at the beginning of the year. The account earns 8% interest (murabaha)* each year.

a. How much money will be in the account after 20 years? (*Hint*: Let $5000(1 + 0.08)^1$ represent the end of the first year.)

b. Is this series *convergent* or *divergent*? Explain.

43. **RECHARGEABLE BATTERIES** A certain rechargeable battery is advertised to recharge back to 99.9% of its previous capacity with every charge. If its initial capacity is 8 hours of life, how many total hours should the battery last?

Find the sum of each infinite series, if it exists.

44. $\frac{7}{5} + \frac{21}{20} + \frac{63}{80} + \ldots$

45. $\frac{15}{4} + \frac{5}{2} + \frac{5}{3} + \ldots$

46. $-\frac{16}{9} + \frac{4}{3} - 1 + \ldots$

47. $\frac{15}{8} + \frac{5}{2} + \frac{10}{3} + \ldots$

48. $\frac{21}{16} + \frac{7}{4} + \frac{7}{3} + \ldots$

49. $-\frac{18}{7} + \frac{12}{7} - \frac{8}{7} + \ldots$

50. **MULTIPLE REPRESENTATIONS** In this problem, you will use a square of paper that is at least 8 centimeters on a side.

a. Concrete Let the square be one unit. Cut away one half of the square. Call this piece Term 1. Next, cut away one half of the remaining sheet of paper. Call this piece Term 2. Continue cutting the remaining paper in half and labeling the pieces with a term number as long as possible. List the fractions represented by the pieces.

b. Numerical If you could cut the squares indefinitely, you would have an infinite series. Find the sum of the series.

c. Verbal How does the sum of the series relate to the original square of paper?

51. **PHYSICS** In a physics experiment, a steel ball on a flat track is accelerated, and then allowed to roll freely. After the first minute, the ball has rolled 120 meters. Each minute the ball travels only 40% as far as it did during the preceding minute. How far does the ball travel?

52. **PENDULUMS** A pendulum travels 12 centimeters on its first swing and 95% of that distance on each swing thereafter. Find the total distance traveled by the pendulum when it comes to rest.

53. **TOYS** If a rubber ball can bounce back to 95% of its original height, what is the total vertical distance that it will travel if it is dropped from an elevation of 30 meters?

54. **CARS** During a maintenance inspection, a tire is removed from a car and spun on a diagnostic machine. When the machine is turned off, the spinning tire completes 20 revolutions the first second and 98% of the revolutions each additional second. How many revolutions does the tire complete before it stops spinning?

*The term interest (murabaha) refers to an amount of money that is paid or received when borrowing or lending money. If a customer borrows money from a bank, the customer pays the bank interest (murabaha) for the use of its money. If a customer saves money in a bank account, the bank pays the customer interest (murabaha) for the use of his or her money.

The amount of money that is initially borrowed or saved is called the principal. The interest rate (murabaha rate) is a percentage earned or charged during a certain time period. Simple interest (murabaha) is the amount of interest (murabaha) charged or earned after the interest rate (murabaha rate) is applied to the principal.

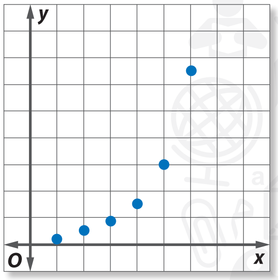
Simple interest (murabaha) (I) is the product of three values: the principal (P), the interest rate (murabaha rate) written as a decimal number (r), and time (t): $I = P \times r \times t$.

55. **ECONOMICS** The government decides to stimulate its economy by giving AED 500 to every adult. The government assumes that everyone who receives the money will spend 80% on consumer goods and that the producers of these goods will in turn spend 80% on consumer goods. How much money is generated for the economy for every AED 500 that the government provides?

56. **SCIENCE MUSEUM** An exhibit at a science museum offers visitors the opportunity to experiment with the motion of an object on a spring. One visitor pulls the object down and lets it go. The object travels 1.2 meters upward before heading back the other way. Each time the object changes direction, it decreases its distance by 20% when compared to the previous direction. Find the total distance traveled by the object.

Match each graph with its corresponding description.

57.



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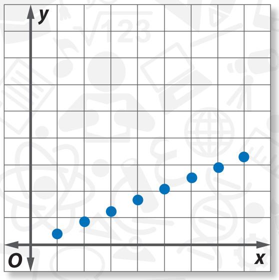

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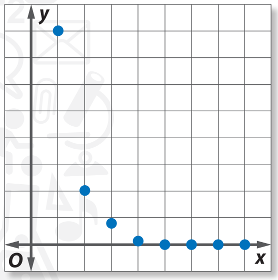


a. converging geometric series

b. diverging geometric series

c. converging arithmetic series

d. diverging arithmetic series

H.O.T. Problems Use Higher-Order Thinking Skills

60. **ERROR ANALYSIS** Mahmoud and Faleh are asked to find the sum of $1 - 1 + 1 - \ldots$. Is either of them correct? Explain your reasoning.

Mahmoud	Faleh
The sum is 0 because the sum of each pair of terms in the sequence is 0.	There is no sum because $\|r\| \geq 1$, and the series diverges.

61. **PROOF** Derive the formula for the sum of an infinite geometric series.

62. **CHALLENGE** For what values of b does $3 + 9b + 27b^2 + 81b^3 + \ldots$ have a sum?

63. **REASONING** When does an infinite geometric series have a sum, and when does it not have a sum? Explain your reasoning.

64. **ARGUMENTS** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

If the absolute value of a term of any geometric series is greater than the absolute value of the previous term, then the series is divergent.

65. **OPEN ENDED** Write an infinite series with a sum that converges to 9.

66. **OPEN ENDED** Write $3 - 6 + 12 - \ldots$ using sigma notation in two different ways.

67. **WRITING IN MATH** Explain why an arithmetic series is always divergent.

Standardized Test Practice

68. SAT/ACT What is the sum of an infinite geometric series with a first term of 27 and a common ratio of $\frac{2}{3}$?

A 18
B 34
C 41
D 65
E 81

69. Hareb, Hamad, Humaid, and Hamdan each simplified the same expression at the board. Each student's work is shown below. The teacher said that while two of them had a correct answer, only one of them had arrived at the correct conclusion using correct steps.

Hareb's work

$$x^2x^{-5} = \frac{x^2}{x^{-5}}$$
$$= x^7, x \neq 0$$

Humaid's work

$$x^2x^{-5} = \frac{x^2}{x^5}$$
$$= \frac{1}{x^3}, x \neq 0$$

Hamad's work

$$x^2x^{-5} = \frac{x^2}{x^{-5}}$$
$$= x^{-3}, x \neq 0$$

Hamdan's work

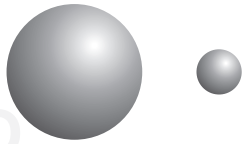
$$x^2x^{-5} = \frac{x^2}{x^5}$$
$$= x^3, x \neq 0$$

Which is a completely accurate simplification?

F Hareb's work
G Hamad's work
H Humaid's work
J Hamdan's work

70. GRIDDED RESPONSE Evaluate $\log_8 60$ to the nearest hundredth.

71. GEOMETRY The radius of a large sphere was multiplied by a factor of $\frac{1}{3}$ to produce a smaller sphere.



Radius $= r$ Radius $= \frac{1}{3}r$

How does the volume of the smaller sphere compare to the volume of the larger sphere?

A The volume of the smaller sphere is $\frac{1}{9}$ as large.
B The volume of the smaller sphere is $\frac{1}{\pi^3}$ as large.
C The volume of the smaller sphere is $\frac{1}{27}$ as large.
D The volume of the smaller sphere is $\frac{1}{3}$ as large.

Spiral Review

72. CONTEST An audition is held for a TV contest. At the end of each round, one half of the prospective contestants are eliminated from the competition. On a particular day, 524 contestants begin the audition. (Lesson 9-3)

a. Write an equation for finding the number of contestants who are left after n rounds.

b. Using this method, will the number of contestants who are to be eliminated always be a whole number? Explain.

73. CLUBS A quilting club consists of 9 members. Every week, each member must bring one completed quilt square. (Lesson 9-2)

a. Find the first eight terms of the sequence that describes the total number of squares that have been made after each meeting.

b. One particular quilt measures 144 centimeters by 168 centimeters and is being designed with 8-centimeter squares. After how many meetings will the quilt be complete?

Skills Review

Find each function value.

74. $f(x) = 5x - 9, f(6)$

75. $g(x) = x^2 - x, g(4)$

76. $h(x) = x^2 - 2x - 1, h(3)$

Graphing Technology Lab
Limits

You may have noticed that in some geometric sequences, the later the term in the sequence, the closer the value is to 0. Another way to describe this is that as n increases, a_n approaches 0. The value that the terms of a sequence approach, in this case 0, is called the **limit** of the sequence. Other types of infinite sequences may also have limits. If the terms of a sequence do not approach a unique value, we say that the limit of the sequence does not exist.

You can use a TI-83/84 Plus graphing calculator to help find the limits of infinite sequences.

Activity

Find the limit of the geometric sequence $1, \frac{1}{4}, \frac{1}{16}, \ldots$.

Step 1 Enter the sequence.

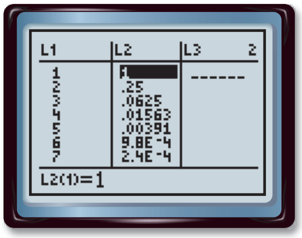
The formula for this sequence is $a_n = \left(\frac{1}{4}\right)^{n-1}$.

- Position the cursor on **L1** in the **STAT EDIT 1: Edit**... screen and enter the formula **seq(N,N,1,10,1)**. This generates the values 1, 2, ..., 10 of the index **N**.

KEYSTROKES: [STAT] [ENTER] [▲] [2nd] [STAT] [▶] 5 [X,T,θ,n] [,] [X,T,θ,n] [,] 1 [,] 10 [,] 1 [)] [ENTER]

- Position the cursor on **L2** and enter the formula **seq((1/4)^(N-1),N,1,10,1)**. This generates the first ten terms of the sequence.

KEYSTROKES: [▶] [▲] [2nd] [STAT] [▶] 5 [(] 1 [÷] 4 [)] [^] [(] [X,T,θ,n] [−] 1 [)] [,] [X,T,θ,n] [,] 1 [,] 10 [,] 1 [)] [ENTER]

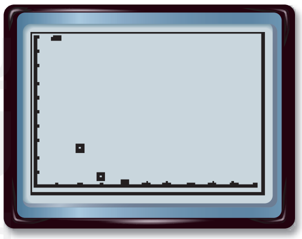


Notice that as n increases, the terms of the given sequence get closer and closer to 0. If you scroll down, you can see that for $n \geq 6$ the terms are so close to 0 that the calculator expresses them in scientific notation. This suggests that the limit of the sequence is 0.

Step 2 Graph the sequence.

Use **STAT PLOT** to graph the sequence. Use **L1** as the **Xlist** and **L2** as the **Ylist**.

The graph also shows that, as n increases, the terms approach 0. In fact, for $n \geq 3$, the marks appear to lie on the horizontal axis. This strongly suggests that the limit of the sequence is 0.



[0, 10] scl: 1 by [0, 1] scl: 0.1

Exercises

Find the limit of each sequence.

1. $a_n = \left(\frac{1}{3}\right)^n$
2. $a_n = \left(-\frac{1}{3}\right)^n$
3. $a_n = 5^n$
4. $a_n = \frac{1}{n^2}$
5. $a_n = \frac{3^n}{3^n + 1}$
6. $a_n = \frac{n^2}{n + 2}$

Mid-Chapter Quiz

CHAPTER 2

Lessons 1-1 through 1-4

Determine whether each sequence is *arithmetic*, *geometric*, or neither. Explain your reasoning. (Lesson 9-1)

Direction Line TK MCQ-DIR (Lesson Ref)

1. Text TK   2. Text TK

1. 5, −3, −12, −22, −33...

2. $\frac{1}{5}, \frac{7}{10}, \frac{6}{5}, \frac{17}{10}, \frac{11}{5}, \ldots$

Direction Line TK MCQ-DIR (Lesson Ref)

3. Text TK

4. **RUNIN HEAD** Text TK (Lesson Ref)

5. **MULTIPLE CHOICE** Text TK (Lesson Ref)

   A Choice   C Choice

   B Choice   D Choice

3. **HOUSING** Suha is a real estate agent. She needs to sell 15 houses in 6 months. (Lesson 9-1)

   a. By the end of the first 2 months she has sold 4 houses. If she sells 2 houses each month for the rest of the 6 months, will she meet her goal? Explain.

   b. If she has sold 5 houses by the end of the first month, how many will she have to sell on average each month in order to meet her goal?

4. **GEOMETRY** The figures below show a pattern of filled squares and white squares. (Lesson 9-1)

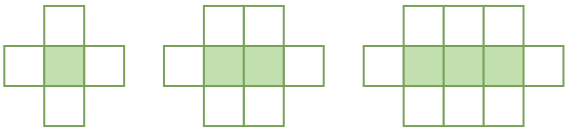


Figure 1   Figure 2   Figure 3

   a. Write an equation representing the nth number in this pattern where n is the number of white squares.

   b. Is it possible to have exactly 84 white squares in an arrangement? Explain.

Find the indicated term of each arithmetic sequence. (Lesson 9-2)

5. $a_1 = 10, d = -5, n = 9$

6. $a_1 = -8, d = 4, n = 99$

Find the sum of each arithmetic series. (Lesson 9-2)

7. $-15 + (-11) + (-7) + \cdots + 53$

8. $a_1 = -12, d = 8, n = 22$

9. $\sum_{k=11}^{50} (-3k + 1)$

10. **MULTIPLE CHOICE** What is the sum of the first 50 odd numbers? (Lesson 9-2)

    A 2550

    B 2500

    C 2499

    D 2401

Find the indicated term for each geometric sequence. (Lesson 9-3)

11. $a_2 = 8, r = 2, a_8 = ?$

12. $a_3 = 0.5, r = 8, a_{10} = ?$

13. **MULTIPLE CHOICE** What are the geometric means of the sequence below? (Lesson 9-3)

    0.5, ____, ____, ____, 2048

    F 512.375, 1024.25, 1536.125

    G 683, 1365.5, 2048

    H 2, 8, 32

    J 4, 32, 256

14. **INCOME** Fahd works for a house building company for 4 months per year. He starts out making AED 9,000 per month. At the end of each month, his salary increases by 5%. How much money will he make in those 4 months? (Lesson 9-3)

Evaluate the sum of each geometric series. (Lesson 9-3)

15. $\sum_{k=1}^{8} 3 \cdot 2^{k-1}$

16. $\sum_{k=1}^{9} 4 \cdot (-1)^{k-1}$

17. $\sum_{k=1}^{20} -2\left(\frac{2}{3}\right)^{k-1}$

Find the sum of each infinite series, if it exists. (Lesson 9-4)

18. $\sum_{n=1}^{\infty} 9 \cdot 2^{n-1}$

19. $\sum_{n=1}^{\infty} (4) \cdot (0.5)^{n-1}$

20. $\sum_{n=1}^{\infty} 12 \cdot \left(\frac{2}{3}\right)^{n-1}$

LESSON 9-6 Recursion and Iteration

Then

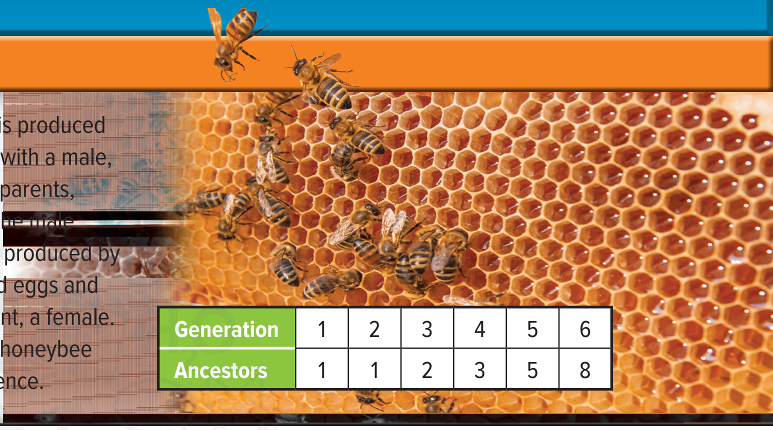
- You explored compositions of functions.

Now

1. Recognize and use special sequences.
2. Recognize recursive functions.

Why?

- The female honeybee is produced after the queen mates with a male, so the female has two parents, a male and a female. The male honeybee, however, is produced by the queen's unfertilized eggs and thus has only one parent, a female. The family tree for the honeybee follows a special sequence.



Generation	1	2	3	4	5	6
Ancestors	1	1	2	3	5	8

NewVocabulary

Fibonacci sequence
recursive sequence
explicit formula
recursive formula
iteration

Mathematical Practices

Attend to precision.
Look for and express regularity in repeated reasoning.

1 Special Sequences

Notice that every term in the list of ancestors is the sum of the previous two terms. This special sequence is called the **Fibonacci sequence**, and it is found in many places in nature. The Fibonacci sequence is an example of a **recursive sequence**. In a recursive sequence, each term is determined by one or more of the previous terms.

The formulas you have used for sequences thus far have been explicit formulas. An **explicit formula** gives a_n as a function of n, such as $a_n = 3n + 1$. The formula that describes the Fibonacci sequence, $a_n = a_{n-2} + a_{n-1}$, is a **recursive formula**, which means that every term will be determined by one or more of the previous terms. An initial term must be given in a recursive formula.

KeyConcept Recursive Formulas for Sequences

Arithmetic Sequence	$a_n = a_{n-1} + d$, where d is the common difference
Geometric Sequence	$a_n = r \cdot a_{n-1}$, where r is the common ratio

Example 1 Use a Recursive Formula

Find the first five terms of the sequence in which $a_1 = -3$ and $a_{n+1} = 4a_n - 2$, if $n \geq 1$.

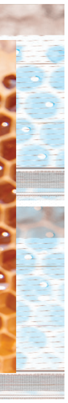
$a_{n+1} = 4a_n - 2$	Recursive formula
$a_{1+1} = 4a_1 - 2$	$n = 1$
$a_2 = 4(-3) - 2$ or -14	$a_1 = -3$
$a_3 = 4(-14) - 2$ or -58	$a_2 = -14$
$a_4 = 4(-58) - 2$ or -234	$a_3 = -58$
$a_5 = 4(-234) - 2$ or -938	$a_4 = -234$

The first five terms of the sequence are $-3, -14, -58, -234$, and -938.

GuidedPractice

1. Find the first five terms of the sequence in which $a_1 = 8$ and $a_{n+1} = -3a_n + 6$, if $n \geq 1$.

In order to find a recursive formula, first determine the initial term. Then evaluate the pattern to generate the later terms. The recursive formula that generates a sequence does not include the value of the initial term.



StudyTip

Sequences and Recursive Formulas Like arithmetic and geometric sequences, recursive formulas define functions in which the domain is the set of positive integers, represented by the term number *n*.

Example 2 Write Recursive Formulas

Write a recursive formula for each sequence.

a. 2, 10, 18, 26, 34, ...

Step 1 Determine whether the sequence is arithmetic or geometric.
The sequence is arithmetic because each term after the first can be found by adding a common difference.

Step 2 Find the common difference.

$d = 10 - 2$ or 8

Step 3 Write the recursive formula.

$a_n = a_{n-1} + d$ Recursive formula for arithmetic sequence

$a_n = a_{n-1} + 8$ $d = 8$

A recursive formula for the sequence is $a_n = a_{n-1} + 8$, $a_1 = 2$.

b. 16, 56, 196, 686, 2401, ...

Step 1 Determine whether the sequence is arithmetic or geometric.
The sequence is geometric because each term after the first can be found after multiplying by a common ratio.

Step 2 Find the common ratio.

$r = \frac{56}{16}$ or 3.5

Step 3 Write the recursive formula.

$a_n = r \cdot a_{n-1}$ Recursive formula for geometric sequence

$a_n = 3.5a_{n-1}$ $r = 3.5$

A recursive formula for the sequence is $a_n = 3.5a_{n-1}$, $a_1 = 16$.

c. $a_4 = 108$ and $r = 3$

Step 1 Determine whether the sequence is arithmetic or geometric.
Because r is given, the sequence is geometric.

Step 2 Write the recursive formula.

$a_n = r \cdot a_{n-1}$ Recursive formula for geometric sequence

$a_n = 3a_{n-1}$ $r = 3$

A recursive formula for the sequence is $a_n = 3a_{n-1}$, $a_1 = 4$.

GuidedPractice

Write a recursive formula for each sequence.

2A. 8, 20, 50, 125, 312.5, ... **2B.** 8, 17, 26, 35, 44, ... **2C.** $a_3 = 16$ and $r = 4$

Real-WorldLink

In 2008, the average credit card debt for college students was about AED 3,173.

Source: *USA Today*

Real-World Example 3 Use a Recursive Formula

FINANCIAL LITERACY **Nasser had AED 15,000 in credit card debt when he graduated from college. The balance increased by 2% each month due to interest (murabaha), and Nasser could only make payments of AED 400 per month. Write a recursive formula for the balance of his account each month. Then determine the balance after five months.**

Step 1 Write the recursive formula.

Let a_n represent the balance of the account in the nth month. The initial balance a_1 is AED 15,000. After one month, interest (murabaha) is added and a payment is made.

$$a_2 = \underbrace{a_1}_{\text{initial balance}} + \underbrace{(a_1 \times 0.02)}_{\text{balance times 2\%}} - \underbrace{400}_{\text{monthly payment}}$$

$$a_2 = 1.02a_1 - 400$$

The formula is $a_n = 1.02a_{n-1} - 400$.

Step 2 Find the next five terms.

$a_n = 1.02a_{n-1} - 400$	Recursive formula
$a_2 = (15{,}000 \times 1.02) - 400$ or $14{,}900$	$a_1 = 15{,}000$
$a_3 = (14{,}900 \times 1.02) - 400$ or $14{,}798$	$a_2 = 14{,}900$
$a_4 = (14{,}798 \times 1.02) - 400$ or $14{,}693.96$	$a_3 = 14{,}798$
$a_5 = (14{,}693.96 \times 1.02) - 400$ or $14{,}587.84$	$a_4 = 14{,}693.96$
$a_6 = (14{,}587.84 \times 1.02) - 400$ or $14{,}479.60$	$a_5 = 14{,}587.84$

After the fifth month, the balance will be AED 14,479.60.

GuidedPractice

3. Write a recursive formula for a AED 10,000 debt, at 2.5% interest (murabaha) per month, with a AED 600 monthly payment. Then find the first five balances.

ReviewVocabulary

Composition of functions
A function is performed, and then a second function is performed on the result of the first function.

2 Iteration

Iteration is the process of repeatedly composing a function with itself. Consider x_0. The first iterate is $f(x_0)$, the second iterate is $f(f(x_0))$, the third iterate is $f(f(f(x_0)))$, and so on.

Iteration can be used to recursively generate a sequence. Start with the initial value x_0. Let $x_1 = f(x_0)$, $x_2 = f(f(x_0))$, and so on.

Example 4 Iterate a Function

Find the first three iterates x_1, x_2, and x_3 of $f(x) = 5x + 4$ for an initial value of $x_0 = 2$.

$x_1 = f(x_0)$	Iterate the function.
$= 5(2) + 4$ or 14	$x_0 = 2$
$x_2 = f(x_1)$	Iterate the function.
$= 5(14) + 4$ or 74	$x_1 = 14$
$x_3 = f(x_2)$	Iterate the function.
$= 5(74) + 4$ or 374	$x_2 = 74$

The first three iterates are 14, 74, and 374.

GuidedPractice

4. Find the first three iterates x_1, x_2, and x_3 of $f(x) = -3x + 8$ for an initial value of $x_0 = 6$.

Check Your Understanding

Example 1 **Find the first five terms of each sequence described.**

1. $a_1 = 16, a_{n+1} = a_n + 4$
2. $a_1 = -3, a_{n+1} = a_n + 8$
3. $a_1 = 5, a_{n+1} = 3a_n + 2$
4. $a_1 = -4, a_{n+1} = 2a_n - 6$

Example 2 **Write a recursive formula for each sequence.**

5. 3, 8, 18, 38, 78, ...
6. 5, 14, 41, 122, 365, ...

Example 3

7. **FINANCING** Faris financed a AED 1,500 rowing machine to help him train for the college rowing team. He could only make a AED 100 payment each month, and his bill increased by 1% due to interest (murabaha) at the end of each month.
   - **a.** Write a recursive formula for the balance owed at the end of each month.
   - **b.** Find the balance owed after the first four months.
   - **c.** How much interest (murabaha) has accumulated after the first six months?

Example 4 **Find the first three iterates of each function for the given initial value.**

8. $f(x) = 5x + 2, x_0 = 8$
9. $f(x) = -4x + 2, x_0 = 5$
10. $f(x) = 6x + 3, x_0 = -4$
11. $f(x) = 8x - 4, x_0 = -6$

Practice and Problem Solving

Example 1 **PERSEVERANCE** **Find the first five terms of each sequence described.**

12. $a_1 = 10, a_{n+1} = 4a_n + 1$
13. $a_1 = -9, a_{n+1} = 2a_n + 8$
14. $a_1 = 12, a_{n+1} = a_n + n$
15. $a_1 = -4, a_{n+1} = 2a_n + n$
16. $a_1 = 6, a_{n+1} = 3a_n - n$
17. $a_1 = -2, a_{n+1} = 5a_n + 2n$
18. $a_1 = 7, a_2 = 10, a_{n+2} = 2a_n + a_{n+1}$
19. $a_1 = 4, a_2 = 5, a_{n+2} = 4a_n - 2a_{n+1}$
20. $a_1 = 4, a_2 = 3x, a_n = a_{n-1} + 4a_{n-2}$
21. $a_1 = 3, a_2 = 2x, a_n = 4a_{n-1} - 3a_{n-2}$
22. $a_1 = 2, a_2 = x + 3, a_n = a_{n-1} + 6a_{n-2}$
23. $a_1 = 1, a_2 = x, a_n = 3a_{n-1} + 6a_{n-2}$

Example 2 **Write a recursive formula for each sequence.**

24. 16, 10, 7, 5.5, 4.75, ...
25. 32, 12, 7, 5.75, ...
26. 4, 15, 224, 50,175, ...
27. 1, 2, 9, 730, ...
28. 9, 33, 129, 513, ...
29. 480, 128, 40, 18, ...
30. 393, 132, 45, 16, ...
31. 68, 104, 176, 320, ...

Example 3

32. **FINANCIAL LITERACY** Mr. Adnan and his company deposit AED 20,000 into his retirement account at the end of each year. The account earns 8% interest (murabaha) before each deposit.
   - **a.** Write a recursive formula for the balance in the account at the end of each year.
   - **b.** Determine how much is in the account at the end of each of the first 8 years.

Example 4 **Find the first three iterates of each function for the given initial value.**

33. $f(x) = 12x + 8, x_0 = 4$
34. $f(x) = -9x + 1, x_0 = -6$
35. $f(x) = -6x + 3, x_0 = 8$
36. $f(x) = 8x + 3, x_0 = -4$
37. $f(x) = -3x^2 + 9, x_0 = 2$
38. $f(x) = 4x^2 + 5, x_0 = -2$
39. $f(x) = 2x^2 - 5x + 1, x_0 = 6$
40. $f(x) = -0.25x^2 + x + 6, x_0 = 8$
41. $f(x) = x^2 + 2x + 3, x_0 = \frac{1}{2}$
42. $f(x) = 2x^2 + x + 1, x_0 = -\frac{1}{2}$

43. FRACTALS Consider the figures at the right. The number of blue triangles increases according to a specific pattern.

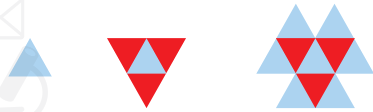
a. Write a recursive formula for the number of blue triangles in the sequence of figures.

b. How many blue triangles will be in the sixth figure?

44. FINANCIAL LITERACY Amer's monthly loan payment is AED 234.85. The recursive formula $b_n = 1.005b_{n-1} - 234.85$ describes the balance left on the loan after n payments. Find the balance of the AED 10,000 loan after each of the first eight payments.

45. CONSERVATION Suppose a lake is populated with 10,000 fish. A year later, 80% of the fish have died or been caught, and the lake is replenished with 10,000 new fish. If the pattern continues, will the lake eventually run out of fish? If not, will the population of the lake converge to any particular value? Explain.

46. GEOMETRY Consider the pattern at the right.



a. Write a sequence of the total number of triangles in the first six figures.

b. Write a recursive formula for the number of triangles.

c. How many triangles will be in the tenth figure?

47. SPREADSHEETS Consider the sequence with $x_0 = 20{,}000$ and $f(x) = 0.3x + 5000$.

a. Enter x_0 in cell A1 of your spreadsheet. Enter "= (0.3)*(A1) + 5000" in cell A2. What answer does it provide?

b. Copy cell A2, highlight cells A3 through A70, and paste. What do you notice about the sequence?

c. How do spreadsheets help analyze recursive sequences?

48. VIDEO GAMES The final monster in Hidaya's video game has 100 health points. During the final battle, the monster regains 10% of its health points after every 10 seconds. If Hidaya can inflict damage to the monster that takes away 10 health points every 10 seconds without getting hurt herself, will she ever kill the monster? If so, when?

H.O.T. Problems Use Higher-Order Thinking Skills

49. CRITIQUE Sultan and Saeed are finding the first three iterates of $f(x) = 5x - 3$ for an initial value of $x_0 = 4$. Is either of them correct? Explain.

Sultan	Saeed
$f(4) = 5(4) - 3$ or 17	$f(4) = 5(4) - 3$ or 17
$f(17) = 5(17) - 3$ or 82	$f(17) = 5(17) - 3$ or 82
The first three iterates are 4, 17, and 82.	$f(82) = 5(82) - 3$ or 407
	The first three iterates are 17, 82, and 407.

50. CHALLENGE Find a recursive formula for 5, 23, 98, 401, … .

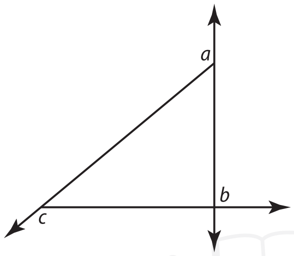
51. REASONING Is the statement *"If the first three terms of a sequence are identical, then the sequence is not recursive" sometimes, always*, or *never* true? Explain your reasoning.

52. OPEN ENDED Write a function for which the first three iterates are 9, 19, and 39.

53. WRITING IN MATH Why is it useful to represent a sequence with an explicit or recursive formula?

Standardized Test Practice

54. **GEOMETRY** In the figure shown, $a + b + c = ?$



A 180°
B 270°
C 360°
D 450°

55. **EXTENDED RESPONSE** Omar launches a model rocket from ground level. The rocket's height h in meters is given by the equation $h = -4.9t^2 + 56t$, where t is the time in seconds after the launch.

a. What is the maximum height the rocket will reach?

b. How long after it is launched will the rocket reach its maximum height? Round to the nearest tenth of a second.

c. How long after it is launched will the rocket land? Round to the nearest tenth of a second.

56. Which of the following is true about the graphs of $y = 3(x - 4)^2 + 5$ and $y = 3(x + 4)^2 + 5$?

F Their vertices are maximums.
G The graphs have the same shape with different vertices.
H The graphs have different shapes with different vertices.
J One graph has a vertex that is a maximum, while the other graph has a vertex that is a minimum.

57. Which factors could represent the length times the width?

$A = 16x^4 - 25y^2$

A $(4x - 5y)(4x - 5y)$
B $(4x + 5y)(4x - 5y)$
C $(4x^2 - 5y)(4x^2 + 5y)$
D $(4x^2 + 5y)(4x^2 + 5y)$

Spiral Review

Write each repeating decimal as a fraction. (Lesson 9-4)

58. $0.\overline{7}$ **59.** $5.\overline{126}$ **60.** $6.\overline{259}$

61. **SPORTS** Obaid is training for a marathon, about 42 kilometers. He begins by running 2 kilometers. Then, when he runs every other day, he runs one and a half times the distance he ran the time before. (Lesson 9-3)

a. Write the first five terms of a sequence describing his training schedule.

b. When will he exceed 26 kilometers in one run?

c. When will he have run 100 total kilometers?

State whether the events are *independent* or *dependent*.

62. tossing a coin and rolling a number cube

63. choosing first and second place in an academic competition

Skills Review

Find each product.

64. $(y + 4)(y + 3)$ **65.** $(x - 2)(x + 6)$ **66.** $(a - 8)(a + 5)$

67. $(4h + 5)(h + 7)$ **68.** $(9p - 1)(3p - 2)$ **69.** $(2g + 7)(5g - 8)$

EXTEND

9-6 Spreadsheet Lab
Amortizing Loans

Mathematical Practices
Use appropriate tools strategically.

When a payment is made on a loan, part of the payment is used to cover the interest (murabaha) that has accumulated since the last payment. The rest is used to reduce the *principal*, or original amount of the loan. This process is called *amortization*. You can use a spreadsheet to analyze the payments, interest (murabaha), and balance on a loan. A table that shows this kind of information is called an *amortization schedule*.

Example

LOANS Najla just bought a new smartphone for AED 695. The store is letting her make monthly payments of AED 60.78 at an interest rate of 9% for one year. How much will she still owe after six months?

Every month, the interest (murabaha) on the remaining balance will be $\frac{9\%}{12}$ or 0.75%. You can find the balance after a payment by multiplying the balance after the previous payment by $1 + 0.0075$ or 1.0075 and then subtracting 60.78.

In a spreadsheet, the column of numbers represents the number of payments, and Column B shows the balance. Enter the interest (murabaha) rate and monthly payment in cells in Column A so that they can be easily updated if the information changes.

The spreadsheet at the right shows the formulas for the balances after each of the first six payments. After six months, Najla still owes AED 355.28.

Smartphone Loan

	A	B	C
1	Interest (murabaha) Rate	=695*(1+A2)–A5	
2	0.0075	=B1*(1+A2)–A5	
3		=B2*(1+A2)–A5	
4	Monthly payment	=B3*(1+A2)–A5	
5	60.78	=B4*(1+A2)–A5	
6		=B5*(1+A2)–A5	
7			

Sheet 1 | Sheet 2 | Sheet 3

Model and Analyze

1. Let b_n be the balance left on Najla's loan after n months. Write an equation relating b_n and b_{n+1}.
2. Payments at the beginning of a loan go more toward interest (murabaha) than payments at the end. What percent of Najla's loan remains to be paid after half a year?
3. Extend the spreadsheet to the whole year. What is the balance after 12 payments? Why is it not 0?
4. Suppose Najla decides to pay AED 70 every month. How long would it take her to pay off the loan?
5. Suppose that, based on how much she can afford, Najla will pay a variable amount each month in addition to the AED 60.78. Explain how the flexibility of a spreadsheet can be used to adapt to this situation.
6. Amer has a three-year, AED 12,000 moped loan. The annual interest (murabaha) rate is 6%, and his monthly payment is AED 365.06. After fifteen months, he receives an inheritance which he wants to use to pay off the loan. How much does he owe at that point?

LESSON 9-7 The Binomial Theorem

Then

- You worked with combinations.

Now

1. Use Pascal's Triangle to write binomial expansions.
2. Use the Binomial Theorem to expand powers of binomials.

Why?

- A manager plans to promote 8 employees. Not wanting to appear biased, the manager wants to promote a combination of senior and junior employees that has at least a 10% chance of occurring randomly. If there are an equal number of senior and junior employees applicants, is the probability of randomly promoting 6 senior and 2 junior employees less than 10%?

NewVocabulary
Pascal's triangle

Mathematical Practices
Model with mathematics.

1 Pascal's Triangle

In the 13th century, the Chinese discovered a pattern of numbers that would later be referred to as **Pascal's triangle**. This pattern can be used to determine the coefficients of an expanded binomial $(a + b)^n$.

$$\begin{array}{lccccccccccc}
(a+b)^0 & & & & & & 1 & & & & & \\
(a+b)^1 & & & & & 1 & & 1 & & & & \\
(a+b)^2 & & & & 1 & & 2 & & 1 & & & \\
(a+b)^3 & & & 1 & & 3 & & 3 & & 1 & & \\
(a+b)^4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(a+b)^5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

For example, the expanded form of
$(a + b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$.

Real-World Example 1 Use Pascal's Triangle

Find the probability of promoting 6 senior and 2 junior employees by expanding $(m + f)^8$.

Write three more rows of Pascal's triangle and use the pattern to write the expansion.

$$\begin{array}{lccccccccccccccccc}
5 & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
6 & & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
7 & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
8 & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}$$

$$(m + f)^8 = m^8 + 8m^7f + 28m^6f^2 + 56m^5f^3 + 70m^4f^4 + 56m^3f^5 + 28m^2f^6 + 8mf^7 + f^8$$

By adding the coefficients of the polynomial, we determine that there are 256 combinations of senior and junior employees that could be promoted.

$28m^6f^2$ represents the number of combinations with 6 senior and 2 junior employees. Therefore, there is a $\frac{28}{256}$ or about an 11% chance of randomly promoting 6 senior and 2 junior employees.

GuidedPractice

1. Expand $(c + d)^9$.

2 The Binomial Theorem

Instead of writing out row after row of Pascal's triangle, you can use the **Binomial Theorem** to expand a binomial. Recall that ${}_nC_r = \frac{n!}{r!(n-r)!}$.

StudyTip

Combinations Recall that both ${}_nC_0$ and ${}_nC_n$ equal 1.

KeyConcept Binomial Theorem

If n is a natural number, then $(a + b)^n =$

$${}_nC_0\, a^n b^0 + {}_nC_1\, a^{n-1} b^1 + {}_nC_2\, a^{n-2} b^2 + \cdots + {}_nC_n\, a^0 b^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, a^{n-k} b^k.$$

To use the theorem, replace n with the value of the exponent. Notice how the terms will follow the pattern of Pascal's triangle, and the coefficients will be symmetric.

Example 2 Use the Binomial Theorem

Expand $(a + b)^7$.

Method 1 Use combinations.

Replace n with 7 in the Binomial Theorem.

$$(a + b)^7 = {}_7C_0\, a^7 + {}_7C_1\, a^6 b + {}_7C_2\, a^5 b^2 + {}_7C_3\, a^4 b^3 + {}_7C_4\, a^3 b^4 + {}_7C_5\, a^2 b^5 + {}_7C_6\, ab^6 + {}_7C_7\, b^7$$

$$= a^7 + \frac{7!}{6!}\, a^6 b + \frac{7!}{2!5!}\, a^5 b^2 + \frac{7!}{3!4!}\, a^4 b^3 + \frac{7!}{4!3!}\, a^3 b^4 + \frac{7!}{5!2!}\, a^2 b^5 + \frac{7!}{6!}\, ab^6 + b^7$$

$$= a^7 + 7a^6 b + 21a^5 b^2 + 35a^4 b^3 + 35a^3 b^4 + 21a^2 b^5 + 7ab^6 + b^7$$

Method 2 Use Pascal's triangle.

Use the Binomial Theorem to determine exponents, but instead of finding the coefficients by using combinations, look at the seventh row of Pascal's triangle.

6		1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

$$(a + b)^7 = a^7 + 7a^6 b + 21a^5 b^2 + 35a^4 b^3 + 35a^3 b^4 + 21a^2 b^5 + 7ab^6 + b^7$$

GuidedPractice

2. Expand $(x + y)^{10}$.

When the binomial to be expanded has coefficients other than 1, the coefficients will no longer be symmetric. In these cases, you may want to use the Binomial Theorem.

Example 3 Coefficients Other Than 1

Expand $(5a - 4b)^4$.

$$(5a - 4b)^4$$

$$= {}_4C_0\, (5a)^4 + {}_4C_1\, (5a)^3(-4b) + {}_4C_2\, (5a)^2(-4b)^2 + {}_4C_3\, (5a)(-4b)^3 + {}_4C_4\, (-4b)^4$$

$$= 625a^4 + \frac{4!}{3!}(125a^3)(-4b) + \frac{4!}{2!2!}(25a^2)(16b^2) + \frac{4!}{3!}(5a)(-64b^3) + 256b^4$$

$$= 625a^4 - 2000a^3 b + 2400a^2 b^2 - 1280ab^3 + 256b^4$$

StudyTip

Graphing Calculator You can calculate ${}_nC_r$ by using a graphing calculator.

Press MATH and choose PRB 3.

GuidedPractice

3. Expand $(3x + 2y)^5$.

Sometimes you may need to find only one term in a binomial expansion. To do this, you can use the summation formula for the Binomial Theorem, $\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, a^{n-k} b^k$.

Example 4 Determine a Single Term

Find the fifth term of $(y + z)^{11}$.

Step 1 Use the Binomial Theorem to write the expansion in sigma notation.

$$(y + z)^{11} = \sum_{k=0}^{11} \frac{11!}{k!(11-k)!} y^{11-k} z^k$$

Step 2

$$\frac{11!}{k!(11-k)!} y^{11-k} z^k = \frac{11!}{4!(11-4)!} y^{11-4} z^4$$ For the fifth term, $k = 4$.

$$= 330y^7z^4$$ $C(11, 4) = 330$

GuidedPractice

4. Find the sixth term of $(c + d)^{10}$.

ConceptSummary Binomial Expansion

In a binomial expansion of $(a + b)^n$,

- there are $n + 1$ terms.
- n is the exponent of a in the first term and b in the last term.
- in successive terms, the exponent of a decreases by 1, and the exponent of b increases by 1.
- the sum of the exponents in each term is n.
- the coefficients are symmetric.

Check Your Understanding

Examples 1–3 **Expand each binomial.**

1. $(c + d)^5$ **2.** $(g + h)^7$ **3.** $(x - 4)^6$

4. $(2y - z)^5$ **5.** $(x + 3)^5$ **6.** $(y - 4z)^4$

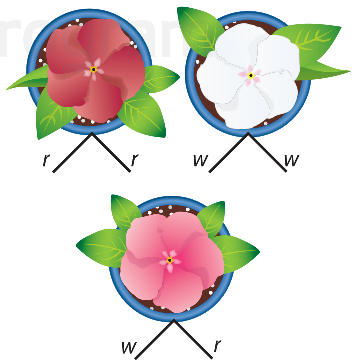
7. **GENETICS** If a woman is equally as likely to have a baby boy or a baby girl, use binomial expansion to determine the probability that 5 of her 6 children are girls. Do not consider identical twins.

Example 4 **Find the indicated term of each expression.**

8. fourth term of $(b + c)^9$ **9.** fifth term of $(x + 3y)^8$ **10.** third term of $(a - 4b)^6$

11. sixth term of $(2c - 3d)^8$ **12.** last term of $(5x + y)^5$ **13.** first term of $(3a + 8b)^5$

14. **MODELING** The color of a particular flower is determined by the combination of two genes, also called *alleles*. If the flower has two red alleles r, the flower is red. If the flower has two white alleles w, the flower is white. If the flower has one allele of each color, the flower will be pink. In a lab, two pink flowers are mated and eventually produce 1000 offspring. How many of the 1000 offspring will be pink?



Practice and Problem Solving

Examples 1–3 **Expand each binomial.**

15. $(a - b)^6$

16. $(c - d)^7$

17. $(x + 6)^6$

18. $(y - 5)^7$

19. $(2a + 4b)^4$

20. $(3a - 4b)^5$

21. **COMMITTEES** If an equal number of seniors and juniors applied to be on a school sports committee and the committee needs a total of 10 people, find the probability that 7 of the members will be juniors. Assume that committee members will be chosen randomly.

22. **BASEBALL** If a pitcher is just as likely to throw a ball as a strike, find the probability that 11 of his first 12 pitches are balls.

Example 4 **Find the indicated term of each expression.**

23. third term of $(x + 2z)^7$

24. fourth term of $(y - 3x)^6$

25. seventh term of $(2a - 2b)^8$

26. sixth term of $(4x + 5y)^6$

27. fifth term of $(x - 4)^9$

28. fourth term of $(c + 6)^8$

Expand each binomial.

29. $\left(x + \frac{1}{2}\right)^5$

30. $\left(x - \frac{1}{3}\right)^4$

31. $\left(2b + \frac{1}{4}\right)^5$

32. $\left(3c + \frac{1}{3}\right)^5$

33. **SENSE-MAKING** In $\frac{n!}{k!(n-k)!}p^k q^{n-k}$, let p represent the likelihood of a success and q represent the likelihood of a failure.

a. If a penalty kick taker scores 70% of his penalties, find the likelihood that he scores 9 of his next 10 attempts.

b. If a midfielder completes 60% of his passes, find the likelihood that he completes 8 of his next 10 attempts.

c. If a team converts 30% of their free kicks, find the likelihood that they convert 2 of their next 5 free kicks.

H.O.T. Problems Use Higher-Order Thinking Skills

34. **CHALLENGE** Find the sixth term of the expansion of $\left(\sqrt{a} + \sqrt{b}\right)^{12}$. Explain your reasoning.

35. **REASONING** Explain how the terms of $(x + y)^n$ and $(x - y)^n$ are the same and how they are different.

36. **REASONING** Determine whether the following statement is *true* or *false*. Explain your reasoning.

The eighth and twelfth terms of $(x + y)^{20}$ have the same coefficients.

37. **OPEN ENDED** Write a power of a binomial for which the second term of the expansion is $6x^4y$.

38. **WRITING IN MATH** Explain how to write out the terms of Pascal's triangle.

Standardized Test Practice

39. PROBABILITY A desk drawer contains 7 sharpened red pencils, 5 sharpened yellow pencils, 3 unsharpened red pencils, and 5 unsharpened yellow pencils. If a pencil is taken from the drawer at random, what is the probability that it is yellow, given that it is one of the sharpened pencils?

A $\frac{5}{12}$

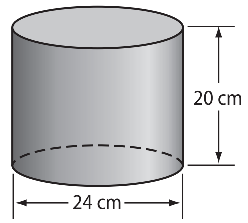
B $\frac{7}{20}$

C $\frac{5}{8}$

D $\frac{1}{5}$

40. GRIDDED RESPONSE Two people are 17.5 kilometers apart. They begin to walk toward each other along a straight line at the same time. One walks at the rate of 4 kilometers per hour, and the other walks at the rate of 3 kilometers per hour. In how many hours will they meet?

41. GEOMETRY Suha has a cylindrical block that she needs to paint for an art project.



What is the surface area of the cylinder in square centimeters rounded to the nearest square centimeter?

F 1960 **H** 5127
G 2413 **J** 6635

42. Which of the following is a linear function?

A $y = \frac{x+3}{x+2}$

B $y = (3x+2)^2$

C $y = \frac{x+3}{2}$

D $y = |3x| + 2$

Spiral Review

Find the first five terms of each sequence. (Lesson 9-5)

43. $a_1 = -2,\ a_{n+1} = a_n + 5$ **44.** $a_1 = 3,\ a_{n+1} = 4a_n - 10$ **45.** $a_1 = 4,\ a_{n+1} = 3a_n - 6$

Find the sum of each infinite geometric series, if it exists. (Lesson 9-4)

46. $-6 + 3 - \frac{3}{2} + \ldots$ **47.** $\frac{3}{4} + \frac{1}{4} + \frac{1}{12} + \ldots$ **48.** $\sqrt{3} + 3 + \sqrt{27} + \ldots$

49. TRAVEL A trip between two towns takes 4 hours under ideal conditions. The first 150 kilometers of the trip is on an interstate, and the last 130 kilometers is on a highway with a speed limit that is 10 kilometers per hour less than on the interstate.

a. If x represents the speed limit on the interstate, write expressions for the time spent at that speed and for the time spent on the other highway.

b. Write and solve an equation to find the speed limits on the two highways.

Skills Review

State whether each statement is *true* or *false* when $n = 1$. Explain.

50. $\frac{(n+1)(n+1)}{2} = 2$ **51.** $3n + 5$ is even. **52.** $n^2 - 1$ is odd.

EXTEND 9-7

Algebra Lab

Combinations and Pascal's Triangle

Recall that an arrangement or selection of objects in which order is not important is called a *combination*. For example, selecting 2 snacks from a choice of 6 is a combination of 6 objects taken 2 at a time and can be written ${}_6C_2$ or $C(6, 2)$.

Activity

A contestant on a tv show has the opportunity to win up to five prizes, one for each of five rounds of the game. If the contestant wins a round, he or she may choose one prize. Determine the number of ways that prizes can be chosen.

Step 1 If a contestant does not win any rounds, he or she receives 0 prizes. This represents 5 items taken 0 at a time.

$${}_nC_r = \frac{n!}{(n-r)!\,r!} \qquad \text{Definition of combination}$$

$${}_5C_0 = \frac{5!}{(5-0)!\,0!} \qquad n = 5 \text{ and } r = 0$$

$$= \frac{120}{120(1)} \text{ or } 1 \qquad 5! = 120 \text{ and } 0! = 1$$

There is 1 way to receive 0 prizes.

If a contestant wins one round, any one of the prizes can be selected. If a contestant wins two rounds, two prizes can be chosen. If three rounds are won, three prizes can be chosen, and so on. In how many ways can 1 prize be chosen? 2 prizes? 3, 4, and 5 prizes? We can determine these answers by examining Pascal's triangle.

Step 2 Examine Pascal's triangle.

List Rows 0 through 5 of Pascal's triangle.

Row 0						1					
Row 1					1		1				
Row 2				1		2		1			
Row 3			1		3		3		1		
Row 4		1		4		6		4		1	
Row 5	1		5		10		10		5		1

The number of ways one prize can be chosen from 5 can be determined by looking at Row 5. The first number in Row 5 represents the number of ways to choose 0 prizes, the second number represents the number of ways to choose 1 prize, and so on.

Analyze the Result

1. Make a conjecture about how the numbers in one of the rows can be used to find the number of ways that 0, 1, 2, 3, 4, ..., n objects can be selected from n objects.
2. Suppose the rules of the game are changed so that there are 6 rounds and 6 prizes from which to choose. Find the number of ways that 0, 1, 2, 3, 4, 5, or 6 prizes can be chosen. Which row of Pascal's triangle can be used to find the answers?
3. Use Pascal's triangle to find ${}_8C_0$, ${}_8C_1$, ${}_8C_2$, ${}_8C_3$, ${}_8C_4$, ${}_8C_5$, ${}_8C_6$, ${}_8C_7$, and ${}_8C_8$. State the row number that you used to find the answers.

LESSON 9-8

Proof by Mathematical Induction

Then

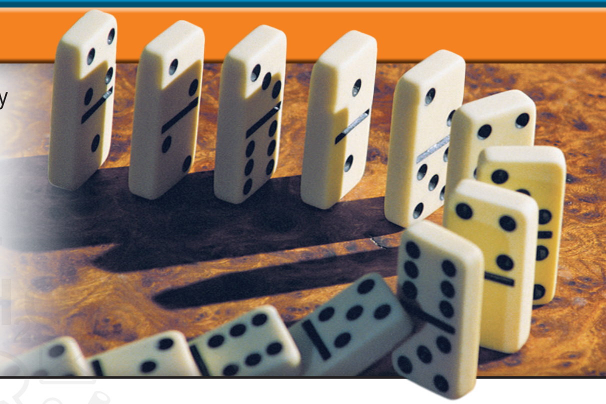
- You have proved the sum of an arithmetic series.

Now

1. Prove statements by using mathematical induction.
2. Disprove statements by finding a counterexample.

Why?

- When dominoes are set up closely and the first domino is knocked down, the rest of the dominoes come tumbling down. All that is needed with this setup is for the first domino to fall, and the rest will follow. The same is true with mathematical induction.



NewVocabulary
mathematical induction
induction hypothesis

Mathematical Practices
Make sense of problems and persevere in solving them.
Construct viable arguments and critique the reasoning of others.

1 Mathematical Induction

Mathematical induction is a method of proving statements involving natural numbers.

KeyConcept Mathematical Induction

To prove that a statement is true for all natural numbers n,

Step 1 Show that the statement is true for $n = 1$.

Step 2 Assume that the statement is true for some natural number k. This assumption is called the **induction hypothesis**.

Step 3 Show that the statement is true for the next natural number $k + 1$.

Example 1 Prove Summation

Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

Step 1 When $n = 1$, the left side of the equation is 1^3 or 1.

The right side is $\dfrac{1^2(1+1)^2}{4}$ or 1. Thus, the statement is true for $n = 1$.

Step 2 Assume that $1^3 + 2^3 + 3^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}$ for a natural number k.

Step 3 Show that the given statement is true for $n = k + 1$.

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4} \qquad \text{Inductive hypothesis}$$

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 \qquad \text{Add } (k+1)^3 \text{ to each side.}$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \qquad \text{The LCD is 4.}$$

$$= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \qquad \text{Factor.}$$

$$= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \qquad \text{Simplify.}$$

$$= \frac{(k+1)^2(k+2)^2}{4} \qquad \text{Factor.}$$

The last expression is the statement to be proved, where n has been replaced by $k + 1$. This proves the conjecture.

GuidedPractice

1. Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Along with summation, mathematical induction can be used to prove divisibility.

StudyTip

Regularity r is a whole number used to show divisibility in proof. When a value equals $4r$, then it must be divisible by 4.

Example 2 Prove Divisibility

Prove that $8^n - 1$ is divisible by 7 for all natural numbers n.

Step 1 When $n = 1$, $8^n - 1 = 8^1 - 1$ or 7. Since 7 is divisible by 7, the statement is true for $n = 1$.

Step 2 Assume that $8^k - 1$ is divisible by 7 for some natural number k. This means that there is a natural number r such that $8^k - 1 = 7r$.

Step 3 Show that the statement is true for $n = k + 1$.

$8^k - 1 = 7r$ Inductive hypothesis

$8^k = 7r + 1$ Add 1 to each side.

$8(8^k) = 8(7r + 1)$ Multiply each side by 8.

$8^{k+1} = 56r + 8$ Simplify.

$8^{k+1} - 1 = 56r + 7$ Subtract 1 from each side.

$8^{k+1} - 1 = 7(8r + 1)$ Factor.

Since r is a natural number, $8r + 1$ is a natural number and $7(8r + 1)$ is divisible by 7. Therefore, $8^{k+1} - 1$ is divisible by 7.

This proves that 8^{n-1} is divisible by 7 for all natural numbers n.

GuidedPractice

2. Prove that $7^n - 1$ is divisible by 6 for all natural numbers n.

2 Counterexamples

Statements can be proved false by using mathematical induction. An easier method is by finding a counterexample, which is a specific case in which the statement is false.

ReviewVocabulary

counterexample One of the synonyms of *counter* is to *contradict*, so a counterexample is an example that contradicts a hypothesis.

Example 3 Use a Counterexample to Disprove

Find a counterexample to disprove the statement that $2^n + 2n^2$ is divisible by 4 for any natural number n.

Test different values of n.

n	$2^n + 2n^2$	Divisible by 4?
1	$2^1 + 2(1)^2 = 2 + 2$ or 4	yes
2	$2^2 + 2(2)^2 = 4 + 8$ or 12	yes
3	$2^3 + 2(3)^2 = 8 + 18$ or 26	no

The value $n = 3$ is a counterexample for the statement.

GuidedPractice

3. Find a counterexample to disprove $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(3n - 1)}{2}$.

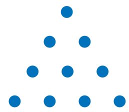
Check Your Understanding

Example 1 **Prove that each statement is true for all natural numbers.**

1. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

2. $1 + 2 + 3 + \cdots + n = \dfrac{n(n + 1)}{2}$

3. **NUMBER THEORY** A number is *triangular* if it can be represented visually by a triangular array.



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a. The first triangular number is 1. Find the next 5 triangular numbers.

b. Write a formula for the nth triangular number.

c. Prove that the sum of the first n triangular numbers equals $\dfrac{n(n + 1)(n + 2)}{6}$.

Example 2 **Prove that each statement is true for all natural numbers.**

4. $10^n - 1$ is divisible by 9.

5. $4^n - 1$ is divisible by 3.

Example 3 **Find a counterexample to disprove each statement.**

6. $3^n + 1$ is divisible by 4.

7. $2^n + 3^n$ is divisible by 4.

Practice and Problem Solving

Example 1 **ARGUMENTS** **Prove that each statement is true for all natural numbers.**

8. $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

9. $2 + 5 + 8 + \cdots + (3n - 1) = \dfrac{n(3n + 1)}{2}$

10. $1 + 2 + 4 + \cdots + 2^{n - 1} = 2^n - 1$

11. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$

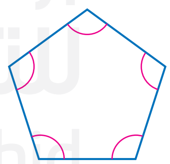
12. $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n - 1)}{2}$

13. $3 + 7 + 11 + \cdots + (4n - 1) = 2n^2 + n$

14. $\dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{12} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$

15. $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \dfrac{n(2n - 1)(2n + 1)}{3}$

16. **GEOMETRY** According to the Interior Angle Sum Formula, if a convex polygon has n sides, then the sum of the measures of the interior angles of a polygon equals $180(n - 2)$. Prove this formula for $n \geq 3$ using mathematical induction and geometry.



Example 2 **Prove that each statement is true for all natural numbers.**

17. $5^n + 3$ is divisible by 4.

18. $9^n - 1$ is divisible by 8.

19. $12^n + 10$ is divisible by 11.

20. $13^n + 11$ is divisible by 12.

Example 3 **Find a counterexample to disprove each statement.**

21. $1 + 2 + 3 + \cdots + n = n^2$

22. $1 + 8 + 27 + \cdots + n^3 = (2n + 2)^2$

23. $n^2 - n + 15$ is prime.

24. $n^2 + n + 23$ is prime.

25. **NATURE** The terms of the Fibonacci sequence are found in many places in nature. The number of spirals of seeds in sunflowers is a Fibonacci number, as is the number of spirals of scales on a pinecone. The Fibonacci sequence begins 1, 1, 2, 3, 5, 8, ... Each element after the first two is found by adding the previous two terms. If f_n stands for the nth Fibonacci number, prove that $f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$.

Prove that each statement is true for all natural numbers or find a counterexample.

26. $7^n + 5$ is divisible by 6.

27. $18^n - 1$ is divisible by 17.

28. $n^2 + 21n + 7$ is a prime number.

29. $n^2 + 3n + 3$ is a prime number.

30. $500 + 100 + 20 + \cdots + 4 \cdot 5^{4-n} = 625\left(1 - \frac{1}{5^n}\right)$

31. $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

32. **PERSEVERANCE** Refer to the figures below.

Figure 1 Figure 2 Figure 3

a. There is a total of 5 squares in the second figure. How many squares are there in the third figure?

b. Write a sequence for the first five figures.

c. How many squares are there in a standard 8 × 8 checkerboard?

d. Write a formula to represent the number of squares in an $n \times n$ grid.

H.O.T. Problems Use Higher-Order Thinking Skills

33. **CHALLENGE** Suggest a formula to represent $2 + 4 + 6 + \cdots + 2n$, and prove your hypothesis using mathematical induction.

REASONING **Determine whether the following statements are *true* or *false*. Explain.**

34. If you cannot find a counterexample to a statement, then it is true.

35. If a statement is true for $n = k$ and $n = k + 1$, then it is also true for $n = 1$.

36. **CHALLENGE** Prove $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$.

37. **REASONING** Find a counterexample to $x^3 + 30 > x^2 + 20x$.

38. **OPEN ENDED** Write a sequence, the formula that produces it, and determine the formula for the sum of the terms of the sequence. Then prove the formula with mathematical induction.

39. **WRITING IN MATH** Explain how the concept of dominoes can help you understand the power of mathematical induction.

40. **WRITING IN MATH** Provide a real-world example other than dominoes that describes mathematical induction.

Standardized Test Practice

41. Which of the following is a counterexample to the statement below?

$$n^2 + n - 11 \text{ is prime.}$$

A $n = -6$
B $n = 4$
C $n = 5$
D $n = 6$

42. **PROBABILITY** Moza wants to create a 7-character password. She wants to use an arrangement of the first 3 letters of her first name (lat), followed by an arrangement of the 4 digits in 1986, the year she was born. How many possible passwords can she create in this way?

F 72
G 144
H 288
J 576

43. **GRIDDED RESPONSE** A gear that is 8 centimeters in diameter turns a smaller gear that is 3 centimeters in diameter. If the larger gear makes 36 revolutions, how many revolutions does the smaller gear make in that time?

44. **SHORT RESPONSE** Write an equation for the nth line. Show how it fits the pattern for each given line in the list.

Line 1: $1 \times 0 = 1 - 1$

Line 2: $2 \times 1 = 4 - 2$

Line 3: $3 \times 2 = 9 - 3$

Line 4: $4 \times 3 = 16 - 4$

Line 5: $5 \times 4 = 25 - 5$

Spiral Review

Find the indicated term of each expansion. (Lesson 9-6)

45. fourth term of $(x + 2y)^6$

46. fifth term of $(a + b)^6$

47. fourth term of $(x - y)^9$

48. **BIOLOGY** In a particular forest, scientists are interested in how the population of wolves will change over the next two years. One model for animal population is the Verhulst population model, $p_{n+1} = p_n + rp_n(1 - p_n)$, where n represents the number of time periods that have passed, p_n represents the percent of the maximum sustainable population that exists at time n, and r is the growth factor. (Lesson 9-5)

a. To find the population of the wolves after one year, evaluate $p_1 = 0.45 + 1.5(0.45)(1 - 0.45)$.

b. Explain what each number in the expression in part **a** represents.

c. The current population of wolves is 165. Find the new population by multiplying 165 by the value in part **a**.

Find the exact solution(s) of each system of equations.

49. $x^2 + y^2 - 18x + 24y + 200 = 0$
$4x + 3y = 0$

50. $4x^2 + y^2 = 16$
$x^2 + 2y^2 = 4$

Skills Review

Evaluate each expression.

51. $P(8, 2)$

52. $P(9, 1)$

53. $P(12, 6)$

54. $C(5, 2)$

55. $C(8, 4)$

56. $C(20, 17)$

57. $P(12, 2)$

58. $P(7, 2)$

59. $C(8, 6)$

60. $C(9, 4) \cdot C(5, 3)$

61. $C(6, 1) \cdot C(4, 1)$

62. $C(10, 5) \cdot C(8, 4)$

LESSON 9-9 Functions as Infinite Series

Then

- You found the nth term of an infinite series expressed using sigma notation.

Now

1. Use a power series to represent a rational function.
2. Use power series representations to approximate values of transcendental functions.

Why?

- The music to which you listen on a digital audio player is first performed by an artist. The waveform of each sound in that performance is then broken down into its component parts and stored digitally. These parts are then retrieved and combined to reproduce each original sound of the performance. The analysis of a special series is an essential ingredient in this process.

NewVocabulary
power series
exponential series
Euler's Formula

1 Power Series

Earlier in this chapter, you saw how some series of numbers can be expressed as functions. In this lesson, you will see that some functions can be broken down into infinite series of component functions.

You learned that the sum of an infinite geometric series,

$$1 + r + r^2 + \cdots + r^n + \cdots,\ a_1 = 1$$

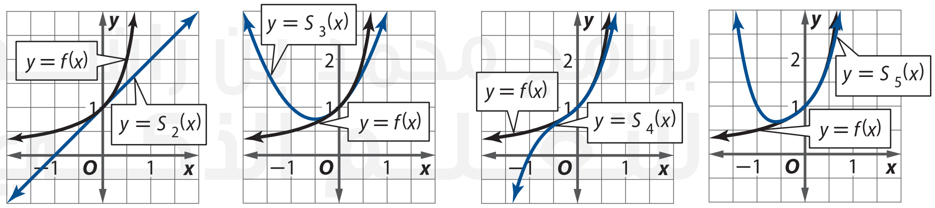
with common ratio r, converges to a sum of $\frac{a_1}{1 - r}$ if $|r| < 1$. Replacing r with x,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1 - x},\ \text{for } |x| < 1.$$

It follows that $f(x) = \frac{1}{1 - x}$ can be expressed as an infinite series. That is,

$$f(x) = \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \text{ or } 1 + x + x^2 + \cdots + x^n + \cdots \text{ for } |x| < 1.$$

The figures below show the graph of $f(x) = \frac{1}{1 - x}$ and the second through fifth partial sums $S_n(x)$ of the series: $S_2(x) = 1 + x$, $S_3(x) = 1 + x + x^2$, $S_4(x) = 1 + x + x^2 + x^3$, and $S_5(x) = 1 + x + x^2 + x^3 + x^4$.



Notice that as n increases, the graph of $S_n(x)$ appears to come closer and closer to the graph of $f(x)$ on the interval $(-1, 1)$ or $|x| < 1$. Notice too that each of the partial sums of the series is a polynomial function, so the series can be thought of as an "infinite" polynomial. An infinite series of this type is called a **power series**.

KeyConcept Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

where x and a_n can take on any values for $n = 0, 1, 2, \ldots$, is called a power series in x.

If you know the power series representation of one function, you can use it to find the power series representations of other related functions.

Example 1 Power Series Representation of a Rational Function

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x) = \frac{1}{3-x}$. Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the sixth partial sum of its power series.

To find the transformation that relates $f(x)$ to $g(x)$, use u-substitution. Substitute u for x in f, equate the two functions, and solve for u as shown.

$$g(x) = f(u)$$
$$\frac{1}{3-x} = \frac{1}{1-u}$$
$$1 - u = 3 - x$$
$$-u = 2 - x$$
$$u = x - 2$$

Therefore, $g(x) = f(x-2)$. Replacing x with $x - 2$ in $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$ yields

$$f(x-2) = \sum_{n=0}^{\infty} (x-2)^n \text{ for } |x-2| < 1.$$

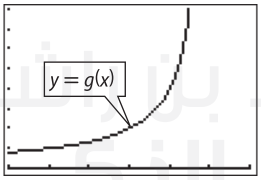
Therefore, $g(x) = \frac{1}{3-x}$ can be represented by the power series $\sum_{n=0}^{\infty} (x-2)^n$.

This series converges for $|x-2| < 1$, which is equivalent to $-1 < x - 2 < 1$ or $1 < x < 3$.

The sixth partial sum $S_6(x)$ of this series is

$$\sum_{n=0}^{5} (x-2)^n \text{ or } 1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4 + (x-2)^5.$$

The graphs of $g(x) = \frac{1}{3-x}$ and $S_6(x) = 1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4 + (x-2)^5$ are shown. Notice that on the interval $(1, 3)$, the graph of $S_6(x)$ comes close to the graph of $g(x)$.



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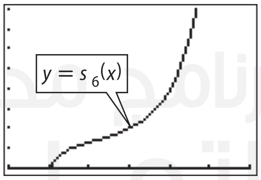

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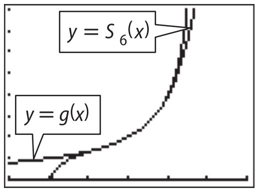

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WatchOut!

When finding the kth partial sum of a series where the lower bound starts at 0 use the series $\sum_{n=0}^{k-1}$. For instance in Example 1, the sixth partial sum is called for, but since the lower bound is 0, the upper bound is 6 − 1 or 5, not 6.

StudyTip

Graphs of Series Notice that the graphs of $f(x)$ and $S_n(x)$ only converge on an interval. The graphs may differ greatly outside of that interval.

GuidedPractice

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x)$. Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the sixth partial sum of its power series.

1A. $g(x) = \frac{1}{1-2x}$

1B. $g(x) = \frac{2}{1-x}$

In calculus, power series representations are often easier to use in calculations than other representations of functions when determining functions called *derivatives* and *integrals*. A more immediate application can be seen by looking at the power series representations of transcendental functions such as $f(x) = e^x$, $f(x) = \sin x$, and $f(x) = \cos x$.

2 Transcendental Functions as Power Series

Previously, you learned that the transcendental number e is given by $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$. Thus, $e^x = \lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^{nx}$. We can use this definition along with the Binomial Theorem to derive a power series representation for $f(x) = e^x$.

ReadingMath

Euler Number The Swiss mathematician Leonhard Euler (pronounced OY ler), published a work in which he developed this irrational number, called e, the Euler number.

If we let $u = \frac{1}{n}$ and $k = nx$, then $\left(1 + \frac{1}{n}\right)^{nx}$ becomes $(1 + u)^k$. Applying the Binomial Theorem,

$$\begin{aligned}
(1+u)^k &= {}_kC_0\,(1)^k u^0 + {}_kC_1\,(1)^{k-1}u + {}_kC_2\,(1)^{k-2}u^2 + {}_kC_3\,(1)^{k-3}u^3 + \cdots \\
&= \frac{k!}{(k-0)!\,0!}(1) + \frac{k!}{(k-1)!\,1!}(1)u + \frac{k!}{(k-2)!\,2!}(1)u^2 + \frac{k!}{(k-3)!\,3!}(1)u^3 + \cdots \\
&= 1 + \frac{k(k-1)!}{(k-1)!}u + \frac{k(k-1)(k-2)!}{(k-2)!\,2!}u^2 + \frac{k(k-1)(k-2)(k-3)!}{(k-3)!\,3!}u^3 + \cdots \\
&= 1 + ku + \frac{k(k-1)}{2!}u^2 + \frac{k(k-1)(k-2)}{3!}u^3 + \cdots
\end{aligned}$$

Now replace u with $\frac{1}{n}$ and k with nx and find the limit as n approaches infinity. Use the fact that as n approaches infinity, the fraction $\frac{1}{n}$ gets increasingly smaller, so $\lim_{n \to \infty} \frac{1}{n} = 0$.

$$\begin{aligned}
\lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^{nx} &= 1 + (nx)\frac{1}{n} + \frac{nx(nx-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{nx(nx-1)(nx-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots \\
&= 1 + x + \frac{x\left(x - \frac{1}{n}\right)}{2!} + \frac{x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right)}{3!} + \cdots \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
\end{aligned}$$

This series is often called the **exponential series**.

StudyTip

Defining e The exponential series provides yet another way to define e. When $x = 1$, $e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!}$ or $\sum_{n=0}^{\infty} \frac{1}{n!}$.

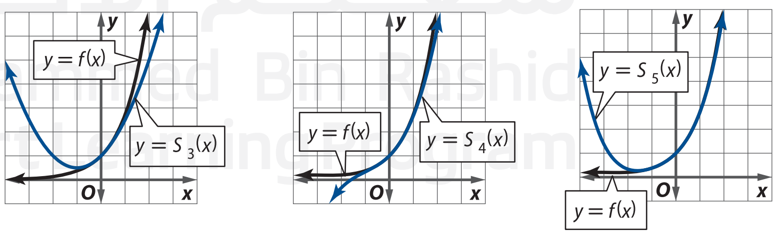
KeyConcept Exponential Series

The power series representing e^x is called the exponential series and is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

which is convergent for all x.

The graph of $f(x) = e^x$ and the partial sums $S_3(x)$, $S_4(x)$, and $S_5(x)$ of the exponential series are shown below.



You can see from the graphs that the partial sums of the exponential series approximate the graph of $f(x) = e^x$ on increasingly wider intervals of the domain for increasingly greater values of n.

Notice that the calculations involved in the exponential series are relatively simple: multiplications (for powers and factorials), divisions, and additions. Because of this, calculators and computer programs use partial sums of the exponential series to evaluate e^x to desired degrees of accuracy.

WatchOut!

Evaluating e^x The fifth partial sum of the exponential series only gives reasonably good approximations of e^x for x on $[-1.5, 2.5]$. Subsequent partial sums, such as the sixth and seventh partial sums, are more accurate for wider intervals of x-values.

Example 2 Exponential Series

Use the fifth partial sum of the exponential series to approximate the value of $e^{1.5}$. Round to three decimal places.

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \qquad e^x \approx \sum_{n=0}^{4} \frac{x^n}{n!}$$

$$e^{1.5} \approx 1 + 1.5 + \frac{1.5^2}{2!} + \frac{1.5^3}{3!} + \frac{1.5^4}{4!} \qquad x = 1.5$$

$$\approx 4.398 \qquad \text{Simplify.}$$

CHECK A calculator, using a partial sum of the exponential series with many more terms, returns an approximation of 4.48 for $e^{1.5}$. Therefore, an approximation of 4.398 is reasonable. ✔

GuidedPractice

Use the fifth partial sum of the exponential series to approximate each value. Round to three decimal places.

2A. $e^{-0.75}$

2B. $e^{0.25}$

Math HistoryLink

Mádhava of Sangamagramma (1340–1425)
An Indian mathematician born near Cochin, Mádhava discovered the series equivalent to the expansions of $\sin x$, $\cos x$, and $\arctan x$ around 1400, two hundred years before their discovery in Europe.

Other transcendental functions have power series representations as well. Calculators and computers use **power series** to approximate the values of cosine and sine functions.

KeyConcept Power Series for Cosine and Sine

The power series representations for $\cos x$ and $\sin x$ are given by

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots, \text{ and}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots,$$

which are convergent for all x.

By replacing x with any angle measure expressed in radians and carrying out the computations, approximate values of the cosine and sine functions can be found to any desired degree of accuracy.

Example 3 Trigonometric Series

a. Use the fifth partial sum of the power series for cosine to approximate the value of $\cos \frac{\pi}{7}$. Round to three decimal places.

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \qquad \cos x \approx \sum_{n=0}^{4} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos \frac{\pi}{7} \approx 1 - \frac{(0.449)^2}{2!} + \frac{(0.449)^4}{4!} - \frac{(0.449)^6}{6!} + \frac{(0.449)^8}{8!} \qquad x = \frac{\pi}{7} \text{ or about } 0.449$$

$$\approx 0.901 \qquad \text{Simplify.}$$

CHECK A calculator, using a partial sum of the power series for cosine with many more terms, returns an approximation of 0.901, to three decimal places, for $\cos \frac{\pi}{7}$. Therefore, an approximation of 0.901 is reasonable. ✔

StudyTip

Fifth Partial Sum While additional partial sums provide a better approximation, the fifth partial sum typically is accurate to three decimal places.

b. **Use the fifth partial sum of the power series for sine to approximate the value of $\sin \frac{\pi}{5}$. Round to three decimal places.**

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \qquad \sin x \approx \sum_{n=0}^{5} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin \frac{\pi}{5} \approx 0.628 - \frac{(0.628)^3}{3!} + \frac{(0.628)^5}{5!} - \frac{(0.628)^7}{7!} + \frac{(0.628)^9}{9!} \qquad x = \frac{\pi}{5} \text{ or about } 0.628$$

$$\approx 0.588 \qquad \text{Simplify.}$$

CHECK Using a calculator, $\sin \frac{\pi}{5} \approx 0.588$. Therefore, an approximation of 0.588 is reasonable. ✔

GuidedPractice

Use the fifth partial sum of the power series for cosine or sine to approximate each value. Round to three decimal places.

3A. $\sin \frac{\pi}{11}$ **3B.** $\cos \frac{2\pi}{17}$

You may have noticed similarities in the power series representations of $f(x) = e^x$ and the power series representations of $f(x) = \sin x$ and $f(x) = \cos x$. A relationship is derived by replacing x by $i\theta$ in the exponential series, where i is the imaginary unit and θ is the measure of an angle in radians.

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots \qquad e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots \qquad i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1,\ i^7 = -i$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - i\frac{\theta^7}{7!} + \cdots\right) \qquad \text{Group real and imaginary terms.}$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \qquad \text{Distributive Property}$$

$$= \cos\theta + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \qquad \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$= \cos\theta + i\sin\theta \qquad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \cdots$$

This relationship is called **Euler's Formula**.

KeyConcept Euler's Formula

For any real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$.

From your work you should recognize the right-hand side of this equation as being part of the polar form of a complex number. Applying Euler's Formula to the polar form of a complex number yields the following result.

$$a + bi = r(\cos\theta + i\sin\theta) \qquad \text{Polar form of a complex number}$$

$$= re^{i\theta} \qquad \text{Euler's Formula}$$

Therefore, Euler's Formula gives us a way of expressing a complex number in exponential form.

KeyConcept Exponential Form of a Complex Number

The exponential form of a complex number $a + bi$ is given by

$$a + bi = re^{i\theta},$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$ for $a > 0$ and $\theta = \tan^{-1}\frac{b}{a} + \pi$ for $a < 0$.

Example 4 Write a Complex Number in Exponential Form

Write $-\sqrt{3} + i$ in exponential form.

Write the polar form of $-\sqrt{3} + \boldsymbol{i}$. In this expression, $a = -\sqrt{3}$, $b = 1$, and $a < 0$. Find r.

$$r = \sqrt{(-\sqrt{3})^2 + 1^2} \qquad r = \sqrt{a^2 + b^2}$$

$$= \sqrt{4} \text{ or } 2 \qquad \text{Simplify.}$$

Now find θ.

$$\theta = \tan^{-1}\frac{1}{-\sqrt{3}} + \pi \qquad \theta = \tan^{-1}\frac{b}{a} + \pi \text{ for } a < 0$$

$$= -\frac{\pi}{6} + \pi \text{ or } \frac{5\pi}{6} \qquad \text{Simplify.}$$

Therefore, because $a + bi = re^{i\theta}$, the exponential form of $-\sqrt{3} + \boldsymbol{i}$ is $2e^{i\frac{5\pi}{6}}$.

GuidedPractice

Write each complex number in exponential form.

4A. $1 + \sqrt{3}\boldsymbol{i}$

4B. $\sqrt{2} + \sqrt{2}\boldsymbol{i}$

From your study of logarithms, you know that no *real* number can be the logarithm of a negative number. We can use Euler's Formula to show that the natural logarithm of a negative number does exist in the *complex* number system.

$$e^{i\theta} = \cos\theta + \boldsymbol{i}\sin\theta \qquad \text{Euler's Formula}$$

$$e^{i\pi} = \cos\pi + \boldsymbol{i}\sin\pi \qquad \text{Let } \theta = \pi.$$

$$e^{i\pi} = -1 + \boldsymbol{i}(0) \qquad \cos\pi = -1 \text{ and } \sin\pi = 0$$

$$e^{i\pi} = -1 \qquad \text{Simplify.}$$

$$\ln e^{i\pi} = \ln(-1) \qquad \text{Take the natural logarithm of each side.}$$

$$\boldsymbol{i}\pi = \ln(-1) \qquad \text{Power Property of Logarithms}$$

This result indicates that the natural logarithm of -1 exists and is the complex number $\boldsymbol{i}\pi$. You can use this result to find the natural logarithm of any negative number $-k$, for $k > 0$.

$$\ln(-k) = \ln[(-1)k] \qquad -k = (-1)k$$

$$= \ln(-1) + \ln k \qquad \text{Product Property of Logarithms}$$

$$= \boldsymbol{i}\pi + \ln k \qquad \ln(-1) = i\pi$$

$$= \ln k + \boldsymbol{i}\pi \qquad \text{Write in the form } a + bi.$$

TechnologyTip

Complex Numbers You can use your calculator to evaluate the natural logarithm of a negative number by changing from REAL to $a + bi$ under MODE.

Example 5 Natural Logarithm of a Negative Number

Find the value of $\ln(-5)$ in the complex number system.

$$\ln(-5) = \ln 5 + \boldsymbol{i}\pi \qquad \ln(-k) = \ln k + i\pi$$

$$\approx 1.609 + \boldsymbol{i}\pi \qquad \text{Use a calculator to compute } \ln 5.$$

GuidedPractice

Find the value of each natural logarithm in the complex number system.

5A. $\ln(-8)$

5B. $\ln(-6.24)$

Exercises

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x)$. **Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the sixth partial sum of its power series.** (Example 1)

1. $g(x) = \dfrac{4}{1-x}$

2. $g(x) = \dfrac{3}{1-2x}$

3. $g(x) = \dfrac{2}{1-x^2}$

4. $g(x) = \dfrac{3}{2-x}$

5. $g(x) = \dfrac{2}{5-3x}$

6. $g(x) = \dfrac{4}{3-2x^2}$

Use the fifth partial sum of the exponential series to approximate each value. Round to three decimal places. (Example 2)

7. $e^{0.5}$

8. $e^{-0.25}$

9. $e^{-2.5}$

10. $e^{0.8}$

11. $e^{-0.3}$

12. $e^{3.5}$

13. ECOLOGY The population density P per square meter of zebra mussels in the Upper Mississippi River can be modeled by $P = 3.5e^{0.08t}$, where t is measured in weeks. Use the fifth partial sum of the exponential series to estimate the zebra mussel population density after 4 weeks, 12 weeks, and 1 year. (Example 2)

Use the fifth partial sum of the power series for cosine or sine to approximate each value. Round to three decimal places. (Example 3)

14. $\sin \dfrac{\pi}{9}$

15. $\cos \dfrac{2\pi}{13}$

16. $\sin \dfrac{5\pi}{13}$

17. $\cos \dfrac{3\pi}{10}$

18. $\cos \dfrac{2\pi}{9}$

19. $\sin \dfrac{3\pi}{19}$

20. AMUSEMENT PARK A ride at an amusement park is in the shape of a giant pendulum that swings riders back and forth in a 240° arc to a maximum height of 41 meters. The pendulum is supported by a tower that is 26 meters tall and dips below ground-level into a pit when swinging below the tower. Use the fifth partial sum of the power series for cosine or sine to approximate the length of the pendulum. (Example 3)

Write each complex number in exponential form. (Example 4)

21. $\sqrt{3} + i$

22. $\sqrt{3} - i$

23. $\sqrt{2} - \sqrt{2}i$

24. $-\sqrt{3} - i$

25. $1 - \sqrt{3}i$

26. $-1 + \sqrt{3}i$

27. $-\sqrt{2} + \sqrt{2}i$

28. $-1 - \sqrt{3}i$

Find the value of each natural logarithm in the complex number system. (Example 5)

29. $\ln(-6)$

30. $\ln(-3.5)$

31. $\ln(-2.45)$

32. $\ln(-7)$

33. $\ln(-4.36)$

34. $\ln(-9.12)$

35. POWER SERIES Use the power series representations of $\sin x$ and $\cos x$ to answer each of the following questions.

a. Graph $f(x) = \sin x$ and the third partial sum of the power series representing $\sin x$. Repeat for the fourth and fifth partial sums. Describe the interval of convergence for each.

b. Repeat part **a** for $f(x) = \cos x$ and the third, fourth, and fifth partial sums of the power series representing $\cos x$. Describe the interval of convergence for each.

c. Describe how the interval of convergence changes as n increases. Then make a conjecture as to the relationship between each trigonometric function and its related power series as $n \to \infty$.

Solve for z over the complex numbers. Round to three decimal places.

36. $2e^z + 5 = 0$

37. $e^{2z} + 12 = 0$

38. $4e^{2z} + 7 = 6$

39. $3(e^z - 1) + 5 = -2$

40. $e^{2z} - e^z = 2$

41. $10e^{2z} + 17e^z = -3$

42. ECONOMICS The total value of an investment of P dirhams compounded continuously at an annual interest rate of r over t years is Pe^{rt}. Use the first five terms of the exponential series to approximate the value of an investment of AED 10,000 compounded continuously at 5.25% for 5 years.

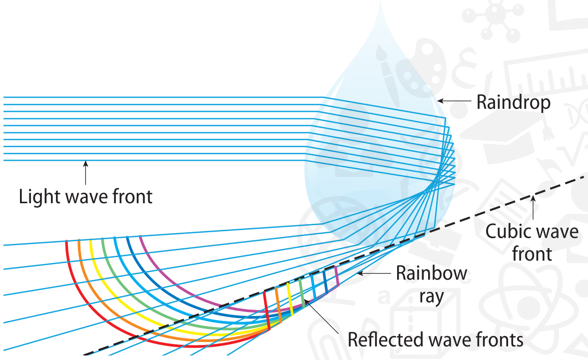
43. RELATIVE ERROR *Relative error* is the absolute error in estimating a quantity divided by its true value. The relative error of an approximation a of a quantity b is given by $\dfrac{|b-a|}{b}$. Find the relative error in approximating $e^{2.1}$ using two, three, and six terms of the exponential series.

Approximate the value of each expression using the first four terms of the power series for sine and cosine. Then find the expected value of each.

44. $\sin^2 \frac{1}{2} + \cos^2 \frac{1}{2}$

45. $\sec^2 1 - \tan^2 1$

46. RAINBOWS *Airy's equation*, which is used in physics to model the diffraction of light, can also be used to explain how a light wave front is converted into a curved wave front in forming rainbows.

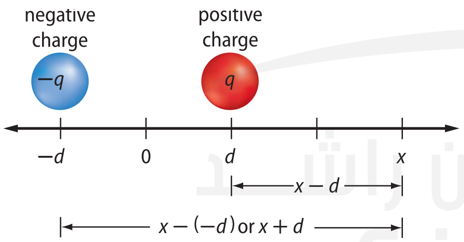


This equation can be represented by the power series shown below.

$$f(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2 \cdot 3)(5 \cdot 6) \cdots [(3k-1) \cdot (3k)]}$$

Use the fifth partial sum of the series to find $f(3)$. Round to the nearest hundredth.

47. ELECTRICITY When an electric charge is accompanied by an equal and opposite charge nearby, such an object is called an *electric dipole*. It consists of charge q at the point $x = d$ and charge $-q$ at $x = -d$, as shown below.



Along the x-axis, the electric field strength at x is the sum of the electric fields from each of the two charges. This is given by $E(x) = \frac{kq}{(x-d)^2} - \frac{kq}{(x+d)^2}$. Find a power series representing $E(x)$ if k is a constant and $d = 1$.

48. SOUND The *Fourier Series* represents a periodic function of time $f(t)$ as a summation of sine waves and cosine waves with frequencies that start at 0 and increase by integer multiples. The series below represents a sound wave from the digital data fed from a CD into a CD player.

$$f(t) = 0.7 + \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n} \cos 270.6nt + \frac{1}{2n-1} \sin 270.6nt \right)$$

Graph the series for $n = 4$. Then analyze the graph.

IDENTITIES Use power series representations from this lesson to verify each trigonometric identity.

49. $\sin(-x) = -\sin x$

50. $\cos(-x) = \cos x$

51. APPROXIMATIONS The infinite series for the inverse tangent function $f(x) = \tan^{-1} x$, is given by $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$.

However, this series is only valid for values of x on the interval $(-1, 1)$.

a. Write the first five terms of the infinite series representation for $f(x) = \tan^{-1} x$.

b. Use the first five terms of the series to approximate $\tan^{-1} 0.1$.

c. On the same coordinate plane, graph $f(x) = \tan^{-1} x$ and the third partial sum of the power series representing $f(x) = \tan^{-1} x$. On another coordinate plane, graph $f(x)$ and the fourth partial sum. Then graph $f(x)$ and the fifth partial sum.

d. Describe what happens on the interval $(-1, 1)$ and in the regions $x \geq 1$ or $x \leq -1$.

H.O.T. Problems Use Higher-Order Thinking Skills

52. WRITING IN MATH Describe how using additional terms in the approximating series for e^x affects the outcome.

53. REASONING Use the power series for sine to explain why, for x-values on the interval $[-0.1, 0.1]$, a close approximation of $\sin x$ is x.

54. CHALLENGE Prove that $2 \sin \theta \cos \theta = \frac{e^{2\theta i} - e^{-2\theta i}}{2i}$

55. REASONING For what values of α and β does $e^{i\alpha} = e^{i\beta}$? Explain.

PROOF Show that for all real numbers x, the following are true.

56. $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

57. $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

58. CHALLENGE The hyperbolic sine and hyperbolic cosine functions are analogs of the trigonometric functions. Just as the points $(\cos x, \sin x)$ form a unit circle, the points $(\cosh t, \sinh t)$ form the right half of an equilateral hyperbola. An equilateral hyperbola has perpendicular asymptotes. The hyperbolic sine (sinh) and hyperbolic cosine (cosh) functions are defined below. Find the power series representations for these functions.

$\sinh x = \frac{e^x - e^{-x}}{2}$ $\qquad$ $\cosh x = \frac{e^x + e^{-x}}{2}$

Spiral Review

Use Pascal's triangle to expand each binomial.

59. $\left(3m + \sqrt{2}\right)^4$

60. $\left(\frac{1}{2}n + 2\right)^5$

61. $(p^2 + q)^8$

62. Prove that $4 + 7 + 10 + \cdots + (3n + 1) = \frac{n(3n + 5)}{2}$ for all positive integers n.

Find each power, and express it in rectangular form.

63. $(-2 + 2i)^3$

64. $\left(1 + \sqrt{3}i\right)^4$

65. $\left(\sqrt{2} + \sqrt{2}i\right)^{-2}$

66. Given $\mathbf{t} = \langle -9, -3, c \rangle$, $\mathbf{u} = \langle 8, -4, 3 \rangle$, $\mathbf{v} = \langle 2, 5, -6 \rangle$, and that the volume of the parallelepiped having adjacent edges $\mathbf{t}$, $\mathbf{u}$, and $\mathbf{v}$ is 93 cubic units, find c.

Use an inverse matrix to solve each system of equations, if possible.

67. $x - 8y = -7$
$2x + 5y = 28$

68. $4x + 7y = 22$
$-9x + 11y = 4$

69. $w + 2x + 3y = 18$
$4w - 8x + 7y = 41$
$-w + 9x - 2y = -4$

Determine whether A and B are inverse matrices.

70. $A = \begin{bmatrix} 1 & -2 \\ 7 & -6 \end{bmatrix}, B = \begin{bmatrix} -6 & 2 \\ -7 & 1 \end{bmatrix}$

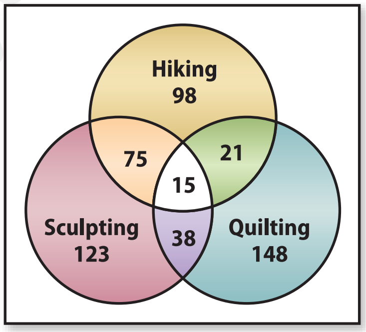
71. $A = \begin{bmatrix} -11 & -5 \\ 9 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 \\ -9 & -11 \end{bmatrix}$

72. $A = \begin{bmatrix} 6 & 2 \\ -2 & 8 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 \\ -3 & -5 \end{bmatrix}$

73. CONFERENCE A university sponsored a conference for 680 women. The Venn diagram shows the number of participants in three of the activities offered. Suppose women who attended the conference were randomly selected for a survey.

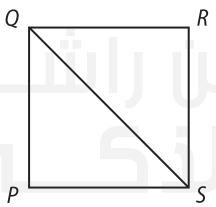
a. What is the probability that a woman selected participated in hiking or sculpting?

b. Describe a set of women such that the probability of being selected is about 0.39.



Skills Review for Standardized Tests

74. SAT/ACT $PQRS$ is a square. What is the ratio of the length of diagonal $\overline{QS}$ to the length of side $\overline{RS}$?



A 2

B $\sqrt{2}$

C 1

D $\frac{\sqrt{2}}{2}$

E $\frac{\sqrt{3}}{2}$

75. REVIEW What is the sum of the infinite geometric series $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \cdots$?

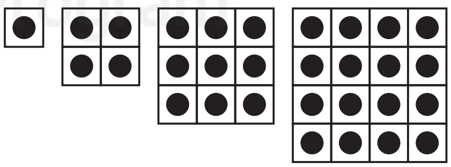
F $\frac{2}{3}$

G 1

H $1\frac{1}{3}$

J $1\frac{2}{3}$

76. FREE RESPONSE Consider the pattern of dots shown.



a. Draw the next figure in this sequence.

b. Write the sequence, starting with 1, that represents the number of dots that must be added to each figure in the sequence to get the number of dots in the next figure.

c. Find the expression for the nth term of the sequence found in part **b**.

d. Find the expression for the number of dots in the nth figure in the original sequence.

e. Prove, through mathematical induction, that the sum of the sequence found in part **b** is equal to the expression found in part **d**.

EXTEND 9-9

Spreadsheet Lab
Detecting Patterns in Data

Objective

- Organize and display data using spreadsheets to detect patterns and departures from patterns.

In Chapter 9, you learned how to detect patterns in a sequence and describe them by using functions.

Pattern in Data Sequence	Pattern in Graph of Data Sequence	Type of Sequence	Function Describing Sequence
common 1st differences	data in a linear pattern	arithmetic	linear
common ratio	data in an exponential pattern	geometric	exponential

In this lab, you will use a spreadsheet to organize and display paired data in order to look for such patterns.

Activity 1 Detect Patterns

DOGS **A certain sheep had a mass of 1.45 kilograms at birth. The table shows the sheep's mass in the first 70 days of its growth. Use a spreadsheet to find a pattern in the data.**

Days after Birth	10	20	30	40	50	60	70
Mass (kg)	1.91	2.46	3.17	4.10	5.22	6.81	8.63

Step 1 Enter the data into the spreadsheet.

Step 2 To determine if the sequence of masses is arithmetic, enter a formula in the next column to find the average daily rate of change in the sheep's mass.

Step 3 To determine if the sequence is geometric, enter the formula shown in the next column to find the average ratio of change in the sheep's mass each day.

	A	B	C	D
1	Days after Birth	Mass (kg)	Average Rate of Change	Average Ratio of Change
2	0	1.45		
3	10	1.91	=(B3-B2)/(A3-A2)	=(B3/B2)^(1/(A3-A2))-1
4	20	2.46		
5	30	3.17		
6	40	4.1		
7	50	5.22		
8	60	6.81		
9	70	8.63		

Sheet 1 Sheet 2 Sheet 3

2. There appears to be a pattern in the average ratio of change between consecutive pairs of data. These values cluster around a common average ratio of 0.026. This suggests that the sequence of mass values is geometric.

Analyze the Results

1. Explain the formulas used in the spreadsheet.
2. Describe any pattern you see in the data. What type of sequence approximates the data? Explain.
3. Use the chart tool to create a scatter plot of the data. Does this graph support your answer to Exercise 2? Explain.
4. Write an equation approximating the sheep's mass y after x days.
5. Use your equation to predict the sheep's mass 25 days after birth and 365 days after birth. Are these predictions reasonable? Explain.

You can also use a spreadsheet to detect and analyze departures from patterns.

Activity 2 Detect Departures from Patterns

HOMEWORK Omar recorded the number of precalculus problems and how long he worked on them for eight nights. Look for a pattern in the data and any departures from that pattern.

Number of Problems	0	3	5	6	8	9	10	15
Time (min)	0	27	44	70	72	82	95	140

Step 1 Enter the data into the spreadsheet.

Step 2 Enter formulas in the adjacent columns to detect whether the sequence of is arithmetic or geometric. Then copy these formulas into the cells below.

Step 3 Look for patterns. Notice that all but two of the rates of change cluster around 9.

◇	A	B	C
1	Number of Problems	Time (min)	Average Rate of Change
2	0	0	
3	3	27	9.00
4	5	44	8.50
5	6	70	26.00
6	8	72	1.00
7	9	81	9.00
8	10	89	8.00
9	15	136	9.40

Sheet 1 Sheet 2 Sheet 3

StudyTip

Series in Data To investigate series in data, you can use the Auto Sum tool. For Activity 2, enter =B2 in cell D2 and =SUM(B2,B3) in cell D3. Copy this second formula into the remaining cells in the column to create a sequence of partial sums.

Analyze the Results

6. Where does the departure in the pattern occur?
7. Write a spreadsheet formula that could model the data if this data value were removed.
8. Create a scatter plot that shows the actual data and the model of the data. Does this graph support your answer to Exercise 7? Explain.
9. Use your formula from Exercise 7 to predict how long it would take Omar to complete 12 problems and 20 problems. Are these predictions reasonable? Explain.

Exercises

Use a spreadsheet to organize and identify a pattern or departure from a pattern in each set of data. Then use a calculator to write an equation to model the data.

10. **INTERNET** The table shows the number of times the main page of a popular blog is read (hits) each month.

Month	2	4	6	8	10	12	15	20
Hits	83	171	266	368	479	732	1405	4017

11. **COLLEGE** The table shows the composite ACT scores and grade-point averages (GPA) of 20 students after their first semester in college. (*Hint:* First use the Sort Ascending tool to organize the data.)

ACT Score	27	16	15	22	20	21	25
College GPA	3.9	2.9	2.7	3.6	3.2	3.4	3.1
ACT Score	26	18	23	19	29	28	17
College GPA	4.0	3.1	3.6	2.6	4.0	3.9	3.0

CHAPTER 9 Study Guide and Review

Study Guide

KeyConcepts

Arithmetic Sequences and Series (Lessons 9-1 and 9-2)

- The nth term a_n of an arithmetic sequence with first term a_1 and common difference d is given by $a_n = a_1 + (n - 1)d$, where n is any positive integer.
- The sum S_n of the first n terms of an arithmetic series is given by $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ or $S_n = \frac{n}{2}(a_1 + a_n)$.

Geometric Sequences and Series (Lessons 9-3 and 9-4)

- The nth term a_n of a geometric sequence with first term a_1 and common ratio r is given by $a_n = a_1 \cdot r^{n-1}$, where n is any positive integer.
- The sum S_n of the first n terms of a geometric series is given by $S_n = \frac{a_1(1 - r^n)}{1 - r}$ or $S_n = \frac{a_1 - a_1 r^n}{1 - r}$, where $r \neq 1$.
- The sum S of an infinite geometric series with $-1 < r < 1$ is given by $S_n = \frac{a_1}{1 - r}$.

Recursion and Iteration (Lesson 9-5)

- In a recursive formula, each term is formulated from one or more previous terms.

The Binomial Theorem (Lesson 9-6)

- The Binomial Theorem:

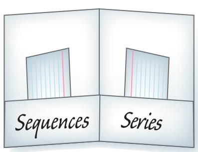
$$(a + b)^n = \sum_{k=0}^{n} \frac{n!}{(n - k)!\,k!} a^{n-k} b^k$$

Mathematical Induction (Lesson 9-7)

- Mathematical induction is a method of proof used to prove statements about the positive integers.

FOLDABLES StudyOrganizer

Be sure the Key Concepts are noted in your Foldable.



KeyVocabulary

arithmetic means	induction hypothesis
arithmetic sequence	infinite geometric series
arithmetic series	infinite sequence
common difference	infinity
common ratio	iteration
convergent series	mathematical induction
divergent series	partial sum
explicit formula	Pascal's triangle
Fibonacci sequence	recursive formula
finite sequence	recursive sequence
geometric means	sequence
geometric sequence	series
geometric series	sigma notation
	term

VocabularyCheck

State whether each sentence is *true* or *false*. If *false*, replace the underlined term to make a true sentence.

1. An infinite geometric series that has a sum is called a convergent series.
2. Mathematical induction is the process of repeatedly composing a function with itself.
3. The arithmetic means of a sequence are the terms between any two non-successive terms of an arithmetic sequence.
4. A term is a list of numbers in a particular order.
5. The sum of the first n terms of a series is called the partial sum.
6. The formula $a_n = a_{n-2} + a_{n-1}$ is a recursive formula.
7. A geometric sequence is a sequence in which every term is determined by adding a constant value to the previous term.
8. An infinite geometric series that does not have a sum is called a partial sum.
9. Eleven and 17 are two geometric means between 5 and 23 in the sequence 5, 11, 17, 23.
10. Using the Binomial Theorem, $(x - 2)^4$ can be expanded to $x^4 - 8x^3 + 24x^2 - 32x + 16$.

Lesson-by-Lesson Review

9-1 Sequences as Functions

Find the indicated term of each arithmetic sequence.

11. $a_1 = 9, d = 3, n = 14$

12. $a_1 = -3, d = 6, n = 22$

13. $a_1 = 10, d = -4, n = 9$

14. $a_1 = -1, d = -5, n = 18$

Example 1

Find the 11th term of an arithmetic sequence if $a_1 = -15$ and $d = 6$.

$a_n = a_1 + (n - 1)d$	Formula for the nth term
$a_{11} = -15 + (11 - 1)6$	$n = 11, a_1 = -15, d = 6$
$a_{11} = 45$	Simplify.

9-2 Arithmetic Sequences and Series

Find the arithmetic means in each sequence.

15. -12, __, __, __, 8

16. 15, __, __, 29

17. 12, __, __, __, __, -8

18. 72, __, __, __, 24

19. **BANKING** Zayed saves AED 150 every 2 months. If he saves at this rate for two years, how much will he have at the end of two years?

Find S_n for each arithmetic series.

20. $a_1 = 16, a_n = 48, n = 6$

21. $a_1 = 8, a_n = 96, n = 20$

22. $9 + 14 + 19 + \cdots + 74$

23. $16 + 7 + -2 + \cdots + -65$

24. **DRAMA** Laila has a drama performance in 12 days. She plans to practice her lines each night. On the first night she rehearses her lines 2 times. The next night she rehearses her lines 4 times. The third night she rehearses her lines 6 times. On the eleventh night, how many times has she rehearsed her lines?

Find the sum of each arithmetic series.

25. $\sum_{k=5}^{21}(3k - 2)$

26. $\sum_{k=0}^{10}(6k - 1)$

27. $\sum_{k=4}^{12}(-2k + 5)$

Example 2

Find the two arithmetic means between 3 and 39.

$a_n = a_1 + (n - 1)d$	Formula for the nth term
$a_4 = 3 + (4 - 1)d$	$n = 4, a_1 = 3$
$39 = 3 + 3d$	$a_4 = 39$
$12 = d$	Simplify.

The arithmetic means are 3 + 12 or 15 and 15 + 12 or 27.

Example 3

Find S_n for the arithmetic series with $a_1 = 18$, $a_n = 56$, and $n = 8$.

$S_n = \frac{n}{2}(a_1 + a_n)$	Sum formula
$S_8 = \frac{8}{2}(18 + 56)$	$n = 8, a_1 = 18, a_n = 56$
$= 296$	Simplify.

Example 4

Evaluate $\sum_{k=3}^{15} 5k + 1$.

Use the formula $S_n = \frac{n}{2}(a_1 + a_n)$. There are 13 terms, $a_1 = 5(3) + 1$ or 16, and $a_{13} = 5(15) + 1$ or 76.

$S_{13} = \frac{13}{2}(16 + 76)$

$= 598$

9-3 Geometric Sequences and Series

Find the indicated term for each geometric sequence.

28. $a_1 = 5, r = 2, n = 7$

29. $a_1 = 11, r = 3, n = 3$

30. $a_1 = 128, r = -\frac{1}{2}, n = 5$

31. a_8 for $\frac{1}{8}, \frac{3}{8}, \frac{9}{8}, \ldots$

Find the geometric means in each sequence.

32. 6, __, __, 162

33. 8, __, __, __, 648

34. −4, __, __, 108

35. **SAVINGS** Najat has a savings account with a current balance of AED 1,500. What would be Najat's account balance after 4 years if he receives 5% interest (murabaha) annually?

Find S_n for each geometric series.

36. $a_1 = 15, r = 2, n = 4$

37. $a_1 = 9, r = 4, n = 6$

38. $5 - 10 + 20 - \cdots$ to 7 terms

39. $243 + 81 + 27 + \cdots$ to 5 terms

Evaluate the sum of each geometric series.

40. $\sum_{k=1}^{7} 3 \cdot (-2)^{k-1}$

41. $\sum_{k=1}^{8} -1\left(\frac{2}{3}\right)^{k-1}$

42. **ADVERTISING** Nabila is handing out fliers to advertise the next student council meeting. She hands out fliers to 4 people. Then, each of those 4 people hand out 4 fliers to 4 other people. Those 4 then hand out 4 fliers to 4 new people. If Nabila is considered the first round, how many people will have been given fliers after 4 rounds?

Example 5

Find the sixth term of a geometric sequence for which $a_1 = 9$ and $r = 4$.

$a_n = a_1 \cdot r^{n-1}$ — Formula for the nth term

$a_6 = 9 \cdot 4^{6-1}$ — $n = 6, a_1 = 9, r = 4$

$a_6 = 9216$

The sixth term is 9216.

Example 6

Find two geometric means between 1 and 27.

$a_n = a_1 \cdot r^{n-1}$ — Formula for the nth term

$a_4 = 1 \cdot r^{4-1}$ — $n = 4$ and $a_1 = 1$

$27 = r^3$ — $a_4 = 27$

$3 = r$ — Simplify.

The geometric means are 1(3) or 3 and 3(3) or 9.

Example 7

Find the sum of a geometric series for which $a_1 = 3$, $r = 5$, and $n = 11$.

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ — Sum formula

$S_{11} = \frac{3 - 3 \cdot 5^{11}}{1 - 5}$ — $n = 11, a_1 = 3, r = 5$

$S_{11} = 36{,}621{,}093$ — Use a calculator.

Example 8

Evaluate $\sum_{k=1}^{6} 2 \cdot (4)^{k-1}$.

$S_6 = \frac{2 - 2 \cdot 4^6}{1 - 4}$ — $n = 6, a_1 = 2, r = 4$

$= \frac{-8190}{-3}$ — Simplify.

$= 2730$ — Simplify.

9-4 Infinite Geometric Series

Find the sum of each infinite series, if it exists.

43. $a_1 = 8, r = \frac{3}{4}$

44. $\frac{5}{6} - \frac{20}{18} + \frac{80}{54} - \frac{320}{162} + \cdots$

45. $\sum_{k=1}^{\infty} 3\left(\frac{1}{2}\right)^{k-1}$

46. PHYSICAL SCIENCE Maysoun drops a ball off of a building that is 20 meters high. Each time the ball bounces, it bounces back to $\frac{2}{3}$ its previous height. If the ball continues to follow this pattern, what will be the total distance that the ball travels?

Example 9

Find the sum of the infinite geometric series for which $a_1 = 15$ and $r = \frac{1}{3}$.

$S = \frac{a_1}{1 - r}$ — Sum formula

$= \frac{15}{1 - \frac{1}{3}}$ — $a_1 = 15, r = \frac{1}{3}$

$= \frac{15}{\frac{2}{3}}$ or 22.5 — Simplify.

9-5 Recursion and Iteration

Find the first five terms of each sequence.

47. $a_1 = -3, a_{n+1} = a_n + 4$

48. $a_1 = 5, a_{n+1} = 2a_n - 5$

49. $a_1 = 1, a_{n+1} = a_n + 5$

50. SAVINGS Shaikha has a savings account with a AED 12,000 balance. She has a 5% interest (murabaha) rate that is compounded monthly. Every month Shaikha adds AED 500 to the account. The recursive formula $b_n = 1.05b_{n-1} + 500$ describes the balance in Shaikha's savings account after n months. Find the balance of Shaikha's account after 3 months. Round your answer to the nearest fil.

Find the first three iterates of each function for the given initial value.

51. $f(x) = 2x + 1, x_0 = 3$

52. $f(x) = 5x - 4, x_0 = 1$

53. $f(x) = 6x - 1, x_0 = 2$

54. $f(x) = 3x + 1, x_0 = 4$

Example 10

Find the first five terms of the sequence in which $a_1 = 1$, $a_{n+1} = 3a_n + 2$.

$a_{n+1} = 3a_n + 2$	Recursive formula
$a_{1+1} = 3a_1 + 2$	$n = 1$
$a_2 = 3(1) + 2$ or 5	$a_1 = 1$
$a_{2+1} = 3a_2 + 2$	$n = 2$
$a_3 = 3(5) + 2$ or 17	$a_2 = 5$
$a_{3+1} = 3a_3 + 2$	$n = 3$
$a_4 = 3(17) + 2$ or 53	$a_3 = 17$
$a_{4+1} = 3a_4 + 2$	$n = 4$
$a_5 = 3(53) + 2$ or 161	$a_4 = 53$

The first five terms of the sequence are 1, 5, 17, 53, and 161.

Example 11

Find the first three iterates of the function $f(x) = 3x - 2$ for the initial value of $x_0 = 2$.

$x_1 = f(x_0)$	$x_2 = f(x_1)$	$x_3 = f(x_2)$
$= f(2)$	$= f(4)$	$= f(10)$
$= 3(2) - 2$	$= 3(4) - 2$	$= 3(10) - 2$
$= 4$	$= 10$	$= 28$

The first three iterates are 4, 10, and 28.

9-6 The Binomial Theorem

Expand each binomial.

55. $(a + b)^3$

56. $(y - 3)^7$

57. $(3 - 2z)^5$

58. $(4a - 3b)^4$

59. $\left(x - \frac{1}{4}\right)^5$

Find the indicated term of each expression.

60. third term of $(a + 2b)^8$

61. sixth term of $(3x + 4y)^7$

62. second term of $(4x - 5)^{10}$

Example 12

Expand $(x - 3y)^4$.

$$(x - 3y)^4$$
$$= x^4 + {}_4C_1x^3(-3y) + {}_4C_2x^2(-3y)^2 + {}_4C_3(-3y)^4 + {}_4C_4(-3y)^4$$
$$= x^4 + \frac{4!}{3!}x^3(-3y) + \frac{4!}{2!2!}x^2(9y^2) + \frac{4!}{3!}x(-27y^3) + 81y^4$$
$$= x^4 + -12x^3y + 54x^2y^2 + -108xy^3 + 81y^4$$

Example 13

Find the fourth term of $(x + y)^8$.

Use the Binomial Theorem to write the expansion in sigma notation.

$$(x + y)^8 = \sum_{k=0}^{8} \frac{8!}{k!(8 - k)!}x^{8-k}y^k$$

For the fourth term, $k = 3$.

$$\frac{8!}{k!(8 - k)!}x^{8-k}y^k = \frac{8!}{3!(8 - 3)!}x^{8-3}y^3$$
$$= 56x^5y^3$$

9-7 Proof by Mathematical Induction

Prove that each statement is true for all positive integers.

63. $2 + 6 + 12 + \cdots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}$

64. $7^n - 1$ is divisible by 6.

65. $5^n - 1$ is divisible by 4.

Find a counterexample for each statement.

66. $8^n + 3$ is divisible by 11.

67. $6^{n+1} - 2$ is divisible by 17.

68. $n^2 + 2n + 4$ is prime.

69. $n + 19$ is prime.

Example 14

Prove that $9^n + 3$ is divisible by 4.

Step 1 When $n = 1$, $9^n + 3 = 9^1 + 3$ or 12. Since 12 divided by 4 is 3, the statement is true for $n = 1$.

Step 2 Assume that $9^k + 3$ is divisible by 4 for some positive integer k. This means that $9^k + 3 = 4r$ for some whole number r.

Step 3
$$9^k + 3 = 4r$$
$$9^k = 4r - 3$$
$$9^{k+1} = 36r - 27$$
$$9^{k+1} + 3 = 36r - 27 + 3$$
$$9^{k+1} + 3 = 36r - 24$$
$$9^{k+1} + 3 = 4(9r - 6)$$

Since r is a whole number, $9r - 6$ is a whole number. Thus, $9^{k+1} + 3$ is divisible by 4, so the statement is true for $n = k + 1$.

Therefore, $9^n + 3$ is divisible by 4 for all positive integers n.

Study Guide and Review

9-9 Functions as Infinite Series

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x)$.
Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the 6th partial sum of its power series.

42. $g(x) = \frac{1}{1-5x}$

43. $g(x) = \frac{3}{1-2x}$

Use the fifth partial sum of the exponential series to approximate each value. Round to three decimal places.

44. $e^{\frac{1}{4}}$

45. $e^{-1.5}$

Find the value of each natural logarithm in the complex number system.

46. $\ln(-4)$

47. $\ln(-7.15)$

Example 6

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x) = \frac{4}{1-x}$. Indicate the interval on which the series converges.

A geometric series converges to $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. Replace x with $\frac{x+3}{4}$ since $g(x)$ is a transformation of $f(x)$ and: $g(x) = f\left(\frac{x+3}{4}\right)$. The result is

$f\left(\frac{x+3}{4}\right) = \sum_{n=0}^{\infty}\left(\frac{x+3}{4}\right)^n$ for $\left|\frac{x+3}{4}\right| < 1$.

Therefore, $g(x) = \frac{4}{1-x}$ can be represented by $\sum_{n=0}^{\infty}\left(\frac{x+3}{4}\right)^n$. This series converges for $\left|\frac{x+3}{4}\right| < 1$, which is equivalent to $-1 < \frac{x+3}{4} < 1$ or $-7 < x < 1$.

CHAPTER 9

Practice Test

1. Find the next 4 terms of the arithmetic sequence 81, 72, 63, ...

2. Find the 25th term of an arithmetic sequence for which $a_1 = 9$ and $d = 5$.

3. **MULTIPLE CHOICE** What is the eighth term in the arithmetic sequence that begins 18, 20.2, 22.4, 24.6, ...?

A 26.8

B 29

C 31.2

D 33.4

4. Find the four arithmetic means between −9 and 11.

5. Find the sum of the arithmetic series for which $a_1 = 11$, $n = 14$, and $a_n = 22$.

6. **MULTIPLE CHOICE** What is the next term in the geometric sequence below?

$$10, \frac{5}{2}, \frac{5}{8}, \frac{5}{32} \dots$$

F $\frac{5}{8}$

G $\frac{5}{32}$

H $\frac{5}{128}$

J $\frac{5}{256}$

7. Find the three geometric means between 6 and 1536.

8. Find the sum of the geometric series for which $a_1 = 15$, $r = \frac{2}{3}$, and $n = 5$.

Find the sum of each series, if it exists.

9. $\sum_{k=2}^{12} (3k - 1)$

10. $\sum_{k=1}^{\infty} \frac{1}{2}(3^k)$

11. $45 + 37 + 29 + \dots + -11$

12. $\frac{1}{8} + \frac{2}{24} + \frac{4}{72} + \dots$

13. Write $0.\overline{65}$ as a fraction.

Find the first five terms of each sequence.

14. $a_1 = -1$, $a_{n+1} = 3a_n + 5$

15. $a_1 = 4$, $a_{n+1} = a_n + n$

16. **MULTIPLE CHOICE** What are the first 3 iterates of $f(x) = -5x + 4$ for an initial value of $x_0 = 3$?

A 3, −11, 59

B −11, 59, −291

C −1, −6, −11

D 59, −291, 1459

17. Expand $(2a - 3b)^4$.

18. What is the coefficient of the fifth term of $(m + 3n)^6$?

19. Find the fourth term of the expansion of $(c + d)^9$.

Prove that each statement is true for all positive integers.

20. $1 + 6 + 36 + \dots + 6^{n-1} = \frac{1}{5}(6^n - 1)$.

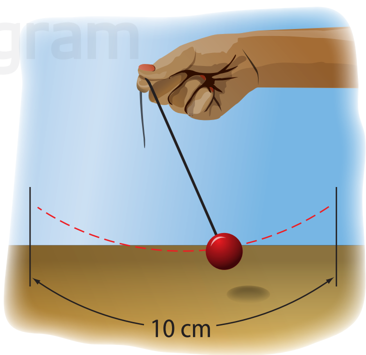
21. $11^n - 1$ is divisible by 10.

22. Find a counterexample for the following statement.

$2^n + 4^n$ is divisible by 4.

23. **SCHOOL** There are an equal number of 15 and 16 year old students in Mr. Khalid's science class. He needs to choose 8 students to represent his class at the science fair. What is the probability that 5 are 15 years old?

24. **PENDULUM** Laila swings a pendulum. The distance traveled per swing decreases by 15% with each swing. If the pendulum initially traveled 10 centimeters, find the total distance traveled when the pendulum comes to a rest.



CHAPTER 9

Preparing for Standardized Tests

Look For a Pattern

One of the most common problem-solving strategies is to look for a pattern. The ability to recognize patterns, model them algebraically, and extend them is a valuable problem-solving tool.

Strategies for Looking For a Pattern

Step 1

Identify the pattern.

- Compare the numbers, shapes, or graphs in the pattern.
- **Ask yourself:** How are the terms of the pattern related?
- **Ask yourself:** Are there any common operations that lead from one term to the next?

Step 2

Generalize the pattern.

- Write a rule using words to describe how the terms of the pattern are generated.
- Assign variables and write an algebraic expression to model the pattern if appropriate.

Step 3

Find missing terms, extend the pattern, and solve the problem.

- Use your pattern or your rule to finding missing terms and/or extend the pattern to solve the problem.
- Check your answer to make sure it makes sense.

Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

Use the sequence of squares shown. How many squares will be needed to make the ninth figure of the sequence?

A 55 C 74

B 65 D 82

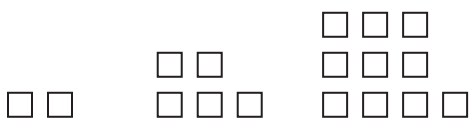
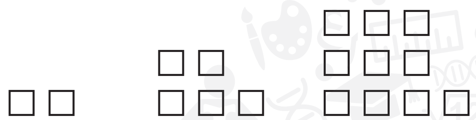


Figure 1 Figure 2 Figure 3

Read the problem statement carefully. You are given three figures of a sequence and asked to find how many squares will be needed to make the ninth figure.

Look for a pattern in the figures of squares. Count the number of squares in each figure.



Write an expression to model this pattern.

Words The number of squares is equal to the square of the figure number plus one.

Variable Let n represent the figure number.

Equation $a_n = n^2 + 1$

Use your expression to extend the pattern and find the number of squares in the ninth figure.

$a_9 = 9^2 + 1 = 82$

So, the ninth figure will have 82 squares. The correct answer is D.

Exercises

Read each problem. Use a pattern to solve the problem.

1. The numbers below form a famous mathematical sequence of numbers known as the Fibonacci sequence. What is the next Fibonacci number in the sequence?

1, 1, 2, 3, 5, 8, 13, 21, ...

A 36

B 34

C 31

D 29

2. What is the missing number in the table?

n	a_n
1	0
2	2
3	6
4	12
5	??
6	30

F 17

G 18

H 20

J 21

CHAPTER 9

Standardized Test Practice

Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Find the next term of the arithmetic sequence.

$$\boxed{7, 13, 19, 25, 31, \ldots}$$

A 36 **C** 38

B 37 **D** 39

2. Maha gets an enlargement of a 10 centimeter by 15 centimeter picture so that the new print has dimensions that are 4 times the dimensions of her original. How does the area of the enlargement compare to the area of the original picture?

F The area is twice as large.

G The area is four times as large.

H The area is eight times as large.

J The area is sixteen times as large.

3. Evaluate $\sum_{k=1}^{15} (8k - 1)$.

A 119 **C** 945

B 826 **D** 1072

4. What is the effect on the graph of the equation $y = 3x^2$ when the equation is changed to $y = 2x^2$?

F The graph of $y = 2x^2$ is a reflection of the graph of $y = 3x^2$ across the y-axis.

G The graph is rotated 90 degrees about the origin.

H The graph is narrower.

J The graph is wider.

5. Write the formula for the nth term of the geometric sequence shown in the table.

n	a_n
1	5
2	10
3	20
4	40
5	80

A $a_n = (5)^n$

B $a_n = 5(2)^{n-1}$

C $a_n = 2(5)^{n-1}$

D $a_n = 5(2)^n$

6. The table shows a dimension of a square tent and the number of people that the tent can fit.

Length of Tent (meters)	Number of People
2	7
5	28
6	39
8	67
12	147

Let ℓ represent the length of the tent and n represent the number of people that can fit in the tent. Identify the equation that best represents the relationship between the length of the tent and the number of people that can fit in the tent.

F $\ell = n^2 + 3$ **H** $\ell = 3n + 1$

G $n = \ell^2 + 3$ **J** $n = 3\ell + 1$

7. An air filter claims to remove 90% of the contaminants in the air each time air is circulated through the filter. If the same volume of air is circulated through the filter three times, what percent of the original contaminants will be removed from the air?

A 0.01% **B** 0.1% **C** 99.0% **D** 99.9%

8. At the movies, the cost of 2 boxes of popcorn and 1 soft drink is AED 34.50. The cost of 3 boxes of popcorn and 4 soft drinks is AED 81.75. Which pair of equations can be used to determine p, the cost of a box of popcorn, and s, the cost of a soft drink?

F $2p + s = 81.75$
$3p + 4s = 34.50$

G $2p - s = 34.50$
$3p - 4s = 81.75$

H $2p + s = 34.50$
$3p + 4s = 81.75$

J $p + s = 34.50$
$p + 4 = 81.75$

9. Which of the following geometric series does *not* converge to a sum?

A $\sum_{k=1}^{\infty} 4 \cdot \left(\frac{9}{10}\right)^{k-1}$ **C** $\sum_{k=1}^{\infty} \frac{7}{6} \cdot \left(\frac{1}{3}\right)^{k-1}$

B $\sum_{k=1}^{\infty} \frac{1}{5} \cdot \left(\frac{3}{2}\right)^{k-1}$ **D** $\sum_{k=1}^{\infty} (-2) \cdot \left(\frac{5}{6}\right)^{k-1}$

Test-Taking Tip

Question 9 Understand the terms used in Algebra and how to apply them. A geometric series converges to a sum if the common ratio r has an absolute value less than 1.

Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

10. What are the dimensions of the matrix that results from the multiplication shown?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix}$$

11. GRIDDED RESPONSE Consider the pattern below. Into how many pieces will the sixth figure of the pattern be divided?

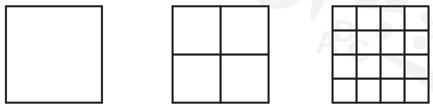
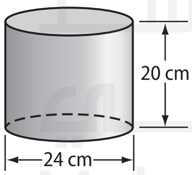


Figure 1 1 piece | Figure 2 4 pieces | Figure 3 16 pieces

12. Use the Binomial Theorem to expand the expression $(c + d)^6$.

13. GRIDDED RESPONSE Suhaila has a cylindrical container that she needs to fill with dirt so she can plant some flowers.



What is the volume of the cylinder in cubic centimeters rounded to the nearest cubic centimeter?

14. Bacteria in a culture are growing exponentially with time, as shown in the table.

Hours	Bacteria
0	1000
1	2000
2	4000

Write an equation to express the number of bacteria, y, with respect to time, t.

15. GRIDDED RESPONSE What is the value of $f[g(6)]$ if $f(x) = 2x + 4$ and $g(x) = x^2 + 5$?

Extended Response

Record your answers on a sheet of paper. Show your work.

16. Prove that the sum of any two odd integers is even.

17. The endpoints of a diameter of a circle are at $(-1, 0)$ and $(5, -8)$.

a. What are the coordinates of the center of the circle? Explain your method.

b. Find the radius of the circle. Explain your method.

c. Write an equation of the circle.

18. A cyclist travels from Dubai to Sharjah in 2.5 hours. If she increases her speed, she can make the trip in 2 hours.

a. Does this situation represent a direct or inverse variation? Explain your reasoning.

b. If the trip from Dubai to Sharjah takes 2.5 hours when traveling at 12 kilometers per hour, what must the speed be to make the trip in 2 hours?

CHAPTER 10 Statistics and Probability

Then

You calculated weighted averages.

Now

You will:

- Evaluate surveys, studies, and experiments.
- Create and use graphs of probability distributions.
- Use the Empirical Rule to find probabilities.
- Compare sample statistics and population statistics.

Why?

EDUCATION Probability and statistics are used in all facets of education. Surveys and experiments are done to find out which teaching methods promote the most learning. Statistics are used to determine grades when classes are curved, or when college professors weight their grades.

Get Ready for the Chapter

QuickCheck

Find the mean, median, and mode for each set of data.

1. number of customers at a store each day during the last two weeks:
78, 80, 101, 66, 73, 92, 97, 125, 110, 76, 89, 90, 82, 87

2. a student's quiz scores for the first grading period:
88, 70, 85, 92, 88, 77, 98, 88, 70, 82

3. the number of goals scored by a player over the last 10 years:
7, 5, 10, 12, 4, 10, 11, 6, 9, 3

A number cube is rolled and a coin is tossed. Find each probability.

4. P (4, heads)

5. P (odds, tails)

6. P (2 or 4, heads)

Expand each binomial.

7. $(a - 2)^4$

8. $(m - a)^5$

9. $(2b - x)^4$

10. $(2a + b)^6$

11. $(3x - 2y)^5$

12. $(3x + 2y)^4$

13. $\left(\frac{a}{2} + 2\right)^5$

14. $\left(3 + \frac{m}{3}\right)^5$

QuickReview

Example 1

The number of days it rained in each month over the past year are shown below. Find the mean, median, and mode.

4, 2, 9, 16, 13, 9, 8, 9, 7, 6, 8, 5

Mean $\bar{x} = \frac{4 + 2 + 9 + 16 + 13 + 9 + 8 + 9 + 7 + 6 + 8 + 5}{12}$ or 8 days

Median 2, 4, 5, 6, 7, 8, 8, 9, 9, 9, 13, 16

$\frac{8 + 8}{2}$ or 8 days

Mode The value that occurs most often in the set is 9, so the mode of the data set is 9 days.

Example 2

A number cube is rolled and a coin is tossed. What is the probability that the number cube shows a 1 and the coin lands tails up?

$P\text{(1, tails)} = \frac{1}{6} \cdot \frac{1}{2}$ or $\frac{1}{12}$

Example 3

Expand $(a + b)^4$.

Replace n with 4 in the Binomial Theorem.

$$(a + b)^4$$
$$= {}_4C_0a^4 + {}_4C_1a^3b + {}_4C_2a^2b^2 + {}_4C_3ab^3 + {}_4C_4b^4$$
$$= \frac{4!}{(4 - 0)! \cdot 0!}a^4 + \frac{4!}{(4 - 1)! \cdot 1!}a^3b + \frac{4!}{(4 - 2)! \cdot 2!}a^2b^2 + \frac{4!}{(4 - 3)! \cdot 3!}ab^3 + \frac{4!}{(4 - 4)! \cdot 4!}b^4$$
$$= \frac{24}{24}a^4 + \frac{24}{6 \cdot 1}a^3b + \frac{24}{2 \cdot 2}a^2b^2 + \frac{24}{1 \cdot 6}ab^3 + \frac{24}{1 \cdot 4}b^4$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Get Started on the Chapter

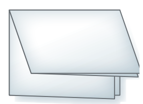
You will learn several new concepts, skills, and vocabulary terms as you study Chapter 10. To get ready, identify important terms and organize your resources.

FOLDABLES® StudyOrganizer

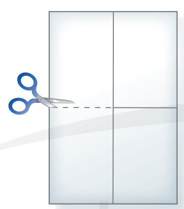
Statistics and Probability Make this Foldable to help you organize your Chapter 10 notes about statistics and probability. Begin with a sheet of $8\frac{1}{2}$" by 11" paper.

1 **Fold** in half lengthwise.

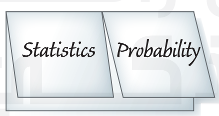
2 **Fold** the top to the bottom.



3 **Open.** Cut along the second fold to make two tabs.



4 **Label** each tab as shown.



NewVocabulary

English

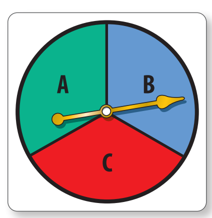
parameter
statistic
survey
experiment
observational study
random variable
probability distribution
expected value
binomial experiment
binomial distribution
normal distribution
z-value
confidence interval
inferential statistics
statistical inference
hypothesis test
null hypothesis
alternative hypothesis

ReviewVocabulary

combination an arrangement or selection of objects in which order is not important

permutation a group of objects or people arranged in a certain order

random Unpredictable, or not based on any predetermined characteristics of the population; when a die is tossed, a coin is flipped, or a spinner is spun, the outcome is a random event.



LESSON 10-1

Designing a Study

Then

- You identified various sampling techniques.

Now

1. Classify study types.
2. Design statistical studies.

Why?

- According to a recent study, 88% of teen cell phone users in the U.S. send text messages, and one in three teens sends more than 100 texts per day.

NewVocabulary
parameter
statistic
bias
random sample
survey
experiment
observational study

Mathematical Practices
Construct viable arguments and critique the reasoning of others.

1 Classifying Studies

In a statistical study, data are collected and used to answer questions about a population characteristic or **parameter**. Due to time and money constraints, it may be impractical or impossible to collect data from each member of a population. Therefore, in many studies, a sample of the population is taken, and a measure called a **statistic** is calculated using the data. The sample statistic, such as the sample mean or sample standard deviation, is then used to make inferences about the population parameter.

The steps in a typical statistical study are shown below.

Identify the objective of the study. → Choose a sample, and collect data. → Organize the data, and calculate sample statistics. → Make inferences, and draw conclusions about the population.

To obtain good information and draw accurate conclusions about a population, it is important to select an *unbiased* sample. A **bias** is an error that results in a misrepresentation of members of a population. A poorly chosen sample can cause biased results. To reduce the possibility of selecting a biased sample, a **random sample** can be taken, in which members of the population are selected entirely by chance.

You will review other sampling methods in Exercise 32.

The following study types can be used to collect sample information.

KeyConcept Study Types

Definition	Example
In a **survey**, data are collected from responses given by members of a population regarding their characteristics, behaviors, or opinions.	To determine whether the student body likes the new cafeteria menu, the student council asks a random sample of students for their opinion.
In an **experiment**, the sample is divided into two groups: • an *experimental group* that undergoes a change, and • a *control group* that does not undergo the change. The effect on the experimental group is then compared to the control group.	A restaurant is considering creating meals with chicken instead of beef. They randomly give half of a group of participants meals with chicken and the other half meals with beef. Then they ask how they like the meals.
In an **observational study**, members of a sample are measured or observed without being affected by the study.	Researchers at an electronics company observe a group of teenagers using different laptops and note their reactions.

Example 1 Classify Study Types

Determine whether each situation describes a *survey*, an *experiment*, or an *observational study*. Then identify the sample, and suggest a population from which it may have been selected.

a. **MUSIC** **A record label wants to test three designs for an album cover. They randomly select 50 teenagers from local high schools to view the covers while they watch and record their reactions.**

This is an observational study, because the company is going to observe the teens without them being affected by the study. The sample is the 50 teenagers selected, and the population is all potential purchasers of this album.

b. **RECYCLING** **The city council wants to start a recycling program. They send out a questionnaire to 200 random citizens asking what items they would recycle.**

This is a survey, because the data are collected from participants' responses in the questionnaire. The sample is the 200 people who received the questionnaire, and the population is all of the citizens of the city.

> **StudyTip**
> **Census**
> A *census* is a survey in which each member of a population is questioned. Therefore, when a census is conducted, there is no sample.

GuidedPractice

1A. **RESEARCH** Scientists study the behavior of one group of cats given a new heartworm treatment and another group of cats given a false treatment or *placebo.*

1B. **YEARBOOKS** The yearbook committee conducts a study to determine whether students would prefer to have a print yearbook or both print and digital yearbooks.

To determine when to use a survey, experiment, or observational study, think about how the data will be obtained and whether or not the participants will be affected by the study.

Example 2 Choose a Study Type

Determine whether each situation calls for a *survey*, an *experiment*, or an *observational study*. Explain your reasoning.

a. **MEDICINE** **A pharmaceutical company wants to test whether a new medicine is effective.**

The treatment will need to be tested on a sample group, which means that the members of the sample will be affected by the study. Therefore, this situation calls for an experiment.

b. **ELECTIONS** **A news organization wants to randomly call citizens to gauge opinions on a presidential election.**

This situation calls for a survey because members of the sample population are asked for their opinion.

GuidedPractice

2A. **RESEARCH** A research company wants to study users and non-users of full-fat dairy products in their diet to determine whether 1 year of non-use affects cholesterol levels.

2B. **PETS** A national pet chain wants to know whether customers would pay a small annual fee to participate in a rewards program. They randomly select 200 customers and send them questionnaires.

2 Designing Studies

The questions chosen for a survey or procedures used in an experiment can also introduce bias, and thus, affect the results of the study.

A survey question that is poorly written may result in a response that does not accurately reflect the opinion of the participant. Therefore, it is important to write questions that are clear and precise. Avoid survey questions that:

- are confusing or wordy
- cause a strong reaction
- encourage a certain response
- address more than one issue

Questions can also introduce bias if there is not enough information given for the participant to give an accurate response.

Example 3 Identify Bias in Survey Questions

Determine whether each survey question is *biased* or *unbiased*. If biased, explain your reasoning.

a. Don't you agree that the cafeteria should serve healthier food?

This question is biased because it encourages a certain response. The phrase "don't you agree" encourages you to agree that the cafeteria should serve healthier food.

b. How often do you exercise?

This question is unbiased because it is clearly stated and does not encourage a certain response.

GuidedPractice

3A. How many glasses of water do you drink a day?

3B. Do you prefer watching exciting action movies or boring documentaries?

When designing a survey, clearly state the objective, identify the population, and carefully choose unbiased survey questions.

Real-World Example 4 Design a Survey

TECHNOLOGY **Nasser is writing an article for his school newspaper about online courses. He wants to conduct a survey to determine how many students at his school would be interested in taking an online course from home. State the objective of the survey, suggest a population, and write two unbiased survey questions.**

Step 1 **State the objective of the survey.**

The objective of the survey is to determine students' interest in taking an online course from home.

Step 2 **Identify the population.**

The population is the student body.

Step 3 **Write unbiased survey questions.**

Possible survey questions:

- "Do you have Internet access at home?"
- "If offered, would you take an online course?"

GuidedPractice

4. TECHNOLOGY In a follow-up article, Nasser decides to conduct a survey to determine how many teachers from his school with at least five years of experience would be interested in teaching an online course. State the objective of the survey, suggest a population, and write two unbiased survey questions.

Real-WorldLink

Online Courses In 2009, about 1.2 million students took at least one online course.

Source: International Association for K-12 Online Learning

To avoid introducing bias in experiments, the experimental and control groups should be randomly selected and the experiment should be designed so that everything about the two groups is alike (except for the treatment or procedure).

Example 5 Identify Flaws in Experiments

Identify any flaws in the design of the experiment, and describe how they could be corrected.

Experiment: **An electronics company wants to test whether using a new graphing calculator increases students' test scores. A random sample is taken. Calculus students in the experimental group are given the new calculator to use, and Algebra 2 students in the control group are asked to use their own calculator.**

Results: **When given the same test, the experimental group scored higher than the control group. The company concludes that the use of this calculator increases test scores.**

Calculus students are more likely to score higher when given the same test as Algebra 2 students. Therefore, the flaw is that the experimental group consists of Calculus students and the control group consists of Algebra 2 students. This flaw could be corrected by selecting a random sample of all Calculus or all Algebra 2 students.

StudyTip

Bias in Experiments An experiment is biased when the participants know which group they are in.

GuidedPractice

5. Experiment: A research firm tests the effectiveness of a de-icer on car locks. They use a random sample of drivers in California and Minnesota for the control and experimental groups.

Results: They concluded that the de-icer is effective.

When designing an experiment, clearly state the objective, identify the population, determine the experimental and control groups, and define the procedure.

Real-World Example 6 Design an Experiment

PLANTS **A research company wants to test the claim of the advertisement shown at the right. State the objective of the experiment, suggest a population, determine the experimental and control groups, and describe a sample procedure.**

Step 1 **State the objective, and identify the population.**

The objective of the experiment is to determine whether tomato plants given the plant food grow taller in three weeks than tomato plants not given the food. The population is all tomato plants.

Step 2 **Determine the experimental and control groups.**

The experimental group is the tomato plants given the food, and the control group is the tomato plants not given the food.

Step 3 **Describe a sample procedure.**

Measure the heights of the plants in each group, and give the experimental group the plant food. Then, wait three weeks, measure the heights of the plants again, and compare the heights for each group to see if the claim was valid.

GuidedPractice

6. **SPORTS** A company wants to determine whether wearing a new tennis shoe improves jogging time. State the objective of the experiment, suggest a population, determine the experimental and control groups, and describe a sample procedure.

Check Your Understanding

Example 1 **Determine whether each situation describes a *survey*, an *experiment*, or an *observational study*. Then identify the sample, and suggest a population from which it may have been selected.**

1. **SCHOOL** A group of high school students is randomly selected and asked to complete the form shown.

Do you agree with the new lunch rules?
- ☐ agree
- ☐ disagree
- ☐ don't care

2. **DESIGN** An advertising company wants to test a new logo design. They randomly select 20 participants and watch them discuss the logo.

Example 2 **ARGUMENTS** **Determine whether each situation calls for a *survey*, an *experiment*, or an *observational study*. Explain your reasoning.**

3. **LITERACY** A literacy group wants to determine whether high school students that participated in a recent national reading program had higher standardized test scores than high school students that did not participate in the program.

4. **RETAIL** The research department of a retail company plans to conduct a study to determine whether a dye used on a new T-shirt will begin fading before 50 washes.

Example 3 **Determine whether each survey question is *biased* or *unbiased*. If biased, explain your reasoning.**

5. Which student council candidate's platform do you support?

6. How long have you lived at your current address?

Example 4

7. **HYBRIDS** A car manufacturer wants to determine what the demand in the U.S. is for hybrid vehicles. State the objective of the survey, suggest a population, and write two unbiased survey questions.

Example 5

8. Identify any flaws in the experiment design, and describe how they could be corrected.

   Experiment: A research company wants to determine whether a new vitamin boosts energy levels and decides to test the vitamin at a college campus. A random sample is taken. The experimental group consists of students who are given the vitamin, and the control group consists of instructors who are given a placebo.

   Results: When given a physical test, the experimental group outperformed the control group. The company concludes that the vitamin is effective.

Example 6

9. **SPORTS** A research company wants to conduct an experiment to test the claim of the protein shake shown. State the objective of the experiment, suggest a population, determine the experimental and control groups, and describe a sample procedure.

Practice and Problem Solving

Example 1 **Determine whether each situation describes a *survey*, an *experiment*, or an *observational study*. Then identify the sample, and suggest a population from which it may have been selected.**

10. **FOOD** A grocery store conducts an online study in which customers are randomly selected and asked to provide feedback on their shopping experience.

11. **GRADES** A research group randomly selects 80 college students, half of whom took a physics course in high school, and compares their grades in a college physics course.

12. **HEALTH** A research group randomly chooses 100 people to participate in a study to determine whether eating blueberries reduces the risk of heart disease for adults.

13. **TELEVISION** A television network mails a questionnaire to randomly selected people across the country to determine whether they prefer watching sitcoms or dramas.

Example 2 **Determine whether each situation calls for a *survey*, an *experiment*, or an *observational study*. Explain your reasoning.**

14. **FASHION** A fashion magazine plans to poll 100 people in the U.S. to determine whether they would be more likely to buy a subscription if given a free issue.

15. **TRAVEL** A travel agency randomly calls 250 U.S. citizens and asks them what their favorite vacation destination is.

16. **FOOD** Ibrahim wants to examine the eating habits of 100 random students at lunch to determine how many students eat in the cafeteria.

17. **ENGINEERING** An engineer is planning to test 50 metal samples to determine whether a new titanium alloy has a higher strength than a different alloy.

Example 3 **Determine whether each survey question is *biased* or *unbiased*. If biased, explain your reasoning.**

18. Do you think that the school needs a new gym and soccer field?

19. Which is your favorite soccer team, Barcelona or Real Madrid?

20. Do you play any extracurricular sports?

21. Don't you agree that students should carpool to school?

Example 4

22. **COLLEGE** A school district wants to conduct a survey to determine the number of grade 11 students in the district who are planning to attend college after high school. State the objective of the survey, suggest a population, and write two unbiased survey questions.

Example 5

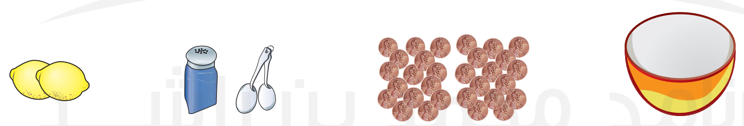
23. Identify any flaws in the experiment design, and describe how they could be corrected.

Experiment: A supermarket chain wants to determine whether shoppers are more likely to buy sunscreen if it is located near the checkout line. The experimental group consists of a group of stores in the midwest in which the sunscreen was moved next to the checkout line, and the control group consists of stores in Arizona in which the sunscreen was not moved.

Results: The Arizona stores sold more sunscreen than the midwest stores. The company concluded that moving the sunscreen closer to the checkout line did not increase sales.

Example 6

24. **ARGUMENTS** In chemistry class, Ahmed learned that copper objects become dull over time because the copper reacts with air to form a layer of copper oxide. He plans to use the supplies shown below to determine whether a mixture of lemon juice and salt will remove copper oxide from coins.



2 lemons 1 teaspoon 30 dull coins plastic bowl

a. State the objective of the experiment, suggest a population, determine the experimental and control groups, and describe a sample procedure.

b. What factors do you think should be considered when selecting coins for the experiment? Explain your reasoning.

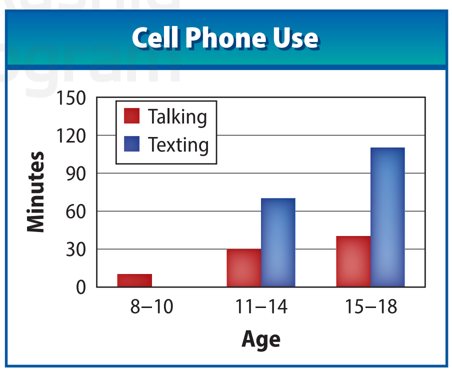
25. **REPORTS** The graph shown is from a report on the average number of minutes 8- to 18-year-olds in the U.S. spend on cell phones each day.

a. Describe the sample and suggest a population.

b. What type of sample statistic do you think was calculated for this report?

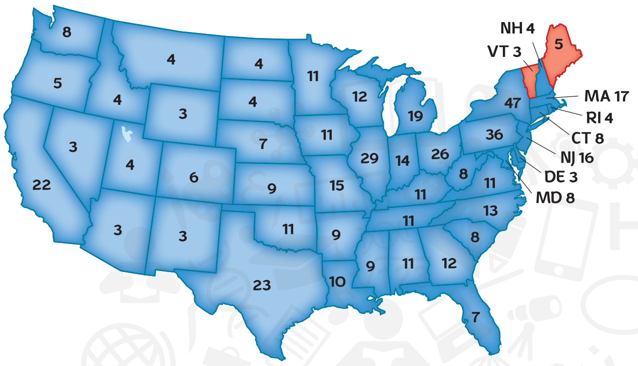
c. Describe the results of the study for each age group.

d. Who do you think would be interested in this type of report? Explain your reasoning.



26. **PERSEVERANCE** In 1936, the *Literary Digest* reported the results of a statistical study used to predict whether Alf Landon or Franklin D. Roosevelt would win the presidential election that year. The sample consisted of 2.4 million Americans, including subscribers to the magazine, registered automobile owners, and telephone users. The results concluded that Landon would win 57% of the popular vote. The actual election results are shown.

ELECTORAL VOTE



POPULAR VOTE

Roosevelt	60.8%
Landon	36.5%

a. Describe the type of study performed, the sample taken, and the population.

b. How do the predicted and actual results compare?

c. Do you think that the survey was biased? Explain your reasoning.

27. **MULTIPLE REPRESENTATIONS** The results of two experiments concluded that Product A is 70% effective and Product B is 80% effective.

a. **NUMERICAL** To simulate the experiment for Product A, use the random number generator on a graphing calculator to generate 30 integers between 0 and 9. Let 0–6 represent an effective outcome and 7–9 represent an ineffective outcome.

b. **TABULAR** Copy and complete the frequency table shown using the results from part **a.** Then use the data to calculate the probability that Product A was effective. Repeat to find the probability for Product B.

Product A	
Number	**Frequency**
0–6	
7–9	

c. **ANALYTICAL** Compare the probabilities that you found in part **b.** Do you think that the difference in the effectiveness of each product is significant enough to justify selecting one product over the other? Explain.

d. **LOGICAL** Suppose Product B costs twice as much as Product A. Do you think the probability of the product's effectiveness justifies the price difference to a consumer? Explain.

H.O.T. Problems Use Higher-Order Thinking Skills

REASONING **Determine whether each statement is *true* or *false*. If false, explain.**

28. To save time and money, population parameters are used to estimate sample statistics.

29. Observational studies and experiments can both be used to study cause-and-effect relationships.

30. **OPEN ENDED** Design an observational study. Identify the objective of the study, define the population and sample, collect and organize the data, and calculate a sample statistic.

31. **CHALLENGE** What factors should be considered when determining whether a given statistical study is reliable?

32. **WRITING IN MATH** Research each of the following sampling methods. Then describe each method and discuss whether using the method could result in bias.

a. convenience sample

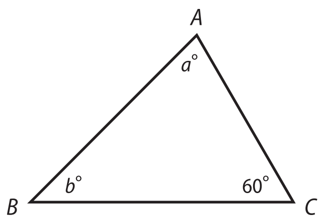
b. self-selected sample

c. stratified sample

d. systematic sample

Standardized Test Practice

33. GEOMETRY In $\triangle ABC$, $BC > AB$. Which of the following must be true?



A $AB = BC$

B $AC < AB$

C $a > 60$

D $a = b$

34. SHORT RESPONSE What is the solution set of $4^{4x^2 - 2x - 4} = 4^{-2}$?

35. SAT/ACT A pie is divided evenly between 3 boys and a man. If one boy gives one half of his share to the man and a second boy keeps two thirds of his share and gives the rest to the man, what portion will the man have in all?

F $\frac{5}{24}$ **H** $\frac{1}{2}$ **K** $\frac{13}{12}$

G $\frac{11}{24}$ **J** $\frac{13}{24}$

36. Which equation represents a hyperbola?

A $y^2 = 49 - x^2$ **C** $y = 49x^2$

B $y = 49 - x^2$ **D** $y = \frac{49}{x}$

Spiral Review

37. Prove that the statement $9^n - 1$ *is divisible by* 8 is true for all natural numbers.

38. INTRAMURALS Eiman is taking ten shots in the intramural free-throw shooting competition. How many sequences of hits and misses are there that result in her making eight shots and missing two?

Solve each system of equations.

39. $y = x + 3$, $y = 2x^2$

40. $x^2 + y^2 = 36$, $y = x + 2$

41. $y^2 + x^2 = 9$, $y = 7 - x$

42. $y + x^2 = 3$, $x^2 + 4y^2 = 36$

43. $x^2 + y^2 = 64$, $x^2 + 64y^2 = 64$

44. $y^2 = x^2 - 25$, $x^2 - y^2 = 7$

Find the distance between each pair of points with the given coordinates.

45. $(9, -2), (12, -14)$

46. $(-4, -10), (-3, -11)$

47. $(1, -14), (-6, 10)$

48. $(-4, 9), (1, -3)$

49. $(2.3, -1.2), (-4.5, 3.7)$

50. $(0.23, 0.4), (0.68, -0.2)$

Simplify. Assume that no variable equals 0.

51. $(5cd^2)(-c^4d)$

52. $(7x^3y^{-5})(4xy^3)$

53. $\frac{a^2n^6}{an^5}$

54. $(n^4)^4$

55. $\frac{-y^5z^7}{y^2z^5}$

56. $(-2r^2t)^3(3rt^2)$

Write a quadratic equation with the given root(s). Write the equation in the form $ax^2 + bx + c = 0$, where a, b, and c are integers.

57. $-3, 9$

58. $-\frac{1}{3}, -\frac{3}{4}$

59. $4, -5$

Skills Review

60. TESTS Ms. Asma's class of 30 students took a biology test. If 20 of her students had an average of 83 on the test and the other students had an average score of 74, what was the average score of the whole class?

61. DRIVING During a 10-hour trip, Ismail drove 4 hours at 60 km per hour and 6 hours at 65 km per hour. What was his average rate, in km/h, for the entire trip?

EXTEND 10-1

Graphing Calculator Lab
Simulations and Margin of Error

The Pew Research Center conducted a survey of a random sample of teens and concluded that 43% of all teens who take their cell phones to school text in class on a daily basis. How accurately did their random sample represent all teens?

Mathematical Practices
Use appropriate tools strategically.

As you learned in the previous lesson, a survey of a random sample is a valuable tool for generalizing information about a larger population. The program in the following activity makes use of a random number generator (randInt(a, b)) to simulate the results of a random sampling survey.

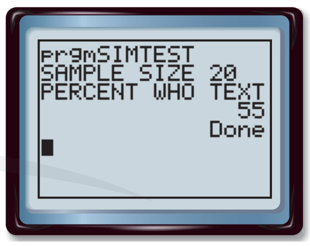
Activity 1 RANDOM SAMPLING SIMULATION

Use the following program that simulates the texting survey to measure the percent of teens who text in class for random sample sizes of 20, 50, and 100 students.

Step 1 Input the following program into a graphing calculator.

```
Program:SIMTEST
:Input "SAMPLE SIZE ",S
:0→A
:0→B
:Lbl Z
:A + 1→A
:randInt(1, 100) →C
:If C ≤ 43
:B + 1→B
:If A<S
:Goto Z
:100B/S→P
:Disp "PERCENT WHO TEXT",P
:Stop
```

Step 2 Run 10 trials of the program for each sample size of 20, 50, and 100. Press ENTER to run the program again each time.



Step 3 Record the percent who text for each trial in the table below.

Sample Size	1	2	3	4	5	6	7	8	9	10
20										
50										
100										

Analyze the Results

1. Discard the percent that is farthest from the Pew survey result of 43% for each sample size. What is the range of the remaining nine percents for each sample size?

2. What is the farthest any of these remaining trials is from the 43% for each sample size?

3. The positive or negative of the result found in Exercise 2 is known as the **margin of error**. For your results, which sample size had the smallest margin of error?

4. What would you expect to happen to the margin of error if we used a sample size of 500?

(continued on the next page)

Simulations and Margin of Error *Continued*

Statisticians have found that for large populations, the margin of error for a random sample of size n can be approximated by the following formula.

KeyConcept Margin of Error Formula

$$\text{Margin of error} = \pm\frac{1}{\sqrt{n}}(100)$$

Since n is in the denominator, the margin of error will decrease as the size of the random sample increases. This expression can also be used to determine the size of a random sample necessary to achieve a desired level of reliability.

Activity 2 MARGIN OF ERROR AND SAMPLE SIZE

You are a member of a research team.

a. You need to decide whether to conduct a survey with a margin of error of ±3% or ±2%. What sample size would be needed to achieve each goal?

Set each percent equal to the margin of error formula and solve for n.

$\pm 3\% = \pm\frac{1}{\sqrt{n}}(100)$	Margin of error formula	$\pm 2\% = \pm\frac{1}{\sqrt{n}}(100)$
$0.03\sqrt{n} = 1$	Multiply by $\frac{\sqrt{n}}{100}$.	$0.02\sqrt{n} = 1$
$\sqrt{n} = 33.333$	Divide.	$\sqrt{n} = 50$
$n = 1111.11$	Square each side.	$n = 2500$

A random sample of about 1100 would have a margin of error of about ±3%, while a random sample of 2500 would have a margin of error of ±2%.

b. Suppose the finance director would like to reduce the cost of the survey by using a random sample of 100. What would be the margin of error for this sample size?

Substitute 100 for n in the margin of error formula.

$$\text{margin of error} = \pm\frac{1}{\sqrt{n}}(100) \qquad \text{Margin of error formula}$$

$$= \pm\frac{1}{\sqrt{100}}(100) \text{ or } \pm 10\% \qquad n = 100$$

A random sample of 100 would have a margin of error of ±10%.

Exercises

5. What random sample size would produce a margin of error of ±1%?
6. What margin of error can be expected when using a sample size of 500?
7. What are some reasons that a research center might decide that a survey with a margin of error of ±3% would be more desirable than one with a margin of error of ±2%?
8. What is the range for the percent of students that text in class that the research center can expect from any random survey they conduct with a sample size of 2500?
9. If a survey with a random sample of 2500 students is conducted, is it possible that only 20% of the students could respond that they text during class? If so, how could this be possible?
10. If Step 2 from Activity 1 is repeated using a sample size of 2500 and the range for the percents is found to be 19%–23%, would this result cause you to question the model?

LESSON 10-2 Distributions of Data

:: Then	:: Now	:: Why?
You calculated measures of central tendency and variation.	1 Use the shapes of distributions to select appropriate statistics. 2 Use the shapes of distributions to compare data.	After four games as a reserve player, Khalid joined the starting lineup and averaged 18 points per game over the *remaining* games. Khalid's scoring average for the *entire* season was less than 18 points per game as a result of the lack of playing time in the first four games.

NewVocabulary
distribution
negatively skewed distribution
symmetric distribution
positively skewed distribution

Mathematical Practices
Make sense of problems and persevere in solving them.

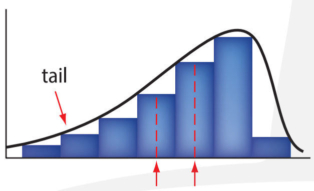
1 Analyzing Distributions

A **distribution** of data shows the observed or theoretical frequency of each possible data value. In an earlier lesson, you described distributions of sample data using statistics. You used the mean or median to describe a distribution's center and standard deviation or quartiles to describe its spread. Analyzing the shape of a distribution can help you decide which measure of center or spread best describes a set of data.

The shape of the distribution for a set of data can be seen by drawing a curve over its histogram.

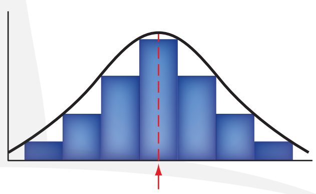
KeyConcept Symmetric and Skewed Distributions

Negatively Skewed Distribution



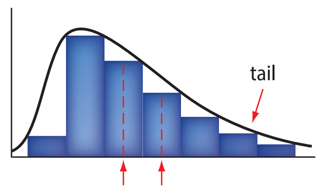
- The mean is less than the median.
- The majority of the data are on the right of the mean.

Symmetric Distribution



- The mean and median are approximately equal.
- The data are evenly distributed on both sides of the mean.

Positively Skewed Distribution

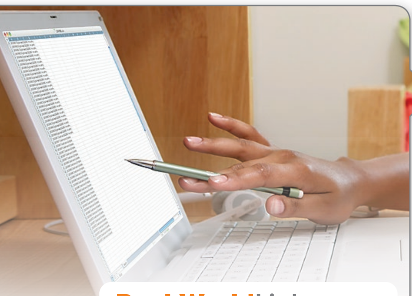


- The mean is greater than the median.
- The majority of the data are on the left of the mean.

When a distribution is symmetric, the mean and standard deviation accurately reflect the center and spread of the data. However, when a distribution is skewed, these statistics are not as reliable. Recall that outliers have a strong effect on the mean of a data set, while the median is less affected. Similarly, when a distribution is skewed, the mean lies away from the majority of the data toward the tail. The median is less affected, so it stays near the majority of the data.

When choosing appropriate statistics to represent a set of data, first determine the skewness of the distribution.

- If the distribution is relatively symmetric, the mean and standard deviation can be used.
- If the distribution is skewed or has outliers, use the five-number summary to describe the center and spread of the data.



Real-WorldLink

The first portable computer, the Osborne I, was available for sale in 1981 for AED 6600. The computer weighed 10 kg and included a 12.5 cm display. Laptops can now be purchased for as little as AED 920 and can weigh as little as 1.3 kg.

Source: Computer History Museum

Real-World Example 1 Describe a Distribution Using a Histogram

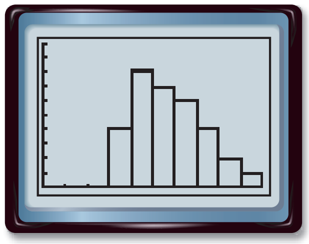
COMPUTERS **The prices for a random sample of tablets are shown.**

Price (AED)							
723	605	847	410	440	386	572	523
374	915	734	472	420	508	613	659
706	463	470	752	671	618	538	425
811	502	490	552	390	512	389	621

a. **Use a graphing calculator to create a histogram. Then describe the shape of the distribution.**

First, press STAT ENTER and enter each data value. Then, press 2nd [STAT PLOT] ENTER ENTER and choose the histogram icon. Finally, adjust the window to the dimensions shown.

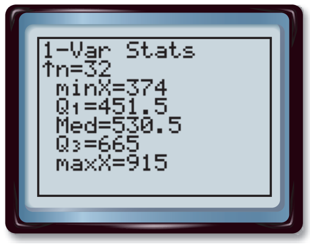
The majority of the tablets cost between AED 400 and AED 700. Some of the tablets are priced significantly higher, forming a tail for the distribution on the right. Therefore, the distribution is positively skewed.



[0, 1000] scl: 100 by [0, 10] scl: 1

b. **Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.**

The distribution is skewed, so use the five-number summary to describe the center and spread. Press STAT ▶ ENTER ENTER and scroll down to view the five-number summary.



The prices for this sample range from AED 374 to AED 915. The median price is AED 530.50, and half of the computers are priced between AED 451.50 and AED 665.

GuidedPractice

1. **RAINFALL** The annual rainfall for a region over a 24-year period is shown below.

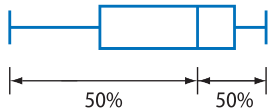
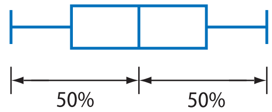
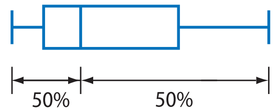
A. Use a graphing calculator to create a histogram. Then describe the shape of the distribution.

B. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

Annual Rainfall (cm)					
69	76	90	66	99	52
73	58	83	68	57	64
75	93	84	72	55	52
62	77	70	79	88	94

A box-and-whisker plot can also be used to identify the shape of a distribution. The position of the line representing the median indicates the center of the data. The "whiskers" show the spread of the data. If one whisker is considerably longer than the other and the median is closer to the shorter whisker, then the distribution is skewed.

KeyConcept Box-and-Whisker Plots as Distributions

Negatively Skewed	Symmetric	Positively Skewed
		
The data to the left of the median are distributed over a wider range than the data to the right. The data have a tail to the left.	The data are equally distributed to the left and right of the median.	The data to the right of the median are distributed over a wider range than the data to the left. The data have a tail to the right.

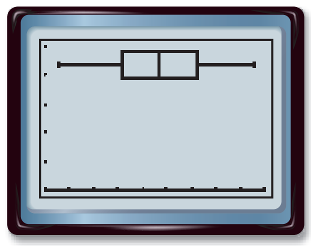
Example 2 Describe a Distribution Using a Box-and-Whisker Plot

HOMEWORK **The students in Mr. Usama's language arts class found the average number of minutes that they each spent on homework each night.**

Minutes per Night					
62	53	46	66	38	45
52	46	73	39	42	56
64	54	48	59	70	60
49	54	48	57	70	33

a. Use a graphing calculator to create a box-and-whisker plot. Then describe the shape of the distribution.

Enter the data as **L1**. Press 2nd [STAT PLOT] ENTER ENTER and choose the box plot icon. Adjust the window to the dimensions shown.



[30, 75] scl: 5 by [0, 5] scl: 1

The lengths of the whiskers are approximately equal, and the median is in the middle of the data. This indicates that the data are equally distributed to the left and right of the median. Thus, the distribution is symmetric.

b. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

The distribution is symmetric, so use the mean and standard deviation to describe the center and spread. The average number of minutes that a student spent on homework each night was 53.5 with standard deviation of about 10.5.

WatchOut!

Standard Deviation Recall that the formulas for standard deviation for a population σ and for a sample s are slightly different. In Example 2, times for all of the students in Mr. Usama's class are being analyzed, so use the population standard deviation.

GuidedPractice

2. **CELL PHONE** Amani's parents have given her a prepaid cell phone. The number of minutes she used each month for the last two years are shown in the table.

Minutes Used per Month			
582	608	670	620
667	598	671	613
537	511	674	627
638	661	642	641
668	673	680	695
658	653	670	688

A. Use a graphing calculator to create a box-and-whisker plot. Then describe the shape of the distribution.

B. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

2 Comparing Distributions

To compare two sets of data, first analyze the shape of each distribution. Use the mean and standard deviation to compare two symmetric distributions. Use the five-number summaries to compare two skewed distributions or a symmetric distribution and a skewed distribution.

Example 3 Compare Data Using Histograms

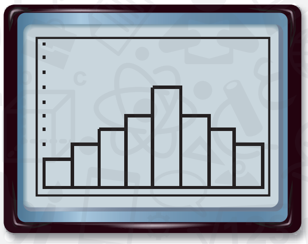
TEST SCORES **Test scores from Mrs. Amal's class are shown below.**

Chapter 3 Test Scores
81, 81, 92, 99, 61, 67, 86, 82, 76, 73, 62, 97, 97, 72, 72, 84, 77, 88, 92, 93, 76, 74, 66, 78, 76, 69, 84, 87, 83, 87, 92, 87, 82

Chapter 4 Test Scores
87, 73, 69, 83, 74, 86, 74, 69, 79, 84, 79, 74, 83, 74, 86, 69, 91, 73, 79, 83, 69, 79, 83, 74, 86, 79, 79, 78, 83, 79, 86, 79, 84

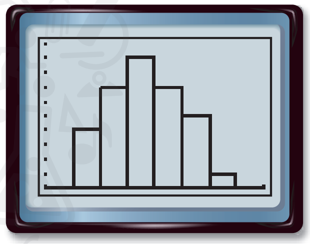
a. Use a graphing calculator to create a histogram for each data set. Then describe the shape of each distribution.

Chapter 3 Test Scores



[60, 100] scl: 5 by [0, 10] scl: 1

Chapter 4 Test Scores



[60, 100] scl: 5 by [0, 10] scl: 1

Both distributions are symmetric.

b. Compare the distributions using either the means and standard deviations or the five-number summaries. Justify your choice.

The distributions are symmetric, so use the means and standard deviations.

Chapter 3 Test Scores

Chapter 4 Test Scores

The Chapter 4 test scores, while lower in average, have a much smaller standard deviation, indicating that the scores are more closely grouped about the mean. Therefore, the mean for the Chapter 4 test scores is a better representation of the data than the mean for the Chapter 3 test scores.

> **StudyTip**
>
> **Tools** To compare two sets of data, enter one set as **L1** and the other as **L2**. In order to calculate statistics for a set of data in **L2**, press
>
> STAT ▶ ENTER 2nd [L2] ENTER.

GuidedPractice

3. **TYPING** The typing speeds of the students in two classes are shown below.

A. Use a graphing calculator to create a histogram for each data set. Then describe the shape of each distribution.

B. Compare the distributions using either the means and standard deviations or the five-number summaries. Justify your choice.

3rd Period (wpm)
23, 38, 27, 28, 40, 45, 32, 33, 34, 27, 40, 22, 26, 34, 29, 31, 35, 33, 37, 38, 28, 29, 39, 42

6th Period (wpm)
38, 26, 43, 46, 23, 24, 27, 36, 22, 21, 26, 27, 31, 32, 27, 25, 23, 22, 28, 29, 28, 33, 23, 24

Box-and-whisker plots can be displayed alongside one another, making them useful for side-by-side comparisons of data.

Example 4 Compare Data Using Box-and-Whisker Plots

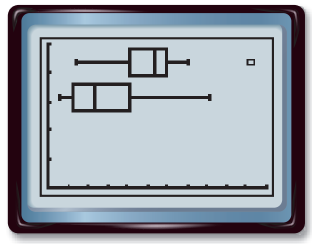
POINTS The points scored per game by a professional rugby team for the 2008 and 2009 football seasons are shown.

2008							
7	51	24	27	17	35	27	33
28	30	27	21	24	30	14	20

2009							
20	9	3	10	6	14	3	10
3	37	7	21	13	41	20	23

a. Use a graphing calculator to create a box-and-whisker plot for each data set. Then describe the shape of each distribution.

Enter the 2008 scores as **L1**. Graph these data as **Plot1** by pressing [2nd] **[STAT PLOT]** [ENTER] [ENTER] and choosing . Enter the 2009 scores as **L2**. Graph these data as **Plot2** by pressing [2nd] **[STAT PLOT]** [▼] [ENTER] [ENTER] and choosing . For **Xlist**, enter **L2**. Adjust the window to the dimensions shown.



[0, 55] scl: 5 by [0, 5] scl: 1

For the 2008 scores, the left whisker is longer than the right and the median is closer to the right whisker. The distribution is negatively skewed.

For the 2009 scores, the right whisker is longer than the left and the median is closer to the left whisker. The distribution is positively skewed.

b. Compare the distributions using either the means and standard deviations or the five-number summaries. Justify your choice.

The distributions are skewed, so use the five-number summaries to compare the data.

The lower quartile for the 2008 season and the upper quartile for the 2009 season are both 20.5. This means that 75% of the scores from the 2008 season were greater than 20.5 and 75% of the scores from the 2009 season were less than 20.5.

The minimum of the 2008 season is approximately equal to the lower quartile for the 2009 season. This means that 25% of the scores from the 2009 season are lower than any score achieved in the 2008 season. Therefore, we can conclude that the team scored a significantly higher amount of points during the 2008 season than the 2009 season.

StudyTip

Outliers Recall from an earlier lesson that outliers are data that are more than 1.5 times the interquartile range beyond the upper or lower quartile. All outliers should be plotted, but the whiskers should be drawn to the least and greatest values that are not outliers.

GuidedPractice

4. GOLF Ayman recorded his golf scores for his grade 10 and grade 11 seasons.

A. Use a graphing calculator to create a box-and-whisker plot for each data set. Then describe the shape of each distribution.

B. Compare the distributions using either the means and standard deviations or the five-number summaries. Justify your choice.

Grade 10 Season
42, 47, 43, 46, 50, 47, 52, 45, 53, 55, 48, 39, 40, 49, 47, 50

Grade 11 Season
44, 38, 46, 48, 42, 41, 42, 46, 43, 40, 43, 43, 44, 45, 39, 44

Check Your Understanding

Example 1

1. **EXERCISE** The amount of time that Badr ran on a treadmill for the first 24 days of his workout is shown.

Time (minutes)											
23	10	18	24	13	27	19	7	25	30	15	22
10	28	23	16	29	26	26	22	12	23	16	27

a. Use a graphing calculator to create a histogram. Then describe the shape of the distribution.

b. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

Example 2

2. **RESTAURANTS** The total number of times that 20 random people either ate at a restaurant or bought fast food in a month are shown.

Restaurants or Fast Food									
4	7	5	13	3	22	13	6	5	10
7	18	4	16	8	5	15	3	12	6

a. Use a graphing calculator to create a box-and-whisker plot. Then describe the shape of the distribution.

b. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

Example 3

3. **TOOLS** The total fundraiser sales for the students in two classes at Al Khalil Secondary School are shown.

Mrs. Muna's Class (AED)					
6	14	17	12	38	15
11	12	23	6	14	28
16	13	27	34	25	32
21	24	21	17	16	

Mrs. Rana's Class (AED)					
29	38	21	28	24	33
14	19	28	15	30	6
31	23	33	12	38	28
18	34	26	34	24	37

a. Use a graphing calculator to create a histogram for each data set. Then describe the shape of each distribution.

b. Compare the distributions using either the means and standard deviations or the five-number summaries. Justify your choice.

Example 4

4. **RECYCLING** The weekly totals of recycled paper for the grade 11 and grade 12 classes are shown.

Grade 11 Class (kg)					
14	24	8	26	19	38
12	15	12	18	9	24
12	21	9	15	13	28

Grade 12 Class (kg)					
25	31	35	20	37	27
22	32	24	28	18	32
25	32	22	29	26	35

a. Use a graphing calculator to create a box-and-whisker plot for each data set. Then describe the shape of each distribution.

b. Compare the distributions using either the means and standard deviations or the five-number summaries. Justify your choice.

Practice and Problem Solving

Examples 1–2 **For Exercises 5 and 6, complete each step.**

a. Use a graphing calculator to create a histogram and a box-and-whisker plot. Then describe the shape of the distribution.

b. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

5. **FANTASY** The weekly total points of Khalid's fantasy soccer team are shown.

Total Points							
165	140	88	158	101	137	112	127
53	151	120	156	142	179	162	79

6. **MOVIES** The students in one of Mr. Jamal's classes recorded the number of movies they saw over the past month.

Movies Seen											
14	11	17	9	6	11	7	8	12	13	10	9
5	11	7	13	9	12	10	9	15	11	13	15

Example 3 **MODELING** **For Exercises 7 and 8, complete each step.**

a. Use a graphing calculator to create a histogram for each data set. Then describe the shape of each distribution.

b. Compare the distributions using either the means and standard deviations or the five-number summaries. Justify your choice.

7. **SAT** A group of students took the SAT their grade 10 year and again their grade 11 year. Their scores are shown.

Grade 10 Year Scores					
1327	1663	1708	1583	1406	1563
1637	1521	1282	1752	1628	1453
1368	1681	1506	1843	1472	1560

Grade 11 Year Scores					
1728	1523	1857	1789	1668	1913
1834	1769	1655	1432	1885	1955
1569	1704	1833	2093	1608	1753

8. **INCOME** The total incomes for 18 households in two neighboring cities are shown.

Yorkshire (thousands of AED)					
68	59	61	78	58	66
56	72	86	58	63	53
68	58	74	60	103	64

Applewood (thousands of AED)					
52	55	60	61	55	65
65	60	45	37	41	71
50	61	65	66	87	55

Example 4 **9.** **TUITION** The annual tuitions for a sample of public colleges and a sample of private colleges are shown. Complete each step.

a. Use a graphing calculator to create a box-and-whisker plot for each data set. Then describe the shape of each distribution.

b. Compare the distributions using either the means and standard deviations or the five-number summaries. Justify your choice.

Public Colleges (AED)					
3773	3992	3004	4223	4821	3880
3163	4416	5063	4937	3321	4308
4006	3508	4498	3471	4679	3612

Private Colleges (AED)					
10,766	13,322	12,995	15,377	16,792	9147
15,976	11,084	17,868	7909	12,824	10,377
14,304	10,055	12,930	16,920	10,004	11,806

10. GRADUATION The total amount of money that a random sample of grade 12 students spent on graduation in a girls school and also in a boys school is shown. Complete each step.

a. Use a graphing calculator to create a box-and-whisker plot for each data set. Then describe the shape of each distribution.

b. Compare the distributions using either the means and standard deviations or the five-number summaries. Justify your choice.

Boys (AED)					
253	288	304	283	348	276
322	368	247	404	450	341
291	260	394	302	297	272

Girls (AED)					
682	533	602	504	635	541
489	703	453	521	472	368
562	426	382	668	352	587

11. BASKETBALL Refer to the beginning of the lesson. The points that Khalid scored in the remaining games are shown.

a. Use a graphing calculator to create a box-and-whisker plot. Describe the center and spread of the data.

b. Khalid scored 0, 2, 1, and 0 points in the first four games. Use a graphing calculator to create a box-and-whisker plot that includes the new data. Then find the mean and median of the new data set.

c. What effect does adding the scores from the first four games have on the shape of the distribution and on how you should describe the center and spread?

Points Scored			
18	10	18	21
9	25	13	17
17	12	24	19
20	17	27	21

12. SCORES Eiman's quiz scores are shown.

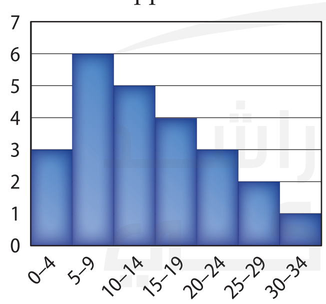
a. Use a graphing calculator to create a box-and-whisker plot. Describe the center and spread.

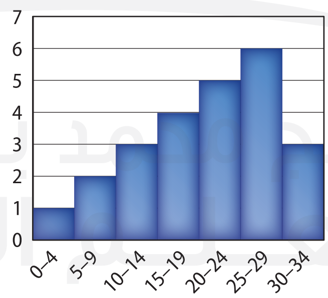
b. Eiman's teacher allows students to drop their two lowest quiz scores. Use a graphing calculator to create a box-and-whisker plot that reflects this change. Then describe the center and spread of the new data set.

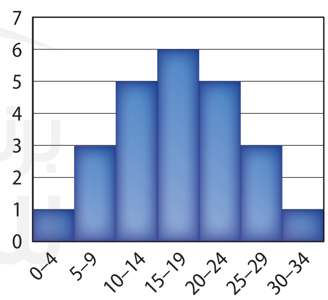
Math Quiz Scores					
83	76	86	82	84	57
86	62	90	96	76	89
76	88	86	86	92	94

H.O.T. Problems Use Higher-Order Thinking Skills

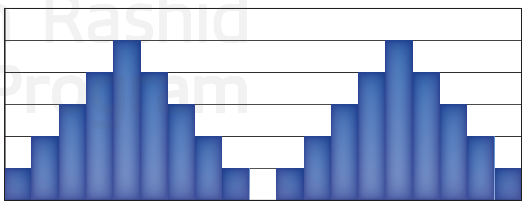
13. CHALLENGE Approximate the mean and median for each distribution of data.

a. 

b. 

c. 

14. ARGUMENTS Distributions of data are not always symmetric or skewed. If a distribution has a gap in the middle, like the one shown, two separate clusters of data may result, forming a *bimodal distribution.* How can the center and spread of a bimodal distribution be described?



15. OPEN ENDED Find a real-world data set that appears to represent a symmetric distribution and one that does not. Describe each distribution. Create a visual representation of each set of data.

16. WRITING IN MATH Explain the difference between positively skewed, negatively skewed, and symmetric sets of data, and give an example of each.

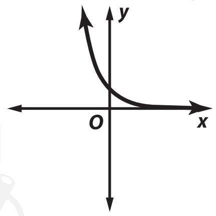
Standardized Test Practice

17. **DISTRIBUTIONS** Which of the following is a characteristic of a negatively skewed distribution?
   - **A The majority of the data are on the left of the mean.**
   - B The mean and median are approximately equal.
   - C The mean is greater than the median.
   - D The mean is less than the median.

18. **SHORT RESPONSE** The average test score of a class of c students is 80, and the average test score of a class of d students is 85. When the scores of both classes are combined, the average score is 82. What is the value of $\frac{c}{d}$?

19. **SAT/ACT** What is the multiplicative inverse of $2i$?
   - F $-2i$
   - G -2
   - H $-\frac{i}{2}$
   - J $\frac{1}{2}$
   - K $\frac{i}{2}$

20. Which equation best represents the graph?
   - A $y = 4x$
   - B $y = x^2 + 4$
   - C $y = 4^{-x}$
   - D $y = -4^x$



Spiral Review

Determine whether each survey question is biased. Explain your reasoning. (Lesson 10-1)

21. What toppings do you prefer on your pizza?

22. What is your favorite class, and what teacher gives the easiest homework?

23. Don't you hate how high airline prices are?

24. **DINNER PARTIES** Suppose each time a new guest arrives at a dinner party, he or she shakes hands with each person already at the dinner party. Prove that after n guests have arrived, a total of $\frac{n(n-1)}{2}$ handshakes have taken place.

25. **ASTRONOMY** The orbit of Pluto can be modeled by $\frac{x^2}{39.5^2} + \frac{y^2}{38.3^2} = 1$, where the units are astronomical units. Suppose a comet is following a path modeled by $x = y^2 + 20$.
   - **a.** Find the point(s) of intersection of the orbits of Pluto and the comet.
   - **b.** Will the comet necessarily hit Pluto? Explain.
   - **c.** Where do the graphs of $y = 2x + 1$ and $2x^2 + y^2 = 11$ intersect?
   - **d.** What are the coordinates of the points that lie on the graphs of both $x^2 + y^2 = 25$ and $2x^2 + 3y^2 = 66$?

Skills Review

Determine whether each situation involves a *permutation* or a *combination*. Then find the number of possibilities.

26. the winner of the first, second, and third prizes in a contest with 8 finalists

27. selecting two of eight employees to attend a business seminar

28. an arrangement of the letters in the word *MATH*

29. placing an algebra book, a geometry book, a chemistry book, an English book, and a health book on a shelf

LESSON 10-3 Probability Distributions

:: Then

- You used statistics to describe symmetrical and skewed distributions of data.

:: Now

1. Construct a probability distribution.
2. Analyze a probability distribution and its summary statistics.

:: Why?

- Mutual funds are professionally managed investments that offer diversity to investors. An accurate analysis of the fund's current and expected performance can help an investor determine if the fund suits their needs.

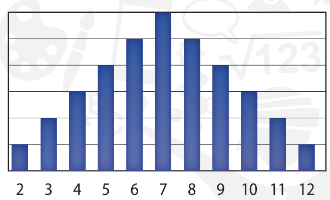
New Vocabulary
random variable
discrete random variable
continuous random variable
probability distribution
theoretical probability distribution
experimental probability distribution
Law of Large Numbers
expected value

Mathematical Practices
Reason abstractly and quantitatively.

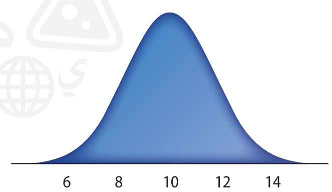
1 Construct a Probability Distribution

A sample space is the set of all possible outcomes in a distribution. Consider a distribution of values represented by the sum of the values on two number cubes and a distribution of the kilometers per liter for a sample of cars.

Sum of Two Number Cubes



Kilometers Per Liter



The sum of the values on the number cubes can be any integer from 2 to 12. So, the sample space is [2, 3, ..., 11, 12]. This distribution is *discrete* because the number of possible values in the sample space can be counted.

The distribution of kilometers per liter is *continuous.* While the sample space includes any positive value less than a certain maximum (around 100), the data can take on an infinite number of values within this range.

The value of a **random variable** is the numerical outcome of a random event. A random variable can be discrete or continuous. **Discrete random variables** represent countable values. **Continuous random variables** can take on any value.

Example 1 Identify and Classify Random Variables

Identify the random variable in each distribution, and classify it as *discrete* or *continuous*. Explain your reasoning.

a. the number of songs found on a random selection of mp3 players

The random variable X is the number of songs on any mp3 player in the random selection of players. The number of songs is countable, so X is discrete.

b. the weights of bowling balls sent by a manufacturer

The random variable X is the weight of any particular bowling ball. The weight of any particular bowling ball can be anywhere within a certain range, typically 6 to 8 kg. Therefore, X is continuous.

Guided Practice

1A. the exact distances of a sample of discus throws

1B. the ages of counselors at a summer camp

StudyTip

Discrete vs. Continuous Variables representing height, weight, and capacity will always be continuous variables because they can take on any positive value.

A **probability distribution** for a particular random variable is a function that maps the sample space to the probabilities of the outcomes in the sample space. Probability distributions can be represented by tables, equations, or graphs. In this lesson, we will focus on discrete probability distributions.

A probability distribution has the following properties.

KeyConcept Probability Distribution

- A probability distribution can be determined theoretically or experimentally.
- A probability distribution can be discrete or continuous.
- The probability of each value of X must be at least 0 and not greater than 1.
- The sum of all the probabilities for all of the possible values of X must equal 1. That is, $\Sigma P(X) = 1$.

ReviewVocabulary

Theoretical and Experimental Probability Theoretical probability is based on assumptions, and experimental probability is based on experiments.

A **theoretical probability distribution** is based on what is expected to happen. For example, the distribution for flipping a fair coin is $P(\text{heads}) = 0.5$, $P(\text{tails}) = 0.5$.

Example 2 Construct a Theoretical Probability Distribution

X represents the sum of the values on two number cubes.

a. Construct a relative-frequency table.

The theoretical probabilities associated with rolling two number cubes can be described using a relative-frequency table. When two number cubes are rolled, 36 total outcomes are possible. To determine the relative frequency, or theoretical probability, of each outcome, divide the frequency by 36.

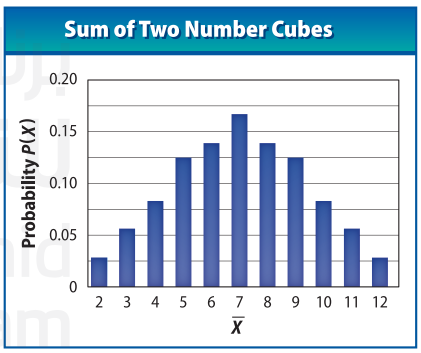
Sum	2	3	4	5	6	7	8	9	10	11	12
Frequency	1	2	3	4	5	6	5	4	3	2	1
Relative Frequency	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

Sum: 36

b. Graph the theoretical probability distribution.

The graph shows the probability distribution for the sum of the values on two number cubes X. The bars are separated on the graph because the distribution is discrete (no other values of X are possible).

Each unique outcome of X is indicated on the horizontal axis, and the probability of each outcome occurring $P(X)$ is indicated on the vertical axis.

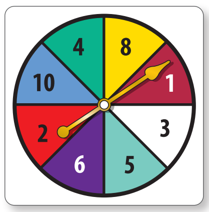


GuidedPractice

2. X represents the sum of the values of two spins of the wheel.

A. Construct a relative-frequency table.

B. Graph the theoretical probability distribution.



An **experimental probability distribution** is a distribution of probabilities estimated from experiments. Simulations can be used to construct an experimental probability distribution. When constructing this type of distribution, use the frequency of occurrences of each observed value to compute its probability.

Example 3 Construct an Experimental Probability Distribution

X represents the sum of the values found by rolling two number cubes.

a. Construct a relative-frequency table.

Roll two number cubes 100 times or use a random number generator to complete the simulation and create a simulation tally sheet.

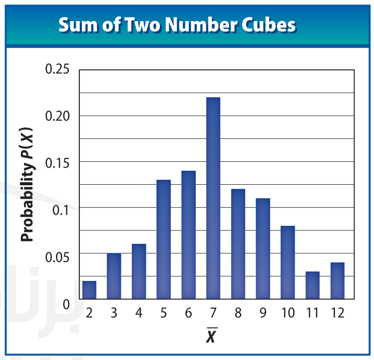
Sum	Tally	Frequency	Sum	Tally	Frequency
2	𝍷𝍷	2	8	𝍸 𝍸 𝍷𝍷	12
3	𝍸	5	9	𝍸 𝍸 𝍷	11
4	𝍸 𝍷	6	10	𝍸 𝍷𝍷𝍷	8
5	𝍸 𝍸 𝍷𝍷𝍷	13	11	𝍷𝍷𝍷	3
6	𝍸 𝍸 𝍷𝍷𝍷𝍷	14	12	𝍷𝍷𝍷𝍷	4
7	𝍸 𝍸 𝍸 𝍸 𝍷𝍷	22			

Calculate the experimental probability of each value by dividing its frequency by the total number of trials, 100.

Sum	2	3	4	5	6	7	8	9	10	11	12
Relative Frequency	0.02	0.05	0.06	0.13	0.14	0.22	0.12	0.11	0.08	0.03	0.04

b. Graph the experimental probability distribution.

The graph shows the discrete probability distribution for the sum of the values shown on two number cubes X.

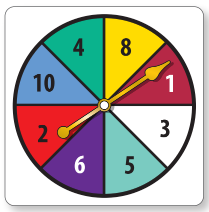


Guided Practice

3. X represents the sum of the values of two spins of the wheel.

A. Construct a relative-frequency table for 100 trials.

B. Graph the experimental probability distribution.



> **Study Tip**
>
> **Random Number Generators and Proportions** When using a random number generator to simulate events with different probabilities, set up a proportion. For example, suppose there are 3 possible outcomes with probabilities of A: 0.25, B:0.35, and C:0.40. Random numbers 1–25 can represent A, 26–60 represent B, and 61–100 represent C.

Notice that this graph is different from the theoretical graph in Example 2. With small sample sizes, experimental distributions may vary greatly from their associated theoretical distributions. However, as the sample size increases, experimental probabilities will more closely resemble their associated theoretical probabilities. This is due to the **Law of Large Numbers**, which states that the variation in a data set decreases as the sample size increases.

Math HistoryLink

Christian Huygens **(1629–1695)** This Dutchman was the first to discuss games of chance. "Although in a pure game of chance the results are uncertain, the chance that one player has to win or to lose depends on a determined value." This became known as the *expected value*.

2 Analyze a Probability Distribution

Probability distributions are often used to analyze financial data. The two most common statistics used to analyze a discrete probability distribution are the mean, or expected value, and the standard deviation. The **expected value** $E(X)$ of a discrete random variable of a probability distribution is the weighted average of the variable.

KeyConcept Expected Value of a Discrete Random Variable

Words The expected value of a discrete random variable is the weighted average of the values of the variable. It is calculated by finding the sum of the products of every possible value of X and its associated probability $P(X)$.

Symbols $E(X) = \Sigma\,[X \cdot P(X)]$

Real-World Example 4 Expected Value

CONTEST A contestant has won one spin of the wheel at the right. Find the expected value of his winnings.

Each prize value represents a value of X and each percent represents the corresponding probability $P(X)$. Find $E(X)$.

$$\begin{aligned}
E(X) &= \Sigma[X \cdot P(X)] \\
&= 0(0.20) + 25{,}000(0.08) + 15{,}000(0.14) + 10{,}000(0.22) + 5000(0.36) \\
&= 0 + 2000 + 2100 + 2200 + 1800 \\
&= 8100
\end{aligned}$$

The expected value of the contestant's winnings is AED 8100.

WatchOut!

Expected Value The expected value is what you *expect* to happen in the long run, not necessarily what *will* happen.

GuidedPractice

4. **PRIZES** Hareb won a ticket for a prize. The distribution of the values of the tickets and their relative frequencies are shown. Find the expected value of his winnings.

Value (AED)	1	10	100	1000	5000	25,000
Frequency	5000	100	25	5	1	1

Sometimes the expected value does not provide enough information to fully analyze a probability distribution. For example, suppose two wheels had roughly the same expected value. Which one would you choose? Which one is *riskier*? The standard deviation can provide more insight into the expected value of a probability distribution.

The formula for calculating the standard deviation of a probability distribution is similar to the one used for a set of data.

KeyConcept Standard Deviation of a Probability Distribution

Words For each value of X, subtract the mean from X and square the difference. Then multiply by the probability of X. The sum of each of these products is the variance. The standard deviation is the square root of the variance.

Symbols
Variance: $\sigma^2 = \Sigma[[X - E(X)]^2 \cdot P(X)]$
Standard Deviation: $\sigma = \sqrt{\sigma^2}$

Real-World Career

Mutual Fund Manager Mutual fund managers buy and sell fund investments according to the investment objective of the fund. Investment management includes financial statement analysis, asset and stock selection, and monitoring of investments. Certification beyond a bachelor's degree is required.

Source: International Financial Services, London

Real-World Example 5 Standard Deviation of a Distribution

DECISION MAKING **Husam is thinking about investing AED 10,000 in two different investment funds. The expected rates of return and the corresponding probabilities for each fund are listed below.**

Fund A	Fund B
50% chance of an AED 800 profit	30% chance of a AED 2400 profit
20% chance of a AED 1200 profit	10% chance of a AED 1900 profit
20% chance of a AED 600 profit	40% chance of a AED 200 loss
10% chance of a AED 100 loss	20% chance of a AED 400 loss

a. Find the expected value of each investment.

Fund A: $E(X) = 0.50(800) + 0.20(1200) + 0.20(600) + 0.10(-100)$ or 750

Fund B: $E(X) = 0.30(2400) + 0.10(1900) + 0.40(-200) + 0.20(-400)$ or 750

An investment of AED 10,000 in Fund A or Fund B will expect to yield AED 750.

b. Find each standard deviation.

Fund A:

Profit, X	$P(X)$	$[X - E(X)]^2$	$[X - E(X)]^2 \cdot P(X)$
800	0.50	$(800 - 750)^2 = 2500$	$2500 \cdot 0.50 = 1250$
1200	0.20	$(1200 - 750)^2 = 202{,}500$	$202{,}500 \cdot 0.20 = 40{,}500$
600	0.20	$(600 - 750)^2 = 22{,}500$	$22{,}500 \cdot 0.20 = 4500$
−100	0.10	$(-100 - 750)^2 = 722{,}500$	$722{,}500 \cdot 0.10 = 72{,}250$
			$\Sigma[[(X - E(X)]^2 \cdot P(X)] = 118{,}500$
			$\sqrt{118{,}500} \approx 344.2$

Fund B:

Profit, X	$P(X)$	$[X - E(X)]^2$	$[X - E(X)]^2 \cdot P(X)$
2400	0.30	$(2400 - 750)^2 = 2{,}722{,}500$	$2{,}722{,}500 \cdot 0.30 = 816{,}750$
1900	0.10	$(1900 - 750)^2 = 1{,}322{,}500$	$1{,}322{,}500 \cdot 0.10 = 132{,}250$
−200	0.40	$(-200 - 750)^2 = 902{,}500$	$902{,}500 \cdot 0.40 = 361{,}000$
−400	0.20	$(-400 - 750)^2 = 1{,}322{,}500$	$1{,}322{,}500 \cdot 0.20 = 264{,}500$
			$\Sigma[[(X - E(X)]^2 \cdot P(X)] = 1{,}574{,}500$
			$\sqrt{1{,}574{,}500} \approx 1254.8$

c. Which investment would you advise Husam to choose, and why?

Husam should choose Fund A. While the funds have identical expected values, the standard deviation of Fund B is almost four times the standard deviation for Fund A. This means that the expected value for Fund B will have about four times the variability than Fund A and will be riskier with a greater chance for gains and losses.

Study Tip

Return on Investment When investing AED 1000 in a product that has a 6% expected return, the investor can expect a 0.06(1000) or AED 60 profit.

Guided Practice

5. **DECISION MAKING** Compare a AED 10,000 investment in the two funds. Which investment would you recommend, and why?

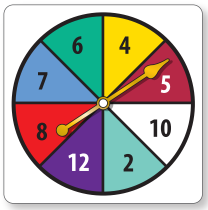
Fund C	Fund D
30% chance of a AED 1000 profit	40% chance of a AED 1000 profit
40% chance of a AED 500 profit	30% chance of a AED 600 profit
20% chance of a AED 100 loss	15% chance of a AED 100 profit
10% chance of a AED 300 loss	15% chance of a AED 200 loss

Check Your Understanding

Example 1 **Identify the random variable in each distribution, and classify it as *discrete* or *continuous*. Explain your reasoning.**

1. the number of pages linked to a Web page
2. the number of stations in a cable package
3. the amount of precipitation in a city per month
4. the number of cars passing through an intersection in a given time interval

Examples 2–5 5. X represents the sum of the values of two spins of the wheel.



a. Construct a relative-frequency table showing the theoretical probabilities.
b. Graph the theoretical probability distribution.
c. Construct a relative-frequency table for 100 trials.
d. Graph the experimental probability distribution.
e. Find the expected value for the sum of two spins of the wheel.
f. Find the standard deviation for the sum of two spins of the wheel.

Practice and Problem Solving

Example 1 **Identify the random variable in each distribution, and classify it as *discrete* or *continuous*. Explain your reasoning.**

6. the number of texts received per week
7. the number of diggs (or “likes”) for a Web page
8. the height of a plant after a specific amount of time
9. the number of files infected by a computer virus

Examples 2–5 10. **PERSEVERANCE** A contestant has won a prize in a competition. The frequency table at the right shows the number of winners for 3200 hypothetical players.

a. Construct a relative-frequency table showing the theoretical probability.
b. Graph the theoretical probability distribution.
c. Construct a relative-frequency table for 50 trials.
d. Graph the experimental probability distribution.
e. Find the expected value.
f. Find the standard deviation.

Prize, X	Winners
AED 100	1120
AED 250	800
AED 500	480
AED 1000	320
AED 2500	256
AED 5000	128
AED 7500	64
AED 10,000	32

11. **SNOW DAYS** The following probability distribution lists the probable number of snow days per school year at Al Nadha Secondary School. Use this information to determine the expected number of snow days per year.

Number of Snow Days Per Year									
Days	0	1	2	3	4	5	6	7	8
Probability	0.1	0.1	0.15	0.15	0.25	0.1	0.08	0.05	0.02

12. **FLASHCARDS** A set of flashcards consists of 52 cards, divided equally between four different colors, red, yellow, green and blue and each color is numbered 1 to 13.

a. What is the expected value of a card that is drawn randomly from the set?
b. If you are dealt 7 cards with replacement, what is the expected number of reds?

13. **COMPETITION** The table shows the probability distribution for a competition if 100 tickets are sold for AED 5 each. There is 1 prize for AED 100, 5 prizes for AED 50, and 10 prizes for AED 25.

Distribution of Prizes				
Prize	no prize	AED 100	AED 50	AED 25
Probability	0.84	0.01	0.05	0.10

a. Graph the theoretical probability distribution.

b. Find the expected value.

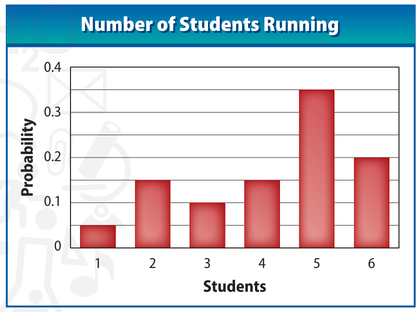
c. Interpret the results you found in part **b**. What can you conclude about the raffle?

14. **TOOLS** Based on previous data, the probability distribution of the number of students running for class president is shown.

a. Determine the expected number of students who will run. Interpret your results.

b. Construct a relative-frequency table for 50 trials.

c. Graph the experimental probability distribution.



15. **BASKETBALL** The distribution below lists the probability of the number of major upsets in the first round of a basketball tournament each year.

Number of Upsets Per Year									
Upsets	0	1	2	3	4	5	6	7	8
Probability	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{3}{32}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{5}{16}$	$\frac{1}{8}$	$\frac{3}{32}$	$\frac{1}{32}$

a. Determine the expected number of upsets. Interpret your results.

b. Find the standard deviation.

c. Construct a relative-frequency table for 50 trials.

d. Graph the experimental probability distribution.

16. **COMPETITION** The French Club sold 500 competition tickets for AED 5 each. The first prize ticket will win AED 500, 2 second prize tickets will each win AED 50, and 5 third prize tickets each win AED 25.

a. What is the expected value of a single ticket?

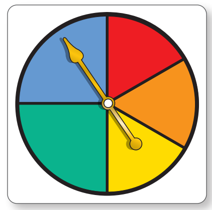
b. Calculate the standard deviation of the probability distribution.

c. **DECISION MAKING** The Glee Club is offering a competition with a similar expected value and a standard deviation of 2.2. In which competition should you participate? Explain your reasoning.

17. **DECISION MAKING** Amal is thinking about investing AED 10,000 in two different investment funds. The expected rates of return and the corresponding probabilities for each fund are listed below. Compare the two investments using the expected value and standard deviation. Which investment would you advise Amal to choose, and why?

Fund A	Fund B
30% chance of a AED 1900 profit	40% chance of a AED 1600 profit
30% chance of a AED 600 profit	10% chance of a AED 900 profit
15% chance of a AED 200 loss	10% chance of a AED 300 loss
25% chance of a AED 500 loss	40% chance of a AED 400 loss

18. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate geometric probability.

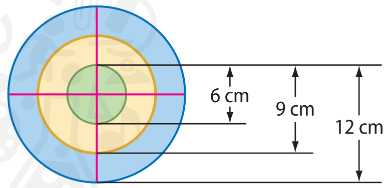


a. Tabular The spinner shown has a radius of 2.5 cm. Copy and complete the table below.

Color	Probability	Sector Area	Total Area	$\frac{\text{Sector Area}}{\text{Total Area}}$
red				
orange				
yellow				
green				
blue				

b. Verbal Make a conjecture about the relationship between the ratio of the area of the sector to the total area and the probability of the spinner landing on each color.

c. Analytical Consider the dartboard shown. Predict the probability of a dart landing in each area of the board. Assume that any dart thrown will land on the board and is equally likely to land at any point on the board.

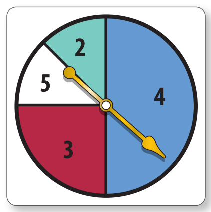


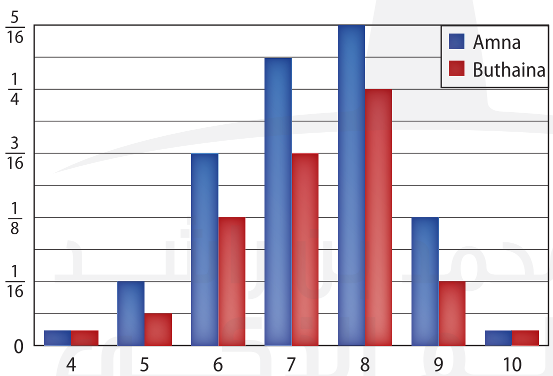
d. Tabular Construct a relative-frequency table for throwing 100 darts.

e. Graphical Graph the experimental probability distribution.

H.O.T. Problems Use Higher-Order Thinking Skills

19. **CRITIQUE** Amna and Buthaina each created a probability distribution for the sum of two spins on the spinner at the right. Is either of them correct? Explain your reasoning.





20. **REASONING** Determine whether the following statement is *true* or *false*. Explain.

If you roll a number cube 10 times, you will roll the expected value at least twice.

21. **OPEN ENDED** Create a discrete probability distribution that shows five different outcomes and their associated probabilities.

22. **REASONING** Determine whether the following statement is *true* or *false*. Explain.

Random variables that can take on an infinite number of values are continuous.

23. **OPEN ENDED** Provide examples of a discrete probability distribution and a continuous probability distribution. Describe the differences between them.

24. **WRITING IN MATH** Compare and contrast two investments that have identical expected values and significantly different standard deviations.

Standardized Test Practice

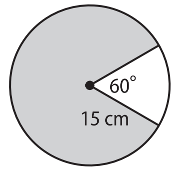
25. GRIDDED RESPONSE The height $f(x)$ of a bouncing ball after x bounces is represented by $f(x) = 140(0.8)^x$. How many times higher is the first bounce than the fifth bounce?

26. PROBABILITY Hamad has a bag that contains 4 red, 6 yellow, 2 blue, and 4 green marbles. If he reaches into the bag and removes a marble without looking, what is the probability that it will not be yellow?

A $\frac{1}{8}$ **C** $\frac{3}{8}$

B $\frac{1}{4}$ **D** $\frac{5}{8}$

27. GEOMETRY Find the area of the shaded portion of the figure to the nearest square centimeter.



F 79 **H** 589

G 94 **J** 707

28. SAT/ACT If x and y are positive integers, which of the following expressions is equivalent to $\frac{(5^x)^y}{5^x}$?

A 1^y **D** 5^{xy-1}

B ± 1 **E** 5^{xy-x}

C 5^y

Spiral Review

29. ARTICLES Hamdan and Humaid each write articles for an online magazine. Their employer keeps track of the number of *likes* received by each article. (Lesson 10-2)

a. Use a graphing calculator to create a histogram for each data set. Then describe the shape of each distribution.

b. Compare the distributions using either the means and standard deviations or the five-number summaries. Justify your choice.

Hamdan's Articles
16, 22, 19, 31, 24, 8, 40, 19, 33, 18, 36, 21, 55, 3, 16, 44, 22, 39, 12, 18, 13, 20, 67, 31, 13, 38, 31, 22, 26, 28

Humaid's Articles
41, 38, 29, 33, 36, 55, 51, 19, 49, 56, 28, 52, 49, 19, 38, 33, 42, 61, 72, 55, 48, 39, 37, 43, 48, 45, 52, 43, 34, 29

Determine whether the situation calls for a *survey*, an *observational study*, or an *experiment*. Explain your reasoning. (Lesson 10-1)

30. You want to test a medicine that reverses male pattern baldness.

31. You want to find voters' opinions on recent legislation.

Find the first five terms of each geometric sequence described.

32. $a_1 = 0.125, r = 1.5$

33. $a_1 = 0.5, r = 2.5$

34. $a_1 = 4, r = 0.5$

35. $a_1 = 12, r = \frac{1}{3}$

36. $a_1 = 21, r = \frac{2}{3}$

37. $a_1 = 80, r = \frac{5}{4}$

38. COMMUNICATION A microphone is placed at the focus of a parabolic reflector to collect sound for the television broadcast of a soccer game. Write an equation for the cross section, assuming that the focus is at the origin, the focus is 6 cm from the vertex, and the parabola opens to the right.

Solve each equation. Check your solutions.

39. $\log_9 x = \frac{3}{2}$

40. $\log_{\frac{1}{10}} x = -3$

41. $\log_b 9 = 2$

Skills Review

Expand each power.

42. $(a - b)^3$

43. $(m + n)^4$

44. $(r + n)^8$

CHAPTER 10

Mid-Chapter Quiz

Lessons 10-1 through 10-3

Determine whether each situation describes a *survey*, an *experiment*, or an *observational study*. Then identify the sample, and suggest a population from which it may have been selected. (Lesson 10-1)

1. A high school principal wants to test five ideas for a new school mascot. He randomly selects 15 high school students to view pictures of the ideas while he watches and records their reactions.

2. Half of the employees of a grocery store are randomly chosen for an extra hour lunch break. The managers then compare their attitudes with their co-workers.

3. Students want to create a school yearbook. They send out a questionnaire to 100 students asking what they would like to showcase in the yearbook.

4. The producers of a sitcom want to determine if a new character that they are planning to introduce will be well received. They show a clip of the show with the new character to 50 randomly chosen participants and then record the participants' reactions.

5. **MULTIPLE CHOICE** Which survey question is *unbiased*? (Lesson 10-1)

   A Do you like days like today?

   B Which is your favorite theme park, Park A or Park B?

   C Don't you think that carrots taste better than celery?

   D How often do you go to the movies?

6. **PARENTS** The table below shows the ages of parents who volunteered to assist in a neighborhood bake sale. (Lesson 10-2)

Ages of Parents (years)				
28	34	33	45	31
33	41	34	36	42
30	29	32	40	36
29	33	29	28	44
47	31	28	27	29

   a. Use a graphing calculator to create a box-and-whisker plot. Then describe the shape of the distribution.

   b. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

7. **TRAINING** Khalid and Khalaf's training times for the 40-meter dash are shown. (Lesson 10-2)

Khalid's 40-Meter Dash Times (seconds)					
4.84	4.94	4.87	4.78	5.04	4.98
4.83	5.03	4.74	5.15	4.82	4.91
4.62	4.83	4.76	4.93	4.85	4.82
4.76	4.98	4.94	5.05	4.94	5.04
4.86	4.85	4.71	4.66	4.91	4.82

Khalaf's 40-Meter Dash Times (seconds)					
5.03	4.76	4.69	4.52	4.81	4.78
4.65	4.66	4.83	4.95	4.64	4.76
4.43	4.64	4.50	4.58	4.68	4.65
4.83	4.78	4.71	4.81	4.76	4.84
4.61	4.63	4.33	4.46	4.74	4.63

   a. Use a graphing calculator to create a histogram for each data set. Then describe the shape of each distribution.

   b. Compare the distributions using either the means and standard deviations or the five-number summaries. Justify your choice.

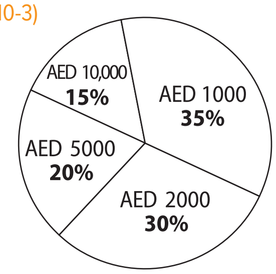
8. **MULTIPLE CHOICE** Find the expected value of winning one of the following prizes. (Lesson 10-3)

   F AED 1950

   G AED 2100

   H AED 3000

   J AED 3450



Identify the random variable in each distribution, and classify it as *discrete* or *continuous*. Explain your reasoning. (Lesson 10-3)

9. the number of calls received by an operator

10. the number of books sold at a yard sale

11. the height of students in a gym class

12. the weight of animals on a farm

LESSON 10-4

The Binomial Distribution

Then

- You used the Binomial Theorem.

Now

1. Identify and conduct a binomial experiment.
2. Find probabilities using binomial distributions.

Why?

- Houriyya forgot to study for her civics quiz. The quiz consists of five multiple-choice questions with each question having four answer choices. Houriyya randomly circles an answer for each question. In order to pass, she needs to answer at least four questions correctly.

NewVocabulary
binomial experiment
binomial distribution

Mathematical Practices
Model with mathematics.

1 Binomial Experiments

Each question on a multiple-choice quiz, like the one described above, can be thought of as a trial with two possible outcomes, correct or incorrect. If Houriyya guesses on each question, the probability that she answers a question correctly is the same for all five questions.

Houriyya's guessing on each question is an example of a binomial experiment. A **binomial experiment** is a probability experiment that satisfies the following conditions.

KeyConcept Binomial Experiments

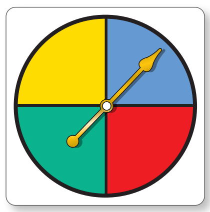
- There is a fixed number of independent trials n.
- Each trial has only two possible outcomes, success or failure.
- The probability of success p is the same in every trial. The probability of failure q is $1 - p$.
- The random variable X is the number of successes in n trials.

Many probability experiments are or can be reduced to binomial experiments.

Example 1 Identify a Binomial Experiment

Determine whether each experiment is a binomial experiment or can be reduced to a binomial experiment. If so, describe a trial, determine the random variable, and state n, p, and q.

a. The spinner at the right is spun 20 times to see how many times it lands on red.



This experiment can be reduced to a binomial experiment with success being that the spinner lands on red and failure being any other outcome. Thus, a trial is a spin, and the random variable X represents the number of reds spun. The number of trials n is 20, the probability of success p is $\frac{1}{4}$ or 0.25, and the probability of failure q is $1 - 0.25$ or 0.75.

b. One hundred students are randomly asked their favorite food.

This is not a binomial experiment because there are many possible outcomes.

GuidedPractice

1A. Seventy-five students are randomly asked if they own a car.

Use the following guidelines when conducting a binomial experiment.

KeyConcept Conducting Binomial Experiments

Step 1 Describe a trial for the situation, and determine the number of trials to be conducted.

Step 2 Define a success, and calculate the theoretical probabilities of success and failure.

Step 3 Describe the random variable X.

Step 4 Design and conduct a simulation to determine the experimental probability.

A binomial experiment can be conducted to compare experimental and theoretical probabilities.

Example 2 Design a Binomial Experiment

Conduct a binomial experiment to determine the probability of drawing an odd-numbered flashcard from a set of flashcards, consisting of 52 cards, divided equally between four different colors and each color is numbered 1 to 13. Then compare the experimental and theoretical probabilities of the experiment.

Step 1 A trial is drawing a flashcards from the set. The number of trials conducted can be any number greater than 0. We will use 52.

Step 2 A success is drawing an odd-numbered flashcard. The odd-numbered flashcards in the set are 1, 3, 5, 7, 9, 11 and 13, and they occur once in each of the four colors. Therefore, there are $4 \cdot 7$ or 28 odd-numbered flashcards in the set. The probability of drawing an odd-numbered flashcard, or the probability of success, is $\frac{28}{52}$ or $\frac{7}{13}$. The probability of failure is $1 - \frac{7}{13}$ or $\frac{6}{13}$.

Step 3 The random variable X represents the number of odd-numbered flashcards drawn in 52 trials.

Step 4 Use the random number generator on a calculator to create a simulation. Assign the integers 0–12 to accurately represent the probability data.

Odd-numbered flashcards	0, 1, 2, 3, 4, 5, 6
Other flashcards	7, 8, 9, 10, 11, 12

Make a frequency table and record the results as you run the generator.

Outcome	Tally	Frequency
Odd-Numbered flashcard	𝍸 𝍸 ǁ	12
Other flashcards	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸	40

An odd-numbered flashcard was drawn 12 times, so the experimental probability is $\frac{12}{52}$ or about 23.1%. This is less than the theoretical probability of $\frac{28}{52}$ or about 53.8%.

StudyTip

Random Number Generator To generate all 52 random numbers using a graphing calculator, press MATH ◀ 5 and then enter the desired range followed by the number of trials. For example, type (0, 12, 52) for Example 2.

GuidedPractice

2. Conduct a binomial experiment to determine the probability of drawing an even-numbered flashcard from a deck of flashcards. Then compare the experimental and theoretical probabilities of the experiment.

2 Binomial Distribution

In the binomial experiment in Example 2, there were 12 successes in 52 trials. If you conducted that same experiment again, there may be any number of successes from 0 to 52. This situation can be represented by a binomial distribution. A **binomial distribution** is a frequency distribution of the probability of each value of X, where the random variable X represents the number of successes in n trials. Because X is a discrete random variable, a binomial distribution is a *discrete probability distribution.*

The probabilities in a binomial distribution can be calculated using the following formula.

> **StudyTip**
>
> **Binomial Probability Formula** In the Binomial Probability Formula, X represents the number of successes in n trials. Thus, the exponent for q, $n - X$, represents the number of failures in n trials.

KeyConcept Binomial Probability Formula

The probability of X successes in n independent trials is

$$P(X) = {}_nC_X\, p^X q^{n-X},$$

where p is the probability of success of an individual trial and q is the probability of failure on that same individual trial ($q = 1 - p$).

Notice that the Binomial Probability Formula is an adaptation of the Binomial Theorem you have already studied. The expression ${}_nC_X\, p^X q^{n-X}$ represents the $p^X q^{n-X}$ term in the binomial expansion of $(p + q)^n$.

Standardized Test Example 3 Find a Probability

Khamis is selling items from a catalog to raise money for school. He has a 40% chance of making a sale each time he solicits a potential customer. Khamis asks 10 people to purchase an item. Find the probability that 6 people make a purchase.

A 8.6% **B** 11.1% **C** 24% **D** 40%

Read the Test Item

We need to find the probability that 6 people purchase an item. A success is making a sale, so $p = 0.4$, $q = 1 - 0.4$ or 0.6, and $n = 10$.

Solve the Test Item

$P(X) = {}_nC_X\, p^X q^{n-X}$ Binomial Probability Formula

$P(6) = {}_{10}C_6\, (0.4)^6\, (0.6)^{10-6}$ $n = 10$, $X = 6$, $p = 0.4$, and $q = 0.6$

≈ 0.111 Simplify.

The probability of Khamis making six sales is about 0.111 or 11.1%. So, the correct answer is B.

> **Test-TakingTip**
>
> **Precision** A common error when using the Binomial Probability Formula is to focus on only the successes and to forget the failures. Notice that $P(6) \neq (0.4)^6$ and $P(6) \neq {}_{10}C_6(0.4)^6$.

GuidedPractice

3. **TELEMARKETING** At Khawla's telemarketing job, 15% of the calls that she makes to potential customers result in a sale. She makes 20 calls in a given hour. What is the probability that 5 calls result in a sale?

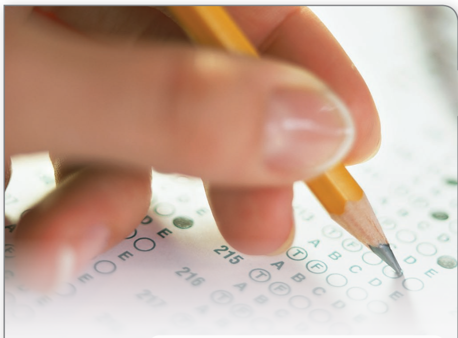
F 6.7% **G** 8.3% **H** 10.3% **J** 11.9%

If, on average, 40% of the people Khamis solicits make a purchase and he solicits 10 people, he can probably expect to make 10(0.40) or 4 sales. This value represents the mean of the binomial distribution. In general, the mean of a binomial distribution can be calculated by the following formula.

KeyConcept Mean of a Binomial Distribution

The mean μ of a binomial distribution is given by $\mu = np$, where n is the number of trials and p is the probability of success.

You can find the probability distribution for a binomial experiment by fully expanding the binomial $(p + q)^n$. A probability distribution can be helpful when solving for problems that allow multiple numbers of successes.



Real-WorldLink

ACT The math portion of the ACT college entrance exam includes 60 multiple-choice questions that each have five answer choices.

Source: ACT

Real-World Example 4 Full Probability Distribution

TEST TAKING Refer to the beginning of the lesson.

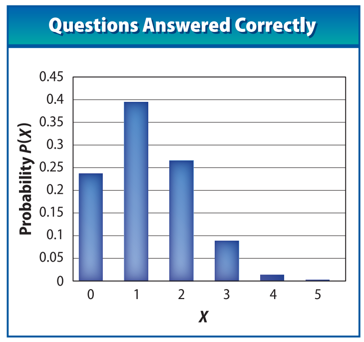
a. Determine the probabilities associated with the number of questions Houriyya answered correctly by calculating the probability distribution.

If there are four answer choices for each question, then the probability that Houriyya guesses and answers a question correctly is $\frac{1}{4}$ or 0.25. In this binomial experiment, $n = 5$, $p = 0.25$, and $q = 1 - 0.25$ or 0.75. Expand the binomial $(p + q)^n$.

$$\begin{aligned}
&(p+q)^n \\
&= 1p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + 1q^5 \\
&= (0.25)^5 + 5(0.25)^4(0.75) + 10(0.25)^3(0.75)^2 + 10(0.25)^2(0.75)^3 + 5(0.25)(0.75)^4 + (0.75)^5 \\
&\approx 0.001 + 0.015 + 0.089 + 0.264 + 0.396 + 0.237
\end{aligned}$$

0.1%	1.5%	8.9%	26.4%	39.6%	23.7%
5 correct	4 correct	3 correct	2 correct	1 correct	0 correct

The graph shows the binomial probability distribution for the number of questions that Houriyya answered correctly.



b. What is the probability that Houriyya passes the quiz?

Houriyya must answer at least four questions correctly to pass the quiz. The probability that Houriyya answers *at least* four correct is the sum of the probabilities that she answers four or five correct and is about 1.5% + 0.1% or 1.6%. So, Houriyya has about a 1.6% chance of passing, which is not likely.

StudyTip

Mean and Expected Value The mean of a binomial distribution can be any positive rational number. The expected value of a binomial distribution, however, should be rounded to the nearest whole number since a fraction of a success is not possible.

c. How many questions should Houriyya expect to answer correctly?

Find the mean.

$\mu = np$ Mean of a Binomial Distribution

$= 5(0.25)$ or 1.25 $n = 5$ and $p = 0.25$

The mean of the distribution is 1.25. On average, Houriyya should expect to answer one question correctly when she guesses on five.

GuidedPractice

4. TEST TAKING Suppose Houriyya's civics quiz consisted of five true-or-false questions instead of multiple-choice questions.

A. Determine the probabilities associated with the number of answers Houriyya answered correctly by calculating the probability distribution.

B. What is the probability that Houriyya passes the quiz?

C. How many questions should Houriyya expect to answer correctly?

Check Your Understanding

Example 1 **Determine whether each experiment is a binomial experiment or can be reduced to a binomial experiment. If so, describe a trial, determine the random variable, and state n, p, and q.**

1. A study finds that 58% of people have pets. You ask 100 people how many pets they have.
2. You roll a number cube 15 times and find the sum of all of the rolls.
3. A poll found that 72% of students plan on going to the graduation ceremony. You ask 30 students if they are going to the graduation ceremony.

Example 2

4. Conduct a binomial experiment to determine the probability of drawing a 1 or 13 from the set of flashcards in Example 2. Then compare the experimental and theoretical probabilities of the experiment.

Example 3

5. **GAMES** Saeed has earned five spins of the wheel on the right. He will receive a prize each time the spinner lands on WIN. What is the probability that he receives three prizes?

A 4.2% **C** 7.1%
B 5.8% **D** 8.8%

Example 4

6. **PRECISION** A poll at Rasheed's high school was taken to see if students are in favor of spending class money to expand the grade 11-grade 12 parking lot. Rasheed surveyed 6 random students from the population.

Expand the Parking Lot	
favor	85%
oppose	15%

a. Determine the probabilities associated with the number of students that Rasheed asked who are in favor of expanding the parking lot by calculating the probability distribution.

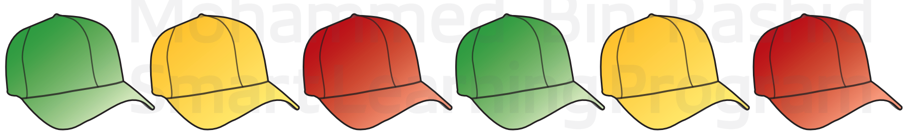
b. What is the probability that no more than 2 people are in favor of expanding the parking lot?

c. How many students should Rasheed expect to find who are in favor of expanding the parking lot?

Practice and Problem Solving

Example 1 **Determine whether each experiment is a binomial experiment or can be reduced to a binomial experiment. If so, describe a trial, determine the random variable, and state n, p, and q.**

7. There is a 35% chance that it rains each day in a given month. You record the number of days that it rains for that month.
8. A survey found that on a scale of 1 to 10, a movie received a 7.8 rating. A movie theater employee asks 200 patrons to rate the movie on a scale of 1 to 10.
9. A ball is hidden under one of the hats shown below. A hat is chosen, one at a time, until the ball is found.



Example 2

10. **NUMBER CUBE** Conduct a binomial experiment to determine the probability of rolling a 7 with two number cubes. Then compare the experimental and theoretical probabilities of the experiment.

11. **MARBLES** Conduct a binomial experiment to determine the probability of pulling a red marble from the bag. Then compare the experimental and theoretical probabilities of the experiment.

12. **SPINNER** Conduct a binomial experiment to determine the probability of the spinner stopping on an even number. Then compare the experimental and theoretical probabilities of the experiment.

13. **FLASHCARDS** Conduct a binomial experiment to determine the probability of drawing an 11, 12 or 13 flashcard out of the set of flashcards in Example 2. Then compare the experimental and theoretical probabilities of the experiment.

Example 3

14. **PERSONAL MEDIA PLAYERS** According to a recent survey, 85% of high school students own a personal media player. What is the probability that 6 out of 10 random high school students own a personal media player?

15. **CARS** According to a recent survey, 92% of high school grade 12 students drive their own car. What is the probability that 10 out of 12 random high school students drive their own car?

16. **GRADE 12 GRADUATION** According to a recent survey, 25% of high school upperclassmen think that the grade 12 graduation is the most important event of the school year. What is the probability that 3 out of 15 random high school upperclassmen think this way?

17. **SOCCER** A certain soccer team has won 75.7% of their games. Find the probability that they win 7 of their next 12 games.

18. **GARDENING** Zayed is planting 24 irises in his front yard. The flowers he bought were a combination of two varieties, blue and white. The flowers are not blooming yet, but Zayed knows that the probability of having a blue flower is 75%. What is the probability that 20 of the flowers will be blue?

19. **RUGBY** A penalty goal kicker is accurate 75% of the time from within 35 m. What is the probability that he makes exactly 7 of his next 10 kicks from within 35 m?

Range (m)	Accuracy (%)
0–35	75
35–45	62
45+	20

20. **BABIES** Mr. and Mrs. Salem are planning to have 3 children. The probability of each child being a boy is 50%. What is the probability that they will have 2 boys?

Example 4

21. **SENSE-MAKING** According to a recent survey, 52% of high school students own a laptop. Ten random students are chosen.

    a. Determine the probabilities associated with the number of students that own a laptop by calculating the probability distribution.

    b. What is the probability that at least 8 of the 10 students own a laptop?

    c. How many students should you expect to own a laptop?

22. **ATHLETICS** A survey was taken to see the percent of students that participate in sports for their school. Six random students are chosen.

Student Athletics	
0 sports	20%
1 sport	55%
2 sports	20%
3+ sports	5%

    a. Determine the probabilities associated with the number of students playing in at least one sport by calculating the probability distribution.

    b. What is the probability that no more than 2 of the students participated in a sport?

    c. How many students should you expect to have participated in at least one sport?

23. **MODELING** An online poll showed that 57% of adults still own vinyl records. Saeed surveyed 8 random adults from the population.

a. Determine the probabilities associated with the number of adults that still own vinyl records by calculating the probability distribution.

b. What is the probability that no less than 6 of the people surveyed still own vinyl records?

c. How many people should Saeed expect to still own vinyl records?

A binomial distribution has a 60% rate of success. There are 18 trials.

24. What is the probability that there will be at least 12 successes?

25. What is the probability that there will be 12 failures?

26. What is the expected number of successes?

27. **DECISION MAKING** Six roommates randomly select someone to wash the dishes each day.

a. What is the probability that the same person has to wash the dishes 3 times in a given week?

b. What method can the roommates use to select who washes the dishes each day?

28. **DECISION MAKING** A committee of five people randomly selects someone to take the notes of each meeting.

a. What is the probability that a person takes notes less than twice in 10 meetings?

b. What method can the committee use to select the notetaker each meeting?

c. If the method described in part **b** results in the same person being notetaker for nine straight meetings, would this result cause you to question the method?

Each binomial distribution has n trials and p probability of success. Determine the most likely number of successes.

29. $n = 8, p = 0.6$

30. $n = 10, p = 0.4$

31. $n = 6, p = 0.8$

32. $n = 12, p = 0.55$

33. $n = 9, p = 0.75$

34. $n = 11, p = 0.35$

35. **COMPETITION** A beverage company is having a competition. The probabilities of winning selected prizes are shown at the right. If Khalid purchases 8 beverages, what is the probability that he wins at least one prize?

Odds of Winning	
beverage	1 in 10
CD	1 in 200
hat	1 in 250
MP3 player	1 in 20,000
car	1 in 25,000,000

Each binomial distribution has n trials and p probability of success. Determine the probability of s successes.

36. $n = 8, p = 0.3, s \geq 2$

37. $n = 10, p = 0.2, s > 2$

38. $n = 6, p = 0.6, s \leq 4$

39. $n = 9, p = 0.25, s \leq 5$

40. $n = 10, p = 0.75, s \geq 8$

41. $n = 12, p = 0.1, s < 3$

H.O.T. Problems Use Higher-Order Thinking Skills

42. **CHALLENGE** A poll of students determined that 88% wanted to go to college. Eight random students are chosen. The probability that at least x students want to go to college is about 0.752 or 75.2%. Solve for x.

43. **WRITING IN MATH** What should you consider when using a binomial distribution to make a decision?

44. **OPEN ENDED** Describe a real-world setting within your school or community activities that seems to fit a binomial distribution. Identify the key components of your setting that connect to binomial distributions.

45. **WRITING IN MATH** Describe how binomial distributions are connected to Pascal's triangle.

46. **WRITING IN MATH** Explain the relationship between a binomial experiment and a binomial distribution.

Standardized Test Practice

47. EXTENDED RESPONSE **Abeer** is taking a 10 question multiple-choice test, in which each question has four choices. If she guesses on each question, what is the probability that she will get

a. 7 questions correct?

b. 9 questions correct?

c. 0 questions correct?

d. 3 questions correct?

48. What is the maximum point of the graph of the equation $y = -2x^2 + 16x + 5$?

A $(-4, -59)$ **C** $(4, 37)$

B $(-4, -91)$ **D** $(4, 101)$

49. GEOMETRY On a number line, point X has coordinate -8 and point Y has coordinate 4. Point P is $\frac{2}{3}$ of the way from X to Y. What is the coordinate of P?

F -4 **H** 0

G -2 **J** 2

50. SAT/ACT The cost of 4 CDs is d AED. At this rate, what is the cost, in AED, of 36 CDs?

A $9d$ **D** $\frac{d}{36}$

B $144d$ **E** $\frac{36}{d}$

C $\frac{9d}{4}$

Spiral Review

Identify the random variable in each distribution, and classify it as *discrete* or *continuous*. Explain your reasoning. (Lesson 10-3)

51. the number of customers at an amusement park

52. the running time of a movie

53. the number of sandwiches sold at a sporting event

54. the distance between two cities

55. FINANCIAL LITERACY The prices of starters offered by a restaurant are shown. (Lesson 10-2)

a. Use a graphing calculator to create a box-and-whisker plot. Then describe the shape of the distribution.

b. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

Prices (AED)			
11.25	14.75	9.00	17.25
19.75	9.75	20.25	15.50
16.50	21.50	10.25	22.75
12.75	18.50	23.00	13.50

Find the missing value for each arithmetic sequence.

56. $a_5 = 12, a_{16} = 133, d = ?$

57. $a_9 = -34, a_{22} = 44, d = ?$

58. $a_4 = 18, a_n = 95, d = 7, n = ?$

59. $a_8 = ?, a_{19} = 31, d = 8$

60. $a_6 = ?, a_{20} = 64, d = 7$

61. $a_7 = -28, a_n = 76, d = 8, n = ?$

62. ASTRONOMY The table at the right shows the closest and farthest distances of Venus and Jupiter from the Sun in millions of kilometers.

a. Write an equation for the orbit of each planet. Assume that the center of the orbit is the origin and the center of the Sun is a focus that lies on the x-axis.

b. Which planet has an orbit that is closer to a circle?

Planet	Closest	Farthest
Venus	107.5	108.9
Jupiter	740.4	816.5

Write an equivalent exponential or logarithmic function.

63. $e^{-x} = 5$

64. $e^2 = 6x$

65. $\ln e = 1$

66. $\ln 5.2 = x$

67. $e^{x+1} = 9$

68. $e^{-1} = x^2$

69. $\ln \frac{7}{3} = 2x$

70. $\ln e^x = 3$

Skills Review

71. READING Wafa owns 11 adventure, 6 history, 16 travel, and 7 art books. Find each probability if she randomly selects 4 books.

a. P(2 travel) **b.** P(1 art) **c.** P(1 travel and 2 history)

LESSON 10-5

The Normal Distribution

Then

- You analyzed probability distributions for discrete random variables.

Now

- **1** Find area under normal distribution curves.
- **2** Find probabilities for normal distributions, and find data values given probabilities.

Why?

- In a recent year, approximately 107 million Americans 20 years and older had a total blood cholesterol level of 200 milligrams per deciliter or higher. Physicians use variables of this type to compare patients' cholesterol levels to *normal* cholesterol ranges. In this lesson, you will determine the probability of a randomly selected person having a specific cholesterol level.

NewVocabulary

normal distribution
empirical rule
z-value
standard normal distribution

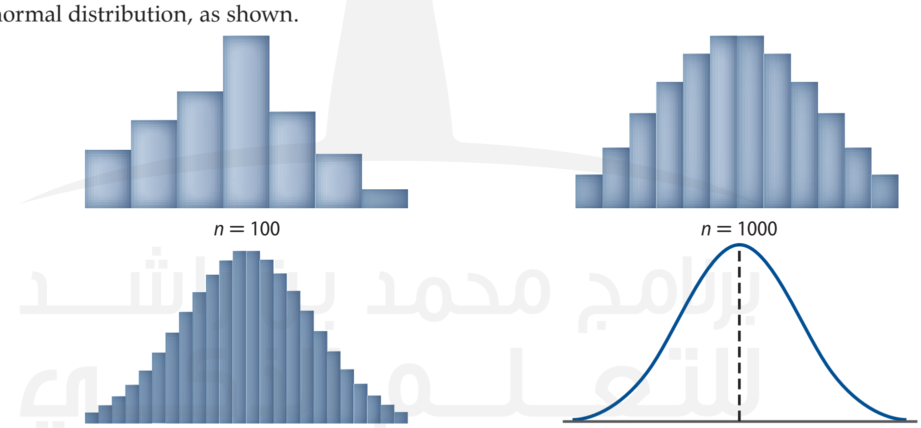
1 The Normal Distribution

The probability distribution for a continuous variable is called a *continuous probability distribution*. The most widely used continuous probability distribution is called the **normal distribution**. The characteristics of the normal distribution are as follows.

KeyConcept Characteristics of the Normal Distribution

- The graph of the curve is bell-shaped and symmetric with respect to the mean.
- The mean, median, and mode are equal and located at the center.
- The curve is continuous.
- The curve approaches, but never touches, the *x*-axis.
- The total area under the curve is equal to 1 or 100%.

Consider a continuous probability distribution of times for a 400-meter run in a random sample of 100 athletes. By increasing sample size and decreasing class width, the distribution becomes more and more symmetrical. If it were possible to sample the entire population, the distribution would approach the normal distribution, as shown.



For every normally distributed random variable, the shape and position of the normal distribution curve are dependent on the mean and standard deviation. For example, in Figure 10.3.1, you can see that a larger standard deviation results in a flatter curve. A change in the mean, as shown in Figure 10.3.2, results in a horizontal translation of the curve.

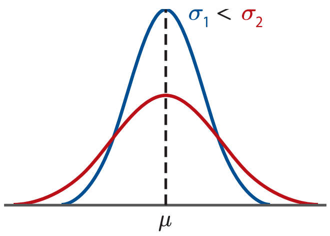


Figure 10.5.1

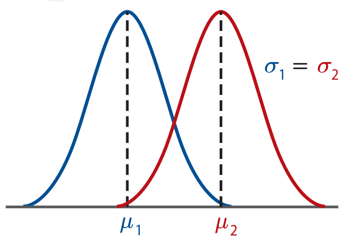


Figure 10.5.2

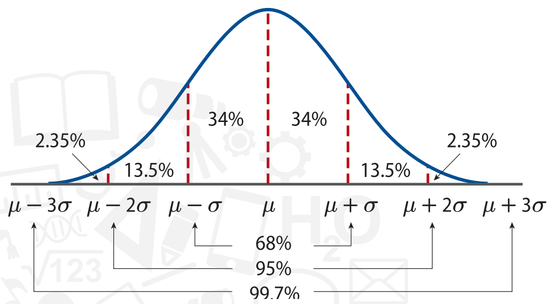
StudyTip

Empirical Rule The empirical rule is also known as the *68–95–99.7 rule.*

The area under the normal distribution curve between two data values represents the percent of data values that fall within that interval. The **empirical rule** can be used to describe areas under the normal curve over intervals that are one, two , or three standard deviations from the mean.

KeyConcept The Empirical Rule

In a normal distribution with mean μ and standard deviation σ:



- approximately 68% of the data values fall between $\mu - \sigma$ and $\mu + \sigma$.
- approximately 95% of the data values fall between $\mu - 2\sigma$ and $\mu + 2\sigma$.
- approximately 99.7% of the data values fall between $\mu - 3\sigma$ and $\mu + 3\sigma$.

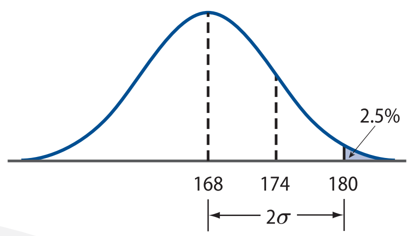
You can solve problems involving approximately normal distributions using the empirical rule.

Example 1 Use the Empirical Rule

HEIGHT The heights of the 880 students at Al-Sharq Secondary School are normally distributed with a mean of 168 cm and a standard deviation of 6 cm.

a. Approximately how many students are more than 180 cm tall?

To determine the number of students that are more than 180 cm tall, find the corresponding area under the curve.

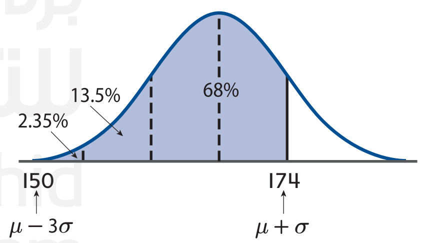


In the figure shown, you can see that 180 is 2σ from the mean. Because 95% of the data values fall within two standard deviations from the mean, each tail represents 2.5% of the data. The area to the right of 180 is 2.5% of 880 or 22.

Thus, about 22 students are more than 180 cm tall.

b. What percent of the students are between 150 and 174 cm tall?

The percent of students between 150 and 174 cm tall is represented by the shaded area in the figure at the right, which is between $\mu - 3\sigma$ and $\mu + \sigma$. The total area under the curve between 150 and 174 is equal to the sum of the areas of each region.



$2.35\% + 13.5\% + 68\% = 83.85\%$

Therefore, about 84% of the students are between 150 and 174 cm tall.

StudyTip

Everything Under the Curve Notice that in Example 1a, we used 2.5%, while in Example 1b, we used 2.35%. When you are asked for *greater than* or *less than*, you need to include everything under that side of the graph.

GuidedPractice

1. **MANUFACTURING** A machine used to fill water bottles dispenses slightly different amounts into each bottle. Suppose the volume of water in 120 bottles is normally distributed with a mean of 1.1 liters and a standard deviation of 0.02 liter.
   - **A.** Approximately how many bottles of water are filled with less than 1.06 liters?
   - **B.** What percent of the bottles have between 1.08 and 1.14 liters?

While the empirical rule can be used to analyze a normal distribution, it is only useful when evaluating specific values, such as $\mu + \sigma$. A normally distributed variable can be transformed into a standard value or z-value, which can be used to analyze any range of values in the normal distribution. This transformation is known as *standardizing*. The **z-value**, also known as the z-score and z test statistic, represents the number of standard deviations that a given data value is from the mean.

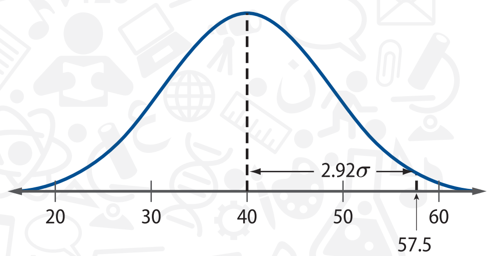
StudyTip

Positive and Negative z-Values If a data value is less than the mean, the corresponding z-value will be negative. Alternately, a data value that is greater than the mean will have a positive z-value.

KeyConcept Formula for z-Values

The z-value for a data value in a set of data is given by $z = \frac{X - \mu}{\sigma}$, where X is the data value, μ is the mean, and σ is the standard deviation.

You can use z-values to determine the position of *any* data value within a set of data. For example, consider a distribution with $\mu = 40$ and $\sigma = 6$. A data value of 57.5 is located about 2.92 standard deviations away from the mean, as shown. Therefore, in this distribution, $X = 57.5$ correlates to a z-value of 2.92.



StudyTip

Relative Position Like percentiles, z-values can be used to compare the relative positions of two values in two different sets of data.

Example 2 Find z-Values

Find each of the following.

a. z if $X = 24$, $\mu = 29$, and $\sigma = 4.2$

$z = \frac{X - \mu}{\sigma}$	Formula for z-values
$= \frac{24 - 29}{4.2}$	$X = 24$, $\mu = 29$, and $\sigma = 4.2$
≈ -1.19	Simplify.

The z-value that corresponds to $X = 24$ is -1.19. Therefore, 24 is 1.19 standard deviations less than the mean in the distribution.

b. X if $z = -1.73$, $\mu = 48$, and $\sigma = 2.3$

$z = \frac{X - \mu}{\sigma}$	Formula for z-values
$-1.73 = \frac{X - 48}{2.3}$	$\mu = 48$, $\sigma = 2.3$, and $z = -1.73$
$-3.979 = X - 48$	Multiply each side by 2.3.
$44.021 = X$	Add 48 to each side.

A z-value of -1.73 corresponds to a data value of approximately 44 in the distribution.

GuidedPractice

2A. z if $X = 32$, $\mu = 28$, and $\sigma = 1.7$

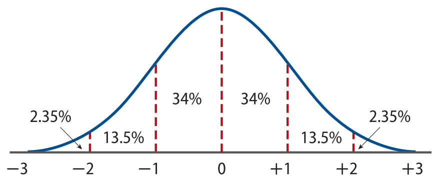
2B. X if $z = 2.15$, $\mu = 39$, and $\sigma = 0.4$

Every normally distributed random variable has a unique mean and standard deviation, which affect the position and shape of the curve. As a result, there are infinitely many normal probability distributions. Fortunately, they can all be related to one distribution known as the *standard* normal distribution. The **standard normal distribution** is a normal distribution of z-values with a mean of 0 and a standard deviation of 1.

The characteristics of the standard normal distribution are summarized below.

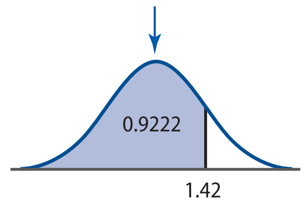
KeyConcept Characteristics of the Standard Normal Distribution

- The total area under the curve is equal to 1 or 100%.
- Almost all of the area is between $z = -3$ and $z = 3$.
- The distribution is symmetric.
- The mean is 0, and the standard deviation is 1.
- The curve approaches, but never touches, the x-axis.



You can solve normal distribution problems by finding the z-value that corresponds to a given X-value, and then finding the approximate area under the standard normal curve. The corresponding area can be found by using a table of z-values that shows the area *to the left* of a given z-value. For example, the area under the curve to the left of a z-value of 1.42 is 0.9222, as shown.

z	0.00	0.01	0.02
0.0	.5000	.5040	.5080
•	•	•	•
•	•	•	•
1.4	.9192	.9207	.9222



You can also find the area under the curve that corresponds to any z-value with a graphing calculator. This method will be used for the remainder of this chapter.

StudyTip
You can use the Normal distribution tables at the end of the book to determine the area of the region corresponding to a given z-value or to determine the z-value corresponding to a given area.

Using the Normal distribution table to determine the area corresponding to $z = 1.42$

- Locate the table of positive z-values.
- Locate in the first column the value 1.4 and in the first row the value 0.02.
- The area corresponding to the z-value of 1.42 is located at the intersection of the row and column, which is 0.9222.

Example 3 Use the Standard Normal Distribution

COMMUNICATION The average number of phone calls received by a customer service representative each day during a 30-day month was 105 with a standard deviation of 12. Find the number of days with fewer than 110 phone calls. Assume that the number of calls is normally distributed.

$$z = \frac{X - \mu}{\sigma} \quad \text{Formula for } z\text{-values}$$

$$= \frac{110 - 105}{12} \text{ or about } 0.42 \quad X = 110,\ \mu = 105, \text{ and } \sigma = 12$$

Although the standard normal distribution extends to negative and positive infinity, when you are finding the area less than or greater than a given value, you can use a lower value of −4 and an upper value of 4.

In this case, enter a lower z-value of −4 and an upper z-value of 0.42. The resulting area is 0.66. Since there were 30 days in the month, there were fewer than 110 calls on 30 • 0.66 or 19.8 days.

```
normalcdf(-4,0.4
2)
          .6627255515
```

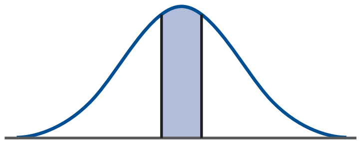
Therefore, there were approximately 20 days with fewer than 110 calls.

TechnologyTip
Area Under the Normal Curve You can use a graphing calculator to find the area under a standard normal curve that corresponds to any pair of z-values by selecting [2nd] [DISTR] and normalcdf (*lower z value, upper z value*).

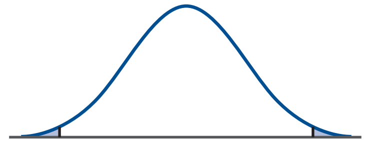
GuidedPractice

3. **BASKETBALL** The average number of points that a basketball team scored during a single season was 63 with a standard deviation of 18. If there were 15 games during the season, find the percentage of games in which the team scored more than 70 points. Assume that the number of points is normally distributed.

In Example 3, you found the area under the normal curve that corresponds to a *z*-value. You can also find *z*-values that correspond to specific areas. For example, you can find the *z*-value that corresponds to a cumulative area of 1%, 20%, or 99%. You can also find intervals of *z*-values that contain or are between a certain percentage of data.



middle 20%



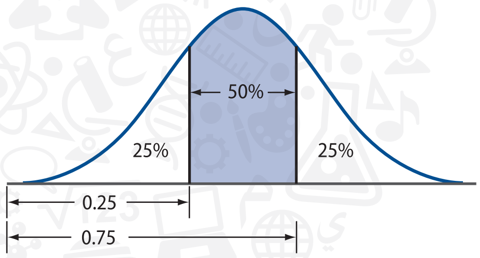
outside 1%

Example 4 Find *z*-Values Corresponding to a Given Area

Find the interval of *z*-values associated with each area.

a. middle 50% of the data

The middle 50% of the data corresponds to the data between 25% and 75% of the distribution, or 0.25 and 0.75, as shown.



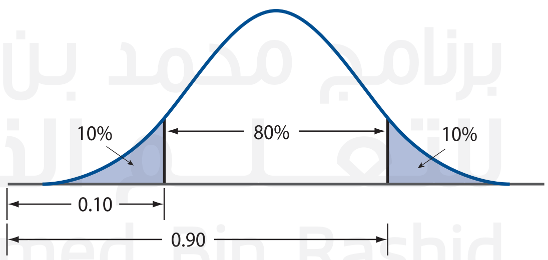
To find the *z*-scores that correspond to 0.25 and 0.75, select 2nd [DISTR] to display the DISTR menu on a graphing calculator. Select invNorm(and enter 0.25. Repeat to find the value corresponding to 0.75. As shown at the right, the *z*-value corresponding to 0.25 is -0.67 and the *z*-value corresponding to 0.75 is 0.67.

```
invNorm(0.25)
        -.6744897495
invNorm(0.75)
         .6744897495
```

Therefore, the interval that represents the middle 50% of the data is $-0.67 < z < 0.67$.

b. the outside 20% of the data

The outside 20% of the data represents the top 10% and the bottom 10% of the distribution or 0.1 and 0.9, as shown.



To find the *z*-value corresponding to 0.10, enter 0.10 into a graphing calculator under invNorm(and repeat for 0.90. As shown, the *z*-value corresponding to 0.10 is -1.28 and the *z*-value corresponding to 0.90 is 1.28.

```
invNorm(0.10)
        -1.281551567
invNorm(0.90)
         1.281551567
```

Therefore, the interval that represents the outside 20% of the data is $-1.28 > z$ or $z > 1.28$.

StudyTip

Symmetry The normal distribution is symmetrical, so when you are asked for the middle or outside set of data, the *z*-values will be opposites.

GuidedPractice

4A. the middle 25% of the data

4B. the outside 60% of the data

StudyTip

Percentage, Proportion, Probability, and Area When a problem asks for a percentage, proportion, or probability, it is asking for the same value—the corresponding area under the normal curve.

2 Probability and the Normal Distribution

You have seen how the area under the normal curve corresponds to the proportion of data values in an interval. The area also corresponds to the probability of data values falling within a given interval. If a z-value is chosen randomly, the probability of choosing a value between 0 and 1 would be equivalent to the area under the curve between 0 and 1.00, which is 0.3413. Therefore, the probability of choosing a value between 0 and 1 would be approximately 34%.

Example 5 Find Probabilities

METEOROLOGY The temperatures for one month for a city in California are normally distributed with $\mu = 81°$ and $\sigma = 6°$. Find each probability, and use a graphing calculator to sketch the corresponding area under the curve.

a. $P(70° < X < 90°)$

The question is asking for the percentage of temperatures that were between 70° and 90°. First, find the corresponding z-values for $X = 70$ and $X = 90$.

$z = \dfrac{X - \mu}{\sigma}$ Formula for z-values

$= \dfrac{70 - 81}{6}$ $X = 70$, $\mu = 81$, and $\sigma = 6$

≈ -1.83 Simplify.

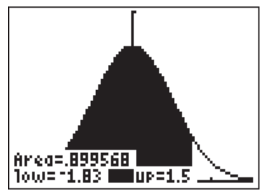
Use 90 to find the other z-value.

$z = \dfrac{X - \mu}{\sigma}$ Formula for z-values

$= \dfrac{90 - 81}{6}$ $X = 90$, $\mu = 81$, and $\sigma = 6$

≈ 1.5 Simplify.

You can use a graphing calculator to display the area that corresponds to any z-value by selecting [2nd] [DISTR]. Then, under the DRAW menu, select ShadeNorm (*lower z value, upper z value*). The area between $z = -1.83$ and $z = 1.5$ is 0.899568, as shown.



[−4, 4] scl: 1 by [0, 0.5] scl: 0.125

Therefore, approximately 90% of the temperatures were between 70 and 90.

StudyTip

Continuity Factors In a continuous distribution, there is no difference between $P(x \geq c)$ and $P(x > c)$ because the probability that x is equal to c is zero.

Using the standard Normal distribution tables to find the area between $z=-1.83$ and $z=1.5$

- In the table of positive z-values, we locate in the first column the value 1.5 and in the first row 0.00, then we locate the area corresponding to $z=1.5$ at the intersection of the row and column. The area is 0.9332.
- In the table of negative z-values, we locate in the first column the value -1.8 and in the first row 0.03, then we locate the area corresponding to $z=-1.83$ at the intersection of the row and column. The area is 0.0336.
- The needed area is the area to the left of the greater z-value minus the area to the left of the smaller z-value; that is,

$0.9332 - 0.0336 = 0.8996$

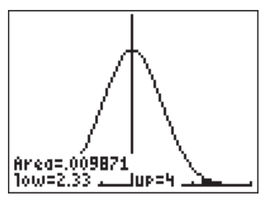
b. $P(X \geq 95°)$

$z = \dfrac{X - \mu}{\sigma}$ Formula for z-values

$= \dfrac{95 - 81}{6}$ $X = 95$, $\mu = 81$, and $\sigma = 6$

≈ 2.33 Simplify.

Using a graphing calculator, you can find the area between $z = 2.33$ and $z = 4$ to be 0.0099.



[−4, 4] scl: 1 by [0, 0.5] scl: 0.125

Therefore, the probability that a randomly selected temperature is at least 95° is about 0.1%.

GuidedPractice

5. **TESTING** The scores on a standardized test are normally distributed with $\mu = 72$ and $\sigma = 11$. Find each probability and use a graphing calculator to sketch the corresponding area under the curve.

A. $P(X < 89)$ **B.** $P(65 < X < 85)$

You can find specific intervals of data for given probabilities or percentages by using the standard normal distribution.

Real-WorldLink

In a recent year, the average national SAT scores were 502 in Critical Reading, 515 in Math, and 494 in Writing. The average national ACT score in that same year was 21.1.

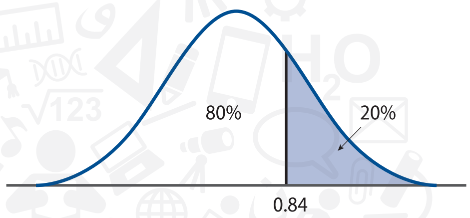
Source: *USA TODAY*

Real-World Example 6 Find Intervals of Data

COLLEGE The scores for the entrance exam for a college's mathematics department is normally distributed with $\mu = 65$ and $\sigma = 8$.

a. If Fatema wants to be in the top 20%, what score must she get?

To find the top 20% of the exam scores, you must find the exam score X that separates the upper 20% of the area under the normal curve, as shown. The top 20% correlates with $1 - 0.2$ or 0.8. Using a graphing calculator, you can find the corresponding z-value to be 0.84.



Now, use the formula for the z-value for a population to find the corresponding exam score.

$z = \dfrac{X - \mu}{\sigma}$ Formula for z-values

$0.84 = \dfrac{X - 65}{8}$ $\mu = 65$, $\sigma = 8$, and $z = 0.84$

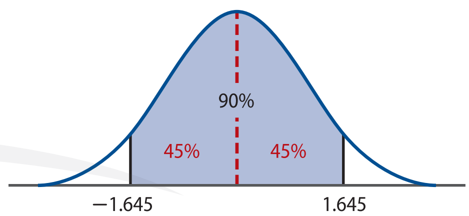
$6.72 = X - 65$ Multiply each side by 8.

$71.72 = X$ Add 65 to each side.

Fatema needs a score of at least 72 to be in the top 20%.

b. Fatema expects to earn a grade in the middle 90% of the distribution. What range of scores fall in this category?

The middle 90% of the exam scores represents 45% on each side of the mean and therefore corresponds to the interval of area from 0.05 to 0.95. Using a graphing calculator, the z-values that correspond to 0.05 and 0.95 are -1.645 and 1.645, respectively.



Use the z-values to find each value of X.

$z = \dfrac{X - \mu}{\sigma}$	Formula for z-values	$z = \dfrac{X - \mu}{\sigma}$
$-1.645 = \dfrac{X - 65}{8}$	$\mu = 65$ and $\sigma = 8$	$1.645 = \dfrac{X - 65}{8}$
$-13.16 = X - 65$	Multiply.	$13.16 = X - 65$
$51.84 = X$	Simplify.	$78.16 = X$

Therefore, Fatema expects to score between 52 and 78.

GuidedPractice

6. RESEARCH As part of a medical study, a researcher selects a study group with a mean weight of 86 kg and a standard deviation of 5.5 kg. Assume that the weights are normally distributed.

A. If the study will mainly focus on participants whose weights are in the middle 80% of the data set, what range of weights will this include?

B. If participants whose weights fall in the outside 5% of the distribution are contacted 2 weeks after the study, people in what weight range will be contacted?

Exercises

1. **NOISE POLLUTION** As part of a noise pollution study, researchers measured the sound level in decibels of a busy city street for 30 days. According to the study, the average noise was 82 decibels with a standard deviation of 6 decibels. Assume that the data are normally distributed. (Example 1)
   - **a.** If a normal conversation is held at about 64 decibels, determine the number of hours during the study that the noise level was this low.
   - **b.** Determine the percent of the study during which the noise was between 76 decibels and 88 decibels.

2. **GAS KILOMETRAGE** Khamis commutes 290 km each week for work. His car averages 29.6 km per liter with a standard deviation of 5.4 km per liter. Assume that the data are normally distributed. (Example 1)
   - **a.** Approximate the number of kilometers that Khamis's car gets a gas kilometrage of 35 km per liter or better.
   - **b.** For what percentage of Khamis's commute does his car have a gas kilometrage between 24.2 km per liter and 40.4 km per liter?

Find each of the following. (Example 2)

3. z if $X = 19$, $\mu = 22$, and $\sigma = 2.6$
4. X if $z = 2.3$, $\mu = 64$, and $\sigma = 1.3$
5. z if $X = 52$, $\mu = 43$, and $\sigma = 3.7$
6. X if $z = 2.5$, $\mu = 27$, and $\sigma = 0.4$
7. z if $X = 32$, $\mu = 38$, and $\sigma = 2.8$
8. X if $z = 1.7$, $\mu = 49$, and $\sigma = 4.1$

9. **ICHTHYOLOGY** As part of a science project, Mazen studied the growth rate of 797 green gold catfish and found the following information. Assume that the data are normally distributed. (Example 3)

   The green gold catfish reaches its maximum length within its first 3 months of life.

   - **Average length at birth 4.69 mm**
   - **Standard deviation 0.258 mm**

   - **a.** Determine the number of fish with a length less than 4.5 mm at birth.
   - **b.** Determine the number of fish with a length greater than 5 mm at birth.

10. **ROLLER COASTER** The average wait in line for the 16,000 daily passengers of a roller coaster is 72 minutes with a standard deviation of 15 minutes. Assume that the data are normally distributed. (Example 3)
    - **a.** Determine the number of passengers who wait less than 60 minutes to ride the roller coaster.
    - **b.** Determine the number of passengers who wait more than 90 minutes to ride the roller coaster.

Find the interval of z-values associated with each area. (Example 4).

11. middle 30%
12. outside 15%
13. outside 40%
14. middle 10%
15. outside 25%
16. middle 84%

17. **BATTERY** The life of a certain brand of AA battery is normally distributed with $\mu = 8$ hours and $\sigma = 1.5$ hours. Find each probability. (Example 5)
    - **a.** The battery will last less than 6 hours.
    - **b.** The battery will last more than 12 hours.
    - **c.** The battery will last between 8 and 9 hours.

18. **HEALTH** The average blood cholesterol level in adult Americans is 203 mg/dL (milligrams per deciliter) with a standard deviation of 38.8 mg/dL. Find each probability. Assume that the data are normally distributed. (Example 5)
    - **a.** a blood cholesterol level below 160 mg/dL, which is considered low and can lead to a higher risk of stroke
    - **b.** a blood cholesterol level above 240 mg/dL, which is considered high and can lead to higher risk of heart disease
    - **c.** a blood cholesterol level between 180 and 200 mg/dL, which is considered normal

19. **SNOWFALL** The average annual snowfall in centimeters for the U.S. and Canada region from 45°N to 55°N is normally distributed with $\mu = 260$ and $\sigma = 27$. (Example 6)
    - **a.** Determine the minimum amount of snowfall occurring in the top 15% of the distribution.
    - **b.** Determine the maximum amount of snowfall occurring in the bottom 30%.
    - **c.** What range of snowfall occurs in the middle 60%?

20. **TRAFFIC SPEED** The speed in kilometers per hour of traffic on North Street is normally distributed with $\mu = 60$ and $\sigma = 9$. (Example 6)
    - **a.** Determine the maximum speed of the slowest 10% of cars driving on North Street.
    - **b.** Determine the minimum speed of the fastest 5% of cars driving on North Street.
    - **c.** At what range of speed do the middle 25% of cars on North Street drive?

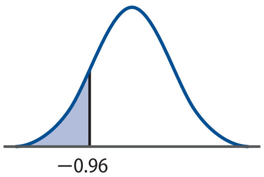
21. **TESTS** Saleh took the ACT and SAT and earned the math scores shown. Which of the scores has a higher relative position? Explain your reasoning.

Test	Saleh's Score	National Average	Standard Deviation
ACT	27	21	4.7
SAT	620	508	111

22. **EXAMS** Asma scored 76 on a physics test that had a mean of 72 and a standard deviation of 10. She also scored 81 on a sociology test that had a mean of 78 and a standard deviation of 9. Compare her relative scores on each test. Assume that the data are normally distributed.

Find the area that corresponds to each shaded region.

23.



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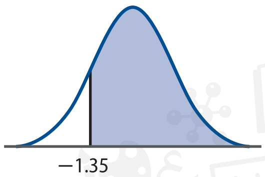

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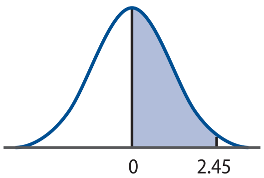

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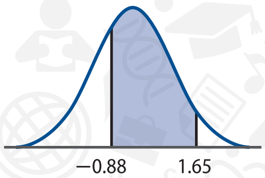


27. **FRACTILES** Quartiles, percentiles, and deciles are three types of fractiles, which divide an ordered set of data into equal groups. Find the *z*-values that correspond to each of the following fractiles.

a. D_{20}, D_{40}, and D_{80}

b. Q_1, Q_2, and Q_3

c. P_{10}, P_{40}, and P_{90}

28. **METEOROLOGY** The humidity observed in the morning during the same day in Chicago, Orlando, and Phoenix is shown. Assume that the data are normally distributed.

City	Humidity	Average Humidity	Standard Deviation
Chicago	85%	82%	12%
Orlando	94%	91%	15%
Phoenix	46%	43%	10%

a. Which city has the highest humidity? the lowest humidity? Explain your reasoning.

b. How would a fourth city compare that has a humidity of 81% and an average humidity of 8% with a standard deviation of 8%?

29. **JOBS** The salaries of employees in the sales department of an advertising agency are normally distributed with a standard deviation of AED 8000. During the holiday season, employees who earn less than AED 35,000 receive a gift basket.

a. Suppose 10% of the employees receive a gift basket. What was the mean salary of the sales department?

b. Suppose employees who make AED 10,000 greater than the mean salary receive an incentive bonus. If 200 employees work in the sales department, how many employees will receive a bonus?

30. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the shape of a normal distribution. Consider a population of 4, 10, 6, 8.

a. **GRAPHICAL** Construct a bar graph, and use it to describe the shape of the distribution. Then find the mean and standard deviation of the data set.

b. **GRAPHICAL** Select eight random samples of size 2, with replacement, from the data set. Construct a bar graph, and use it to describe the shape of the distribution. Find the mean and standard deviation of the sample means.

c. **TABULAR** The table includes all samples of size 2 that can be taken, with replacement, from the data set. Find the mean of each sample and the mean and standard deviation of all of the sample means.

Sample	Mean	Sample	Mean
4, 4		8, 4	
4, 6		8, 6	
4, 8		8, 8	
4, 10		8, 10	
6, 4		10, 4	
6, 6		10, 6	
6, 8		10, 8	
6, 10		10, 10	

d. **GRAPHICAL** Construct a bar graph of the sample means from part **c** and use it to describe the shape of the distribution. What happens to the shape of a distribution of data as the sample size increases?

e. **ANALYTICAL** Divide the standard deviation of the population that you found in part **a** by the square root of the sample size. What do you think happens to the mean and standard deviation of a distribution of data as the sample size increases?

H.O.T. Problems Use Higher-Order Thinking Skills

31. **ERROR ANALYSIS** Husam and Salem are finding the *z* interval associated with the outside 35% of a distribution of data. Husam thinks it is the interval $z < -0.39$ or $z > 0.39$, while Salem thinks it is the interval $z < -0.93$ or $z > 0.93$. Is either of them correct? Explain your reasoning.

32. **REASONING** In real-life applications, *z*-values usually fall between −3 and +3 in the standard normal distribution. Why do you think this is the case? Explain your reasoning.

33. **CHALLENGE** Find two *z*-values, one positive and one negative, so that the combined area of the two equivalent tails is equal to each of the following.

a. 1% **b.** 5% **c.** 10%

34. **REASONING** Continuous variables *sometimes*, *always*, or *never* have normal distributions. Explain your reasoning.

35. **WRITING IN MATH** Compare and contrast the characteristics of a normal distribution and the standard normal distribution.

Spiral Review

36. **BASEBALL** The number of hits by each Wildcats player during a doubleheader is shown in the frequency distribution.

a. Construct and graph a probability distribution for the random variable X.

b. Find and interpret the mean in the context of the situation.

c. Find the variance and standard deviation.

Hits, X	Frequency
0	3
1	1
2	8
3	2
4	3

37. **FOOTBALL** The number of penalties a professional football team received for each game during two recent seasons is shown. Construct side-by-side box plots of the data sets. Then use this display to compare the distributions.

Season 1				Season 2			
8	11	6	13	9	1	3	5
9	18	16	11	8	3	6	4
15	14	14	9	10	6	3	1
8	5	10	5	5	5	3	2

Find the sum of each arithmetic series.

38. S_{51} of $-92 + (-88) + (-84) + \ldots$

39. 24th partial sum of $-13 + 2 + 17 + \ldots$

40. S_{46} of $295 + 281 + 267 + \ldots$

Find rectangular coordinates for each point with the given polar coordinates.

41. $\left(\frac{1}{4}, \frac{\pi}{2}\right)$

42. $\left(3, \frac{\pi}{3}\right)$

43. $(-2, \pi)$

Given v and u • v, find u. There may be more than one answer.

44. $\mathbf{v} = \langle -4, 2, -7 \rangle$, $\mathbf{u} \cdot \mathbf{v} = 17$

45. $\mathbf{v} = \langle 2, 8, 5 \rangle$, $\mathbf{u} \cdot \mathbf{v} = -6$

46. $\mathbf{v} = \left\langle \frac{2}{3}, -3, \frac{1}{3} \right\rangle$, $\mathbf{u} \cdot \mathbf{v} = 10$

Find the direction angle of each vector.

47. $6\mathbf{i} + 3\mathbf{j}$

48. $-3\mathbf{i} + 4\mathbf{j}$

49. $2\mathbf{i} - 8\mathbf{j}$

Write an equation of an ellipse with each set of characteristics.

50. vertices $(-3, 11)$, $(-3, -9)$; foci $(-3, 7)$, $(-3, -5)$

51. co-vertices $(-1, -6)$, $(-3, -6)$; length of major axis equals 10

52. vertices $(-4, 2)$, $(8, 2)$; length of minor axis equals 8

Skills Review for Standardized Tests

53. **SAT/ACT** If X is the sum of the first 1000 positive even integers and Y is the sum of the first 500 positive odd integers, about what percent greater is X than Y?

A 100% **C** 300% **E** 500%

B 200% **D** 400%

54. **REVIEW** In a recent year, the mean and standard deviation for scores on the ACT was 21.0 and 4.7. Assume that the scores were normally distributed. What is the approximate probability that a test taker scored higher than 30.2?

F 1% **H** 2%

G 1.5% **J** 2.5%

55. The length of each song in a music collection is normally distributed with $\mu = 4.12$ minutes and $\sigma = 0.68$ minutes. Find the probability that a song selected from the collection at random is longer than 5 minutes.

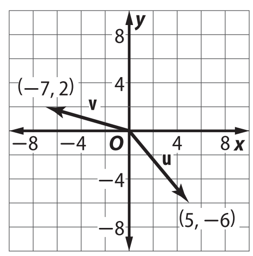
A 10% **C** 39%

B 19% **D** 89%

56. **REVIEW** Find $\mathbf{u} \cdot \mathbf{v}$.

F -47 **H** -6

G -24 **J** 47



EXTEND 10-5 Graphing Technology Lab
Transforming Skewed Data

Objective

- Use a graphing calculator to transform skewed data into data that resemble a normal distribution.

It is common for biological, medical, and other data to be positively skewed. It can sometimes be helpful to *transform* the original data so that it better resembles a normal distribution. This allows for the data to be spread out as opposed to being bunched at one end of a display.

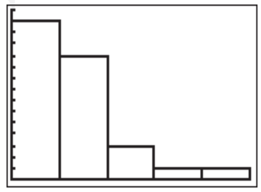
Activity Transform Data Using Natural Logarithms

Use the data to construct a histogram, and describe the shape of the distribution. Then transform the data by calculating the common logarithm of each entry. Graph the new data, and describe the shape of the distribution.

Data									
15	7	2	5	8	17	15	8	3	4
9	18	13	10	9	8	10	23	26	10
7	14	25	7	6	13	35	48	14	6

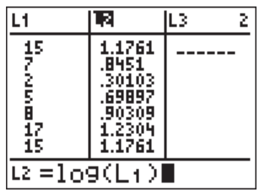
Step 1 Input the data into L1. Construct a histogram for the data using the intervals and scales shown.

The data appear to be positively skewed.



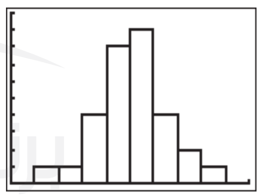
[0, 50] scl: 10 by [0, 15] scl: 1

Step 2 Input the common logarithm for each value into L2. Place the cursor on L2. Press LOG and enter L1. Press ENTER.



Step 3 Construct a histogram for the new data using the intervals and scales shown.

The data appear to have a normal distribution.



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Data may also be transformed by calculating the square roots or powers of the entries. When data are transformed, the type of operation performed should always be specified. A transformation will not always result in the new data being normally distributed.

Exercise

Use the data to construct a histogram, and describe the shape of the distribution. Then transform the data by calculating the square root of each entry. Graph the new data, and describe the shape of the distribution. Explain how the transformation affected the summary statistics.

Data									
23	30	36	39	36	24	31	33	42	36
26	32	46	45	27	34	52	41	28	33
43	20	24	34	30	40	29	35	61	35

LESSON 10-6 The Central Limit Theorem

Then

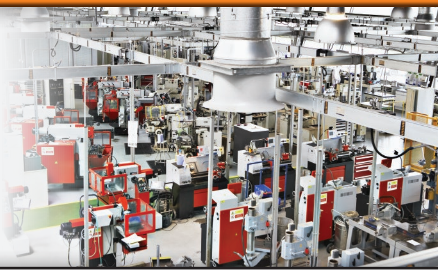
- You used the normal distribution to find probabilities for intervals of data values in distributions.

Now

1. Use the Central Limit Theorem to find probabilities.
2. Find normal approximations of binomial distributions

Why?

- In manufacturing processes, quality control systems are used to determine when a process is outside of upper and lower control limits or “out of control.” The mean of the process is controlled; therefore, successive sample means should be normally distributed around the actual mean.



NewVocabulary

sampling distribution
standard error of the mean
sampling error
continuity correction factor

1 The Central Limit Theorem

Sampling is an important statistical tool in which subgroups of a population are selected so that inferences can be made about the entire population. The means of these subgroups, or sample means, can be compared to the mean of the population by using a sampling distribution. A **sampling distribution** is a distribution of the means of random samples of a certain size that are taken from a population.

Consider a population consisting of 16, 18, 20, 20, 22, and 24, with $\mu = 20$ and $\sigma = 2.582$. Suppose 12 random samples of size 2 are taken, with replacement. The mean $\bar{x}$ of each sample is shown.

Sample	$\bar{x}$	Sample	$\bar{x}$	Sample	$\bar{x}$
20,22	21	20,18	19	22,22	22
22,18	20	16,22	19	18,18	18
20,24	22	24,16	20	20,16	18
20,20	20	20,24	22	24,22	23

The distribution of the means of the 12 random samples, shown in Figure 10.4.1, does not appear to be normal. However, if all 36 samples of size of 2 from the population are found, the distribution of sample means will approach the normal distribution, as shown in Figure 10.4.2.

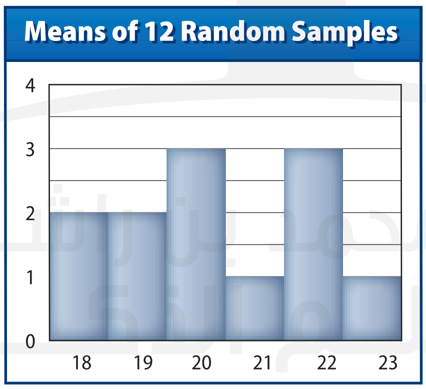


Figure 10.6.1

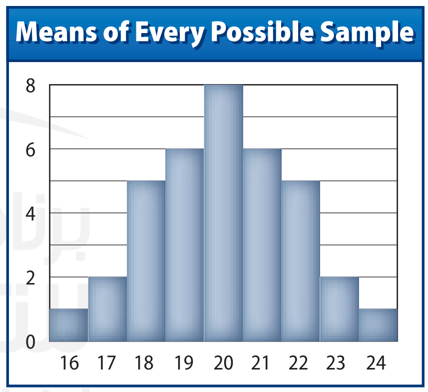


Figure 10.6.2

The mean of the means of every possible sample of size 2 from the population is

$$\mu_{\bar{x}} = \frac{16 + 17 + \ldots + 24}{36} = \frac{720}{36} \text{ or } 20.$$

Notice that this value is equal to the population mean $\mu = 20$. So, when the mean of the means of every possible sample of size 2 are found, $\mu_{\bar{x}} = \mu$. The standard deviation of the sample means $\sigma_{\bar{x}}$ and the standard deviation of the population σ when divided by the square root of the sample of size n are

$$\sigma_{\bar{x}} = \frac{\sqrt{(16-20)^2 + (17-20)^2 + \ldots + (24-20)^2}}{36} \approx 1.826 \qquad \text{and} \qquad \frac{\sigma}{\sqrt{n}} = \frac{2.582}{\sqrt{2}} \approx 1.826.$$

Since these two values are equal, the standard deviation of the sample means, also known as the **standard error of the mean**, can be found by using the formula $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$.

In general, randomly selected samples will have sample means that differ from the population mean. These differences are caused by **sampling error**, which occurs because the sample is not a complete representation of the population. However, if *all* possible samples of size n are taken from a population with mean μ and a standard deviation σ, the distribution of sample means will have:

- a mean $\mu_{\bar{x}}$ that is equal to μ and
- a standard deviation $\sigma_{\bar{x}}$ that is equal to $\frac{\sigma}{\sqrt{n}}$.

When the sample size n is large, regardless of the shape of the original distribution, the Central Limit Theorem states that the shape of the distribution of the sample means will approach a normal distribution.

StudyTip

Normally Distributed Variables If the original variable is not normally distributed, then n must be greater than 30 in order to use the standard normal distribution to approximate a distribution of sample means.

KeyConcept Central Limit Theorem

As the sampling size n increases:

- the shape of the distribution of the sample means of a population with mean μ and standard deviation σ will approach a normal distribution and
- the distribution will have a mean μ and standard deviation $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$.

The Central Limit Theorem can be used to answer questions about sample means in the same way that the normal distribution was used to answer questions about individual values. In this case, we can use a formula for the z-value of a sample mean.

KeyConcept *z*-Value of a Sample Mean

The z-value for a sample mean in a population is given by $z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$, where $\bar{x}$ is the sample mean, μ is the mean of the population, and $\sigma_{\bar{x}}$ is the standard error.

Example 1 Use the Central Limit Theorem

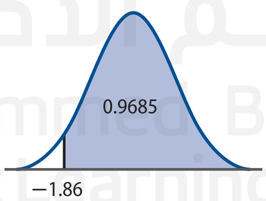
AGE According to a recent study, the average age that an American adult leaves home is 26 years old. Assume that this variable is normally distributed with a standard deviation of 2.4 years. If a random sample of 20 adults is selected, find the probability that the mean age the participants left home is greater than 25 years old.

Since the variable is normally distributed, the distribution of the sample means will be approximately normal with $\mu = 26$ and $\sigma_{\bar{x}} = \frac{2.4}{\sqrt{20}}$ or about 0.537. Find the z-value.

$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$ — *z*-value for a sample mean

$= \frac{25 - 26}{0.537}$ — $\bar{x} = 25, \mu = 26$, and $\sigma_{\bar{x}} = 0.537$

≈ -1.86 — Simplify.




```
normalcdf(-1.86,
4)
        .9685256139
```

The area to the right of a z-value of -1.86 is 0.9685. Therefore, the probability that the mean age of the sample is greater than 25 or $P(\bar{x} > 25)$ is about 96.85%.

GuidedPractice

1. **TORNADOES** The average number of tornadoes in Kansas is 47 per year, with a standard deviation of approximately 14.2 tornadoes. If a random sample of 15 years is selected, find the probability that the mean number of tornadoes is less than 50.

Real-WorldLink

In 1994, a nonprofit organization called The Rechargeable Battery Recycling Corporation was formed to promote the recycling of rechargeable batteries in North America. It provides information for over 50,000 collection locations nationwide where rechargeable batteries can be recycled for free.

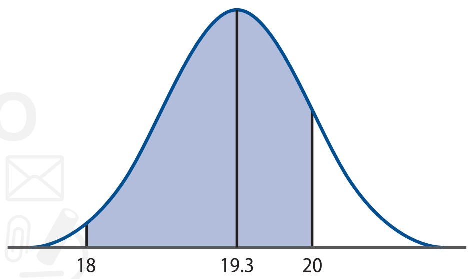
Source: Battery University

You can also determine the probability that a sample mean will fall within a given interval of the sampling distribution.

Real-World Example 2 Find the Area Between Two Sample Means

BATTERY LIFE A company that produces rechargeable batteries is designing a battery that will need to be recharged after an average of 19.3 hours of use. Assume that the distribution is normal with a standard deviation of 2.4 hours. If a random sample of 20 batteries is selected, find the probability that the mean life of the batteries before recharging is between 18 and 20 hours.

The area that corresponds to an interval of 18 to 20 hours is shown at the right.



First, find the standard deviation of the sample means.

$\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$ Standard deviation of a sample mean

$= \dfrac{2.4}{\sqrt{20}}$ $\sigma = 2.4$ and $n = 20$

≈ 0.536 Simplify.

Use the z-value formula for a sample mean to find the corresponding z-values for 18 and 20.

z-value for $\bar{x} = 18$:

$z = \dfrac{\bar{x} - \mu}{\sigma_{\bar{x}}}$ z-value formula for a sample mean

$= \dfrac{18 - 19.3}{0.536}$ $\bar{x} = 18$, $\mu = 19.3$, and $\sigma_{\bar{x}} = 0.536$

≈ -2.42 Simplify.

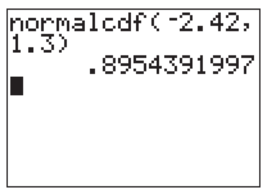
z-value for $\bar{x} = 20$:

$z = \dfrac{\bar{x} - \mu}{\sigma_{\bar{x}}}$ z-value formula for a sample mean

$= \dfrac{20 - 19.3}{0.536}$ $\bar{x} = 20$, $\mu = 19.3$, and $\sigma_{\bar{x}} = 0.536$

≈ 1.30 Simplify.

Using a graphing calculator, select normalcdf(to find the area between $z = -2.42$ and $z = 1.30$.



The area between z-values of -2.42 and 1.30 is 0.8954. Therefore, $P(18 < \mu < 20)$ is 89.54%. So, the probability that the mean life of the batteries is between 18 and 20 hours is 89.54%.

GuidedPractice

2. **DAIRY** The average cost of a liter of milk in a U.S. city is AED 3.49 with a standard deviation of AED 0.24. If a random sample of 40 1-liter containers of milk is selected, find the probability that the mean of the sample will be between AED 3.40 and AED 3.60.

Example 3 Analyze Individual Values and Sample Means

CLASS SIZE **According to a recent study, the average class size in high schools nationwide is 24.7 students per class. Assume that the distribution is normal with a standard deviation of 3.6 students.**

a. Find the probability that a randomly selected class will have fewer than 23 students.

The question is asking for an individual value in which $P(x < 23)$. Use the z-value formula for an individual data value to find the corresponding z-value.

$$z = \frac{X - \mu}{\sigma} \qquad \text{z-value formula for an individual value}$$

$$= \frac{23 - 24.7}{3.6} \text{ or about } -0.47 \qquad X = 23,\ \mu = 24.7, \text{ and } \sigma = 3.6$$

The area associated with $z < -0.47$, or $P(z < -0.47)$, is 0.3192. Therefore, the probability that a randomly selected class has fewer than 23 students is 31.9%.

b. If a sample of 15 classes is selected, find the probability that the mean of the sample will be fewer than 23 students per class.

This question deals with a sample mean, so use the z-value formula for a sample mean to find the corresponding z-value. First, find the standard error of the mean.

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \qquad \text{Standard error of the mean}$$

$$= \frac{3.6}{\sqrt{15}} \text{ or about } 0.93 \qquad \sigma = 3.6 \text{ and } n = 15$$

Next, find the z-value using the z-value formula for a sample mean.

$$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}} \qquad \text{z-value formula for a sample mean}$$

$$= \frac{23 - 24.7}{0.93} \text{ or about } -1.83 \qquad \bar{x} = 23,\ \mu = 24.7, \text{ and } \sigma_{\bar{x}} = 0.93$$

The area associated with $z < -1.83$, or $P(z < -1.83)$, is 0.0336. Therefore, the probability that a sample of 15 classes will have a mean class size of fewer than 23 students is 3.36%.

StudyTip

***z*-Value Formulas** Notice that the difference between the z-value formula for an individual data value and the z-value formula for a sample mean is that $\bar{x}$ is substituted for X and $\sigma_{\bar{x}}$ is substituted for σ in the formula for an individual value.

GuidedPractice

3. **APPLES** Consumers in the U.S. eat an average of 19 kg of apples per year. Assume that the standard deviation is 4 kg and the distribution is approximately normal.

A. Find the probability that a randomly selected person consumes more than 21 kg of apples per year.

B. If a sample of 30 people is selected, find the probability that the mean of the sample would be more than 21 kg of apples per year.

Notice in Figure 10.4.3 that the probability that an individual class has fewer than 23 students is much greater than the probability associated with the mean of a sample being fewer than 23 shown in Figure 10.4.4. This means that as the sample size increases, the distribution becomes narrower and the variability decreases.

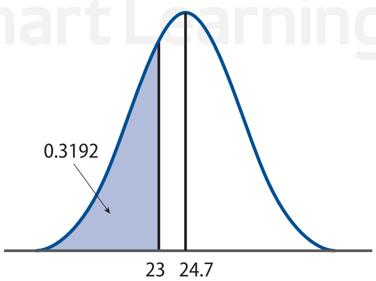


Figure 10.6.3

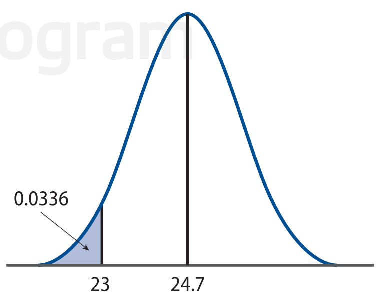
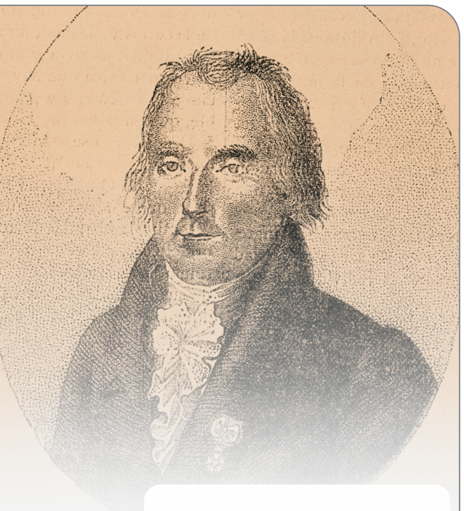


Figure 10.6.4



Math HistoryLink

Pierre-Simon Laplace
(1749–1827)
A French mathematician and astronomer, Pierre-Simon Laplace was born in Beaumont-en Auge, France. Laplace first approximated the binomial distribution with the normal distribution in his 1812 work *Théorie Analytique des Probabilités*.

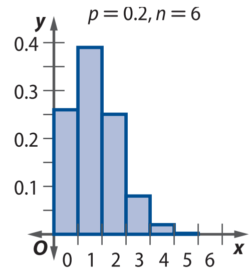
2 The Normal Approximation

According to the Central Limit Theorem, any sampling distribution can approach the normal distribution as n increases. As a result, other distributions such as the binomial distribution can be approximated with the normal distribution. The binomial distribution can be determined by using the equation

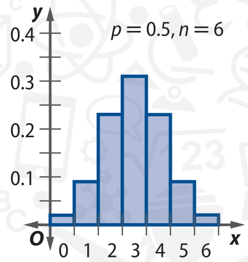
$$P(X) = {}_nC_x\, p^x q^{n-x},$$

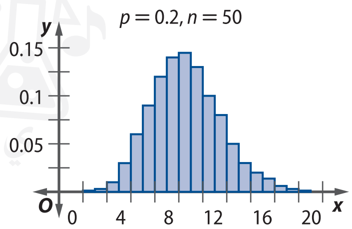
where n is the number of trials, p is the probability of success, and q is the probability of failure.

If the number of trials increases or the probability of success gets close to 0.5, the shape of the binomial distribution begins to resemble the normal distribution. For example, consider the binomial distribution at the right. When $p = 0.2$ and $n = 6$, the distribution is positively skewed.



However, when $p = 0.5$ and $n = 6$ or when $p = 0.2$ and $n = 50$, as shown below, the distribution is approximately normal.





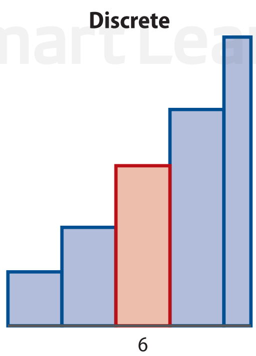
When the probability of success is close to 0 or 1 and the number of trials is relatively small, the normal approximation is not accurate. Therefore, as a rule, the normal approximation is typically used only when $np \geq 5$ and $nq \geq 5$.

KeyConcept Approximation Rule for Binomial Distributions

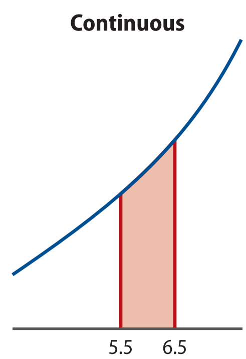
Words	The normal distribution can be used to approximate a binomial distribution when $np \geq 5$ and $nq \geq 5$.
Example	If p is 0.4 and n is 5, then $np = 5(0.4)$ or 2. Since $2 < 5$, the normal distribution should not be used to approximate the binomial distribution.

It also is important to remember that the normal distribution should only be used to approximate a binomial distribution if the original variable is normally distributed or $n \geq 30$.

Since binomial distributions are *discrete* and normal distributions are *continuous*, a correction for continuity called the **continuity correction factor** must be used when approximating a binomial distribution. To use the correction factor, 0.5 unit is added to or subtracted from a given discrete boundary. For example, to find $P(X = 6)$ in a discrete distribution, the correction would be to find $P(5.5 < X < 6.5)$ for a continuous distribution, as shown below.



$\longrightarrow P(6 - 0.5 < X < 6 + 0.5) \longrightarrow$



Use the following steps to approximate a binomial distribution with the normal distribution.

StudyTip

Binomial Formulas The mean μ and standard deviation σ of a binomial distribution are found using $\mu = np$ and $\sigma = \sqrt{npq}$, respectively.

KeyConcept Normal Approximation of a Binomial Distribution

The procedure for the normal approximation of a binomial distribution is as follows.

Step 1 Find the mean μ and standard deviation σ.

Step 2 Write the problem in probability notation using X.

Step 3 Find the continuity correction factor, and rewrite the problem to show the corresponding area under the normal distribution.

Step 4 Find any corresponding z-values for X.

Step 5 Use a graphing calculator to find the corresponding area.

Example 4 Normal Approximation of a Binomial Distribution

COLLEGE **A school newspaper reported that 20% of the current senior class would be attending an out-of-state college. If 35 seniors are selected at random, find the probability that fewer than 5 of the seniors will be attending an out-of-state college.**

In this binomial experiment, $n = 35$, $p = 0.2$, and $q = 0.8$.

Step 1 Find the mean μ and standard deviation σ.

	Mean and standard deviation of a binomial distribution	
$\mu = np$	Mean and standard deviation of a binomial distribution	$\sigma = \sqrt{npq}$
$= 35 \cdot 0.2$	$n = 35$, $p = 0.2$, and $q = 0.8$	$= \sqrt{35 \cdot 0.2 \cdot 0.8}$
$= 7$	Simplify.	≈ 2.37

Since $np = 35(0.2)$ or 7 and $nq = 35(0.8)$ or 28, which are both greater than 5, the normal distribution can be used to approximate the binomial distribution.

Step 2 Write the problem in probability notation using X.

The probability that fewer than 5 of the seniors will be attending an out-of-state college is $P(X < 5)$.

Step 3 Rewrite the problem with the continuity factor included.

Since the question is asking for the probability that *fewer than* 5 will be attending, subtract 0.5 unit from 5.

$P(X < 5) = P(X < 5 - 0.5)$ or $P(X < 4.5)$

WatchOut!

z-Value Formula When approximating the binomial distribution using the normal distribution, remember to use the z-value formula for an individual data value, not the formula for a sample mean.

Step 4 Find the corresponding z-value for X.

$$z = \frac{X - \mu}{\sigma} \quad z\text{-value formula}$$

$$= \frac{4.5 - 7}{2.37} \quad X = 4.5,\ \mu = 7, \text{ and } \sigma = 2.37$$

$$\approx -1.05 \quad \text{Simplify.}$$

Step 5 Use a graphing calculator to find the area to the left of z.

The approximate area to the left of $z = -1.05$ is 0.147, as shown at the right. Therefore, the probability that fewer than 5 seniors will be attending an out-of-state college in a random sample of 35 seniors is about 14.7%.

```
normalcdf(-4,-1.
05)
        .1468273946
```

GuidedPractice

4. **ADVERTISING** According to the results of an advertising survey sent to customers selected at random, 65% of the customers had not seen a recent television advertisement. Find the probability that from a sample of 50 customer responses, 15 or more did not see the advertisement.

Real-WorldLink

Product recalls occur when a manufacturer sends out a request to the consumers to return a product after discovering a safety issue. Recalls are costly, but are done to limit the liability of the manufacturer.

Source: National Highway Traffic Safety Administration

Real-World Example 5 Normal Approximation of a Binomial Distribution

MANUFACTURING An automaker has discovered a defect in a new model. The defect is expected to affect 30% of the cars that were produced. What is the probability that there are at least 10 and at most 15 cars with the defect in a random sample of 40 cars?

In this binomial experiment, $n = 40$, $p = 0.3$, and $q = 0.7$.

Step 1 Begin by finding the mean μ and standard deviation σ.

$\mu = np$	Mean and standard deviation of a binomial distribution	$\sigma = \sqrt{npq}$
$= 40 \cdot 0.3$	$n = 40, p = 0.3$, and $q = 0.7$	$= \sqrt{40 \cdot 0.3 \cdot 0.7}$
$= 12$	Simplify.	≈ 2.9

Since $np = 40(0.3)$ or 12 and $nq = 40(0.7)$ or 28, which are both greater than 5, the normal distribution can be used to approximate the binomial distribution.

Step 2 Write the problem in probability notation: $P(10 \leq X \leq 15)$.

Step 3 Rewrite the problem with the continuity factor included.

$P(10 \leq X \leq 15) = P(10 - 0.5 < X < 15 + 0.5)$ or $P(9.5 \leq X \leq 15.5)$

Step 4 Find the corresponding z-values for $X = 9.5$ and $X = 15.5$.

$z = \dfrac{X - \mu}{\sigma}$	z-value formula	$z = \dfrac{X - \mu}{\sigma}$
$= \dfrac{9.5 - 12}{2.9}$	Substitute	$= \dfrac{15.5 - 12}{2.9}$
≈ -0.86	Simplify.	≈ 1.21

Step 5 Use a graphing calculator to find the area between $z = -0.86$ and $z = 1.21$.

```
normalcdf(-0.86,
1.21)
         .6919660179
```

The approximate area that corresponds to $-0.86 < z < 1.21$ is 0.692, as shown at the right. Therefore, the probability of there being at least 10 and at most 15 cars with the defect in a random sample of 40 cars is about 69.2%.

GuidedPractice

5. **MANUFACTURING** Suppose a defect in a second model by the same automaker is expected to affect 20% of the cars that were produced. What is the probability that there are at least 8 and at most 10 defects in a random sample of 30 cars?

It may seem difficult to know whether to add or subtract 0.5 unit from a discrete data value to find the continuity correction factor. The table below shows each case.

WatchOut!

Writing Inequalities When a problem is asking for a probability *between* two values, write the inequality as $P(c_1 < X < c_2)$, not $P(c_1 \leq X \leq c_2)$. For instance, in Example 5, the probability that there are *between* 10 and 15 defects would be $P(10 < X < 15)$.

ConceptSummary Binomial Distribution Correction Factors

When using the normal distribution to approximate a binomial distribution, the following correction factors should be used, where c is a given data value in the binomial distribution.

Binomial	Normal
$P(X = c)$	$P(c - 0.5 < X < c + 0.5)$
$P(X > c)$	$P(X > c + 0.5)$
$P(X \geq c)$	$P(X > c - 0.5)$
$P(X < c)$	$P(X < c - 0.5)$
$P(X \leq c)$	$P(X < c + 0.5)$

Exercises

1. **VIDEO GAMES** The average prices for three video games at an online auction site are shown. Assume that the variable is normally distributed. (Examples 1 and 2)

Game	Average Price (AED)
Column Craze	35
Dungeon Attack!	45
Space Race	52

   a. For a sample of 35 online prices for Column Craze, find the probability that the mean price is more than AED 38, if the standard deviation is AED 9.

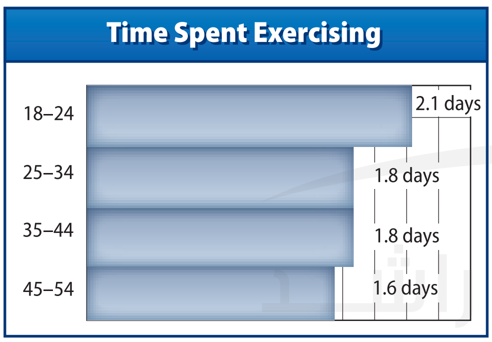
   b. For a random sample of 40 online prices for Space Race, find the probability that the mean price will be between AED 50 and AED 55 if the standard deviation is AED 12.

2. **CHEWING GUM** Americans chew an average of 182 sticks of gum per year. Assume a standard deviation of 13 sticks for each question. Assume that the variable is normally distributed. (Examples 1 and 2)

   a. Find the probability that 50 randomly selected people chew an average of 175 sticks or more per year.

   b. If a random sample of 45 people is selected, find the probability that the mean number of sticks of gum they chew per year is between 180 and 185.

3. **EXERCISE** The average number of days per week that Americans from four different age groups spent exercising during a recent year is shown. Assume that the variable is normally distributed. (Examples 1 and 2)



   a. Find the probability that a random sample of 30 Americans ages 45 to 54 spent more than 1.5 days a week exercising, if the standard deviation is 0.5 day.

   b. Assuming a standard deviation of 1.2 days, in a random sample of 30 Americans ages 18 to 24, find the probability that the average time spent exercising is between 2 to 2.5 days per week.

4. **MEDICINE** The mean recovery time for patients with a certain virus is 4.5 days with a standard deviation of 2 days. Assume that the variable is normally distributed. (Examples 1 and 2)

   a. Find the probability of an average recovery time of less than 4 days for a random sample of 75 people.

   b. In a random sample of 80 people, find the probability that average recovery time is between 4.4 and 4.8 days.

5. **TOURISM** The average number of tourists that visit a national monument every month is 55,000, with a standard deviation of 8000. Assume that the variable is normally distributed. (Example 3)

   a. If a random month is selected, find the probability that there would be fewer than 50,000 visiting tourists.

   b. If a sample of 10 months is selected, find the probability that there would be fewer than 50,000 visiting tourists.

6. **NUTRITION** The average protein content of a certain brand of energy bar is 12 grams with a standard deviation of 2 grams. Assume that the variable is normally distributed. (Example 3)

   a. Find the probability that a randomly selected bar will have more than 10 grams of protein.

   b. In a sample of 15 bars, find the probability that the average protein content will be greater than 10 grams.

7. **WORLD CUP** In a recent year, 33% of Americans said that they were planning to watch the World Cup soccer tournament. What is the probability that in a random sample of 45 people, fewer than 14 people plan to watch the World Cup? Assume that the variable is normally distributed. (Example 4)

8. **MOVIES** According to a national poll, in a recent year, 27% of Americans saw 5 or more movies in theaters. What is the probability that in a random sample of 40 people, between 6 and 11 people saw more than 5 movies in a movie theater that year? Assume that the variable is normally distributed. (Example 5)

9. **LIBRARY** A poll was conducted at a library to approximate the percent of books, CDs, magazines, and movies that were checked out during one month. The results are shown. Assume that the variable is normally distributed. (Examples 4 and 5)

Resources	Percent
books	45
CDs	20
magazines	3
movies	32

   a. What is the probability that of 65 randomly selected resources, exactly 35 were books?

   b. Find the probability that of 85 randomly selected resources, at least 15 and at most 18 were CDs.

10. **DRIVING** A driving instructor has found that 12% of students cancel or forget about lessons. Assume that the variable is normally distributed. (Examples 4 and 5)

   a. If the instructor has 60 students, what is the probability that more than 10 of the students will miss a lesson?

   b. What is the probability that of 80 students, exactly 7 students will miss a lesson?

11. **TESTS** A multiple-choice test consists of 50 questions, with possible answers A, B, C, and D. Find the probability that, with random guessing, the number of correct answers will be each of the following.

a. less than 18

b. exactly 12

c. at least 14

d. between 10 and 15

Find the minimum sample size needed for each probability so that the normal distribution can be used to approximate the binomial distribution.

12. $p = 0.1$

13. $p = 0.4$

14. $p = 0.5$

15. $p = 0.8$

16. **BASKETBALL** The average points per game scored by four different basketball players are shown.

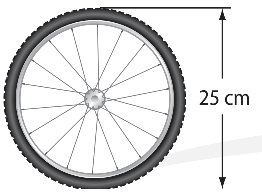
Player	A	B	C	D
Average	8.1	6.3	4.9	10.3

a. Find the mean and standard deviation of the averages.

b. Identify each possible combination of 3 players' averages, and find the mean of each combination.

c. Find the mean of each of the means that you found in part **b.** How does this compare to the mean that you found in part **a**?

17. **BICYCLES** Consider the bicycle rim shown, where $\mu = 25$ cm and $\sigma = 0.125$ cm.



The diameters for 10 random samples of 3 bicycle rims from a company's assembly line are shown.

Sample	Diameter	Sample	Diameter
1	25.2, 24.9, 25	6	24.9, 25.1, 24.8
2	25.1, 25, 24.8	7	25.3, 24.9, 25.1
3	25.3, 24.9, 24.8	8	25.4, 24.8, 25.3
4	24.9, 25.3, 25.2	9	24.8, 24.9, 25.2
5	25, 25.2, 24.7	10	25, 25.3, 24.7

a. Find $\bar{x}$ and s for each sample.

b. Construct a scatter plot with the sample number on the x-axis and the sample means on the y-axis.

c. In this process, the upper control limit is $\bar{x} + \frac{3\sigma}{\sqrt{n}}$ and the lower control limit is $\bar{x} - \frac{3\sigma}{\sqrt{n}}$. If the process is in control, all values should fall within the control limits. Use the graph from part **b** to determine whether the process is in control. Explain your reasoning.

18. **BLOOD TYPES** The distributions of blood types of U.S. and Canadian citizens are shown.

U.S.		Canada	
Type	Distribution	Type	Distribution
O	44%	O	46%
A	42%	A	42%
B	10%	B	9%
AB	4%	AB	3%

a. If 50 U.S. citizens are selected at random, find the probability that fewer than 20 of those chosen will have type O blood.

b. Find the probability that out of 100 randomly selected Canadian citizens, between 80 and 90 of those chosen will have types O or A blood.

c. What is the probability that two randomly chosen people from the U.S. or Canada will have the same blood type?

H.O.T. Problems Use Higher-Order Thinking Skills

19. **ERROR ANALYSIS** Halima and Hana are calculating results for a survey that they are taking as part of a summer internship. They found that of the residents they surveyed, 65% do not recycle. Halima found the probability that fewer than 30 out of 50 random residents do not recycle is 18.7%, while Hana found that it would be 27.7%. Is either of them correct? Explain your reasoning.

20. **WRITING IN MATH** Explain how the Central Limit Theorem can be used to describe the shape, center, and spread of a distribution of sample means.

21. **CHALLENGE** In the United States, 7% of the male population and 0.4% of the female population are color-blind. Suppose random samples of 100 men and 1500 women are selected. Is there a greater probability that the men's or women's sample will include at least 10 people who are color-blind? Explain your reasoning.

22. **OPEN ENDED** Give an example of a population and a sample of the population. Explain what is meant by the corresponding sampling distribution.

REASONING **Determine whether each statement is *true* or *false*. Explain your reasoning.**

23. As the number of samples increases, a sampling distribution of sample means will approach the normal distribution.

24. In a binomial distribution, $P(X \geq c) \neq P(X > c)$.

25. **WRITING IN MATH** Explain why the normal distribution can be used to approximate a binomial distribution, what conditions are necessary to do so, and why a correction for continuity is needed.

Spiral Review

26. **COMMUNITY SERVICE** A recent study of 1286 high school seniors revealed that the students completed an average of 38 hours of volunteer work over the summer with a standard deviation of 6.7 hours. Determine the number of seniors who completed more than 42 hours of community service. Assume that the variable is normally distributed.

27. **GAMES** Managers of a fitness club randomly surveyed 56 members and recorded the number of days that each member attended the club in a given week. Use the frequency distribution shown to construct a probability distribution for the random variable X. Then find the mean, variation, and standard deviation of the probability distribution.

Days, X	Frequency	Days, X	Frequency
0	3	4	11
1	5	5	9
2	10	6	3
3	14	7	1

Find each sum.

28. $\sum_{n=1}^{19} -50 + 5n$

29. $\sum_{n=12}^{68} 5 - \frac{n}{4}$

30. $\sum_{n=10}^{16} 24n - 90$

Find the specified term of each sequence.

31. 7th term, $a_n = (a_{n-1} - 6)^2$, $a_1 = 4$

32. 6th term, $a_n = 3n^2 - 4n$

33. 4th term, $a_n = (a_{n-1})^2 - 11$, $a_1 = 3$

Find rectangular coordinates for each point with the given polar coordinates.

34. $\left(2, \frac{\pi}{2}\right)$

35. $\left(\frac{1}{4}, \frac{\pi}{4}\right)$

36. $(6, 210^\circ)$

Find each of the following for $\mathbf{p} = \langle 4, 0\rangle$, $\mathbf{q} = \langle -2, -3\rangle$, and $\mathbf{t} = \langle -4, 2\rangle$.

37. $\mathbf{p} - \mathbf{t} - 2\mathbf{q}$

38. $\mathbf{q} - 4\mathbf{p} + 3\mathbf{t}$

39. $4\mathbf{p} + 3\mathbf{q} - 6\mathbf{t}$

Write an equation for and graph each parabola with focus F and the given characteristics.

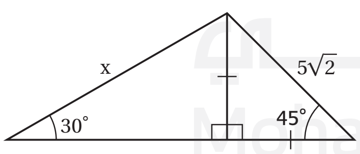
40. $F(-6, 8)$; opens up; contains $(0, 16)$

41. $F(2, -5)$; opens down; contains $(10, -11)$

42. **YOGURT** The Frozen Yogurt Shack sells cones in three sizes: small, AED 2.89; medium, AED 3.19; and large, AED 3.39. On Friday, 78 cones were sold totaling AED 246.42. The Shack sold six more medium cones than small cones that day. Use Cramer's Rule to determine the number of each type of cone sold on Friday.

Skills Review for Standardized Tests

43. **SAT/ACT** What is the value of x?



A $2\sqrt{2}$ **B** 5 **C** $5\sqrt{3}$ **D** 10 **E** $5\sqrt{6}$

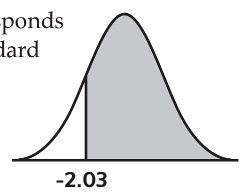
44. **REVIEW** In a study, 62% of registered voters said they voted in the 2008 presidential election. If 6 registered voters are chosen at random, what is the probability that at least 4 of them voted?

F 32% **G** 41.2% **H** 58.6% **J** 73.2%

45. The average number of patients who are seen every week at a certain hospital is normally distributed. The average per week is 12,423, with a standard deviation of 3269. If a week is selected at random, find the probability that there would be fewer than 4000 patients.

A 0.50% **B** 2.37% **C** 32.20% **D** 36.73%

46. **REVIEW** Find the area that corresponds to the shaded region of this standard normal distribution.



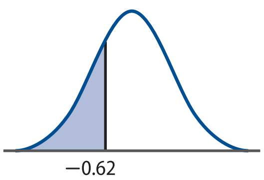
F 0.02
G 0.04
H 0.96
J 0.98

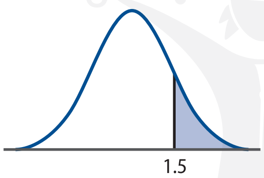
CHAPTER 10 Mid-Chapter Quiz

1. **SHAMPOO** The amount of water in milliliters in a particular shampoo is normally distributed with $\mu = 125$ and $\sigma = 7$. Find each of the following. (Lesson 10-5)
   a. $P(X < 105)$
   b. $P(X > 140)$
   c. $P(115 < X < 130)$

2. **GOLF** A random sample of 130 golfers resulted in an average score of 78 with a standard deviation of 6.3. Find the number of golfers with an average of 70 or lower. (Lesson 10-5)

Find the area that corresponds with the shaded region. (Lesson 10-5)

3.


4.


5. **PROJECTS** The scores on a science project for one class are normally distributed with $\mu = 78$ and $\sigma = 8$. Find each probability. (Lesson 10-5)
   a. $P(X \geq 96)$
   b. $P(60 < X < 85)$

Find the probability of each sample mean. (Lesson 10-5)

6. $P(\overline{x} < 38)$; $\mu = 40$, $\sigma = 5.5$, $n = 25$

7. $P(\overline{x} > 82.2)$; $\mu = 82.5$, $\sigma = 4.1$, $n = 50$

8. **EMPLOYMENT** According to a recent study, the average age that a person starts his or her first job is 16.8 years old. Assume that this variable is normally distributed with a standard deviation of 1.7 years. If a random sample of 25 people is selected, find the probability that the mean age the participants started their first jobs is greater than 17 years old. (Lesson 10-5)

CHAPTER 10 Study Guide and Review

Study Guide

KeyConcepts

Designing a Study (Lesson 10-1)

- A survey, an experiment, or an observational study can be used to collect information.
- A bias is an error that results in a misrepresentation of members of a population.

Distributions of Data (Lesson 10-2)

- Use the mean and standard deviation to describe a symmetric distribution.
- Use the five-number summary to describe a skewed distribution.

Probability Distributions (Lesson 10-3)

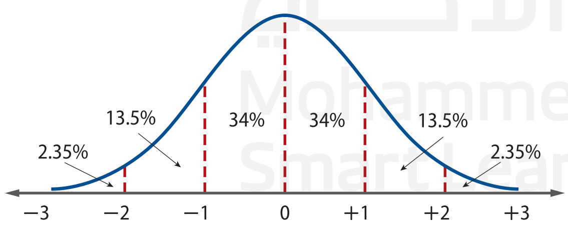
- A theoretical probability distribution is based on what is expected to happen. An experimental probability distribution is a distribution of probabilities estimated from experiments.
- The expected value of a discrete random variable is the weighted average of the values of the variable.

Binomial Distributions (Lesson 10-4)

- A binomial experiment has a fixed number of independent trials, only two possible outcomes for each trial, and the same probability of success for every trial.
- A binomial distribution is a frequency distribution of the probability of each value of a random variable.

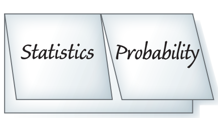
Normal Distributions (Lesson 10-5)

- The z-value represents the number of standard deviations that a given data value is from the mean, and is given by $z = \frac{X - \mu}{\sigma}$.
- The standard normal distribution is a distribution of z values with mean 0 and standard deviation 1.



FOLDABLES StudyOrganizer

Be sure the following Key Concepts are noted in your Foldable.



KeyVocabulary

alternative hypothesis H_a
bias
binomial distribution
confidence interval
continuous random variable
discrete random variable
Empirical Rule
expected value $E(X)$
experiment
experimental probability distribution
hypothesis test
inferential statistics
maximum error of estimate
negatively skewed distribution
normal distribution
null hypothesis H_0
observational study
parameter
positively skewed distribution
probability distribution
random variable
standard normal distribution
statistic
statistical inference
survey
symmetric distribution
theoretical probability distribution
z-value

VocabularyCheck

Choose a term from the list above that best completes each statement.

1. A(n) ____________ for a particular random variable is a function that maps the sample space to the probabilities of the outcomes of the sample space.
2. A(n) ____________ is an error that results in a misrepresentation of members of a population.
3. In a statistical study, data are collected and used to answer questions about a population characteristic or ____________.
4. The ____________ can be used to determine the area under the normal curve at specific intervals.
5. In a(n) ____________, members of a sample are measured or observed without being affected by the study.
6. A ____________ is an estimate of a population parameter stated as a range with a specific degree of certainty.

Lesson-by-Lesson Review

10-1 Designing a Study

Determine whether each situation describes a *survey,* an *experiment,* or an *observational study.* Then identify the sample, and suggest a population from which it may have been selected.

7. **SHOPPING** Every tenth shopper coming out of a store is asked questions about his or her satisfaction with the store.

8. **MILK SHAKE** A fast food restaurant gives 25 of their customers a sample of a new milk shake and employees monitor their reactions as they taste it.

9. **SCHOOL** Every fifth person coming out of a high school is asked what their favorite class is.

Example 1

A dealership wants to test different promotions. They randomly select 100 customers and ask them which promotion they prefer. Does this situation describe a *survey*, an *experiment*, or an *observational study*? Identify the sample, and suggest a population from which it may have been selected.

This is a survey, because the data are collected from participants' responses. The sample is the 100 customers that were selected, and the population is all potential customers.

10-2 Distributions of Data

10. **RACING** The Iditarod is a race across Alaska. The table shows the winning times, in days, for recent years.

Iditarod Winning Times

9.1, 9.4, 10.3, 9.3, 9.6, 8.7, 9.5, 9.4, 9.2, 17.3, 15.4,
15.5, 14.2, 12.0, 16.6, 13.5, 13.0, 18.1, 12.4, 11.6,
11.5, 11.3, 11.3, 13.1, 11.2, 11.6, 11.6, 9.7

   a. Use a graphing calculator to create a histogram. Then describe the shape of the distribution.

   b. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

11. **SWIMMING** Hessa's practice times in the 400-meter individual medley are shown in the table.

Times in Seconds

301, 311, 320, 308, 312, 307, 303, 305, 309, 308,
304, 302, 311, 313, 313, 316, 314, 306, 329, 326,
319, 310, 306, 309, 320, 318, 315, 318, 314, 309

   a. Use a graphing calculator to create a box-and-whisker plot. Then describe the shape of the distribution.

   b. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

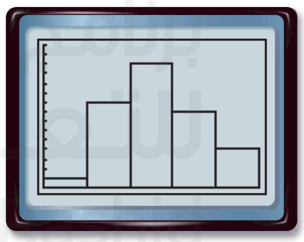
Example 2

Data collected from a group of third graders is shown.

Number of Years Playing an Instrument

2.5, 2.4, 3.1, 2.9, 4.2, 1.3, 2.6, 2.4, 3.3, 1.9, 3.4, 4.8,
2.3, 1.7, 3.2, 2.3, 3.5, 2.2, 3.6, 1.2, 4.4, 2.1, 3.4, 4.5,
1.9, 1.5, 1.4, 0.7, 1.2, 2.5, 1.9, 2.0, 2.4, 2.5, 3.4

a. Use a graphing calculator to create a histogram. Then describe the shape of the distribution.



[0, 5] scl: 1 by [0, 15] scl: 1

The distribution is symmetric.

b. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

The distribution is symmetric, so use the mean and standard deviation. The mean number of years is about 2.6 with standard deviation of about 1 year.

10-3 Probability Distributions

Identify the random variable in each distribution, and classify it as *discrete* or *continuous*. Explain your reasoning.

12. the number of ice cream sandwiches sold at an ice cream shop

13. the time it takes to run a 5-kilometer race

14. **POETRY RECITALS** The probability distribution lists the probable number of poetry recitals per year at Muna's Poetry Class. Determine the expected number of poetry recitals per year.

Number of Poetry Recitals Per Year					
Recitals	0	1	2	3	4
Probability	0.3	0.3	0.13	0.13	0.14

15. **SNOW DAYS** The distribution lists the number of snow days per year at Washington Elementary over the past 26 years. Determine the expected number of snow days this year.

Number of Snow Days Per Year					
Snow Days	0	1	2	3	4
Frequency	4	8	6	3	5

Example 3

Identify the random variable in a distribution of the number of DVDs on display at a store. Classify it as *discrete* or *continuous*. Explain your reasoning.

The random variable X is the number of DVDs on display. The number of DVDs is countable, so X is discrete.

Example 4

MEDICINE **The probability distribution lists the probable number of drops of medicine that a veterinarian administers to her sick patients. Find the expected number of drops of medicine.**

Number of Drops of Medicine				
Drops	1	2	3	4
Probability	0.5	0.3	0.1	0.1

Each drop of medicine represents a value of X, and each decimal represents the corresponding probability $P(X)$.

$$E(X) = \Sigma[X \cdot P(X)]$$
$$= 1(0.5) + 2(0.3) + 3(0.1) + 4(0.1)$$
$$= 0.5 + 0.6 + 0.3 + 0.4 \text{ or } 1.8$$

The expected number of drops of medicine is 1.8.

10-4 The Binomial Distribution

Determine whether each experiment is a binomial experiment or can be reduced to a binomial experiment. If so, describe a trial, determine the random variable, and state *n, p,* and *q*.

16. A survey found that 30% of adults like chocolate ice cream more than any other flavor. You ask 35 adults if they prefer chocolate ice cream more than any other flavor.

17. Thirty random guests from Moza's graduation party are asked their favorite song.

18. **WATCHES** According to an online poll, 74% of adults wear watches. Obaid surveyed 25 random adults. What is the probability that 20 of the adults surveyed wear a watch?

19. **SEASONS** Of 1108 people surveyed, 68% say that summer is their favorite season. What is the probability that at least 15 of 20 randomly selected people will prefer summer?

Example 5

WORK **According to an online poll, 40% of adults feel that the standard 40-hour workweek should be increased. Moza conducts a survey of 10 random adults. What is the probability that 3 of the surveyed adults feel that the standard 40-hour workweek should be increased?**

A success is an adult agreeing that the 40-hour workweek should be increased, so $p = 0.4$, $X = 3$, $q = 1 - 0.4$ or 0.6, and $n = 10$.

$$P(X) = {}_nC_x p^X q^{n-X}$$
$$P(6) = {}_{10}C_3(0.4)^3(0.6)^{10-3}$$
$$\approx 0.215$$

The probability that three surveyed adults will feel that the standard 40-hour workweek should be increased is about 0.215 or 21.5%.

10-5 The Normal Distribution

Find each of the following.

16. z if $X = 1.5$, $\mu = 1.1$, and $\sigma = 0.3$

17. X if $z = 2.34$, $\mu = 105$, and $\sigma = 18$

18. z if $X = 125$, $\mu = 100$, and $\sigma = 15$

19. X if $z = -1.12$, $\mu = 35$, and $\sigma = 3.4$

Find the interval of z-values associated with each area.

20. outside 55%

21. middle 24%

22. middle 96%

23. outside 49%

Example 3

Find z if $X = 36$, $\mu = 31$, and $\sigma = 1.3$.

$z = \dfrac{X - \mu}{\sigma}$ — Formula for z-values

$= \dfrac{36 - 31}{1.3}$ — $X = 36$, $\mu = 31$, and $\sigma = 1.3$

≈ 3.85 — Simplify.

10-6 The Central Limit Theorem

24. **GRADES** The average grade-point average or GPA in a particular school is 2.88 with a standard deviation of approximately 0.67. Find each probability for a random sample of 50 students from that school.

a. the probability that the mean GPA will be less than 2.75

b. the probability that the mean GPA will be greater than 3.05

c. the probability that the mean GPA will be greater than 3.0 but less than 3.75

25. **PHOTOGRAPHY** A local photographer reported that 55% of seniors had their senior photos taken outdoors. If 15 seniors are selected at random, find the probability that fewer than 5 of the seniors will get their pictures taken outdoors.

Find each of the following if z is the z-value, $\bar{x}$ is the sample mean, μ is the mean of the population, n is the sample size, and σ is the standard deviation.

26. z if $\bar{x} = 5.8$, $\mu = 5.5$, $n = 18$, and $\sigma = 0.2$

27. μ if $\bar{x} = 14.8$, $z = 4.49$, $n = 14$, and $\sigma = 1.5$

28. n if $z = 1.5$, $\bar{x} = 227$, $\mu = 224$, and $\sigma = 10$

29. σ if $z = -2.67$, $\bar{x} = 38.2$, $\mu = 40$, and $n = 16$

Example 4

WEATHER **The average annual snowfall for Albany, New York, is 62 cm with a standard deviation of approximately 20 cm. Find the probability that the mean snowfall will be between 60 and 70 cm using a random sample of data for 7 years.**

z-value for $\bar{x} = 60$:

$= \dfrac{60 - 62}{7.56}$ — $\bar{x} = 60$, $\mu = 62$, and $\sigma_{\bar{x}} = \dfrac{20}{\sqrt{7}} \approx 7.56$

≈ -0.26 — Simplify.

z-value for $\bar{x} = 70$:

$= \dfrac{70 - 62}{7.56}$ — $\bar{x} = 70$, $\mu = 62$, and $\sigma_{\bar{x}} = \dfrac{20}{\sqrt{7}} \approx 7.56$

≈ 1.06 — Simplify.

There is a 45.8% probability that the snowfall will be between 60 and 70 cm.

```
normalcdf(-0.26,
1.06)
       .4579957305
```

CHAPTER 10

Practice Test

1. **BUTTERFLIES** Students in a biology class are learning about the monarch butterfly's life cycle. Each student is given a caterpillar. When a caterpillar turns into chrysalis, it is placed in a glass enclosure with food and a heat lamp and examined.

   a. Determine whether the situation describes a *survey*, an *experiment* or an *observational study*.

   b. Identify the sample, and suggest a population from which it was selected.

2. **HEIGHTS** **The heights of Ms. Najla's students are shown.**

Height (cm)				
152	162	157	175	162
160	165	162	167	185
187	160	157	165	162
172	177	167	160	154

   a. Use a graphing calculator to create a box-and-whisker plot. Then describe the shape of the distribution.

   b. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

3. A binomial distribution has a 65% rate of success. There are 15 trials.

   a. What is the probability that there will be exactly 12 successes?

   b. What is the probability that there will be at least 10 successes?

Determine whether each experiment is a binomial experiment or can be reduced to a binomial experiment. If so, describe a trial, determine the random variable, and state n, p, and q.

4. A poll found that 65% of high school teachers own a pet. You ask 15 high school teachers if they own a pet.

5. A study finds that 20% of families in a town have a land line telephone. You ask 55 families how many land line telephones they have.

6. A survey found that on a scale of 1 to 5, a sandwich received a 3.5 rating. A restaurant manager asks 150 customers to rate the sandwich on a scale of 1 to 5.

Identify the random variable in each distribution, and classify it as *discrete* or *continuous*. Explain your reasoning.

7. the number of laps that Manal swims

8. the body temperatures of patients in a hospital

9. the weights of pets in a pet shelter

10. **MULTIPLE CHOICE** The table shows the number of gift cards previously won in a mall contest. What is the expected value of the gift card that is won?

   A AED 250.00

   B AED 223.15

   C AED 143.25

   D AED 100.23

Amount, X	Winners
AED 100	495
AED 125	405
AED 150	285
AED 200	180
AED 250	90
AED 300	45

11. **WEIGHTS** The weights of 1500 bodybuilders are normally distributed with a mean of 86 kg and a standard deviation of 2.5 kg.

   a. About how many bodybuilders are between 81 and 86 kg?

   b. What is the probability that a bodybuilder selected at random has a weight greater than 88 kg?

A normal distribution has a mean of 16.4 and a standard deviation of 2.6.

12. Find the range of values that represent the middle 95% of the distribution.

13. What percent of the data will be less than 19?

CHAPTER 10 Preparing for Standardized Tests

Solve Multi-Step Problems

Some problems that you will encounter on standardized tests require you to solve multiple parts in order to come up with the final solution. Use this lesson to practice these types of problems.

Strategies for Solving Multi-Step Problems

Step 1

Read the problem statement carefully.

Ask yourself:

- What am I being asked to solve? What information is given?
- Are there any intermediate steps that need to be completed before I can solve the problem?

Step 2

Organize your approach.

- List the steps you will need to complete in order to solve the problem.
- Remember that there may be more than one possible way to solve the problem.

Step 3

Solve and check.

- Work as efficiently as possible to complete each step and solve.
- If time permits, check your answer.

Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

There are 15 grade 12 students and 12 grade 11 students in Mrs. Hala's homeroom. Suppose a committee is to be made up of 6 randomly selected students. What is the probability that the committee will contain 3 grade 12 students and 3 grade 11 students? Round your answer to the nearest tenth of a percent.

A 27.2%
B 29.6%
C 31.5%
D 33.8%

Read the problem statement carefully. You are asked to find the probability that a committee will be made up of 3 grade 12 students and 3 grade 11 students. Finding this probability involves successfully completing several steps.

Step 1 Find the number of possible successes.

There are C(15, 3) ways to choose 3 grade 12 students from 15, and there are C(12, 3) to choose 3 grade 11 students from 12. Use the Fundamental Counting Principle to find s,
the number of possible successes.

$s = C(15, 3) \times C(12, 3) = \frac{15!}{12!3!} \times \frac{12!}{9!3!}$ or 100,100

Step 2 Find the total number of possible outcomes.

Compute the number of ways 6 people can be chosen from a group of 27 students.

$C(27, 6) = 296{,}010$

Step 3 Compute the probability.

Find the probability by comparing the number of successes to the number of possible outcomes.

$P(\text{3 grade 12 students, 3 grade 11 students}) = \frac{100{,}100}{296{,}010} \approx 0.33816$

So, there is about a 33.8% chance of selecting 3 grade 12 students and 3 grade 11 students for the committee. The answer is D.

Exercises

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

1. There are 52 cards in a set of flashcards divided equally between four different colors and each color is numbered 1 to 13. Of these, 4 of the cards are numbered with 1. What is the probability of randomly choosing 5 flashcards that include 2 that are numbered 1? Round your answer to the nearest whole percent.

   A 4%

   B 5%

   C 6%

   D 7%

2. According to the table, what is the probability that a randomly selected camper went on the horse ride, given that the camper is an 8th grader?

Camp Activities

Grade	Canoe Trip	Horse Ride	Nature Hike
6th	8	6	3
7th	5	4	7
8th	11	9	6

   F 0.731 **H** 0.346

   G 0.441 **J** 0.153

CHAPTER 10 Standardized Test Practice

Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Suppose the test scores on a final exam are normally distributed with a mean of 74 and a standard deviation of 3. What is the probability that a randomly selected test has a score higher than 77?

A 2.5% **C** 16%

B 13.5% **D** 34%

2. A diameter of a circle has endpoints $A(4, 6)$ and $B(-3, -1)$. Find the approximate length of the radius.

F 2.5 units **H** 5.1 units

G 4.9 units **J** 9.9 units

3. An equation can be used to find the total cost of a pizza with a certain diameter. Using the table below, find the equation that best represents y, the total cost, as a function of x, the diameter in centimeters.

Diameter, x (cm)	Total Cost, y
9	AED 10.80
12	AED 14.40
20	AED 24.00

A $y = 1.2x$ **C** $y = 0.83x$

B $x = 1.2y$ **D** $y = x + 1.80$

4. Which shows the functions correctly listed in order from widest to narrowest graph?

F $y = 8x^2, y = 2x^2, y = \frac{1}{2}x^2, y = -\frac{4}{5}x^2$

G $y = -\frac{4}{5}x^2, y = \frac{1}{2}x^2, y = 2x^2, y = 8x^2$

H $y = \frac{1}{2}x^2, y = -\frac{4}{5}x^2, y = 2x^2, y = 8x^2$

J $y = 8x^2, y = 2x^2, y = -\frac{4}{5}x^2, y = \frac{1}{2}x^2$

Test-Taking Tip

Question 5 You can use a scientific calculator to find the standard deviation. Enter the data values as a list and calculate the 1-Var statistics.

5. The table at the right shows the grades earned by students on a science test. Calculate the standard deviation of the test scores.

76	84	91	75	83
82	65	94	90	71
92	84	83	88	80
78	84	89	95	93

A 7.82 **B** 8.03 **C** 8.23 **D** 8.75

6. Which expression is equivalent to $(6a - 2b) - \frac{1}{4}(4a + 12b)$?

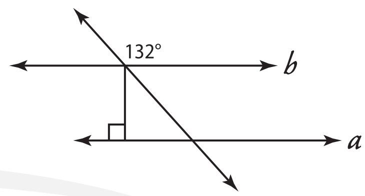
F $5a + 10b$ **H** $5a + b$

G $10a + 10b$ **J** $5a - 5b$

7. Simplify $\sqrt[6]{27x^3}$.

A $3x^2$ **B** $3x$ **C** $\sqrt{3x}$ **D** $\sqrt{3}x$

8. In the figure below, lines a and b are parallel. What are the measures of the angles in the triangle?



F 42, 48, 90 **H** 48, 52, 90

G 42, 90, 132 **J** 48, 90, 132

9. Which of the following functions represents exponential decay?

A $y = 0.2(7)^x$ **C** $y = 4(9)^x$

B $y = (0.5)^x$ **D** $y = 5\left(\frac{4}{3}\right)^x$

10. Using the table below, which expression can be used to determine the nth term of the sequence?

n	1	2	3	4
y	6	10	14	18

F $y = 6n$ **H** $y = 2n + 1$

G $y = n + 5$ **J** $y = 2(2n + 1)$

Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

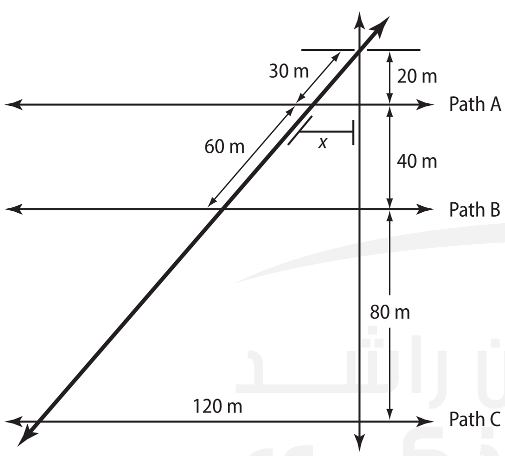
11. Determine whether each of the following situations calls for a *survey*, an *observational study*, or an *experiment*. Explain the process.

a. Laila wants to find out if a particular plant food helps plants to grow faster than just water.

b. Lamis wants to find opinions on favorite candidates in the upcoming student counsel elections.

c. Lamya wants to find out if people who get regular exercise sleep better at night.

12. **GRIDDED RESPONSE** Fawzia received a map of some walking paths through her college campus. Paths A, B, and C are parallel. What is the length x to the nearest tenth of a meter?



13. Adnan wants to find the area of a triangle. He draws the triangle on a coordinate plane and finds that it has vertices at (2, 1), (3, 4), and (1, 4). Find the area of the triangle using determinants.

14. **GRIDDED RESPONSE** Ahmed drove to the gym at an average rate of 30 kilometers per hour. It took him 45 minutes. Going home, he took the same route, but drove at a rate of 45 kilometers per hour. How many kilometers is it to his house from the gym?

Extended Response

Record your answers on a sheet of paper. Show your work.

15. Faris is taking a multiple choice test that has 8 questions. Each question has four possible answers: A, B, C, or D. Faris forgot to study for the test, so he must guess at each answer.

a. What is the probability of guessing a correct answer on the test?

b. What is the expected number of correct answers if Faris guesses at each question?

c. What is the probability that Faris will get at least half of the questions correct? Round your answer to the nearest tenth of a percent.

16. Maha had one dress and three sweaters cleaned at the dry cleaner and the charge was AED 71.50. The next week, she had two dresses and two sweaters cleaned for a total charge of AED 85.00.

a. Let d represent the price of cleaning a dress and s represent the price of cleaning a sweater. Write a system of linear equations to represent the prices of cleaning each item.

b. Solve the system of equations using substitution or elimination. Explain your choice of method.

c. What will the charge be if Maha takes two dresses and four sweaters to be cleaned?

Student Handbook

Symbols, Formulas, and Key Concepts

Statistical Tables

Symbols

Algebra

$\neq$	is not equal to
$\approx$	is approximately equal to
$\sim$	is similar to
$>, \geq$	is greater than, is greater than or equal to
$<, \leq$	is less than, is less than or equal to
$-a$	opposite or additive inverse of a
$\lvert a \rvert$	absolute value of a
$\sqrt{a}$	principal square root of a
$a : b$	ratio of a to b
(x, y)	ordered pair
(x, y, z)	ordered triple
i	the imaginary unit
$b^{\frac{1}{n}} = \sqrt[n]{b}$	nth root of b
$\mathbb{Q}$	rational numbers
$\mathbb{I}$	irrational numbers
$\mathbb{Z}$	integers
$\mathbb{W}$	whole numbers
$\mathbb{N}$	natural numbers
∞	infinity
$-\infty$	negative infinity
[ ]	endpoint included
( )	endpoints not included
$\log_b x$	logarithm base b of x
$\log x$	common logarithm of x
$\ln x$	natural logarithm of x
ω	omega, angular speed
α	alpha, angle measure
β	beta, angle measure
γ	gamma, angle measure
θ	theta, angle measure
λ	lambda, wavelength
ϕ	phi, angle measure
$\mathbf{a}$	vector **a**
$\lvert \mathbf{a} \rvert$	magnitude of vector **a**

Sets and Logic

$\in$	is an element of
$\subset$	is a subset of
$\cap$	intersection
$\cup$	union
$\varnothing$	empty set
$\sim p$	negation of p, not p
$p \wedge q$	conjunction of p and q
$p \vee q$	disjunction of p and q
$p \longrightarrow q$	conditional statement, if p then q
$p \longleftrightarrow q$	biconditional statement, p if and only if q

Geometry

$\angle$	angle
$\triangle$	triangle
$^\circ$	degree
π	pi
$\measuredangle s$	angles
$m\angle A$	degree measure of $\angle A$
$\overleftrightarrow{AB}$	line containing points A and B
$\overline{AB}$	segment with endpoints A and B
$\overrightarrow{AB}$	ray with endpoint A containing B
AB	measure of $\overline{AB}$, distance between points A and B
$\parallel$	is parallel to
$\nparallel$	is not parallel to
$\perp$	is perpendicular to
$\triangle$	triangle
$\square$	parallelogram
n-gon	polygon with n sides
$\vec{a}$	vector a
$\overrightarrow{AB}$	vector from A to B
$\lvert \overrightarrow{AB} \rvert$	magnitude of the vector from A to B
A'	the image of preimage A
$\rightarrow$	is mapped onto
$\odot A$	circle with center A
$\widehat{AB}$	minor arc with endpoints A and B
$\widehat{ABC}$	major arc with endpoints A and C
$m\widehat{AB}$	degree measure of arc AB

Trigonometry

$\sin x$	sine of x
$\cos x$	cosine of x
$\tan x$	tangent of x
$\sin^{-1} x$	Arcsin x
$\cos^{-1} x$	Arccos x
$\tan^{-1} x$	Arctan x

Symbols

Functions

$f(x)$	f of x, the value of f at x
$f(x) = \{$	piecewise-defined function
$f(x) = \|x\|$	absolute value function
$f(x) = [\![x]\!]$	function of greatest integer not greater than x
$f(x, y)$	f of x and y, a function with two variables, x and y
$[f \circ g](x)$	f of g of x, the composition of functions f and g
$f^{-1}(x)$	inverse of $f(x)$

Calculus

$\lim\limits_{x \to c}$	limit as x approaches c
m_{sec}	slope of a secant line
$f'(x)$	derivative of $f(x)$
Δ	delta, change
$\int$	indefinite integral
$\int_a^b$	definite integral
$F(x)$	antiderivative of $f(x)$

Probability and Statistics

$P(a)$	probability of a
$P(n, r)$ or ${}_nP_r$	permutation of n objects taken r at a time
$C(n, r)$ or ${}_nC_r$	combination of n objects taken r at a time
$P(A)$	probability of A
$P(A\|B)$	the probability of A given that B has already occurred
$n!$	Factorial of n (n being a natural number)
Σ	sigma (uppercase), summation
μ	mu, population mean
σ	sigma (lowercase), population standard deviation
σ^2	population variance
s	sample standard deviation
s^2	sample variance
$\sum\limits_{n=1}^{k}$	summation from $n = 1$ to k
$\bar{x}$	x-bar, sample mean
H_0	null hypothesis
H_a	alternative hypothesis

Measures

Metric	Customary
Length	
1 kilometer (km) = 1000 meters (m) 1 meter = 100 centimeters (cm) 1 centimeter = 10 millimeters (mm)	1 mile (mi) = 1760 yards (yd) 1 mile = 5280 feet (ft) 1 yard = 3 feet 1 foot = 12 inches (in) 1 yard = 36 inches
Volume and Capacity	
1 liter (L) = 1000 milliliters (mL) 1 kiloliter (kL) = 1000 liters	1 gallon (gal) = 4 quarts (qt) 1 gallon = 128 fluid ounces (fl oz) 1 quart = 2 pints (pt) 1 pint = 2 cups (c) 1 cup = 8 fluid ounces
Weight and Mass	
1 kilogram (kg) = 1000 grams (g) 1 gram = 1000 milligrams (mg) 1 metric ton (t) = 1000 kilograms	1 ton (T) = 2000 pounds (lb) 1 pound = 16 ounces (oz)

Arithmetic Operations and Relations

Identity	For any number a, $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
Substitution (=)	If $a = b$, then a may be replaced by b.
Reflexive (=)	$a = a$
Symmetric (=)	If $a = b$, then $b = a$.
Transitive (=)	If $a = b$ and $b = c$, then $a = c$.
Commutative	For any numbers a and b, $a + b = b + a$ and $a \cdot b = b \cdot a$.
Associative	For any numbers a, b, and c, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
Distributive	For any numbers a, b, and c, $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$.
Additive Inverse	For any number a, there is exactly one number $-a$ such that $a + (-a) = 0$.
Multiplicative Inverse	For any number $\frac{a}{b}$, where $a, b \neq 0$, there is exactly one number $\frac{b}{a}$ such that $\frac{a}{b} \cdot \frac{b}{a} = 1$.
Multiplicative (0)	For any number a, $a \cdot 0 = 0 \cdot a = 0$.
Addition (=)	For any numbers a, b, and c, if $a = b$, then $a + c = b + c$.
Subtraction (=)	For any numbers a, b, and c, if $a = b$, then $a - c = b - c$.
Multiplication and Division (=)	For any numbers a, b, and c, with $c \neq 0$, if $a = b$, then $ac = bc$ and $\frac{a}{c} = \frac{b}{c}$.
Addition (>)*	For any numbers a, b, and c, if $a > b$, then $a + c > b + c$.
Subtraction (>)*	For any numbers a, b, and c, if $a > b$, then $a - c > b - c$.
Multiplication and Division (>)*	For any numbers a, b, and c, 1. if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$. 2. if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
Zero Product	For any real numbers a and b, if $ab = 0$, then $a = 0$, $b = 0$, or both a and b equal 0.

* *These properties are also true for $<$, $\geq$, and $\leq$.*

Algebraic Formulas and Key Concepts

	Matrices		
Adding	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$	**Multiplying by a Scalar**	$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$
Subtracting	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$	**Multiplying**	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$
	Polynomials		
Quadratic Formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$	**Square of a Difference**	$(a - b)^2 = (a - b)(a - b)$ $= a^2 - 2ab + b^2$
Square of a Sum	$(a + b)^2 = (a + b)(a + b)$ $= a^2 + 2ab + b^2$	**Product of Sum and Difference**	$(a + b)(a - b) = (a - b)(a + b)$ $= a^2 - b^2$
	Logarithms		
Product Property	$\log_x ab = \log_x a + \log_x b$	**Power Property**	$\log_b m^p = p \log_b m$
Quotient Property	$\log_x \frac{a}{b} = \log_x a - \log_x b, b \neq 0$	**Change of Base**	$\log_a n = \frac{\log_b n}{\log_b a}$

Algebraic Formulas and Key Concepts

Exponential and Logarithmic Functions

Compound Interest	$A = P\left(1 + \frac{r}{n}\right)^{nt}$	**Exponential Growth or Decay**	$N = N_0(1 + r)^t$
Continuous Compound Interest	$A = Pe^{rt}$	**Continuous Exponential Growth or Decay**	$N = N_0 e^{kt}$
Product Property	$\log_b xy = \log_b x + \log_b y$	**Power Property**	$\log_b x^p = p \log_b x$
Quotient Property	$\log_b \frac{x}{y} = \log_b x - \log_b y$	**Change of Base**	$\log_b x = \frac{\log_a x}{\log_a b}$
Logistic Growth	$f(t) = \frac{c}{1 + ae^{-bt}}$		

Sequences and Series

***n*th term, Arithmetic**	$a_n = a_1 + (n - 1)d$	***n*th term, Geometric**	$a_n = a_1 r^{n-1}$
Sum of Arithmetic Series	$S_n = n\left(\frac{a_1 + a_2}{2}\right)$ or $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$	**Sum of Geometric Series**	$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ or $S_n = \frac{a_1 - a_n r}{1 - r}, r \neq 1$
Sum of Infinite Geometric Series	$S = \frac{a_1}{1 - r}, \lvert r \rvert < 1$	**Euler's Formula**	$e^{i\theta} = \cos\theta + \boldsymbol{i}\sin\theta$
Power Series	$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$	**Exponential Series**	$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
Binomial Theorem	$(a + b)^n = {}_nC_0\, a^n b^0 + {}_nC_1\, a^{n-1} b^1 + {}_nC_2\, a^{n-2} b^2 + \cdots + {}_nC_r\, a^{n-r} b^r + \cdots + {}_nC_n\, a^0 b^n$		
Cosine and Sine Power Series	$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$		$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Vectors

Addition in Plane	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$	**Addition in Space**	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
Subtraction in Plane	$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$	**Subtraction in Space**	$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$
Scalar Multiplication in Plane	$k\mathbf{a} = \langle ka_1, ka_2 \rangle$	**Scalar Multiplication in Space**	$k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$
Dot Product in Plane	$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$	**Dot Product in Space**	$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
Angle Between Two Vectors	$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\lvert\mathbf{a}\rvert \lvert\mathbf{b}\rvert}$	**Projection of u onto v**	$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\lvert\mathbf{v}\rvert^2}\right)\mathbf{v}$
Magnitude of a Vector	$\lvert\mathbf{v}\rvert = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	**Triple Scalar Product**	$\mathbf{t} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} t_1 & t_2 & t_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

Equations of a Line on a Coordinate Plane

Slope-intercept form of a line	$y = mx + b$
Point-slope form of a line	$y - y_1 = m(x - x_1)$

Algebraic Formulas and Key Concepts

Conic Sections			
Parabola	$(x-h)^2 = 4p(y-k)$ or $(y-k)^2 = 4p(x-h)$	**Circle**	$x^2 + y^2 = r^2$ or $(x-h)^2 + (y-k)^2 = r^2$
Ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$	**Hyperbola**	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Rotation of Conics	$x' = x\cos\theta + y\sin\theta$ and $y' = y\cos\theta - x\sin\theta$		

Parametric Equations			
Vertical Position	$y = tv_0 \sin\theta - \frac{1}{2}gt^2 + h_0$	**Horizontal Distance**	$x = tv_0\cos\theta$

Complex Numbers			
Product Formula	$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + \boldsymbol{i}\sin(\theta_1+\theta_2)]$	**Quotient Formula**	$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + \boldsymbol{i}\sin(\theta_1-\theta_2)]$
Distinct Roots Formula	$r^{\frac{1}{p}}\left(\cos\frac{\theta+2n\pi}{p} + \boldsymbol{i}\sin\frac{\theta+2n\pi}{p}\right)$	**De Moivre's Theorem**	$z^n = [r(\cos\theta + \boldsymbol{i}\sin\theta)]^n$ or $r^n(\cos n\theta + \boldsymbol{i}\sin n\theta)$

Geometric Formulas and Key Concepts

Coordinate Geometry			
Slope	$m = \frac{y_2 - y_1}{x_2 - x_1}, x_2 \neq x_1$	**Distance on a number line**	$d = \lvert a - b\rvert$
Distance on a coordinate plane	$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$	**Arc length**	$\ell = \frac{x}{360} \cdot 2\pi r$
Midpoint on a coordinate plane	$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$	**Midpoint on a number line**	$M = \frac{a+b}{2}$
Midpoint in space	$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$	**Pythagorean Theorem**	$a^2 + b^2 = c^2$
Distance in space	$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$		

Perimeter and Circumference					
Square	$P = 4s$	**Rectangle**	$P = 2\ell + 2w$	**Circle**	$C = 2\pi r$ or $C = \pi d$

Lateral Surface Area			
Prism	$L = Ph$	**Pyramid**	$L = \frac{1}{2}P\ell$
Cylinder	$L = 2\pi rh$	**Cone**	$L = \pi r\ell$

Total Surface Area					
Prism	$S = Ph + 2B$	**Cone**	$S = \pi r\ell + \pi r^2$	**Cylinder**	$S = 2\pi rh + 2\pi r^2$
Pyramid	$S = \frac{1}{2}P\ell + B$	**Sphere**	$S = 4\pi r^2$	**Cube**	$S = 6s^2$

Volume					
Prism	$V = Bh$	**Cone**	$V = \frac{1}{3}\pi r^2 h$	**Cylinder**	$V = \pi r^2 h$
Pyramid	$V = \frac{1}{3}Bh$	**Sphere**	$V = \frac{4}{3}\pi r^3$	**Cube**	$V = s^3$
Rectangular prism	$V = \ell wh$				

Trigonometric Functions and Identities

Trigonometric Functions

Trigonometric Functions	$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin\theta}$	$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\cos\theta}$	$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\cos\theta}{\sin\theta}$
Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos A$	$b^2 = a^2 + c^2 - 2ac\cos B$	$c^2 = a^2 + b^2 - 2ab\cos C$
Law of Sines	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	**Heron's Formula**	$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$
Linear Speed	$v = \frac{s}{t}$	**Angular Speed**	$\omega = \frac{\theta}{t}$

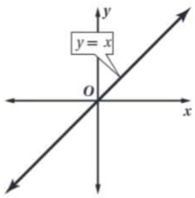
Trigonometric Identities

Reciprocal	$\sin\theta = \frac{1}{\csc\theta}$ $\csc\theta = \frac{1}{\sin\theta}$	$\cos\theta = \frac{1}{\sec\theta}$ $\sec\theta = \frac{1}{\cos\theta}$	$\tan\theta = \frac{1}{\cot\theta}$ $\cot\theta = \frac{1}{\tan\theta}$
Pythagorean	$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$\cot^2\theta + 1 = \csc^2\theta$
Cofunction	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$	$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$	$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$
Odd-Even	$\sin(-\theta) = -\sin\theta$ $\csc(-\theta) = -\csc\theta$	$\cos(-\theta) = \cos\theta$ $\sec(-\theta) = \sec\theta$	$\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$
Sum & Difference	$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$	$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$	
Double-Angle	$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = 2\cos^2\theta - 1$ $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$	$\cos 2\theta = 1 - 2\sin^2\theta$
Power-Reducing	$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$	$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$	$\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$
Half-Angle	$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}$ $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}$	$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$ $\tan\frac{\theta}{2} = \frac{1 - \cos\theta}{\sin\theta}$	$\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta}$
Product-to-Sum	$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$ $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$	$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$ $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$	
Sum-to-Product	$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$	$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$	

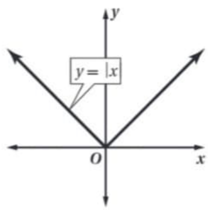
Parent Functions and Function Operations

Parent Functions

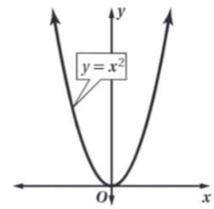
Linear Functions



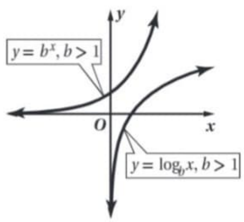
Absolute Value Functions



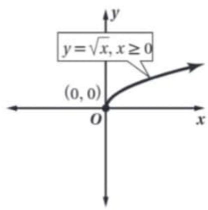
Quadratic Functions



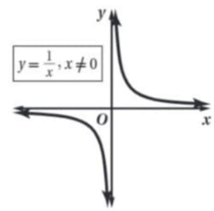
Exponential and Logarithmic Functions



Square Root Functions



Reciprocal and Rational Functions



Function Operations

Addition	$(f+g)(x) = f(x) + g(x)$	**Multiplication**	$(f \cdot g)(x) = f(x) \cdot g(x)$
Subtraction	$(f-g)(x) = f(x) - g(x)$	**Division**	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$

Calculus

Limits

Sum	$\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$	**Difference**	$\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
Scalar Multiple	$\lim_{x \to c} [k\,f(x)] = k \lim_{x \to c} f(x)$	**Product**	$\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
Quotient	$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, if $\lim_{x \to c} g(x) \neq 0$	**Power**	$\lim_{x \to c} [f(x)^n] = \left[\lim_{x \to c} f(x)\right]^n$
***n*th Root**	$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$, if $\lim_{x \to c} f(x) > 0$ when n is even	**Velocity**	**Average** $v_{avg} = \frac{f(b) - f(a)}{b - a}$ **Instantaneous** $v(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$

Derivatives

Power Rule	If $f(x) = x^n$, $f'(x) = nx^{n-1}$	**Sum or Difference**	If $f(x) = g(x) \pm h(x)$, then $f'(x) = g'(x) \pm h'(x)$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$	**Quotient Rule**	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Integrals

Indefinite Integral	$\int f(x)\,dx = F(x) + C$	**Fundamental Theorem of Calculus**	$\int_a^b f(x)\,dx = F(b) - F(a)$

Statistics Formulas and Key Concepts

***z*-Values**	$z = \dfrac{X - \mu}{\sigma}$	***z*-Value of a Sample Mean**	$z = \dfrac{\overline{x} - \mu}{\sigma_{\overline{x}}}$
Binomial Probability	$P(X) = {}_nC_x\, p^x q^{n-x} = \dfrac{n!}{(n-x)!\,x!}\, p^x q^{n-x}$	**Maximum Error of Estimate**	$E = z \cdot \sigma_{\overline{x}}$ or $z \cdot \dfrac{\sigma}{\sqrt{n}}$
Confidence Interval, Normal Distribution	$CI = \overline{x} \pm E$ or $\overline{x} \pm z \cdot \dfrac{\sigma}{\sqrt{n}}$	**Confidence Interval, *t*-Distribution**	$CI = \overline{x} \pm t \cdot \dfrac{s}{\sqrt{n}}$
Correlation Coefficient	$r = \dfrac{1}{n-1}\sum\left(\dfrac{x_i - \overline{x}}{s_x}\right)\left(\dfrac{y_i - \overline{y}}{s_y}\right)$	***t*-Test for the Correlation Coefficient**	$t = r\sqrt{\dfrac{n-2}{1-r^2}}$, degrees of freedom: $n - 2$

TABLE A The Standard Normal Distribution

Cumulative Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

For z values less than −3.49, use 0.0001.

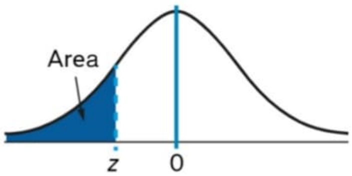


TABLE A (continued)

Cumulative Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

For *z* values greater than 3.49, use 0.9999.

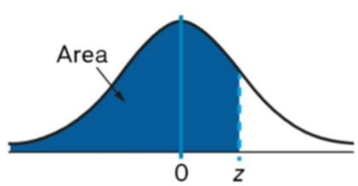


TABLE B Student's *t*-Distribution (the *t* Distribution)

Degrees of Freedom	Confidence intervals	80%	90%	95%	98%	99%
	One tail, α	0.10	0.05	0.025	0.01	0.005
	Two tails, α	0.20	0.10	0.05	0.02	0.01
1		3.078	6.314	12.706	31.821	63.657
2		1.886	2.920	4.303	6.965	9.925
3		1.638	2.353	3.182	4.541	5.841
4		1.533	2.132	2.776	3.747	4.604
5		1.476	2.015	2.571	3.365	4.032
6		1.440	1.943	2.447	3.143	3.707
7		1.415	1.895	2.365	2.998	3.499
8		1.397	1.860	2.306	2.896	3.355
9		1.383	1.833	2.262	2.821	3.250
10		1.372	1.812	2.228	2.764	3.169
11		1.363	1.796	2.201	2.718	3.106
12		1.356	1.782	2.179	2.681	3.055
13		1.350	1.771	2.160	2.650	3.012
14		1.345	1.761	2.145	2.624	2.977
15		1.341	1.753	2.131	2.602	2.947
16		1.337	1.746	2.120	2.583	2.921
17		1.333	1.740	2.110	2.567	2.898
18		1.330	1.734	2.101	2.552	2.878
19		1.328	1.729	2.093	2.539	2.861
20		1.325	1.725	2.086	2.528	2.845
21		1.323	1.721	2.080	2.518	2.831
22		1.321	1.717	2.074	2.508	2.819
23		1.319	1.714	2.069	2.500	2.807
24		1.318	1.711	2.064	2.492	2.797
25		1.316	1.708	2.060	2.485	2.787
26		1.315	1.706	2.056	2.479	2.779
27		1.314	1.703	2.052	2.473	2.771
28		1.313	1.701	2.048	2.467	2.763
29		1.311	1.699	2.045	2.462	2.756
30		1.310	1.697	2.042	2.457	2.750
32		1.309	1.694	2.037	2.449	2.738
34		1.307	1.691	2.032	2.441	2.728
36		1.306	1.688	2.028	2.434	2.719
38		1.304	1.686	2.024	2.429	2.712
40		1.303	1.684	2.021	2.423	2.704
45		1.301	1.679	2.014	2.412	2.690
50		1.299	1.676	2.009	2.403	2.678
55		1.297	1.673	2.004	2.396	2.668
60		1.296	1.671	2.000	2.390	2.660
65		1.295	1.669	1.997	2.385	2.654
70		1.294	1.667	1.994	2.381	2.648
75		1.293	1.665	1.992	2.377	2.643
80		1.292	1.664	1.990	2.374	2.639
90		1.291	1.662	1.987	2.368	2.632
100		1.290	1.660	1.984	2.364	2.626
500		1.283	1.648	1.965	2.334	2.586
1000		1.282	1.646	1.962	2.330	2.581
(z) ∞		1.282	1.645	1.960	2.326	2.576

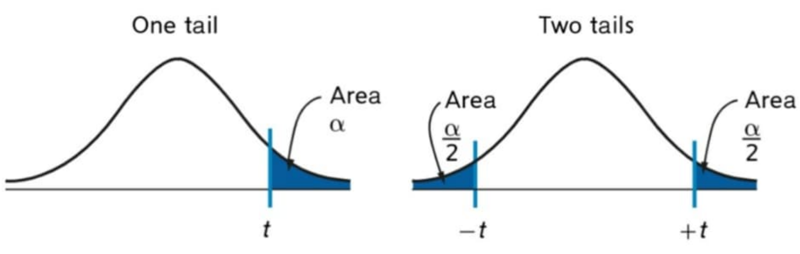
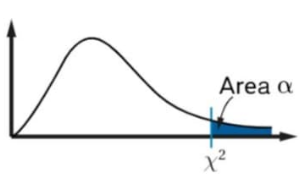


TABLE C The Chi-Square Distribution (χ^2)

Degrees of Freedom	α									
	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
1	—	—	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.071	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.299
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.042	30.813	33.924	36.781	40.289	42.796
23	9.262	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.257	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.954	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169



Hybrid education in the Emirati school

Within the strategic dimension of the Ministry of Education's development plans and its endeavor to diversify education channels and overcome all the challenges that may prevent it, and to ensure continuity in all circumstances, the Ministry has implemented a hybrid education plan for all students at all levels of education.

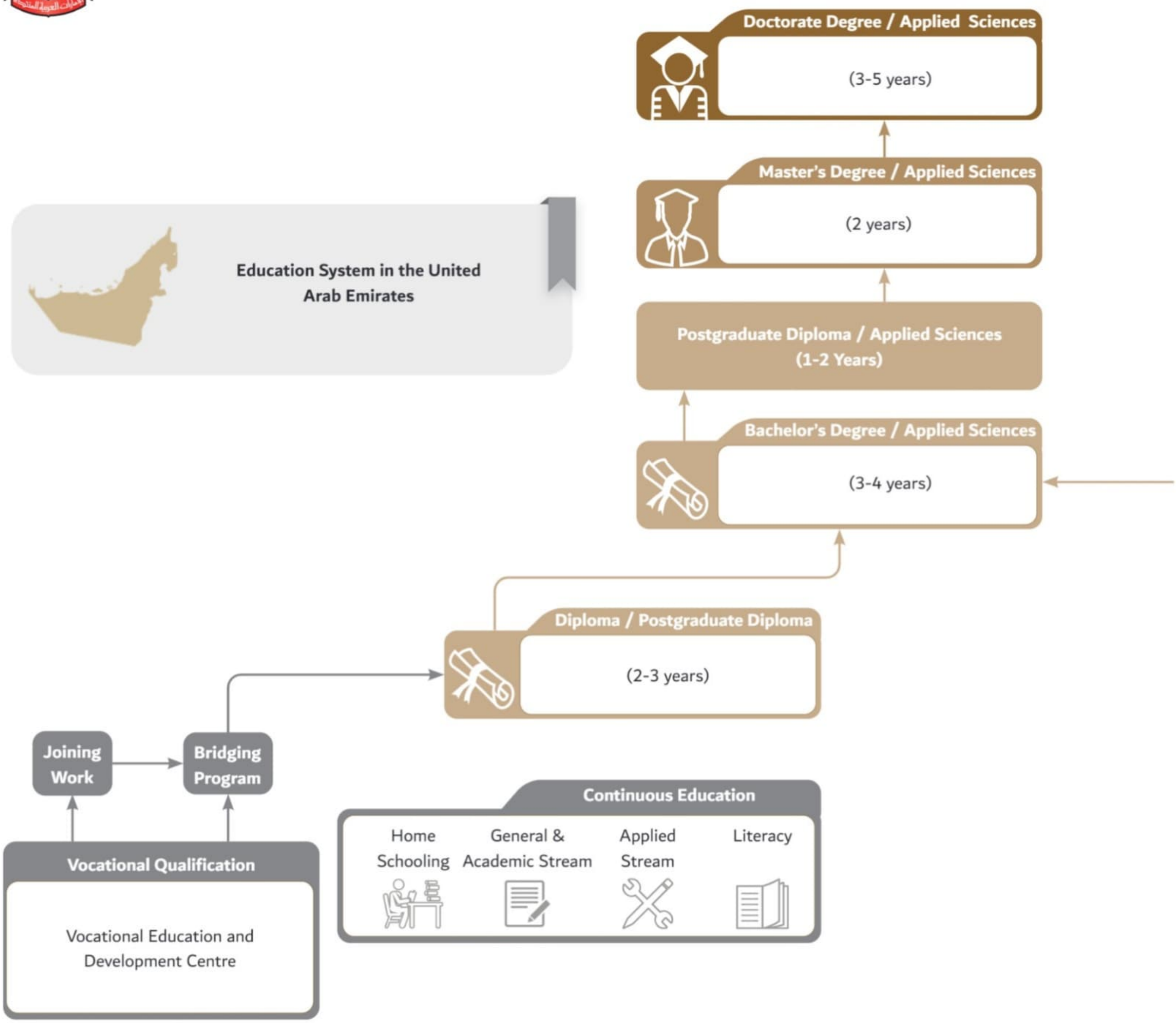
Study Plan		Pre-kindergarten/ kindergarten	Cycle 1	Cycle 2	Cycle 3
School learning		✓	✓	✓	✓
e-learning	Direct	–	–	✓	✓
	Self	✓	✓	✓	✓

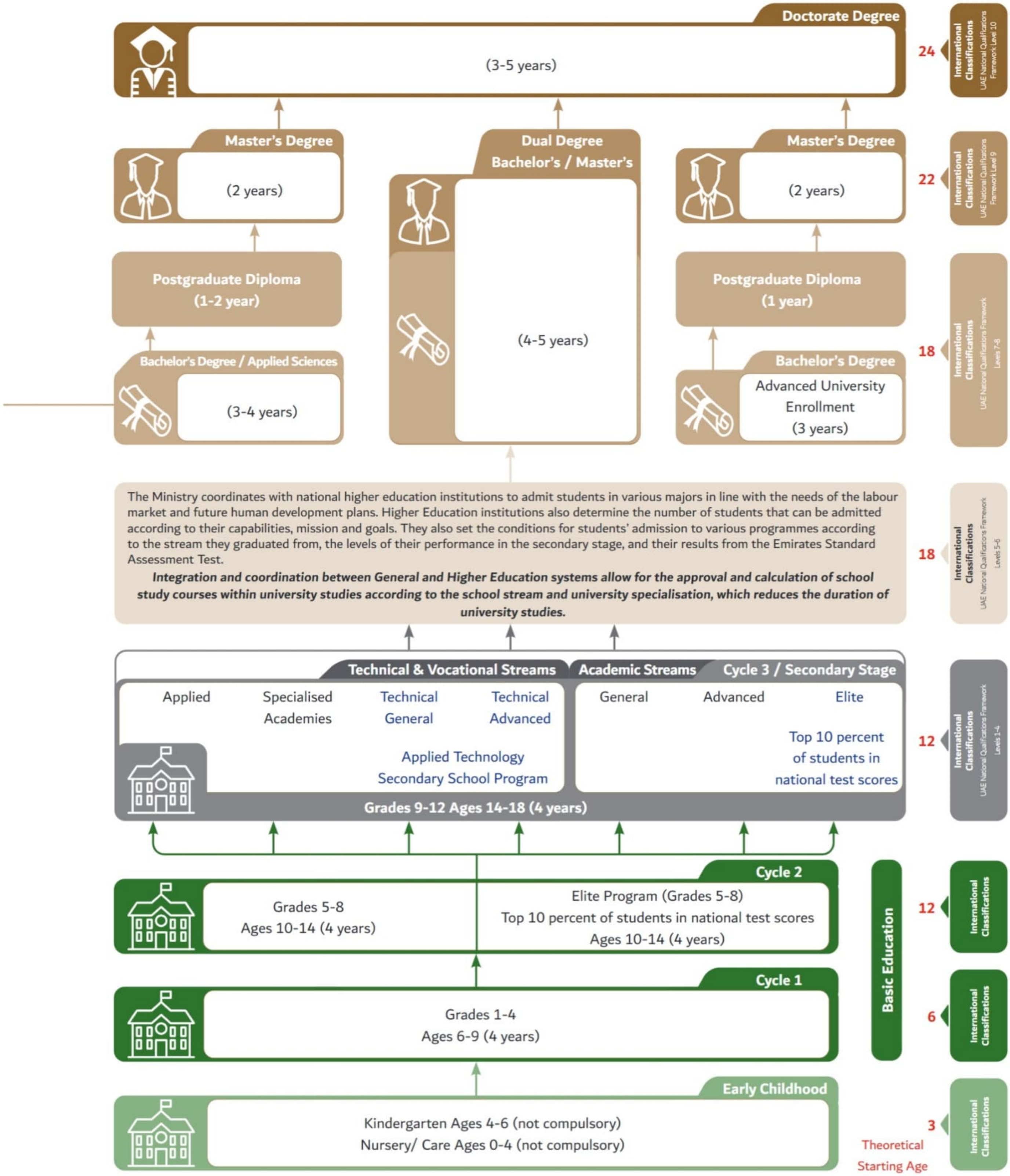
Channels for obtaining a textbook:

برنامج محمد بن راشد للتعلم الذكي
Mohammed Bin Rashid Smart Learning Program

Electronic units

UNITED ARAB EMIRATES
MINISTRY OF EDUCATION




Doctorate Degree
(3-5 years)
24
International Classifications
UAE National Qualifications Framework Level 10
Master's Degree
(2 years)
Dual Degree
Bachelor's / Master's
(4-5 years)
Master's Degree
(2 years)
22
International Classifications
UAE National Qualifications Framework Level 9
Postgraduate Diploma
(1-2 year)
Postgraduate Diploma
(1 year)
Bachelor's Degree / Applied Sciences
(3-4 years)
Bachelor's Degree
Advanced University Enrollment
(3 years)
18
International Classifications
UAE National Qualifications Framework Levels 7-8
The Ministry coordinates with national higher education institutions to admit students in various majors in line with the needs of the labour market and future human development plans. Higher Education institutions also determine the number of students that can be admitted according to their capabilities, mission and goals. They also set the conditions for students' admission to various programmes according to the stream they graduated from, the levels of their performance in the secondary stage, and their results from the Emirates Standard Assessment Test.
Integration and coordination between General and Higher Education systems allow for the approval and calculation of school study courses within university studies according to the school stream and university specialisation, which reduces the duration of university studies.
18
International Classifications
UAE National Qualifications Framework Levels 5-6
Technical & Vocational Streams
Academic Streams
Cycle 3 / Secondary Stage
Applied
Specialised Academies
Technical General
Technical Advanced
Applied Technology Secondary School Program
General
Advanced
Elite
Top 10 percent of students in national test scores
Grades 9-12 Ages 14-18 (4 years)
12
International Classifications
UAE National Qualifications Framework Levels 1-4
Cycle 2
Grades 5-8
Ages 10-14 (4 years)
Elite Program (Grades 5-8)
Top 10 percent of students in national test scores
Ages 10-14 (4 years)
12
International Classifications
Cycle 1
Grades 1-4
Ages 6-9 (4 years)
Basic Education
6
International Classifications
Early Childhood
Kindergarten Ages 4-6 (not compulsory)
Nursery/ Care Ages 0-4 (not compulsory)
3
Theoretical Starting Age
International Classifications

Ministry of Education
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