# CHAPTER 3 Trigonometric Functions

## Then

You have studied exponential and logarithmic functions, which are two types of transcendental functions.

## Now

You will learn to

- Use trigonometric ratios to solve right triangles.
- Find values of trigonometric ratios for any angle.
- Graph trigonometric and inverse trigonometric functions.

## Why?

**SATELLITE NAVIGATION** Satellite navigation systems operate by receiving signals from satellites in orbit, determining the distance to each of the satellites, and then using trigonometry to establish the location on Earth's surface. These techniques are also used when navigating cars, planes, ships, and spacecraft.

PREREAD Use the prereading strategy of previewing to make two or three predictions of what Chapter 3 is about.

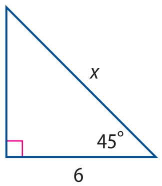
# **Get Ready** for the Chapter

**Take the Quick Check Below**

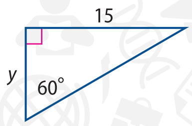
## QuickCheck

**Find the missing value in each figure.**
(Prerequisite Skill)

**1.**




**2.**

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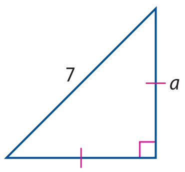


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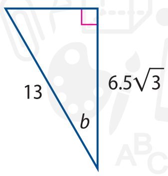



**Determine whether each of the following could represent the measures of three sides of a triangle. Write *yes* or *no*.** (Prerequisite Skill)

**5.** 4, 8, 12

**6.** 12, 15, 18

**7.** **ALGEBRA** The sides of a triangle have lengths $x$, $x + 17$, and 25. If the length of the longest side is 25, what value of $x$ makes the triangle a right triangle? (Prerequisite Skill)

**Find the equations of any vertical or horizontal asymptotes.**

**8.** $f(x) = \dfrac{x^2 - 4}{x^2 + 8}$

**9.** $h(x) = \dfrac{x^3 - 27}{x + 5}$

**10.** $f(x) = \dfrac{x(x - 1)^2}{(x - 2)(x + 4)}$

**11.** $g(x) = \dfrac{x + 5}{(x - 3)(x - 5)}$

**12.** $h(x) = \dfrac{x^2 + x - 20}{x + 5}$

**13.** $f(x) = \dfrac{2x^2 + 5x - 12}{2x - 3}$

## NewVocabulary

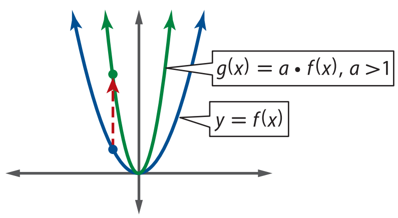
**English**

trignometric functions
sine
cosine
tangent
cosecant
secant
cotangent
reciprocal function
inverse sine
inverse cosine
inverse tangent
radian
coterminal angles
reference angle
unit circle
circular function
period
sinusoid
amplitude
frequency
phase shift
Law of Sines
Law of Cosines

## ReviewVocabulary

reflection p. 48 the mirror image of the graph of a function with respect to a specific line

dilation p. 49 a nonrigid transformation that has the effect of compressing (shrinking) or expanding (enlarging) the graph of a function vertically or horizontally




**Vertical dilation**

LESSON 3-1

# Right Triangle Trigonometry

| Then | Now | Why? |
|---|---|---|
| You evaluated functions. | **1** Find values of trigonometric functions for acute angles of right triangles.<br>**2** Solve right triangles. | Large helium-filled balloons are a tradition of many holiday parades. Long cables attached to the balloon are used by volunteers to lead the balloon along the parade route.<br><br>Suppose two of these cables are attached to a balloon at the same point, and the volunteers holding these cables stand so that the ends of the cables lie in the same vertical plane. If you know the measure of the angle that each cable makes with the ground and the distance between the volunteers, you can use right triangle trigonometry to find the height of the balloon above the ground. |

## NewVocabulary

trigonometric ratios
trigonometric functions
sine
cosine
tangent
cosecant
secant
cotangent
reciprocal function
inverse trigonometric function
inverse sine
inverse cosine
inverse tangent
angle of elevation
angle of depression
solve a right triangle

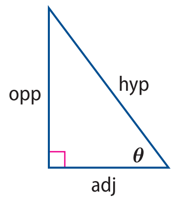
## 1 Values of Trigonometric Ratios

The word *trigonometry* means *triangle measure.* In this chapter, you will study trigonometry as the relationships among the sides and angles of triangles and as a set of functions defined on the real number system. In this lesson, you will study *right triangle trigonometry.*

Using the side measures of a right triangle and a reference angle labeled $\theta$, we can form the six **trigonometric ratios** that define six **trigonometric functions**.

### KeyConcept Trigonometric Functions

Let $\theta$ be an acute angle in a right triangle and the abbreviations opp, adj, and hyp refer to the length of the side opposite $\theta$, the length of the side adjacent to $\theta$, and the length of the hypotenuse, respectively.




Then the six trigonometric functions of $\theta$ are defined as follows.

**sine** $(\theta) = \sin\theta = \dfrac{\text{opp}}{\text{hyp}}$ **cosecant** $(\theta) = \csc\theta = \dfrac{\text{hyp}}{\text{opp}}$

**cosine** $(\theta) = \cos\theta = \dfrac{\text{adj}}{\text{hyp}}$ **secant** $(\theta) = \sec\theta = \dfrac{\text{hyp}}{\text{adj}}$

**tangent** $(\theta) = \tan\theta = \dfrac{\text{opp}}{\text{adj}}$ **cotangent** $(\theta) = \cot\theta = \dfrac{\text{adj}}{\text{opp}}$

The cosecant, secant, and cotangent functions are called **reciprocal functions** because their ratios are reciprocals of the sine, cosine, and tangent ratios, respectively. Therefore, the following statements are true.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

From the definitions of the sine, cosine, tangent, and cotangent functions, you can also derive the following relationships. You will prove these relationships in Exercise 83.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{and} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

**Study**Tip

**Memorizing Trigonometric Ratios**

The mnemonic device **SOH-CAH-TOA** is most commonly used to remember the ratios for sine, cosine, and tangent.

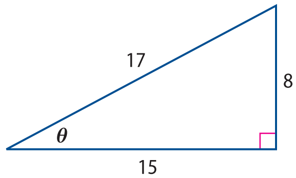
$\sin\theta = \frac{\text{opp}}{\text{hyp}}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$

$\tan\theta = \frac{\text{opp}}{\text{adj}}$

## Example 1 Find Values of Trigonometric Ratios

**Find the exact values of the six trigonometric functions of $\theta$.**




The length of the side opposite $\theta$ is 8, the length of the side adjacent to $\theta$ is 15, and the length of the hypotenuse is 17.

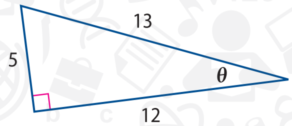
$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ or $\frac{8}{17}$    opp = 8 and hyp = 17    $\csc\theta = \frac{\text{hyp}}{\text{opp}}$ or $\frac{17}{8}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ or $\frac{15}{17}$    adj = 15 and hyp = 17    $\sec\theta = \frac{\text{hyp}}{\text{adj}}$ or $\frac{17}{15}$

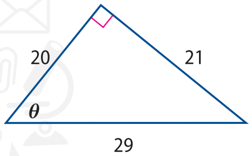
$\tan\theta = \frac{\text{opp}}{\text{adj}}$ or $\frac{8}{15}$    opp = 8 and adj = 15    $\cot\theta = \frac{\text{adj}}{\text{opp}}$ or $\frac{15}{8}$

### **Guided**Practice

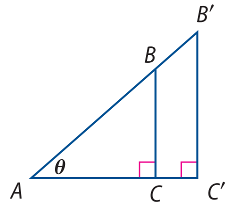
**1A.**




**1B.**




Consider $\sin\theta$ in the figure.




Using $\triangle ABC$: $\sin\theta = \frac{BC}{AB}$     Using $\triangle AB'C'$: $\sin\theta = \frac{B'C'}{AB'}$

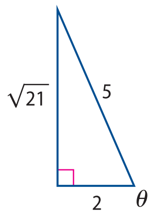
Notice that the triangles are similar because they are two right triangles that share a common angle, $\theta$. Because the triangles are similar, the ratios of the corresponding sides are equal. So, $\frac{BC}{AB} = \frac{B'C'}{AB'}$.

Therefore, $\sin\theta$ has the same value regardless of the triangle used. The values of the functions are constant for a given angle measure. They do not depend on the size of the right triangle.

## Example 2 Use One Trigonometric Value to Find Others

**If $\cos\theta = \frac{2}{5}$, find the exact values of the five remaining trigonometric functions for the acute angle $\theta$.**

Begin by drawing a right triangle and labeling one acute angle $\theta$.




Because $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{2}{5}$, label the adjacent side 2 and the hypotenuse 5.

By the Pythagorean Theorem, the length of the leg opposite $\theta$ is $\sqrt{5^2 - 2^2}$ or $\sqrt{21}$.

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{21}}{5}$    $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{21}}{2}$    $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{2}$

$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{\sqrt{21}}$ or $\frac{5\sqrt{21}}{21}$    $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{2}{\sqrt{21}}$ or $\frac{2\sqrt{21}}{21}$

**Watch**Out!

**Common Misconception**

In Example 2, the adjacent side of the triangle could also have been labeled 4 and the hypotenuse 10. This is because $\cos\theta = \frac{2}{5}$ gives the ratio of the adjacent side and hypotenuse, not their specific measures.

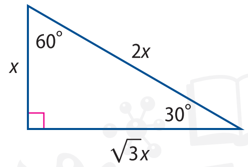
### **Guided**Practice

**2.** If $\tan\theta = \frac{1}{2}$, find the exact values of the five remaining trigonometric functions for the acute angle $\theta$.

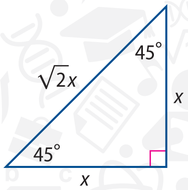
You will often be asked to find the trigonometric functions of specific acute angle measures. The table below gives the values of the six trigonometric functions for three common angle measures: 30°, 45°, and 60°. To remember these values, you can use the properties of 30°-60°-90° and 45°-45°-90° triangles.

## KeyConcept Trigonometric Values of Special Angles

**30°-60°-90° Triangle**




**45°-45°-90° Triangle**




| $\theta$ | 30° | 45° | 60° |
|---|---|---|---|
| $\sin\theta$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ |
| $\cos\theta$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ |
| $\tan\theta$ | $\frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ |
| $\csc\theta$ | 2 | $\sqrt{2}$ | $\frac{2\sqrt{3}}{3}$ |
| $\sec\theta$ | $\frac{2\sqrt{3}}{3}$ | $\sqrt{2}$ | 2 |
| $\cot\theta$ | $\sqrt{3}$ | 1 | $\frac{\sqrt{3}}{3}$ |

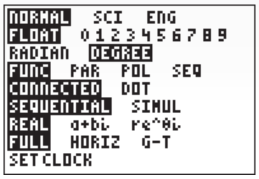
You will verify some of these values in Exercises 57–62.

## 2 Solving Right Triangles

Trigonometric functions can be used to find missing side lengths and angle measures of right triangles.

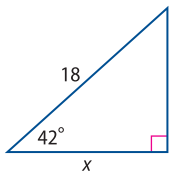
### TechnologyTip

**Degree Mode** To evaluate a trigonometric function of an angle measured in degrees, first set the calculator to *degree mode* by selecting DEGREE on the MODE feature of the graphing calculator.




### Example 3 Find a Missing Side Length

**Find the value of $x$. Round to the nearest tenth, if necessary.**




Because you are given an acute angle measure and the length of the hypotenuse of the triangle, use the cosine function to find the length of the side adjacent to the given angle.

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ — Cosine function

$\cos 42° = \frac{x}{18}$ — $\theta = 42°$, adj $= x$, and hyp $= 18$

$18\cos 42° = x$ — Multiply each side by 18.

$13.4 \approx x$ — Use a calculator.

Therefore, $x$ is about 13.4.

**CHECK** You can check your answer by substituting $x = 13.4$ into $\cos 42° = \frac{x}{18}$.

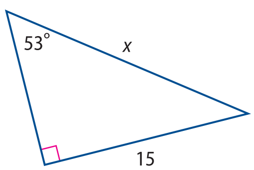
$\cos 42° = \frac{x}{18}$

$\cos 42° = \frac{13.4}{18}$ — $x = 13.4$

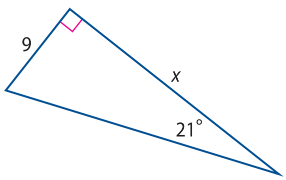
$0.74 = 0.74$ ✓ — Simplify.

### GuidedPractice

**3A.**




**3B.**



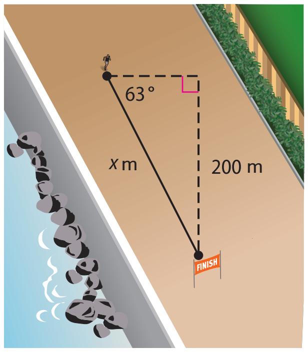
### Real-WorldLink

The Ironman Triathlon held in Kailua-Kona Bay, Hawaii, consists of three endurance events, including a 3.9 km swim, a 180 km bike ride, and a 42.2 km marathon.

**Source:** World Triathlon Corporation

## Real-World Example 4 Finding a Missing Side Length

**TRIATHLONS A competitor in a triathlon is running along the course shown. Determine the length in feet that the runner must cover to reach the finish line.**



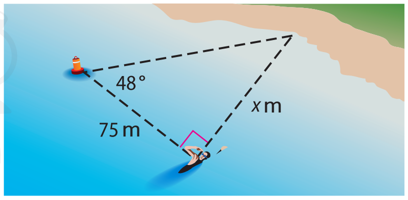

An acute angle measure and the opposite side length are given, so the sine function can be used to find the hypotenuse.

| | |
|---|---|
| $\sin \theta = \frac{\text{opp}}{\text{hyp}}$ | Sine function |
| $\sin 63^\circ = \frac{200}{x}$ | $\theta = 63°$, opp $= 200$, and hyp $= x$ |
| $x \sin 63^\circ = 200$ | Multiply each side by $x$. |
| $x = \frac{200}{\sin 63^\circ}$ or about 224.47 | Divide each side by sin 63°. |

So, the competitor must run about 224.5 m to finish the triathlon.

### GuidedPractice

**4. TRIATHLONS** Suppose a competitor in the swimming portion of the race is swimming along the course shown. Find the distance the competitor must swim to reach the shore.




When a trigonometric value of an acute angle is known, the corresponding **inverse trigonometric function** can be used to find the measure of the angle.

### ReadingMath

**Inverse Trigonometric Ratios**

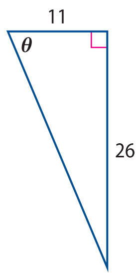
The expression $\sin^{-1} x$ is read *the inverse sine of x*. Be careful not to confuse this notation with the notation for negative exponents: $\sin^{-1} x \neq \frac{1}{\sin x}$. Instead, this notation is similar to the notation for an inverse function, $f^{-1}(x)$.

### KeyConcept Inverse Trigonometric Functions

| | |
|---|---|
| **Inverse Sine** | If $\theta$ is an acute angle and the sine of $\theta$ is $x$, then the **inverse sine** of $x$ is the measure of angle $\theta$. That is, if $\sin \theta = x$, then $\sin^{-1} x = \theta$. |
| **Inverse Cosine** | If $\theta$ is an acute angle and the cosine of $\theta$ is $x$, then the **inverse cosine** of $x$ is the measure of angle $\theta$. That is, if $\cos \theta = x$, then $\cos^{-1} x = \theta$. |
| **Inverse Tangent** | If $\theta$ is an acute angle and the tangent of $\theta$ is $x$, then the **inverse tangent** of $x$ is the measure of angle $\theta$. That is, if $\tan \theta = x$, then $\tan^{-1} x = \theta$. |

## Example 5 Find a Missing Angle Measure

**Use a trigonometric function to find the measure of $\theta$. Round to the nearest degree, if necessary.**

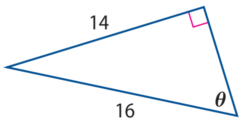



Because the measures of the sides opposite and adjacent to $\theta$ are given, use the tangent function.

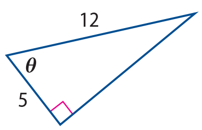
| | |
|---|---|
| $\tan \theta = \frac{\text{opp}}{\text{adj}}$ | Tangent function |
| $\tan \theta = \frac{26}{11}$ | opp $= 26$ and adj $= 11$ |
| $\theta = \tan^{-1} \frac{26}{11}$ or about 67° | Definition of inverse tangent |

### GuidedPractice

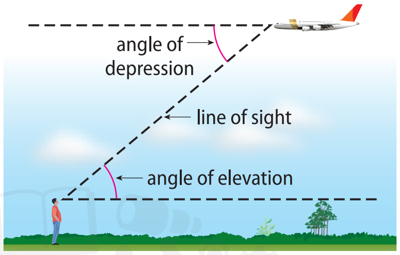
**5A.**




**5B.**



Some applications of trigonometry use an angle of elevation or depression. An **angle of elevation** is the angle formed by a horizontal line and an observer's line of sight to an object above. An **angle of depression** is the angle formed by a horizontal line and an observer's line of sight to an object below.




In the figure, the angles of elevation and depression are congruent because they are alternate interior angles of parallel lines.

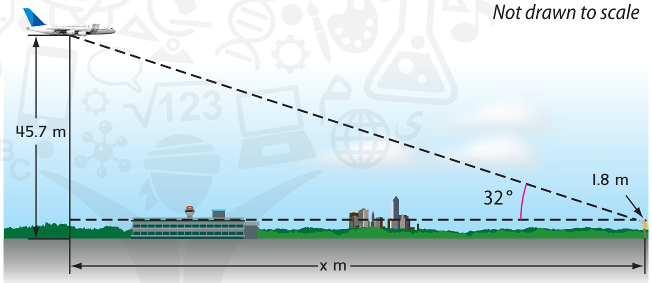
**Real-World Career**

**Airport Ground Crew**
Ground crewpersons operate ramp-servicing vehicles, handle cargo/baggage, and marshal or tow aircraft. Crewpersons must have a high school diploma, a valid driver's license, and a good driving record.

## Real-World Example 6 Use an Angle of Elevation

**AIRPLANES** **A ground crew worker who is 1.8 m tall is directing a plane on a runway. If the worker sights the plane at an angle of elevation of 32°, what is the horizontal distance from the worker to the plane?**

*Not drawn to scale*



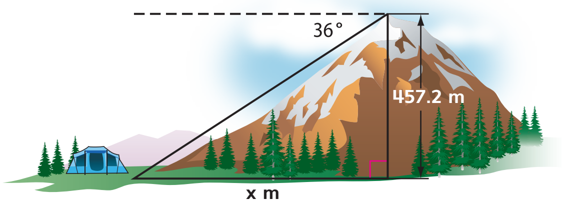

Because the worker is 1.8 m tall, the vertical distance from the worker to the plane is 45.7 − 1.8, or 43.9 m. Because the measures of an angle and opposite side are given in the problem, you can use the tangent function to find $x$.

| | |
|---|---|
| $\tan \theta = \dfrac{\text{opp}}{\text{adj}}$ | Tangent function |
| $\tan 32° = \dfrac{43.9}{x}$ | $\theta = 32°$, opp $= 43.9$, and adj $= x$ |
| $x \tan 32° = 43.9$ | Multiply each side by $x$. |
| $x = \dfrac{43.9}{\tan 32°}$ | Divide each side by tan 32°. |
| $x \approx 70.2$ | Use a calculator. |

So, the horizontal distance from the worker to the plane is approximately 70.2 m.

### Guided Practice

**6.** **CAMPING** A group of hikers on a camping trip climb to the top of a 457.2 m mountain. When the hikers look down at an angle of depression of 36°, they can see the campsite in the distance. What is the horizontal distance between the campsite and the group to the nearest meter?



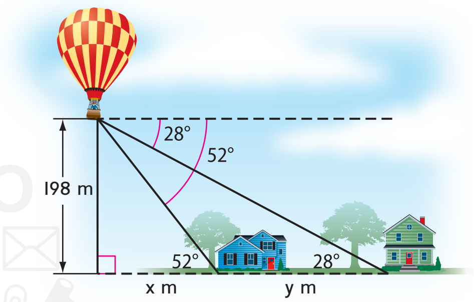
Angles of elevation or depression can be used to estimate the distance between two objects, as well as the height of an object when given two angles from two different positions of observation.

> **StudyTip**
> **Indirect Measurement** When calculating the distance between two objects using angles of depression, it is important to remember that the objects must lie in the same horizontal plane.

## Real-World Example 7 Use Two Angles of Elevation or Depression

**BALLOONING A hot air balloon that is moving above a neighborhood has an angle of depression of 28° to one house and 52° to a house down the street. If the height of the balloon is 198 m, estimate the distance between the two houses.**

Draw a diagram to model this situation. Because the angle of elevation from a house to the balloon is congruent to the angle of depression from the balloon to that house, you can label the angles of elevation as shown. Label the horizontal distance from the balloon to the first house $x$ and the distance between the two houses $y$.




From the smaller right triangle, you can use the tangent function to find $x$.

$$\tan \theta = \frac{\text{opp}}{\text{adj}}$$ Tangent function

$$\tan 52^\circ = \frac{198}{x}$$ $\theta = 52°$, opp $= 198$, and adj $= x$

$$x \tan 52^\circ = 198$$ Multiply each side by $x$.

$$x = \frac{198}{\tan 52^\circ}$$ Divide each side by tan 52°.

From the larger triangle, you can use the tangent function to find $y$.

$$\tan \theta = \frac{\text{opp}}{\text{adj}}$$ Tangent function

$$\tan 28^\circ = \frac{198}{x + y}$$ $\theta = 28°$, opp $= 198$, and adj $= x + y$

$$(x + y) \tan 28^\circ = 198$$ Multiply each side by $x + y$.

$$x + y = \frac{198}{\tan 28^\circ}$$ Divide each side by tan 28°.

$$\frac{198}{\tan 52^\circ} + y = \frac{198}{\tan 28^\circ}$$ Substitute $x = \frac{198}{\tan 52^\circ}$.

$$y = \frac{198}{\tan 28^\circ} - \frac{198}{\tan 52^\circ}$$ Subtract $\frac{198}{\tan 52^\circ}$ from each side.

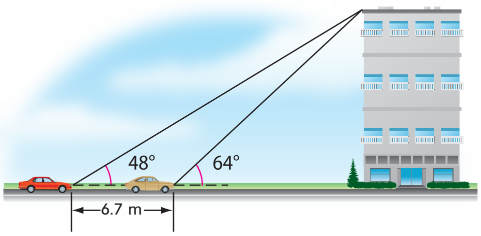
$$y \approx 217.8$$ Use a calculator.

Therefore, the houses are about 217.8 m apart.

> **TechnologyTip**
> **Using Parentheses** When evaluating a trigonometric expression using a graphing calculator, be careful to close parentheses. While a calculator returns the same value for the expressions tan(26 and tan(26), it does not for expressions tan(26 + 50 and tan(26) + 50.

### GuidedPractice

7. **BUILDINGS** The angle of elevation from a car to the top of an apartment building is 48°. If the angle of elevation from another car that is 6.7 m directly in front of the first car is 64°, how tall is the building?



Trigonometric functions and inverse relations can be used to **solve a right triangle**, which means to find the measures of all of the sides and angles of the triangle.

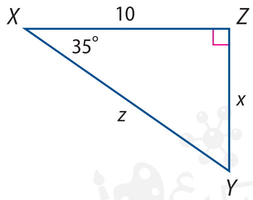
**ReadingMath**

**Labeling Triangles**
Throughout this chapter, a capital letter will be used to represent both a vertex of a triangle and the measure of the angle at that vertex. The same letter in lowercase will be used to represent both the side opposite that angle and the length of that side.

## Example 8 Solve a Right Triangle

**Solve each triangle. Round side lengths to the nearest tenth and angle measures to the nearest degree.**

**a.**




Find $x$ and $z$ using trigonometric functions.

| | | | |
|---|---|---|---|
| $\tan 35^\circ = \frac{x}{10}$ | Substitute. | $\cos 35^\circ = \frac{10}{z}$ | Substitute. |
| $10 \tan 35^\circ = x$ | Multiply. | $z \cos 35^\circ = 10$ | Multiply. |
| $7.0 \approx x$ | Use a calculator. | $z = \frac{10}{\cos 35^\circ}$ | Divide. |
| | | $z \approx 12.2$ | Use a calculator. |

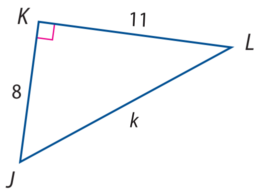
Because the measures of two angles are given, $Y$ can be found by subtracting $X$ from $90^\circ$.

$35^\circ + Y = 90^\circ$ Angles $X$ and $Y$ are complementary.

$Y = 55^\circ$ Subtract.

Therefore, $Y = 55^\circ$, $x \approx 7.0$, and $z \approx 12.2$.

**b.**




Because two side lengths are given, you can use the Pythagorean Theorem to find that $k = \sqrt{185}$ or about 13.6. You can find $J$ by using any of the trigonometric functions.

$\tan J = \frac{11}{8}$ Substitute.

$J = \tan^{-1} \frac{11}{8}$ Definition of inverse tangent

$J \approx 53.97^\circ$ Use a calculator.

Because $J$ is now known, you can find $L$ by subtracting $J$ from $90^\circ$.

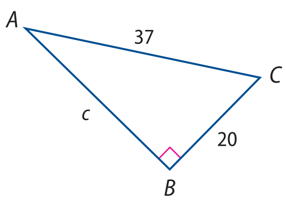
$53.97^\circ + L \approx 90^\circ$ Angles $J$ and $L$ are complementary.

$L \approx 36.03^\circ$ Subtract.

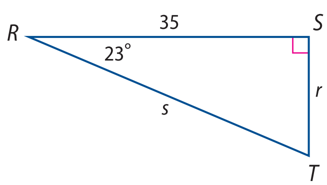
Therefore, $J \approx 54^\circ$, $L \approx 36^\circ$, and $k \approx 13.6$.

**GuidedPractice**

**8A.**

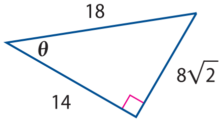



**8B.**



## Exercises

**Find the exact values of the six trigonometric functions of $\theta$.** (Example 1)

**1.**
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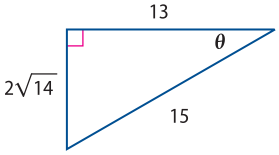


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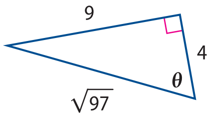


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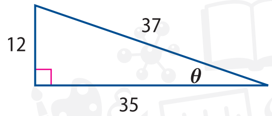


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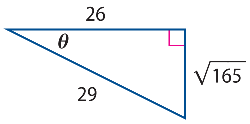


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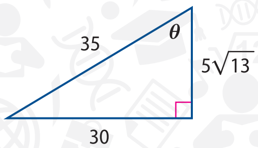


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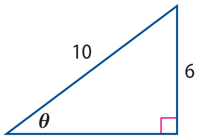
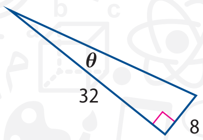



**8.**



**Use the given trigonometric function value of the acute angle $\theta$ to find the exact values of the five remaining trigonometric function values of $\theta$.** (Example 2)

**9.** $\sin\theta = \frac{4}{5}$

**10.** $\cos\theta = \frac{6}{7}$

**11.** $\tan\theta = 3$

**12.** $\sec\theta = 8$

**13.** $\cos\theta = \frac{5}{9}$

**14.** $\tan\theta = \frac{1}{4}$

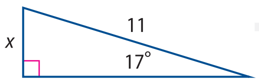
**15.** $\cot\theta = 5$

**16.** $\csc\theta = 6$

**17.** $\sec\theta = \frac{9}{2}$

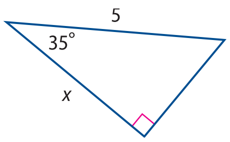
**18.** $\sin\theta = \frac{8}{13}$

**Find the value of $x$. Round to the nearest tenth, if necessary.** (Example 3)

**19.**



**20.**



**21.**
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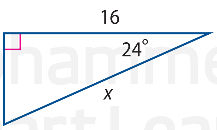


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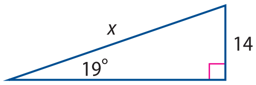
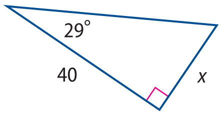
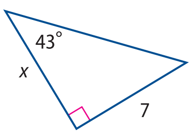
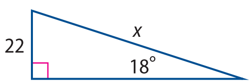
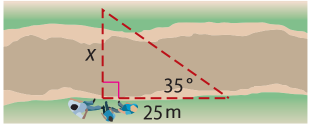



**24.**



**25.**



**26.**



**27. MOUNTAIN CLIMBING** A team of climbers must determine the width of a ravine in order to set up equipment to cross it. If the climbers walk 25 m along the ravine from their crossing point, and sight the crossing point on the far side of the ravine to be at a 35° angle, how wide is the ravine? (Example 4)




**28. SNOWBOARDING** Ahmed built a snowboarding ramp with a height of 3.5 ft and an 18° incline. (Example 4)

**a.** Draw a diagram to represent the situation.

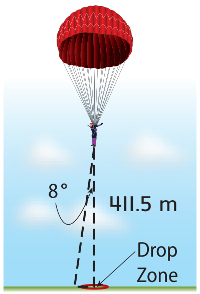
**b.** Determine the length of the ramp.

**29. DETOUR** Traffic is detoured from Nasser Avenue, left 0.8 mile on Etihad Street, and then right on Hessa Street, which intersects Nasser Avenue at a 32° angle. (Example 4)

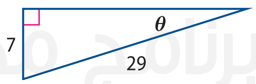
**a.** Draw a diagram to represent the situation.

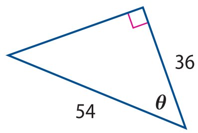
**b.** Determine the length of Nasser Ave. that is detoured.

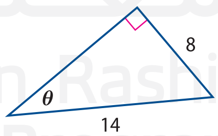
**30. PARACHUTING** A paratrooper encounters stronger winds than anticipated while parachuting from 411.5 m, causing him to drift at an 8° angle. How far from the drop zone will the paratrooper land? (Example 4)

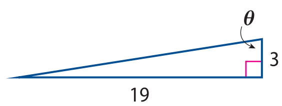


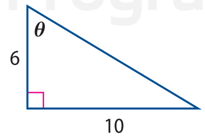

**Find the measure of angle $\theta$. Round to the nearest degree, if necessary.** (Example 5)

**31.**



**32.**



**33.**



**34.**



**35.**
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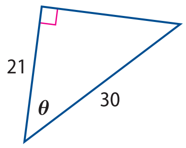
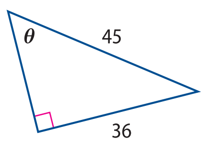
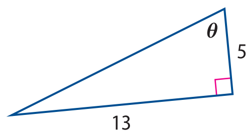
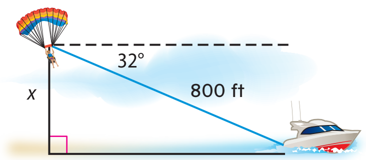


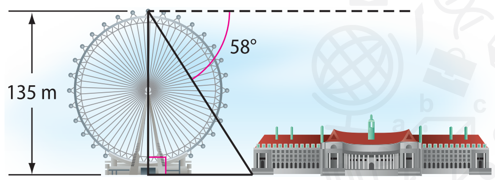

**37.**



**38.**


**39. PARASAILING** Eiman decided to try parasailing. She was strapped into a parachute towed by a boat. An 800 ft line connected her parachute to the boat, which was at a 32° angle of depression below her. How high above the water was Eiman? (Example 6)




**40. OBSERVATION WHEEL** The London Eye is a 135 m-tall observation wheel. If a passenger at the top of the wheel sights the London Aquarium at a 58° angle of depression, what is the distance between the aquarium and the London Eye? (Example 6)




**41. ROLLER COASTER** On a roller coaster, 114.3 ft of track ascend at a 55° angle of elevation to the top before the first and highest drop. (Example 6)

**a.** Draw a diagram to represent the situation.

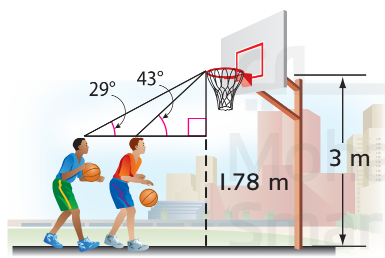
**b.** Determine the height of the roller coaster.

**42. SKI LIFT** A company is installing a new ski lift on a 225 m-high mountain that will ascend at a 48° angle of elevation. (Example 6)

**a.** Draw a diagram to represent the situation.

**b.** Determine the length of cable the lift requires to extend from the base to the peak of the mountain.

**43. BASKETBALL** Both Ahmed and Ali are 1.78 m-tall. Ahmed looks at a 3 m basketball goal with an angle of elevation of 29°, and Ali looks at the goal with an angle of elevation of 43°. If Ali is directly in front of Ahmed, how far apart are the boys standing? (Example 7)




**44. PARIS** A tourist on the first observation level of the Eiffel Tower sights the Musée D'Orsay at a 1.4° angle of depression. A tourist on the third observation level, located 219 m directly above the first, sights the Musée D'Orsay at a 6.8° angle of depression. (Example 7)

**a.** Draw a diagram to represent the situation.

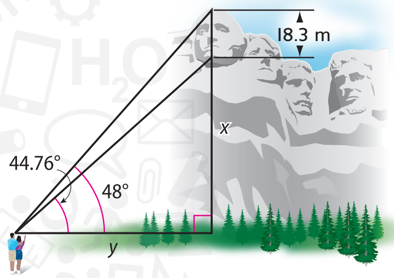
**b.** Determine the distance between the Eiffel Tower and the Musée D'Orsay.

**45. LIGHTHOUSE** Two ships are spotted from the top of a 47.5 m lighthouse. The first ship is at a 27° angle of depression, and the second ship is directly behind the first at a 7° angle of depression. (Example 7)

**a.** Draw a diagram to represent the situation.

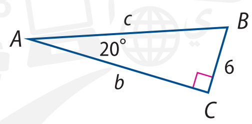
**b.** Determine the distance between the two ships.

**46. MOUNT RUSHMORE** The faces of the presidents at Mount Rushmore are 18.3 m-tall. A visitor sees the top of George Washington's head at a 48° angle of elevation and his chin at a 44.76° angle of elevation. Find the height of Mount Rushmore. (Example 7)

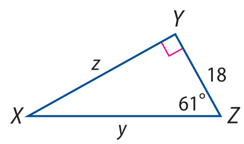



**Solve each triangle. Round side lengths to the nearest tenth and angle measures to the nearest degree.** (Example 8)

**47.**




**48.**

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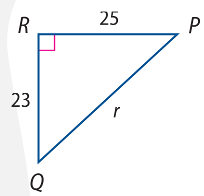


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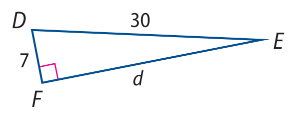
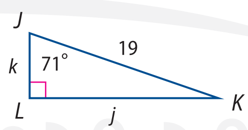



**51.**

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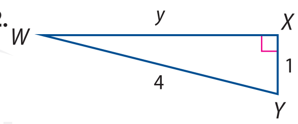


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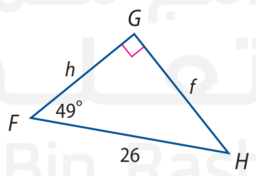


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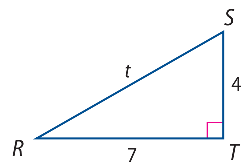



**55. BASEBALL** Ahmed's seat at a game is 19.8 m behind home plate. His line of vision is 3 m above the field.

**a.** Draw a diagram to represent the situation.

**b.** What is the angle of depression to home plate?

**56. HIKING** Rana is standing 2 km from the center of the base of Pikes Peak and looking at the summit of the mountain, which is 1.4 km from the base.

**a.** Draw a diagram to represent the situation.

**b.** With what angle of elevation is Rana looking at the summit of the mountain?

**Find the exact value of each expression without using a calculator.**

**57.** $\sin 60°$ **58.** $\cot 30°$ **59.** $\sec 30°$

**60.** $\cos 45°$ **61.** $\tan 60°$ **62.** $\csc 45°$

**Without using a calculator, find the measure of the acute angle $\theta$ in a right triangle that satisfies each equation.**

**63.** $\tan \theta = 1$

**64.** $\cos \theta = \frac{\sqrt{3}}{2}$

**65.** $\cot \theta = \frac{\sqrt{3}}{3}$

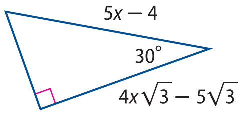
**66.** $\sin \theta = \frac{\sqrt{2}}{2}$

**67.** $\csc \theta = 2$

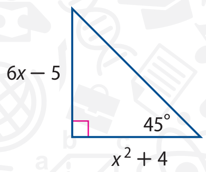
**68.** $\sec \theta = 2$

**Without using a calculator, determine the value of $x$.**

**69.**




**70.**



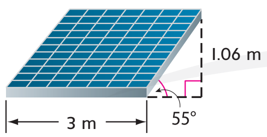

**71.** **SCUBA DIVING** A scuba diver located 6.1 m below the surface of the water spots a shipwreck at a 70° angle of depression. After descending to a point 13.7 m above the ocean floor, the diver sees the shipwreck at a 57° angle of depression. Draw a diagram to represent the situation, and determine the depth of the shipwreck.

**Find the value of $\cos \theta$ if $\theta$ is the measure of the smallest angle in each type of right triangle.**

**72.** 3-4-5

**73.** 5-12-13

**74.** **SOLAR POWER** Find the total area of the solar panel shown below.




**Without using a calculator, insert the appropriate symbol >, <, or = to complete each equation.**

**75.** $\sin 45°$ ◯ $\cot 60°$

**76.** $\tan 60°$ ◯ $\cot 30°$

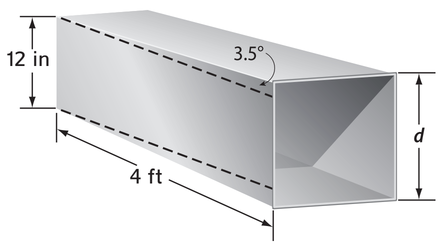
**77.** $\cos 30°$ ◯ $\csc 45°$

**78.** $\cos 30°$ ◯ $\sin 60°$

**79.** $\sec 45°$ ◯ $\csc 60°$

**80.** $\tan 45°$ ◯ $\sec 30°$

**81.** **ENGINEERING** Determine the depth of the shaft at the large end $d$ of the air duct shown below if the taper of the duct is 3.5°.




**82.** **MULTIPLE REPRESENTATIONS** In this problem, you will investigate trigonometric functions of acute angles and their relationship to points on the coordinate plane.

**a.** **GRAPHICAL** Let $P(x, y)$ be a point in the first quadrant. Graph the line through point $P$ and the origin. Form a right triangle by connecting the points $P$, $(x, 0)$, and the origin. Label the lengths of the legs of the triangle in terms of $x$ or $y$. Label the length of the hypotenuse as $r$ and the angle the line makes with the $x$-axis $\theta$.

**b.** **ANALYTICAL** Express the value of $r$ in terms of $x$ and $y$.

**c.** **ANALYTICAL** Express $\sin \theta$, $\cos \theta$, and $\tan \theta$ in terms of $x$, $y$, and/or $r$.

**d.** **VERBAL** Under what condition can the coordinates of point $P$ be expressed as $(\cos \theta, \sin \theta)$?

**e.** **ANALYTICAL** Which trigonometric ratio involving $\theta$ corresponds to the slope of the line?

**f.** **ANALYTICAL** Find an expression for the slope of the line perpendicular to the line in part **a** in terms of $\theta$.

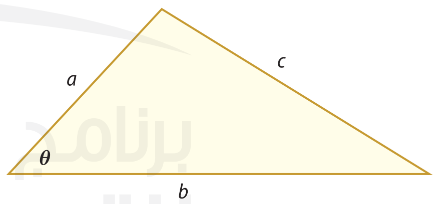
## H.O.T. Problems Use Higher-Order Thinking Skills

**83.** **PROOF** Prove that if $\theta$ is an acute angle of a right triangle, then $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and $\cot \theta = \frac{\cos \theta}{\sin \theta}$.

**84.** **ERROR ANALYSIS** Khalid and Mohammed know the value of $\sin \theta = a$ and are asked to find $\csc \theta$. Khalid says that this is not possible, but Mohammed disagrees. Is either of them correct? Explain your reasoning.

**85.** **WRITING IN MATH** Explain why the six trigonometric functions are transcendental functions.

**86.** **CHALLENGE** Write an expression in terms of $\theta$ for the area of the scalene triangle shown. Explain.




**88.** **PROOF** Prove that if $\theta$ is an acute angle of a right triangle, then $(\sin \theta)^2 + (\cos \theta)^2 = 1$.

**REASONING** **If $A$ and $B$ are the acute angles of a right triangle and $m\angle A < m\angle B$, determine whether each statement is true or false. If false, give a counterexample.**

**88.** $\sin A < \sin B$

**89.** $\cos A < \cos B$

**90.** $\tan A < \tan B$

**91.** **WRITING IN MATH** Notice on a graphing calculator that there is no key for finding the secant, cosecant, or cotangent of an angle measure. Explain why you think this might be so.

## Spiral Review

**92. ECONOMICS** The Consumer Price Index (CPI) measures inflation. It is based on the average prices of goods and services, with the annual average for the years 1982–1984 set at an index of 100. The table shown gives some annual average CPI values from 1955 to 2005. Find an exponential model relating this data (year, CPI) by linearizing the data. Let $x = 0$ represent 1955. Then use your model to predict the CPI for 2025.

| Year | CPI |
|---|---|
| 1955 | 26.8 |
| 1965 | 31.5 |
| 1975 | 53.8 |
| 1985 | 107.6 |
| 1995 | 152.4 |
| 2005 | 195.3 |

**Solve each equation. Round to the nearest hundredth.**

**93.** $e^{5x} = 24$

**94.** $2e^{x-7} - 6 = 0$

**Sketch and analyze the graph of each function. Describe its domain, range, intercepts, asymptotes, end behavior, and where the function is increasing or decreasing.**

**95.** $f(x) = -3^{x-2}$

**96.** $f(x) = 2^{3x-4} + 1$

**97.** $f(x) = -4^{-x+6}$

**Solve each equation.**

**98.** $\dfrac{x^2 - 16}{(x+4)(2x-1)} = \dfrac{4}{x+4} - \dfrac{1}{2x-1}$

**99.** $\dfrac{x^2 - 7}{(x+1)(x-5)} = \dfrac{6}{x+1} + \dfrac{3}{x-5}$

**100.** $\dfrac{2x^2 + 3}{3x^2 + 5x + 2} = \dfrac{5}{3x+2} - \dfrac{1}{x+1}$

**101. NEWSPAPERS** The circulation in thousands of papers of a national newspaper is shown.

| Year | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 | 2008 |
|---|---|---|---|---|---|---|---|
| Circulation (in thousands) | 904.3 | 814.7 | 773.9 | 725.5 | 716.2 | 699.1 | 673.0 |

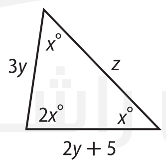
**a.** Let $x$ equal the number of years after 2001. Create a scatter plot of the data.

**b.** Determine a power function to model the data.

**c.** Use the function to predict the circulation of the newspaper in 2015.

## Skills Review for Standardized Tests

**102. SAT/ACT** In the figure, what is the value of $z$?




Note: Figure not drawn to scale.

**A** 15

**B** $15\sqrt{2}$

**C** $15\sqrt{3}$

**D** $30\sqrt{2}$

**E** $30\sqrt{3}$

**103. REVIEW** Mowmed uses a ladder to reach a window 3 m above the ground. If the ladder is 0.9 m away from the wall, how long should the ladder be?

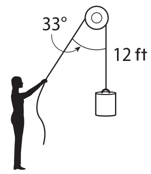
**F** 2.86 m

**G** 3.2 m

**H** 3.4 m

**J** 3.7 m

**104.** A person holds one end of a rope that runs through a pulley and has a weight attached to the other end. Assume that the weight is at the same height as the person's hand. What is the distance from the person's hand to the weight?




**A** 7.8 ft

**B** 10.5 ft

**C** 12.9 ft

**D** 14.3 ft

**105. REVIEW** A kite is being flown at a 45° angle. The string of the kite is 120 ft long. How high is the kite above the point at which the string is held?

**F** 60 ft

**G** $60\sqrt{2}$ ft

**H** $60\sqrt{3}$ ft

**J** 120 ft

# LESSON 3-2 Degrees and Radians

**Then**

- You used the measures of acute angles in triangles given in degrees.

**Now**

1. Convert degree measures of angles to radian measures, and vice versa.
2. Use angle measures to solve real-world problems.

**Why?**

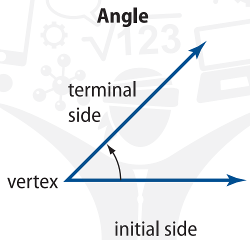
- In Lesson 3-1, you worked only with acute angles, but angles can have *any* real number measurement. For example, in skateboarding, a 540 is an aerial trick in which a skateboarder and the board rotate through an angle of 540°, or one and a half complete turns, in midair.

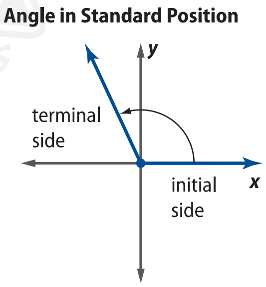
**NewVocabulary**

vertex
initial side
terminal side
standard position
radian
coterminal angles
linear speed
angular speed
sector

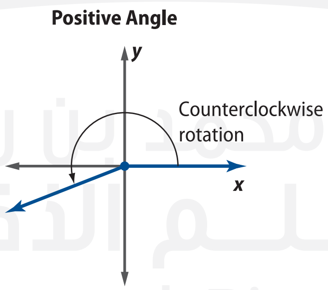
## 1 Angles and Their Measures

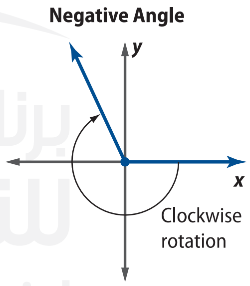
From geometry, you may recall an angle being defined as two noncollinear rays that share a common endpoint known as a **vertex**. An angle can also be thought of as being formed by the action of rotating a ray about its endpoint. From this dynamic perspective, the starting position of the ray forms the **initial side** of the angle, while the ray's position after rotation forms the angle's **terminal side**. In the coordinate plane, an angle with its vertex at the origin and its initial side along the positive x-axis is said to be in **standard position**.



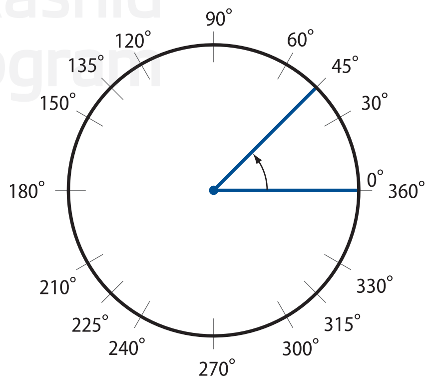

The measure of an angle describes the amount and direction of rotation necessary to move from the initial side to the terminal side of the angle. A *positive angle* is generated by a counterclockwise rotation and a *negative angle* by a clockwise rotation.






The most common angular unit of measure is the *degree* (°), which is equivalent to $\frac{1}{360}$ of a full rotation (counterclockwise) about the vertex. From the diagram shown, you can see that 360° corresponds to 1 complete rotation, 180° to a $\frac{1}{2}$ rotation, 90° to a $\frac{1}{4}$ rotation, and so on, as marked along the circumference of the circle.



**StudyTip**

**Base 60** The concept of degree measurement dates back to the ancient Babylonians, who made early astronomical calculations using their number system, which was based on 60 (sexagesimal) rather than on 10 (decimal) as we do today.

Degree measures can also be expressed using a decimal degree form or a degree-minute-second (DMS) form where each degree is subdivided into 60 minutes (′) and each minute is subdivided into 60 seconds (″).

## Example 1 Convert Between DMS and Decimal Degree Form

**Write each decimal degree measure in DMS form and each DMS measure in decimal degree form to the nearest thousandth.**

**a. 56.735°**

First, convert 0.735° into minutes and seconds.

$$\begin{aligned} 56.735^\circ &= 56^\circ + 0.735^\circ\left(\frac{60'}{1^\circ}\right) && 1^\circ = 60' \\ &= 56^\circ + 44.1' && \text{Simplify.} \end{aligned}$$

Next, convert 0.1′ into seconds.

$$\begin{aligned} 56.735^\circ &= 56^\circ + 44' + 0.1'\left(\frac{60''}{1'}\right) && 1' = 60'' \\ &= 56^\circ + 44' + 6'' && \text{Simplify.} \end{aligned}$$

Therefore, 56.735° can be written as 56° 44′ 6″.

**b. 32° 5′ 28″**

Each minute is $\frac{1}{60}$ of a degree and each second is $\frac{1}{60}$ of a minute, so each second is $\frac{1}{3600}$ of a degree.

$$\begin{aligned} 32^\circ\, 5'\, 28'' &= 32^\circ + 5'\left(\frac{1^\circ}{60'}\right) + 28''\left(\frac{1^\circ}{3600''}\right) && 1' = \frac{1}{60}(1^\circ) \text{ and } 1'' = \frac{1}{3600}(1^\circ) \\ &\approx 32^\circ + 0.083 + 0.008 && \text{Simplify.} \\ &\approx 32.091^\circ && \text{Add.} \end{aligned}$$

Therefore, 32° 5′ 28″ can be written as about 32.091°.

**GuidedPractice**

**1A.** 213.875° **1B.** 89° 56′ 7″

**TechnologyTip**

**DMS Form** You can use some calculators to convert decimal degree values to degrees, minutes, and seconds using the DMS function under the Angle menu.

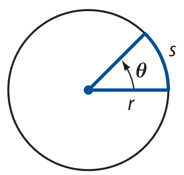
Measuring angles in degrees is appropriate when applying trigonometry to solve many real-world problems, such as those in surveying and navigation. For other applications with trigonometric functions, using an angle measured in degrees poses a significant problem. A degree has no relationship to any linear measure; inch-degrees or $\frac{\text{inch}}{\text{degree}}$ has no meaning. Measuring angles in **radians** provides a solution to this problem.

## KeyConcept Radian Measure

**Words** The measure $\theta$ in radians of a central angle of a circle is equal to the ratio of the length of the intercepted arc $s$ to the radius $r$ of the circle.

**Symbols** $\theta = \frac{s}{r}$, where $\theta$ is measured in radians (rad)

**Example** A central angle has a measure of 1 radian if it intercepts an arc with the same length as the radius of the circle.




$\theta = 1$ radian when $s = r$.

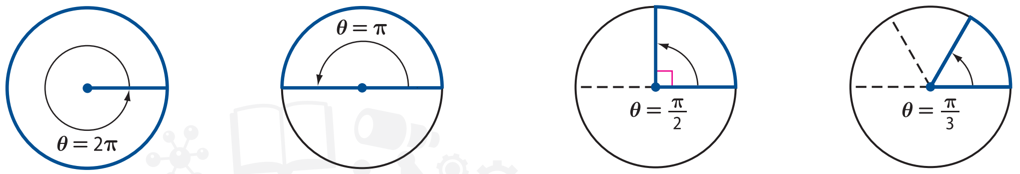
Notice that as long as the arc length $s$ and radius $r$ are measured using the same linear units, the ratio $\frac{s}{r}$ is unitless. For this reason, the word *radian* or its abbreviation *rad* is usually omitted when writing the radian measure of an angle.

**StudyTip**

**Degree-Radian Equivalences** From the equivalence statement shown, you can determine that $1° \approx 0.017$ rad and $1 \text{ rad} \approx 57.296°$.

The central angle representing one full rotation counterclockwise about a vertex corresponds to an arc length equivalent to the circumference of the circle, $2\pi r$. From this, you can obtain the following radian measures.

$$1 \text{ rotation} = \frac{2\pi r}{r} = 2\pi \text{ rad} \qquad \frac{1}{2} \text{ rotation} = \frac{1}{2} \cdot 2\pi = \pi \text{ rad} \qquad \frac{1}{4} \text{ rotation} = \frac{1}{4} \cdot 2\pi = \frac{\pi}{2} \text{ rad} \qquad \frac{1}{6} \text{ rotation} = \frac{1}{6} \cdot 2\pi = \frac{\pi}{3} \text{ rad}$$




Because $2\pi$ radians and $360°$ both correspond to one complete revolution, you can write $360° = 2\pi$ radians or $180° = \pi$ radians. This last equation leads to the following equivalence statements.

$$1° = \frac{\pi}{180} \text{ radians} \qquad \text{and} \qquad 1 \text{ radian} = \left(\frac{180}{\pi}\right)^{\circ}$$

Using these statements, we obtain the following conversion rules.

**ReadingMath**

**Angle Measure** If no units of angle measure are specified, radian measure is implied. If degrees are intended, the degree symbol (°) must be used.

## KeyConcept Degree/Radian Conversion Rules

1. To convert a degree measure to radians, multiply by $\frac{\pi \text{ radians}}{180°}$.
2. To convert a radian measure to degrees, multiply by $\frac{180°}{\pi \text{ radians}}$.

## Example 2 Convert Between Degree and Radian Measure

**Write each degree measure in radians as a multiple of π and each radian measure in degrees.**

**a. 120°**

$120° = 120°\left(\frac{\pi \text{ radians}}{180°}\right)$ Multiply by $\frac{\pi \text{ radians}}{180°}$.

$= \frac{2\pi}{3}$ radians or $\frac{2\pi}{3}$ Simplify.

**b. −45°**

$-45° = -45°\left(\frac{\pi \text{ radians}}{180°}\right)$ Multiply by $\frac{\pi \text{ radians}}{180°}$.

$= -\frac{\pi}{4}$ radians or $-\frac{\pi}{4}$ Simplify.

**c.** $\frac{5\pi}{6}$

$\frac{5\pi}{6} = \frac{5\pi}{6}$ radians Multiply by $\frac{180°}{\pi \text{ radians}}$.

$= \frac{5\pi}{6} \text{ radians}\left(\frac{180°}{\pi \text{ radians}}\right)$ or $150°$ Simplify.

**d.** $-\frac{3\pi}{2}$

$-\frac{3\pi}{2} = -\frac{3\pi}{2}$ radians Multiply by $\frac{180°}{\pi \text{ radians}}$.

$= -\frac{3\pi}{2} \text{ radians}\left(\frac{180°}{\pi \text{ radians}}\right)$ or $-270°$ Simplify.

**GuidedPractice**

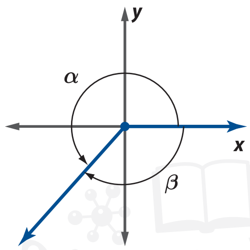
**2A.** $210°$ **2B.** $-60°$ **2C.** $\frac{4\pi}{3}$ **2D.** $-\frac{\pi}{6}$

**ReadingMath**

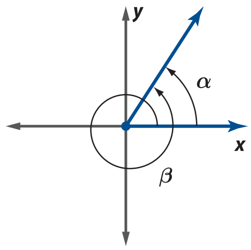
**Naming Angles** In trigonometry, angles are often labeled using Greek letters, such as $\alpha$ (alpha), $\beta$ (beta), and $\theta$ (theta).

By defining angles in terms of their rotation about a vertex, two angles can have the same initial and terminal sides but different measures. Such angles are called **coterminal angles**. In the figures below, angles $\alpha$ and $\beta$ are coterminal.

**Positive and Negative Coterminal Angles**




**Positive Coterminal Angles**




The two positive coterminal angles shown differ by one full rotation. A given angle has infinitely many coterminal angles found by adding or subtracting integer multiples of 360° or $2\pi$ radians.

## KeyConcept Coterminal Angles

| Degrees | Radians |
|---|---|
| If $\alpha$ is the degree measure of an angle, then all angles measuring $\alpha + 360n°$, where $n$ is an integer, are coterminal with $\alpha$. | If $\alpha$ is the radian measure of an angle, then all angles measuring $\alpha + 2n\pi$, where $n$ is an integer, are coterminal with $\alpha$. |

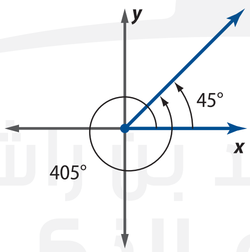
## Example 3 Find and Draw Coterminal Angles

**Identify all angles that are coterminal with the given angle. Then find and draw one positive and one negative angle coterminal with the given angle.**

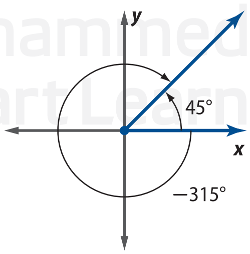
**a.** **45°**

All angles measuring $45° + 360n°$ are coterminal with a $45°$ angle. Let $n = 1$ and $-1$.

$45° + 360(\mathbf{1})° = 45° + 360°$ or $405°$



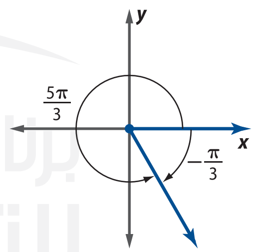

$45° + 360(\mathbf{-1})° = 45° - 360°$ or $-315°$



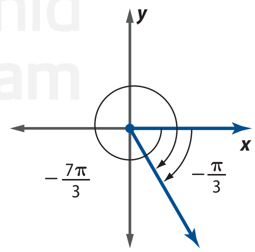

**b.** $-\frac{\pi}{3}$

All angles measuring $-\frac{\pi}{3} + 2n\pi$ are coterminal with a $-\frac{\pi}{3}$ angle. Let $n = 1$ and $-1$.

$-\frac{\pi}{3} + 2(\mathbf{1})\pi = -\frac{\pi}{3} + 2\pi$ or $\frac{5\pi}{3}$




$-\frac{\pi}{3} + 2(\mathbf{-1})\pi = -\frac{\pi}{3} - 2\pi$ or $-\frac{7\pi}{3}$




**GuidedPractice**

**3A.** $-30°$

**3B.** $\frac{3\pi}{4}$

## 2 Applications with Angle Measure

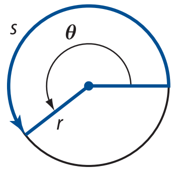
Solving $\theta = \frac{s}{r}$ for the arc length $s$ yields a convenient formula for finding the length of an arc of a circle.

### KeyConcept Arc Length

If $\theta$ is a central angle in a circle of radius $r$, then the length of the intercepted arc $s$ is given by

$$s = r\theta,$$

where $\theta$ is measured in radians.




When $\theta$ is measured in degrees, you could also use the equation $s = \frac{\pi r \theta}{180}$, which already incorporates the degree-radian conversion.

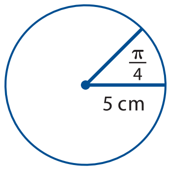
**StudyTip**

**Operating with Radians** Notice in Example 4a that when $r = 5$ centimeters and $\theta = \frac{\pi}{4}$ radians, $s = \frac{5\pi}{4}$ centimeters, not $\frac{5\pi}{4}$ centimeter-radians. This is because a radian is a unitless ratio.

### Example 4 Find Arc Length

**Find the length of the intercepted arc in each circle with the given central angle measure and radius. Round to the nearest tenth.**

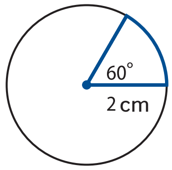
**a.** $\frac{\pi}{4}$, $r = 5$ cm




| | |
|---|---|
| $s = r\theta$ | Arc length |
| $= 5\left(\frac{\pi}{4}\right)$ | $r = 5$ and $\theta = \frac{\pi}{4}$ |
| $= \frac{5\pi}{4}$ | Simplify. |

The length of the intercepted arc is $\frac{5\pi}{4}$ or about 3.9 cm.

**b.** $60°$, $r = 2$ cm




**Method 1** Convert 60° to radian measure, and then use $s = r\theta$ to find the arc length.

| | |
|---|---|
| $60° = 60°\left(\frac{\pi \text{ radians}}{180°}\right)$ | Multiply by $\frac{\pi \text{ radians}}{180°}$. |
| $= \frac{\pi}{3}$ | Simplify. |

Substitute $r = 2$ and $\theta = \frac{\pi}{3}$.

| | |
|---|---|
| $s = r\theta$ | Arc length |
| $= 2\left(\frac{\pi}{3}\right)$ | $r = 2$ and $\theta = \frac{\pi}{3}$ |
| $= \frac{2\pi}{3}$ | Simplify. |

**Method 2** Use $s = \frac{\pi r \theta}{180°}$ to find the arc length.

| | |
|---|---|
| $s = \frac{\pi r \theta}{180°}$ | Arc length |
| $= \frac{\pi(2)(60°)}{180°}$ | $r = 2$ and $\theta = 60°$ |
| $= \frac{2\pi}{3}$ | Simplify. |

The length of the intercepted arc is $\frac{2\pi}{3}$ or about 2.1 cm.

**GuidedPractice**

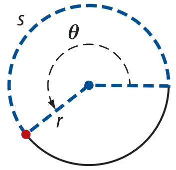
**4A.** $\frac{2\pi}{3}$, $r = 2$ m

**4B.** $135°$, $r = 0.5$ m

The formula for arc length can be used to analyze circular motion. The rate at which an object moves along a circular path is called its **linear speed**. The rate at which the object *rotates* about a fixed point is called its **angular speed**. Linear speed is measured in units like miles per hour, while angular speed is measured in units like revolutions per minute.

## KeyConcept Linear and Angular Speed

Suppose an object moves at a constant speed along a circular path of radius $r$.

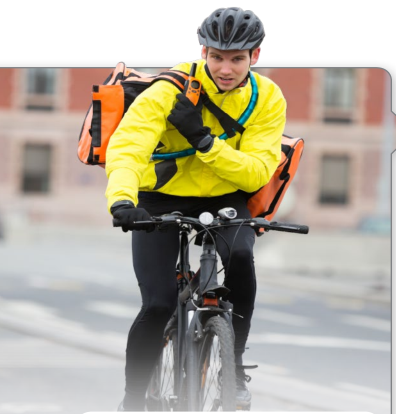



If $s$ is the arc length traveled by the object during time $t$, then the object's *linear speed* $v$ is given by $v = \frac{s}{t}$.

If $\theta$ is the angle of rotation (in radians) through which the object moves during time $t$, then the *angular speed* $\omega$ of the object is given by $\omega = \frac{\theta}{t}$.

**ReadingMath**

**Omega** The lowercase Greek letter omega $\omega$ is usually used to denote angular speed.



**Real-WorldLink**

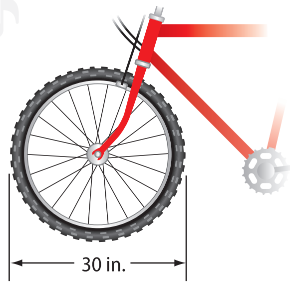
In some U.S. cities, it is possible for bicycle messengers to ride an average of 48 to 56 kilometers a day while making 30 to 45 deliveries.

**Source:** New York Bicycle Messenger Association

## Real-World Example 5 Find Angular and Linear Speeds

**BICYCLING** **A bicycle messenger rides the bicycle shown.**

**a. During one delivery, the tires rotate at a rate of 140 revolutions per minute. Find the angular speed of the tire in radians per minute.**




Because each rotation measures $2\pi$ radians, 140 revolutions correspond to an angle of rotation $\theta$ of $140 \times 2\pi$ or $280\pi$ radians.

$\omega = \frac{\theta}{t}$     Angular speed

$= \frac{280\pi \text{ radians}}{1 \text{ minute}}$     $\theta = 280\pi$ radians and $t = 1$ minute

Therefore, the angular speed of the tire is $280\pi$ or about 879.6 radians per minute.

**b. On part of the trip to the next delivery, the tire turns at a constant rate of 2.5 revolutions per second. Find the linear speed of the tire in kilometers per hour.**

A rotation of 2.5 revolutions corresponds to an angle of rotation $\theta$ of $2.5 \times 2\pi$ or $5\pi$.

$v = \frac{s}{t}$     Linear speed

$= \frac{r\theta}{t}$     $s = r\theta$

$= \frac{38.1(5\pi) \text{ centimeters}}{1 \text{ second}}$ or $\frac{190.5\pi \text{ centimeters}}{1 \text{ second}}$     $r = 38.1$ centimeters, $\theta = 5\pi$ radians, and $t = 1$ second

Use dimensional analysis to convert this speed from centimeters per second to kilometers per hour.

$$\frac{190.5\pi \cancel{\text{ centimeters}}}{1 \cancel{\text{ second}}} \times \frac{60 \cancel{\text{ seconds}}}{1 \cancel{\text{ minute}}} \times \frac{60 \cancel{\text{ minutes}}}{1 \text{ hour}} \times \frac{1 \cancel{\text{ meter}}}{100 \cancel{\text{ centimeters}}} \times \frac{1 \text{ kilometer}}{1000 \text{ meter}} \approx \frac{21.6 \text{ kilometers}}{\text{hour}}$$

Therefore, the linear speed of the tire is about 21.6 km/h.

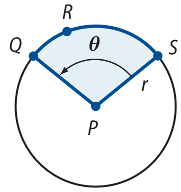
### GuidedPractice

**MEDIA** **Consider the DVD shown.**


**5A.** Find the angular speed of the DVD in radians per second if the disc rotates at a rate of 3.5 revolutions per second.

**5B.** If the DVD player overheats and the disc begins to rotate at a slower rate of 3 revolutions per second, find the disc's linear speed in meters per minute.

Recall from geometry that a **sector** of a circle is a region bounded by a central angle and its intercepted arc. For example, the shaded portion in the figure is a sector of circle $P$. The ratio of the area of a sector to the area of a whole circle is equal to the ratio of the corresponding arc length to the circumference of the circle. Let $A$ represent the area of the sector.




$$\frac{A}{\pi r^2} = \frac{\text{length of } \widehat{QRS}}{2\pi r}$$ $\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{arc length}}{\text{circumference of circle}}$

$$\frac{A}{\pi r^2} = \frac{r\theta}{2\pi r}$$ The length of $\widehat{QRS}$ is $r\theta$.

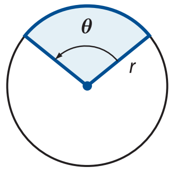
$$A = \frac{1}{2}r^2\theta$$ Solve for $A$.

## KeyConcept Area of a Sector

The area $A$ of a sector of a circle with radius $r$ and central angle $\theta$ is

$$A = \frac{1}{2}r^2\theta,$$

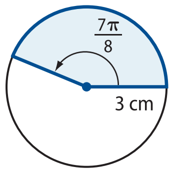
where $\theta$ is measured in radians.




## Example 6 Find Areas of Sectors

**a. Find the area of the sector of the circle.**

The measure of the sector's central angle $\theta$ is $\frac{7\pi}{8}$, and the radius is 3 cm.

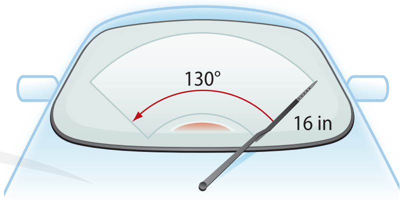



$$A = \frac{1}{2}r^2\theta$$ Area of a sector

$$= \frac{1}{2}(3)^2\left(\frac{7\pi}{8}\right) \text{ or } \frac{63\pi}{16}$$ $r = 3$ and $\theta = \frac{7\pi}{8}$

Therefore, the area of the sector is $\frac{63\pi}{16}$ or about 12.4 $\text{cm}^2$.

**b. WIPERS Find the approximate area swept by the wiper blade shown, if the total length of the windshield wiper mechanism is 26 inches.**




The area swept by the wiper blade is the difference between the areas of the sectors with radii 26 inches and 26 − 16 or 10 inches.

Convert the central angle measure to radians.

$$130° = 130°\left(\frac{\pi \text{ radians}}{180°}\right) = \frac{13\pi}{18}$$

Then use the radius of each sector to find the area swept. Let $A_1$ = the area of the sector with a 26-inch radius, and let $A_2$ = the area of the sector with a 10-inch radius.

$$A = A_1 - A_2$$ Swept area

$$= \frac{1}{2}(26)^2\left(\frac{13\pi}{18}\right) - \frac{1}{2}(10)^2\left(\frac{13\pi}{18}\right)$$ Area of a sector

$$= \frac{2197\pi}{9} - \frac{325\pi}{9}$$ Simplify.

$$= 208\pi \text{ or about } 653.5$$ Simplify.

Therefore, the swept area is about 653.5 $\text{in}^2$.

### Real-WorldLink

A typical wipe angle for a front windshield wiper of a passenger car is about 67°. Windshield wiper blades are generally 30 to 76 centimeters long.

**Source:** *Car and Driver*

### GuidedPractice

**Find the area of the sector of a circle with the given central angle $\theta$ and radius $r$.**

**6A.** $\theta = \frac{3\pi}{4}$, $r = 1.5$ ft

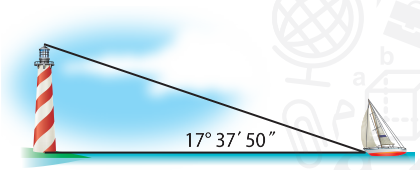
**6B.** $\theta = 50°$, $r = 6$ m

## Exercises

**Write each decimal degree measure in DMS form and each DMS measure in decimal degree form to the nearest thousandth.** (Example 1)

1. $11.773°$
2. $58.244°$
3. $141.549°$
4. $273.396°$
5. $87° 53' 10''$
6. $126° 6' 34''$
7. $45° 21' 25''$
8. $301° 42' 8''$

9. **NAVIGATION** A sailing enthusiast uses a sextant, an instrument that can measure the angle between two objects with a precision to the nearest 10 seconds, to measure the angle between his sailboat and a lighthouse. If his reading is $17° 37' 50''$, what is the measure in decimal degree form to the nearest hundredth? (Example 1)



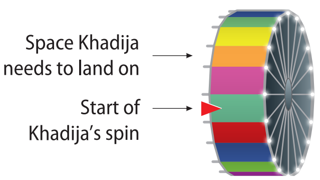

**Write each degree measure in radians as a multiple of π and each radian measure in degrees.** (Example 2)

10. $30°$
11. $225°$
12. $-165°$
13. $-45°$
14. $\frac{2\pi}{3}$
15. $\frac{5\pi}{2}$
16. $-\frac{\pi}{4}$
17. $-\frac{7\pi}{6}$

**Identify all angles that are coterminal with the given angle. Then find and draw one positive and one negative angle coterminal with the given angle.** (Example 3)

18. $120°$
19. $-75°$
20. $225°$
21. $-150°$
22. $\frac{\pi}{3}$
23. $-\frac{3\pi}{4}$
24. $-\frac{\pi}{12}$
25. $\frac{3\pi}{2}$

26. **GAME SHOW** Khadija is spinning a wheel on a game show. There are 20 values in equal-sized spaces around the circumference of the wheel. The value that Khadija needs to win is two spaces above the space where she starts her spin, and the wheel must make at least one full rotation for the spin to count. Describe a spin rotation in degrees that will give Khadija a winning result. (Example 3)




**Find the length of the intercepted arc with the given central angle measure in a circle with the given radius. Round to the nearest tenth.** (Example 4)

27. $\frac{\pi}{6}$, $r = 2.5$ m
28. $\frac{2\pi}{3}$, $r = 3$ cm
29. $\frac{5\pi}{12}$, $r = 4$ m
30. $105°$, $r = 18.2$ cm
31. $45°$, $r = 5$ km
32. $150°$, $r = 79$ mm

33. **AMUSEMENT PARK** A carousel at an amusement park rotates $3024°$ per ride. (Example 4)
    a. How far would a rider seated 4 m from the center of the carousel travel during the ride?
    b. How much farther would a second rider seated 5.5 m from the center of the carousel travel during the ride than the rider in part **a**?

**Find the rotation in revolutions per minute given the angular speed and the radius given the linear speed and the rate of rotation.** (Example 5)

34. $\omega = \frac{2}{3}\pi$ rad/s
35. $\omega = 135\pi$ rad/h
36. $\omega = 104\pi$ rad/min
37. $v = 82.3$ m/s, 131 rev/min
38. $v = 144.2$ m/min, 10.9 rev/min
39. $v = 553$ cm/h, 0.09 rev/min

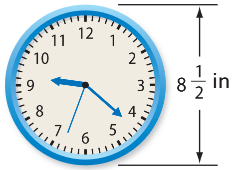
40. **MANUFACTURING** A company manufactures several circular saws with the blade diameters and motor speeds shown below. (Example 5)

| Blade Diameter (cm) | Motor Speed (rps) |
|---|---|
| 3 | 2800 |
| 5 | 5500 |
| $5\frac{1}{2}$ | 4500 |
| $6\frac{1}{8}$ | 5500 |
| $7\frac{1}{4}$ | 5000 |

    a. Determine the angular and linear speeds of the blades in each saw. Round to the nearest tenth.
    b. How much faster is the linear speed of the $6\frac{1}{8}$ cm saw compared to the 3 cm saw?

41. **CARS** On a stretch of interstate, a vehicle's tires range between 646 and 840 revolutions per minute. The diameter of each tire is 66 cm. (Example 5)
    a. Find the range of values for the angular speeds of the tires in radians per minute.
    b. Find the range of values for the linear speeds of the tires in kilometers per hour.

**42.** **TIME** A wall clock has a face diameter of $8\frac{1}{2}$ in. The length of the hour hand is 2.4 in, the length of the minute hand is 3.2 in, and the length of the second hand is 3.4 in. (Example 5)

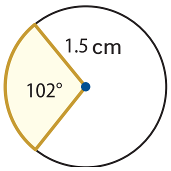



**a.** Determine the angular speed in radians per hour and the linear speed in centimeters per hour for each hand.

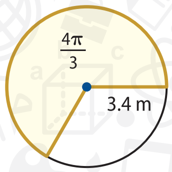
**b.** If the linear speed of the second hand is 20 in per minute, is the clock running fast or slow? How much time would it gain or lose per day?

**GEOMETRY** **Find the area of each sector.** (Example 6)

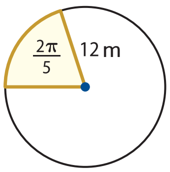
**43.**



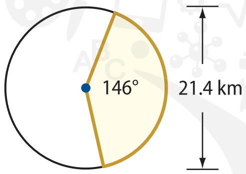

**44.**



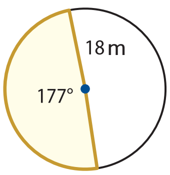

**45.**



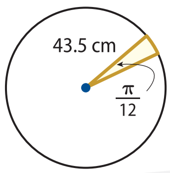

**46.**



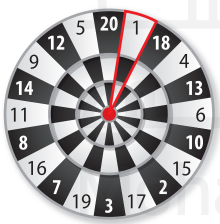

**47.**




**48.**




**49.** **GAMES** The dart board shown is divided into twenty equal sectors. If the diameter of the board is 18 in, what area of the board does each sector cover? (Example 6)



**50.** **LAWN CARE** A sprinkler waters an area that forms one third of a circle. If the stream from the sprinkler extends 6 ft, what area of the grass does the sprinkler water? (Example 6)

**The area of a sector of a circle and the measure of its central angle are given. Find the radius of the circle.**

**51.** $A = 29\text{ ft}^2$, $\theta = 68°$

**52.** $A = 808\text{ cm}^2$, $\theta = 210°$

**53.** $A = 377\text{ in}^2$, $\theta = \frac{5\pi}{3}$

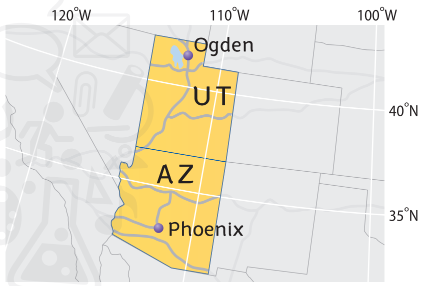
**54.** $A = 75\text{ m}^2$, $\theta = \frac{3\pi}{4}$

**55.** Describe the radian measure between 0 and $2\pi$ of an angle $\theta$ that is in standard position with a terminal side that lies in:

**a.** Quadrant I
**b.** Quadrant II
**c.** Quadrant III
**d.** Quadrant IV

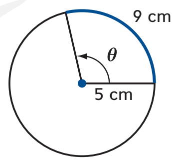
**56.** If the terminal side of an angle that is in standard position lies on one of the axes, it is called a *quadrantal angle.* Give the radian measures of four quadrantal angles.

**57.** **GEOGRAPHY** Phoenix, Arizona, and Ogden, Utah, are located on the same line of longitude, which means that Ogden is directly north of Phoenix. The latitude of Phoenix is 33° 26′ N, and the latitude of Ogden is 41° 12′ N. If Earth's radius is approximately 6378 kilometers, about how far apart are the two cities?

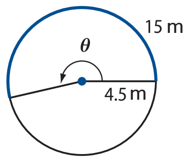



**Find the measure of angle $\theta$ in radians and degrees.**

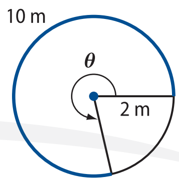
**58.**



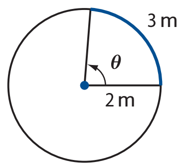

**59.**



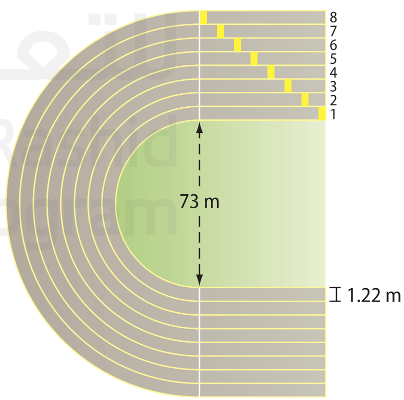

**60.**




**61.**




**62.** **TRACK** The curve of a standard 8-lane track is semicircular as shown.



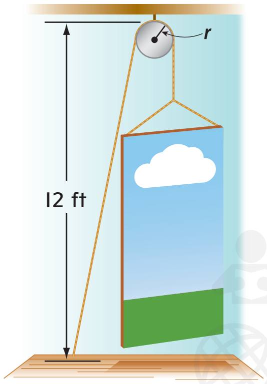

**a.** What is the length of the outside edge of Lane 4 in the curve?

**b.** How much longer is the inside edge of Lane 7 than the inside edge of Lane 3 in the curve?

63. **DRAMA** A pulley with radius $r$ is being used to remove part of the set of a play during intermission. The height of the pulley is 12 ft.

**a.** If the radius of the pulley is 6 in, and it rotates 180°, how high will the object be lifted?

**b.** If the radius of the pulley is 4 in, and it rotates 900°, how high will the object be lifted?




64. **ENGINEERING** A pulley like the one in Exercise 63 is being used to lift a crate in a warehouse. Determine which of the following scenarios could be used to lift the crate a distance of 4.6 m the fastest. Explain how you reached your conclusion.

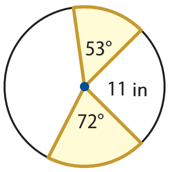
**I.** The radius of the pulley is 12.7 cm rotating at 65 revolutions per minute.

**II.** The radius of the pulley is 11.4 cm rotating at 70 revolutions per minute.

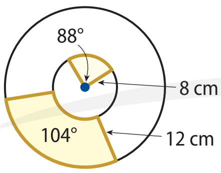
**III.** The radius of the pulley is 15.3 cm rotating at 60 revolutions per minute.

**GEOMETRY** **Find the area of each shaded region.**

65.




66.




67. **CARS** The speedometer shown measures the speed of a car in miles per hour.


**a.** If the angle between 25 mi/h and 60 mi/h is 81.1°, about how many miles per hour are represented by each degree?

**b.** If the angle of the speedometer changes by 95°, how much did the speed of the car increase?

**Find the complement and supplement of each angle, if possible. If not possible, explain your reasoning.**

68. $\frac{2\pi}{5}$

69. $\frac{5\pi}{6}$

70. $\frac{3\pi}{8}$

71. $-\frac{\pi}{3}$

72. **SKATEBOARDING** A physics class conducted an experiment to test three different wheel sizes on a skateboard with constant angular speed.

**a.** Write an equation for the linear speed of the skateboard in terms of the radius and angular speed. Explain your reasoning.

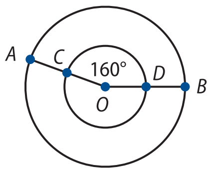
**b.** Using the equation you wrote in part **a**, predict the linear speed in meters per second of a skateboard with an angular speed of 3 revolutions per second for wheel diameters of 52, 56, and 60 mm.

**c.** Based on your results in part **b**, how do you think wheel size affects linear speed?

## H.O.T. Problems Use Higher-Order Thinking Skills

73. **ERROR ANALYSIS** Rana and Khadija are told that the perimeter of a sector of a circle is 10 times the length of the circle's radius. Rana thinks that the radian measure of the sector's central angle is 8 radians. Khadija thinks that there is not enough information given to solve the problem. Is either of them correct? Explain your reasoning.

74. **CHALLENGE** The two circles shown are concentric. If the length of the arc from $A$ to $B$ measures $8\pi$ in, and $DB = 2$ in, find the arc length from $C$ to $D$ in terms of $\pi$.




**REASONING** **Describe how the linear speed would change for each parameter below. Explain.**

75. a decrease in the radius

76. a decrease in the unit of time

77. an increase in the angular speed

78. **PROOF** If $\frac{s_1}{r_1} = \frac{s_2}{r_2}$, prove that $\theta_1 = \theta_2$.

79. **REASONING** What effect does doubling the radius of a circle have on each of the following measures? Explain your reasoning.

**a.** the perimeter of the sector of the circle with a central angle that measures $\theta$ radians

**b.** the area of a sector of the circle with a central angle that measures $\theta$ radians

80. **WRITING IN MATH** Compare and contrast degree and radian measures. Then create a diagram similar to the one on page 231. Label the diagram using degree measures on the inside and radian measures on the outside of the circle.

## Spiral Review

**Use the given trigonometric function value of the acute angle $\theta$ to find the exact values of the five remaining trigonometric function values of $\theta$.**

**81.** $\sin\theta = \frac{8}{15}$

**82.** $\sec\theta = \frac{4\sqrt{7}}{10}$

**83.** $\cot\theta = \frac{17}{19}$

**84. BANKING** An account that Wafa's grandmother opened in 1955 earned continuously compounded interest. The table shows the balances of the account from 1955 to 1959.

**a.** Use regression to find a function that models the amount in the account. Use the number of years after Jan. 1, 1955, as the independent variable.

**b.** Write the equation from part **a** in terms of base $e$.

**c.** What was the interest rate on the account if no deposits or withdrawals were made during the period in question?

| | Date | Balance |
|---|---|---|
| 1 | Jan. 1, 1955 | AED 2137.52 |
| 2 | Jan. 1, 1956 | AED 2251.61 |
| 3 | Jan. 1, 1957 | AED 2371.79 |
| 4 | Jan. 1, 1958 | AED 2498.39 |
| 5 | Jan. 1, 1959 | AED 2631.74 |

**Express each logarithm in terms of ln 2 and ln 5.**

**85.** $\ln\frac{25}{16}$

**86.** $\ln 250$

**87.** $\ln\frac{10}{25}$

**List all possible rational zeros of each function. Then determine which, if any, are zeros.**

**88.** $f(x) = x^4 - x^3 - 12x - 144$

**89.** $g(x) = x^3 - 5x^2 - 4x + 20$

**90.** $g(x) = 6x^4 + 35x^3 - x^2 - 7x - 1$

**Describe the end behavior of each function.**

**91.** $f(x) = 4x^5 + 2x^4 - 3x - 1$

**92.** $g(x) = -x^6 + x^4 - 5x^2 + 4$

**93.** $h(x) = -\frac{1}{x^3} + 2$

**Write each set in set-builder and interval notation, if possible.**

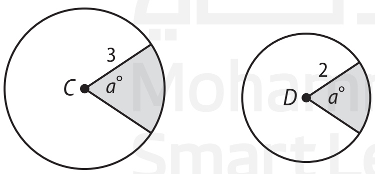
**94.** $n > -7$

**95.** $-4 \le x < 10$

**96.** $y < 1$ or $y \ge 11$

## Skills Review for Standardized Tests

**97. SAT/ACT** In the figure, $C$ and $D$ are the centers of the two circles with radii of 3 and 2, respectively. If the larger shaded region has an area of 9, what is the area of the smaller shaded region?




Note: Figure not drawn to scale.

**A** 3
**B** 4
**C** 5
**D** 7
**E** 8

**98. REVIEW** If $\cot\theta = 1$, then $\tan\theta =$

**F** $-1$
**G** 0
**H** 1
**J** 3

**99. REVIEW** If $\sec\theta = \frac{25}{7}$ and $\theta$ is acute, then $\sin\theta =$

**A** $\frac{7}{25}$

**B** $\frac{24}{25}$

**C** $-\frac{24}{25}$

**D** $\frac{25}{7}$

**100.** Which of the following radian measures is equal to 56°?

**F** $\frac{\pi}{15}$

**G** $\frac{7\pi}{45}$

**H** $\frac{14\pi}{45}$

**J** $\frac{\pi}{3}$

# LESSON 3-3 Trigonometric Functions on the Unit Circle

**Then**

- You found values of trigonometric functions for acute angles using ratios in right triangles.

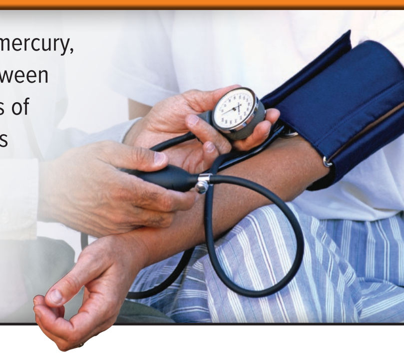
**Now**

1. Find values of trigonometric functions for any angle.
2. Find values of trigonometric functions using the unit circle.

**Why?**

- A blood pressure of 120 over 80, measured in millimeters of mercury, means that a person's blood pressure oscillates or cycles between 20 millimeters above and below a pressure of 100 millimeters of mercury for a given time $t$ in seconds. A complete cycle of this oscillation takes about 1 second.

  If the pressure exerted by the blood at time $t = 0.25$ second is 120 millimeters of mercury, then at time $t = 1.25$ seconds the pressure is also 120 millimeters of mercury.



**NewVocabulary**
quadrantal angle
reference angle
unit circle
circular function
periodic function
period

## 1 Trigonometric Functions of Any Angle

In Lesson 3-1, the definitions of the six trigonometric functions were restricted to positive acute angles. In this lesson, these definitions are extended to include *any* angle.

### KeyConcept Trigonometric Functions of Any Angle

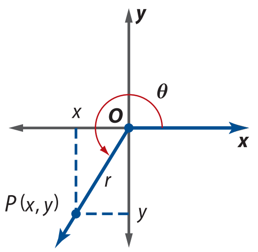
Let $\theta$ be any angle in standard position and point $P(x, y)$ be a point on the terminal side of $\theta$. Let $r$ represent the nonzero distance from $P$ to the origin.

That is, let $r = \sqrt{x^2 + y^2} \neq 0$. Then the trigonometric functions of $\theta$ are as follows.

$\sin \theta = \frac{y}{r}$ $\qquad$ $\csc \theta = \frac{r}{y}, y \neq 0$

$\cos \theta = \frac{x}{r}$ $\qquad$ $\sec \theta = \frac{r}{x}, x \neq 0$

$\tan \theta = \frac{y}{x}, x \neq 0$ $\qquad$ $\cot \theta = \frac{x}{y}, y \neq 0$




### Example 1 Evaluate Trigonometric Functions Given a Point

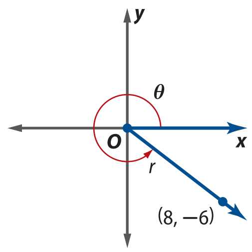
**Let $(8, -6)$ be a point on the terminal side of an angle $\theta$ in standard position. Find the exact values of the six trigonometric functions of $\theta$.**

Use the values of $x$ and $y$ to find $r$.

$r = \sqrt{x^2 + y^2}$ $\qquad$ Pythagorean Theorem

$= \sqrt{8^2 + (-6)^2}$ $\qquad$ $x = 8$ and $y = -6$

$= \sqrt{100}$ or 10 $\qquad$ Take the positive square root.




Use $x = 8$, $y = -6$, and $r = 10$ to write the six trigonometric ratios.

$\sin \theta = \frac{y}{r} = \frac{-6}{10}$ or $-\frac{3}{5}$ $\qquad$ $\cos \theta = \frac{x}{r} = \frac{8}{10}$ or $\frac{4}{5}$ $\qquad$ $\tan \theta = \frac{y}{x} = \frac{-6}{8}$ or $-\frac{3}{4}$

$\csc \theta = \frac{r}{y} = \frac{10}{-6}$ or $-\frac{5}{3}$ $\qquad$ $\sec \theta = \frac{r}{x} = \frac{10}{8}$ or $\frac{5}{4}$ $\qquad$ $\cot \theta = \frac{x}{y} = \frac{8}{-6}$ or $-\frac{4}{3}$

**GuidedPractice**

**The given point lies on the terminal side of an angle $\theta$ in standard position. Find the values of the six trigonometric functions of $\theta$.**

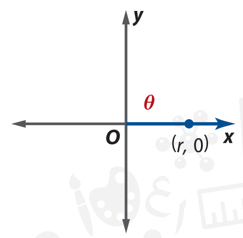
**1A.** $(4, 3)$ $\qquad$ **1B.** $(-2, -1)$

In Example 1, you found the trigonometric values of $\theta$ without knowing the measure of $\theta$. Now we will discuss methods for finding these function values when only $\theta$ is known. Consider trigonometric functions of quadrantal angles. When the terminal side of an angle $\theta$ that is in standard position lies on one of the coordinate axes, the angle is called a **quadrantal angle**.

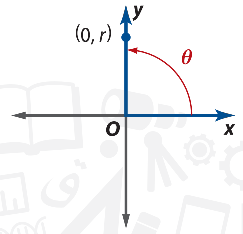
**StudyTip**

**Quadrantal Angles** There are infinitely many quadrantal angles that are coterminal with the quadrantal angles listed at the right. The measure of a quadrantal angle is a multiple of 90° or $\frac{\pi}{2}$.

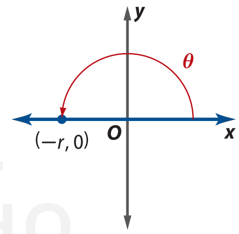
## KeyConcept Common Quadrantal Angles



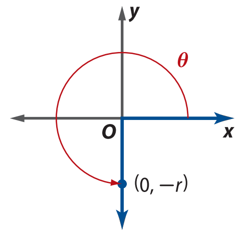

$\theta = 0°$ or 0 radians




$\theta = 90°$ or $\frac{\pi}{2}$ radians




$\theta = 180°$ or $\pi$ radians




$\theta = 270°$ or $\frac{3\pi}{2}$ radians

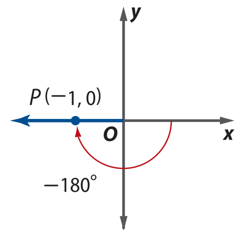
You can find the values of the trigonometric functions of quadrantal angles by choosing a point on the terminal side of the angle and evaluating the function at that point. Any point can be chosen. However, to simplify calculations, pick a point for which $r$ equals 1.

## Example 2 Evaluate Trigonometric Functions of Quadrantal Angles

**Find the exact value of each trigonometric function, if defined. If not defined, write *undefined*.**

**a. sin (−180°)**

The terminal side of −180° in standard position lies on the negative $x$-axis. Choose a point $P$ on the terminal side of the angle. A convenient point is (−1, 0) because $r = 1$.

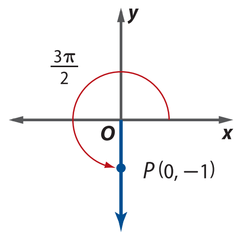



$\sin(-180°) = \frac{y}{r}$ — Sine function

$= \frac{0}{1}$ or 0 — $y = 0$ and $r = 1$

**b. tan $\frac{3\pi}{2}$**

The terminal side of $\frac{3\pi}{2}$ in standard position lies on the negative $y$-axis. Choose a point $P(0, -1)$ on the terminal side of the angle because $r = 1$.

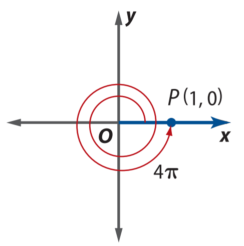



$\tan\frac{3\pi}{2} = \frac{y}{x}$ — Tangent function

$= \frac{-1}{0}$ or undefined — $y = -1$ and $x = 0$

**c. sec 4π**

The terminal side of $4\pi$ in standard position lies on the positive $x$-axis. The point (1, 0) is convenient because $r = 1$.




$\sec 4\pi = \frac{r}{x}$ — Secant function

$= \frac{1}{1}$ or 1 — $r = 1$ and $x = 1$

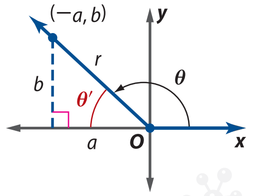
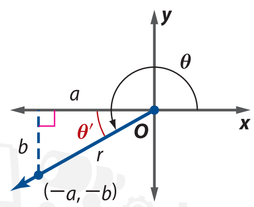
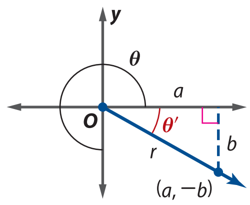
**GuidedPractice**

**2A.** cos 270°

**2B.** csc $\frac{\pi}{2}$

**2C.** cot (−90°)

To find the values of the trigonometric functions of angles that are neither acute nor quadrantal, consider the three cases shown below in which $a$ and $b$ are positive real numbers. Compare the values of sine, cosine, and tangent of $\theta$ and $\theta'$.

| Quadrant II | Quadrant III | Quadrant IV |
|---|---|---|
|  |  |  |
| $\sin\theta = \frac{b}{r}$ $\quad \sin\theta' = \frac{b}{r}$ | $\sin\theta = -\frac{b}{r}$ $\quad \sin\theta' = \frac{b}{r}$ | $\sin\theta = -\frac{b}{r}$ $\quad \sin\theta' = \frac{b}{r}$ |
| $\cos\theta = -\frac{a}{r}$ $\quad \cos\theta' = \frac{a}{r}$ | $\cos\theta = -\frac{a}{r}$ $\quad \cos\theta' = \frac{a}{r}$ | $\cos\theta = \frac{a}{r}$ $\quad \cos\theta' = \frac{a}{r}$ |
| $\tan\theta = -\frac{b}{a}$ $\quad \tan\theta' = \frac{b}{a}$ | $\tan\theta = \frac{b}{a}$ $\quad \tan\theta' = \frac{b}{a}$ | $\tan\theta = -\frac{b}{a}$ $\quad \tan\theta' = \frac{b}{a}$ |

**StudyTip**

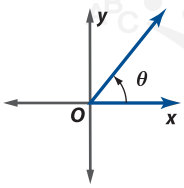
**Reference Angles** Notice that in some cases, the three trigonometric values of $\theta$ and $\theta'$ (read *theta prime*) are the same. In other cases, they differ only in sign.

This angle $\theta'$, called a **reference angle**, can be used to find the trigonometric values of any angle $\theta$.

## KeyConcept Reference Angle Rules

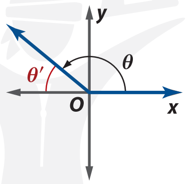
If $\theta$ is an angle in standard position, its reference angle $\theta'$ is the acute angle formed by the terminal side of $\theta$ and the $x$-axis. The reference angle $\theta'$ for any angle $\theta$, $0° < \theta < 360°$ or $0 < \theta < 2\pi$, is defined as follows.

**Quadrant I**



$\theta' = \theta$

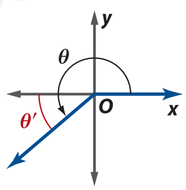
**Quadrant II**



$\theta' = 180° - \theta$

$\theta' = \pi - \theta$

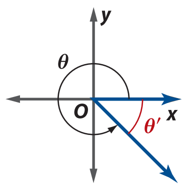
**Quadrant III**



$\theta' = \theta - 180°$

$\theta' = \theta - \pi$

**Quadrant IV**



$\theta' = 360° - \theta$

$\theta' = 2\pi - \theta$

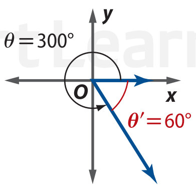
To find a reference angle for angles outside the interval $0° < \theta < 360°$ or $0 < \theta < 2\pi$, first find a corresponding coterminal angle in this interval.

## Example 3 Find Reference Angles

**Sketch each angle. Then find its reference angle.**

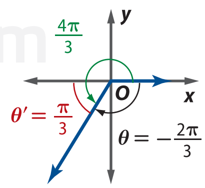
**a.** $300°$

The terminal side of $300°$ lies in Quadrant IV. Therefore, its reference angle is $\theta' = 360° - 300°$ or $60°$.




**b.** $-\frac{2\pi}{3}$

A coterminal angle is $2\pi - \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. The terminal side of $\frac{4\pi}{3}$ lies in Quadrant III, so its reference angle is $\frac{4\pi}{3} - \pi$ or $\frac{\pi}{3}$.




**GuidedPractice**

**3A.** $\frac{5\pi}{4}$ **3B.** $-240°$ **3C.** $390°$

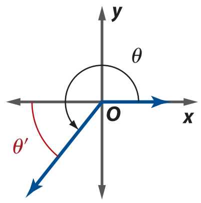
Because the trigonometric values of an angle and its reference angle are equal or differ only in sign, you can use the following steps to find the value of a trigonometric function of any angle $\theta$.

## KeyConcept Evaluating Trigonometric Functions of Any Angle

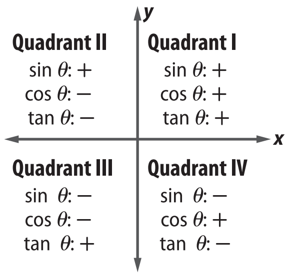
**Step 1** Find the reference angle $\theta'$.

**Step 2** Find the value of the trigonometric function for $\theta'$.

**Step 3** Using the quadrant in which the terminal side of $\theta$ lies, determine the sign of the trigonometric function value of $\theta$.




The signs of the trigonometric functions in each quadrant can be determined using the function definitions given on page 242. For example, because $\sin\theta = \frac{y}{r}$, it follows that $\sin\theta$ is negative when $y < 0$, which occurs in Quadrants III and IV. Using this same logic, you can verify each of the signs for $\sin\theta$, $\cos\theta$, and $\tan\theta$ shown in the diagram. Notice that these values depend only on $x$ and $y$ because $r$ is always positive.




**StudyTip**

**Memorizing Trigonometric Values** To memorize the exact values of sine for 0°, 30°, 45°, 60°, and 90°, consider the following pattern.

$\sin 0° = \frac{\sqrt{0}}{2}$, or 0

$\sin 30° = \frac{\sqrt{1}}{2}$, or $\frac{1}{2}$

$\sin 45° = \frac{\sqrt{2}}{2}$

$\sin 60° = \frac{\sqrt{3}}{2}$

$\sin 90° = \frac{\sqrt{4}}{2}$, or 1

A similar pattern exists for the cosine function, except the values are given in reverse order.

Because you know the exact trigonometric values of 30°, 45°, and 60° angles, you can find the exact trigonometric values of *all* angles for which these angles are reference angles. The table lists these values for $\theta$ in both degrees and radians.

| $\theta$ | 30° or $\frac{\pi}{6}$ | 45° or $\frac{\pi}{4}$ | 60° or $\frac{\pi}{3}$ |
|---|---|---|---|
| $\sin\theta$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ |
| $\cos\theta$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ |
| $\tan\theta$ | $\frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ |

### Example 4 Use Reference Angles to Find Trigonometric Values

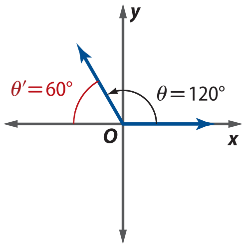
**Find the exact value of each expression.**

**a. cos 120°**

Because the terminal side of $\theta$ lies in Quadrant II, the reference angle $\theta'$ is $180° - 120°$ or $60°$.

$\cos 120° = -\cos 60°$     In Quadrant II, cos $\theta$ is negative.

$= -\frac{1}{2}$     $\cos 60° = \frac{1}{2}$

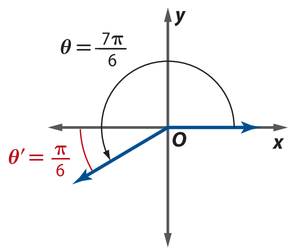



**b. $\tan\frac{7\pi}{6}$**

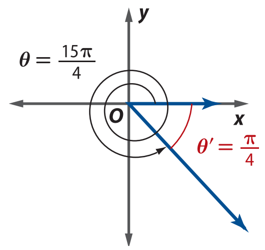
Because the terminal side of $\theta$ lies in Quadrant III, the reference angle $\theta'$ is $\frac{7\pi}{6} - \pi$ or $\frac{\pi}{6}$.

$\tan\frac{7\pi}{6} = \tan\frac{\pi}{6}$     In Quadrant III, tan $\theta$ is positive.

$= \frac{\sqrt{3}}{3}$     $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$



**c.** $\csc \frac{15\pi}{4}$




A coterminal angle of $\theta$ is $\frac{15\pi}{4} - 2\pi$ or $\frac{7\pi}{4}$, which lies in Quadrant IV. So, the reference angle $\theta'$ is $2\pi - \frac{7\pi}{4}$ or $\frac{\pi}{4}$. Because sine and cosecant are reciprocal functions and $\sin \theta$ is negative in Quadrant IV, it follows that $\csc \theta$ is also negative in Quadrant IV.

$\csc \frac{15\pi}{4} = -\csc \frac{\pi}{4}$ — In Quadrant IV, $\csc \theta$ is negative.

$= -\dfrac{1}{\sin \frac{\pi}{4}}$ — $\csc \theta = \frac{1}{\sin \theta}$

$= -\dfrac{1}{\frac{\sqrt{2}}{2}}$ or $-\sqrt{2}$ — $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

**CHECK** You can check your answer by using a graphing calculator.

$\csc \frac{15\pi}{4} \approx -1.414$ ✔

$-\sqrt{2} \approx -1.414$ ✔

### GuidedPractice

**Find the exact value of each expression.**

**4A.** $\tan \frac{5\pi}{3}$ **4B.** $\sin \frac{5\pi}{6}$ **4C.** $\sec(-135°)$

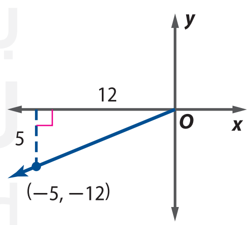
If the value of one or more of the trigonometric functions and the quadrant in which the terminal side of $\theta$ lies is known, the remaining function values can be found.

## Example 5 Use One Trigonometric Value to Find Others

**Let $\tan \theta = \frac{5}{12}$, where $\sin \theta < 0$. Find the exact values of the five remaining trigonometric functions of $\theta$.**

To find the other function values, you must find the coordinates of a point on the terminal side of $\theta$. You know that $\tan \theta$ is positive and $\sin \theta$ is negative, so $\theta$ must lie in Quadrant III. This means that both $x$ and $y$ are negative.

Because $\tan \theta = \frac{y}{x}$ or $\frac{5}{12}$, use the point $(-12, -5)$ to find $r$.




$r = \sqrt{x^2 + y^2}$ — Pythagorean Theorem

$= \sqrt{(-12)^2 + (-5)^2}$ — $x = -12$ and $y = -5$

$= \sqrt{169}$ or 13 — Take the positive square root.

Use $x = -12$, $y = -5$, and $r = 13$ to write the five remaining trigonometric ratios.

$\sin \theta = \frac{y}{r}$ or $-\frac{5}{13}$ $\cos \theta = \frac{x}{r}$ or $-\frac{12}{13}$ $\cot \theta = \frac{x}{y}$ or $\frac{12}{5}$

$\csc \theta = \frac{r}{y}$ or $-\frac{13}{5}$ $\sec \theta = \frac{r}{x}$ or $-\frac{13}{12}$

### GuidedPractice

**Find the exact values of the five remaining trigonometric functions of $\theta$.**

**5A.** $\sec \theta = \sqrt{3}$, $\tan \theta < 0$ **5B.** $\sin \theta = \frac{5}{7}$, $\cot \theta > 0$

**WatchOut!**

**Rationalizing the Denominator** Be sure to rationalize the denominator, if necessary.

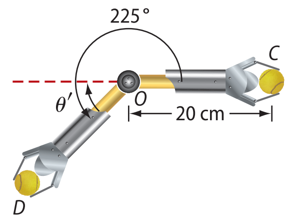
## Real-WorldLink

RoboCup is an international competition in which teams compete in a series of soccer matches, depending on the size and intelligence of their robots. The aim of the project is to advance artificial intelligence and robotics research.

**Source:** RoboCup

## Real-World Example 6 Find Coordinates Given a Radius and an Angle

**ROBOTICS As part of the range of motion category in a high school robotics competition, a student programmed a 20 cm long robotic arm to pick up an object at point *C* and rotate through an angle of exactly 225° in order to release it into a container at point *D*. Find the position of the object at point *D*, relative to the pivot point *O*.**




With the pivot point at the origin and the angle through which the arm rotates in standard position, point $C$ has coordinates $(20, 0)$. The reference angle $\theta'$ for 225° is $225° - 180°$ or 45°.

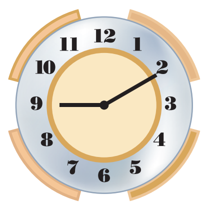
Let the position of point $D$ have coordinates $(x, y)$. The definitions of sine and cosine can then be used to find the values of $x$ and $y$. The value of $r$, 20 cm, is the length of the robotic arm. Since $D$ is in Quadrant III, the sine and cosine of 225° are negative.

| | | | |
|---|---|---|---|
| $\cos\theta = \frac{x}{r}$ | Cosine ratio | $\sin\theta = \frac{y}{r}$ | Sine ratio |
| $\cos 225° = \frac{x}{20}$ | $\theta = 225°$ and $r = 20$ | $\sin 225° = \frac{y}{20}$ | $\theta = 225°$ and $r = 20$ |
| $-\cos 45° = \frac{x}{20}$ | $\cos 225° = -\cos 45°$ | $-\sin 45° = \frac{y}{20}$ | $\sin 225° = -\sin 45°$ |
| $-\frac{\sqrt{2}}{2} = \frac{x}{20}$ | $\cos 45° = \frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2} = \frac{y}{20}$ | $\sin 45° = \frac{\sqrt{2}}{2}$ |
| $-10\sqrt{2} = x$ | Solve for $x$. | $-10\sqrt{2} = y$ | Solve for $y$. |

The exact coordinates of $D$ are $\left(-10\sqrt{2}, -10\sqrt{2}\right)$. Since $10\sqrt{2}$ is about 14.14, the object is about 14.14 cm to the left of the pivot point and about 14.14 cm below the pivot point.

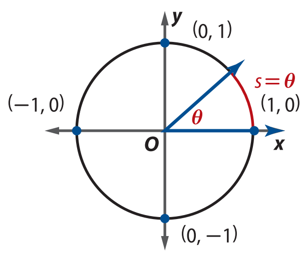
### GuidedPractice

**6. CLOCKWORK** A 3 in-long minute hand on a clock shows a time of 45 minutes past the hour. What is the new position of the end of the minute hand relative to the pivot point at 10 minutes past the next hour?



## 2 Trigonometric Functions on the Unit Circle

A **unit circle** is a circle of radius 1 centered at the origin. Notice thaton a unit circle, the radian measure of a central angle $\theta = \frac{s}{1}$ or $s$, so the arc length intercepted by $\theta$ corresponds exactly to the angle's radian measure. This provides a way of mapping a real number input value for a trigonometric function to a real number output value.

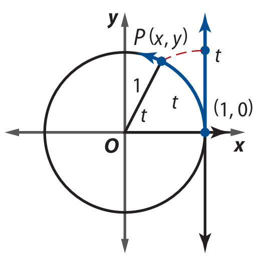



### StudyTip

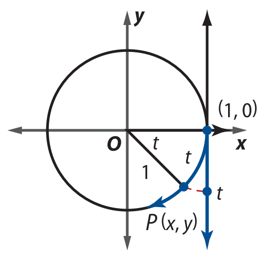
**Wrapping Function** The association of a point on the number line with a point on a circle is called the *wrapping function*, $w(t)$. For example, if $w(t)$ associates a point $t$ on the number line with a point $P(x, y)$ on the unit circle, then $w(\pi) = (-1, 0)$ and $w(2\pi) = (1, 0)$.

Consider the real number line placed vertically tangent to the unit circle at $(1, 0)$ as shown below. If this line were wrapped about the circle in both the positive (counterclockwise) and negative (clockwise) direction, each point $t$ on the line would map to a unique point $P(x, y)$ on the circle. Because $r = 1$, we can define the trigonometric ratios of angle $t$ in terms of just $x$ and $y$.

**Positive Values of $t$**

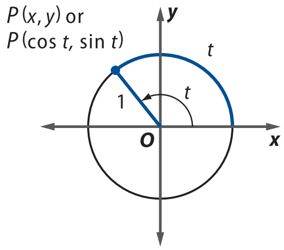



**Negative Values of $t$**



## KeyConcept Trigonometric Functions on the Unit Circle

Let $t$ be any real number on a number line and let $P(x, y)$ be the point on $t$ when the number line is wrapped onto the unit circle. Then the trigonometric functions of $t$ are as follows.




$\sin t = y$ $\qquad$ $\cos t = x$ $\qquad$ $\tan t = \frac{y}{x}, x \neq 0$

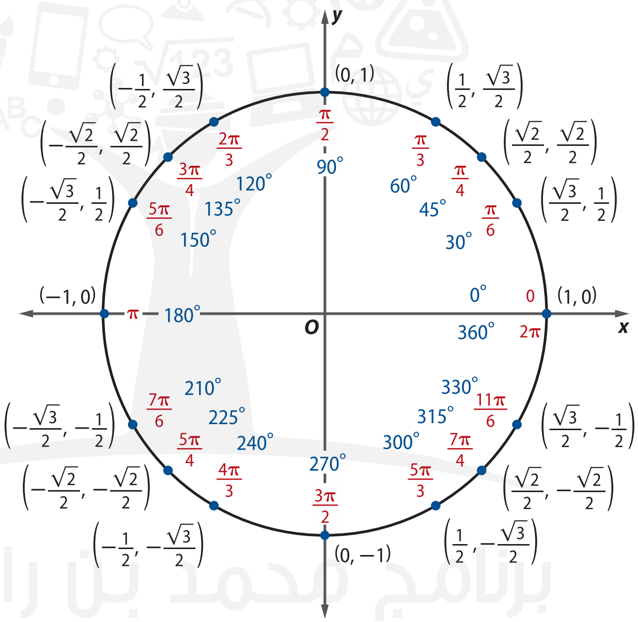
$\csc t = \frac{1}{y}, y \neq 0$ $\qquad$ $\sec t = \frac{1}{x}, x \neq 0$ $\qquad$ $\cot t = \frac{x}{y}, y \neq 0$

Therefore, the coordinates of $P$ corresponding to the angle $t$ can be written as $P(\cos t, \sin t)$.

Notice that the input value in each of the definitions above can be thought of as an angle measure or as a real number $t$. When defined as functions of the real number system using the unit circle, the trigonometric functions are often called **circular functions**.

Using reference angles or quadrantal angles, you should now be able to find the trigonometric function values for all integer multiples of 30°, or $\frac{\pi}{6}$ radians, and 45°, or $\frac{\pi}{4}$ radians. These special values wrap to 16 special points on the unit circle, as shown below.

**16-Point Unit Circle**




**StudyTip**

**16-Point Unit Circle** You have already memorized these values in the first quadrant. The remaining values can be determined using the $x$-axis, $y$-axis, and origin symmetry of the unit circle along with the signs of $x$ and $y$ in each quadrant.

Using the $(x, y)$ coordinates in the 16-point unit circle and the definitions in the Key Concept Box at the top of the page, you can find the values of the trigonometric functions for common angle measures. It is helpful to memorize these exact function values so you can quickly perform calculations involving them.



### Example 7 Find Trignometric Values Using the Unit Circle

**Find the exact value of each expression. If undefined, write *undefined*.**

**a.** $\sin \frac{\pi}{3}$

$\frac{\pi}{3}$ corresponds to the point $(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle.

$\sin t = y$ $\qquad$ Definition of $\sin t$

$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\qquad$ $y = \frac{\sqrt{3}}{2}$ when $t = \frac{\pi}{3}$.

**b. cos 135°**

135° corresponds to the point $(x, y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on the unit circle.

$\cos t = x$     Definition of cos $t$

$\cos 135° = -\frac{\sqrt{2}}{2}$     $x = -\frac{\sqrt{2}}{2}$ when $t = 135°$.

**c. tan 270°**

270° corresponds to the point $(x, y) = (0, -1)$ on the unit circle.

$\tan t = \frac{y}{x}$     Definition of tan $t$

$\tan 270° = \frac{-1}{0}$     $x = 0$ and $y = -1$, when $t = 270°$.

Therefore, tan 270° is undefined.

**d. csc $\frac{11\pi}{6}$**

$\frac{11\pi}{6}$ corresponds to the point $(x, y) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

$\csc t = \frac{1}{y}$     Definition of csc $t$

$\csc \frac{11\pi}{6} = \frac{1}{-\frac{1}{2}}$     $y = -\frac{1}{2}$ when $t = \frac{11\pi}{6}$.

$= -2$     Simplify.

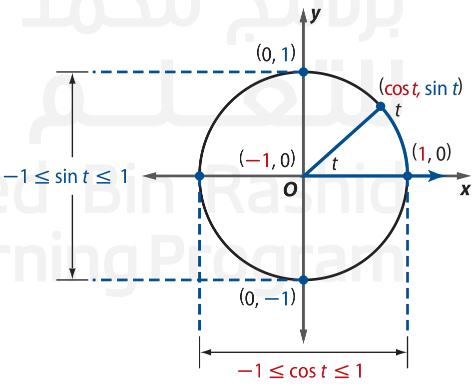
## GuidedPractice

**7A.** $\cos \frac{\pi}{4}$    **7B.** $\sin 120°$    **7C.** $\cot 210°$    **7D.** $\sec \frac{7\pi}{4}$

**StudyTip**

**Radians vs. Degrees** While we could also discuss one wrapping as corresponding to an angle measure of 360°, this measure is not related to a distance. On the unit circle, one wrapping corresponds to both the angle measuring $2\pi$ and the distance $2\pi$ around the circle.

As defined by wrapping the number line around the unit circle, the domain of both the sine and cosine functions is the set of all real numbers $(-\infty, \infty)$. Extending infinitely in either direction, the number line can be wrapped multiple times around the unit circle, mapping more than one $t$-value to the same point $P(x, y)$ with each wrapping, positive or negative.




Because $\cos t = x$, $\sin t = y$, and one wrapping corresponds to a distance of $2\pi$,

$$\cos(t + 2n\pi) = \cos t \quad \text{and} \quad \sin(t + 2n\pi) = \sin t,$$

for any integer $n$ and real number $t$.

**StudyTip**

**Periodic Functions** The other three circular functions are also periodic. The periods of these functions will be discussed in Lesson 3-5.

The values for the sine and cosine function therefore lie in the interval $[-1, 1]$ and repeat for every integer multiple of $2\pi$ on the number line. Functions with values that repeat at regular intervals are called **periodic functions**.

## KeyConcept Periodic Functions

A function $y = f(t)$ is periodic if there exists a positive real number $c$ such that $f(t + c) = f(t)$ for all values of $t$ in the domain of $f$.

The smallest number $c$ for which $f$ is periodic is called the **period** of $f$.

The sine and cosine functions are periodic, repeating values after $2\pi$, so these functions have a period of $2\pi$. It can be shown that the values of the tangent function repeat after a distance of $\pi$ on the number line, so the tangent function has a period of $\pi$ and

$$\tan t = \tan (t + n\pi),$$

for any integer $n$ and real number $t$, unless both $\tan t$ and $\tan (t + n\pi)$ are undefined. You can use the periodic nature of the sine, cosine, and tangent functions to evaluate these functions.

## Example 8 Use the Periodic Nature of Circular Functions

**Find the exact value of each expression.**

**a.** $\cos \frac{11\pi}{4}$

$\cos \frac{11\pi}{4} = \cos\left(\frac{3\pi}{4} + 2\pi\right)$ — Rewrite $\frac{11\pi}{4}$ as the sum of a number and $2\pi$.

$= \cos \frac{3\pi}{4}$ — $\frac{3\pi}{4}$ and $\frac{3\pi}{4} + 2\pi$ map to the same point $(x, y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on the unit circle.

$= -\frac{\sqrt{2}}{2}$ — $\cos t = x$ and $x = -\frac{\sqrt{2}}{2}$ when $t = \frac{3\pi}{4}$.

**b.** $\sin\left(-\frac{2\pi}{3}\right)$

$\sin\left(-\frac{2\pi}{3}\right) = \sin\left(\frac{4\pi}{3} + 2(-1)\pi\right)$ — Rewrite $-\frac{2\pi}{3}$ as the sum of a number and an integer multiple of $2\pi$.

$= \sin \frac{4\pi}{3}$ — $\frac{4\pi}{3}$ and $\frac{4\pi}{3} - 2(-1)\pi$ map to the same point $(x, y) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ on the unit circle.

$= -\frac{\sqrt{3}}{2}$ — $\sin t = y$ and $y = -\frac{\sqrt{3}}{2}$ when $t = \frac{4\pi}{3}$.

**c.** $\tan \frac{19\pi}{6}$

$\tan \frac{19\pi}{6} = \tan\left(\frac{\pi}{6} + 3\pi\right)$ — Rewrite $\frac{19\pi}{6}$ as the sum of a number and an integer multiple of $\pi$.

$= \tan \frac{\pi}{6}$ — $\frac{\pi}{6}$ and $\frac{\pi}{6} + 3\pi$ map to points on the unit circle with the same tangent values.

$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$ or $\frac{\sqrt{3}}{3}$ — $\tan t = \frac{y}{x}$; $x = \frac{\sqrt{3}}{2}$ and $y = \frac{1}{2}$ when $t = \frac{\pi}{6}$.

**GuidedPractice**

**8A.** $\sin \frac{13\pi}{4}$ **8B.** $\cos\left(-\frac{4\pi}{3}\right)$ **8C.** $\tan \frac{15\pi}{6}$

Recall from Lesson 1-2 that a function $f$ is *even* if for every $x$ in the domain of $f$, $f(-x) = f(x)$ and *odd* if for every $x$ in the domain of $f$, $f(-x) = -f(x)$. You can use the unit circle to verify that the cosine function is even and that the sine and tangent functions are odd. That is,

$$\cos(-t) = \cos t \qquad \sin(-t) = -\sin t \qquad \tan(-t) = -\tan t.$$

## Exercises

**The given point lies on the terminal side of an angle $\theta$ in standard position. Find the values of the six trigonometric functions of $\theta$.** (Example 1)

**1.** $(3, 4)$

**2.** $(-6, 6)$

**3.** $(-4, -3)$

**4.** $(2, 0)$

**5.** $(1, -8)$

**6.** $(5, -3)$

**7.** $(-8, 15)$

**8.** $(-1, -2)$

**Find the exact value of each trigonometric function, if defined. If not defined, write *undefined*.** (Example 2)

**9.** $\sin \frac{\pi}{2}$

**10.** $\tan 2\pi$

**11.** $\cot (-180°)$

**12.** $\csc 270°$

**13.** $\cos (-270°)$

**14.** $\sec 180°$

**15.** $\tan \pi$

**16.** $\sec \left(-\frac{\pi}{2}\right)$

**Sketch each angle. Then find its reference angle.** (Example 3)

**17.** $135°$

**18.** $210°$

**19.** $\frac{7\pi}{12}$

**20.** $\frac{11\pi}{3}$

**21.** $-405°$

**22.** $-75°$

**23.** $\frac{5\pi}{6}$

**24.** $\frac{13\pi}{6}$

**Find the exact value of each expression.** (Example 4)

**25.** $\cos \frac{4\pi}{3}$

**26.** $\tan \frac{7\pi}{6}$

**27.** $\sin \frac{3\pi}{4}$

**28.** $\cot (-45°)$

**29.** $\csc 390°$

**30.** $\sec (-150°)$

**31.** $\tan \frac{11\pi}{6}$

**32.** $\sin 300°$

**Find the exact values of the five remaining trigonometric functions of $\theta$.** (Example 5)

**33.** $\tan \theta = 2$, where $\sin \theta > 0$ and $\cos \theta > 0$

**34.** $\csc \theta = 2$, where $\sin \theta > 0$ and $\cos \theta < 0$

**35.** $\sin \theta = -\frac{1}{5}$, where $\cos \theta > 0$

**36.** $\cos \theta = -\frac{12}{13}$, where $\sin \theta < 0$

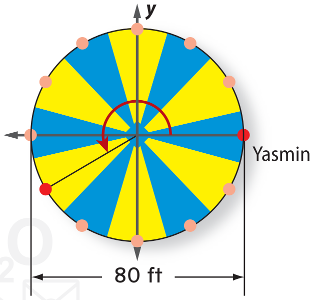
**37.** $\sec \theta = \sqrt{3}$, where $\sin \theta < 0$ and $\cos \theta > 0$

**38.** $\cot \theta = 1$, where $\sin \theta < 0$ and $\cos \theta < 0$

**39.** $\tan \theta = -1$, where $\sin \theta < 0$

**40.** $\cos \theta = -\frac{1}{2}$, where $\sin \theta > 0$

**41.** **CAROUSEL** Yasmin is on a carousel at the carnival. The diameter of the carousel is 80 ft. Find the position of her seat from the center of the carousel after a rotation of 210°. (Example 6)




**42.** **COIN FUNNEL** A coin is dropped into a funnel where it spins in smaller circles until it drops into the bottom of the bank. The diameter of the first circle the coin makes is 24 cm. Before completing one full circle, the coin travels 150° and falls over. What is the new position of the coin relative to the center of the funnel? (Example 6)

**Find the exact value of each expression. If undefined, write *undefined*.** (Examples 7 and 8)

**43.** $\sec 120°$

**44.** $\sin 315°$

**45.** $\cos \frac{11\pi}{3}$

**46.** $\tan \left(-\frac{5\pi}{4}\right)$

**47.** $\csc 390°$

**48.** $\cot 510°$

**49.** $\csc 5400°$

**50.** $\sec \frac{3\pi}{2}$

**51.** $\cot \left(-\frac{5\pi}{6}\right)$

**52.** $\csc \frac{17\pi}{6}$

**53.** $\tan \frac{5\pi}{3}$

**54.** $\sec \frac{7\pi}{6}$

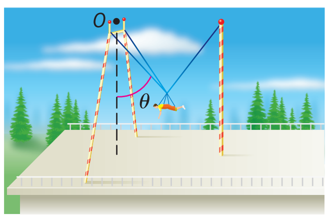
**55.** $\sin \left(-\frac{5\pi}{3}\right)$

**56.** $\cos \frac{7\pi}{4}$

**57.** $\tan \frac{14\pi}{3}$

**58.** $\cos \left(-\frac{19\pi}{6}\right)$

**59.** **RIDES** Mazen and Ayoub are on a ride at an amusement park. After the first several swings, the angle the ride makes with the vertical is modeled by $\theta = 22 \cos \pi t$, with $\theta$ measured in radians and $t$ measured in seconds. Determine the measure of the angle in radians for $t = 0$, 0.5, 1, 1.5, 2, and 2.5. (Example 8)



**Complete each trigonometric expression.**

**60.** $\cos 60^\circ = \sin$ ___

**61.** $\tan \frac{\pi}{4} = \sin$ ___

**62.** $\sin \frac{2\pi}{3} = \cos$ ___

**63.** $\cos \frac{7\pi}{6} = \sin$ ___

**64.** $\sin(-45^\circ) = \cos$ ___

**65.** $\cos \frac{5\pi}{3} = \sin$ ___

**66.** **ICE CREAM** The monthly sales in thousands of dirhams for Ahmed's Fine Ice Cream shop can be modeled by $y = 71.3 + 59.6 \sin \frac{\pi(t-4)}{6}$, where $t = 1$ represents January, $t = 2$ represents February, and so on.

**a.** Estimate the sales for January, March, July, and October.

**b.** Describe why the ice cream shop's sales can be represented by a trigonometric function.

**Use the given values to evaluate the trigonometric functions.**

**67.** $\cos(-\theta) = \frac{8}{11}$; $\cos \theta = ?$; $\sec \theta = ?$

**68.** $\sin(-\theta) = \frac{5}{9}$; $\sin \theta = ?$; $\csc \theta = ?$

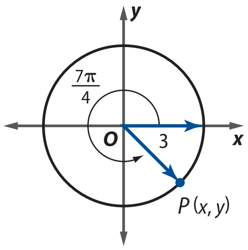
**69.** $\sec \theta = \frac{13}{12}$; $\cos \theta = ?$; $\cos(-\theta) = ?$

**70.** $\csc \theta = \frac{19}{17}$; $\sin \theta = ?$; $\sin(-\theta) = ?$

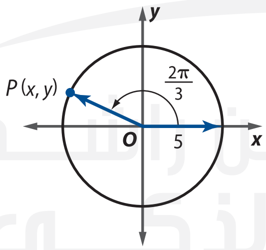
**71.** **GRAPHS** Suppose the terminal side of an angle $\theta$ in standard position coincides with the graph of $y = 2x$ in Quadrant III. Find the six trigonometric functions of $\theta$.

**Find the coordinates of *P* for each circle with the given radius and angle measure.**

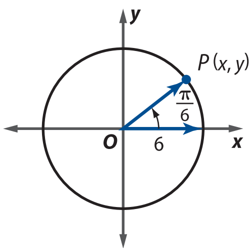
**72.**



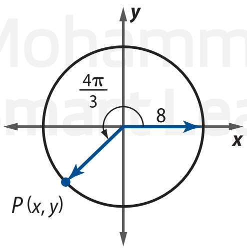

**73.**




**74.**




**75.**




**76.** **COMPARISON** Suppose the terminal side of an angle $\theta_1$ in standard position contains the point $(7, -8)$, and the terminal side of a second angle $\theta_2$ in standard position contains the point $(-7, 8)$. Compare the sines of $\theta_1$ and $\theta_2$.

**77.** **TIDES** The depth $y$ in meters of the tide on a beach varies as a sine function of $x$, the hour of the day. On a certain day, that function was $y = 3 \sin\left[\frac{\pi}{6}(x-4)\right] + 8$, where $x = 0, 1, 2, \ldots, 24$ corresponds to 12:00 midnight, 1:00 A.M., 2:00 A.M., ..., 12:00 midnight the next night.

**a.** What is the maximum depth, or high tide, that day?

**b.** At what time(s) does the high tide occur?

**78.** **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the period of the sine function.

**a.** **TABULAR** Copy and complete a table similar to the one below that includes all 16 angle measures from the unit circle.

| $\theta$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\cdots$ | $2\pi$ |
|---|---|---|---|---|---|---|
| $\sin \theta$ | | | | | | |
| $\sin 2\theta$ | | | | | | |
| $\sin 4\theta$ | | | | | | |

**b.** **VERBAL** After what values of $\theta$ do $\sin \theta$, $\sin 2\theta$, and $\sin 4\theta$, repeat their range values? In other words, what are the periods of these functions?

**c.** **VERBAL** Make a conjecture as to how the period of $y = \sin n\theta$ is affected for different values of $n$.

## H.O.T. Problems Use Higher-Order Thinking Skills

**79.** **CHALLENGE** For each statement, describe $n$.

**a.** $\cos\left(n \cdot \frac{\pi}{2}\right) = 0$

**b.** $\csc\left(n \cdot \frac{\pi}{2}\right)$ is undefined.

**REASONING Determine whether each statement is *true* or *false*. Explain your reasoning.**

**80.** If $\cos \theta = 0.8$, $\sec \theta - \cos(-\theta) = 0.45$.

**81.** Since $\tan(-t) = -\tan t$, the tangent of a negative angle is a negative number.

**82.** **WRITING IN MATH** Explain why the attendance at a year-round theme park could be modeled by a periodic function. What issues or events could occur over time to alter this periodic depiction?

**REASONING Use the unit circle to verify each relationship.**

**83.** $\sin(-t) = -\sin t$

**84.** $\cos(-t) = \cos t$

**85.** $\tan(-t) = -\tan t$

**86.** **WRITING IN MATH** Make a conjecture as to the periods of the secant, cosecant, and cotangent functions. Explain your reasoning.

## Spiral Review

**Write each decimal degree measure in DMS form and each DMS measure in decimal degree form to the nearest thousandth.**

**87.** 168.35°

**88.** 27.465°

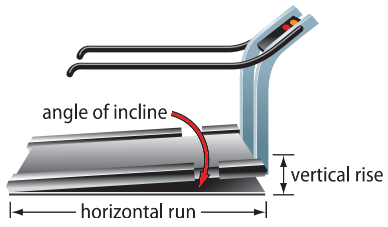
**89.** 14° 5′ 20″

**90.** 173° 24′ 35″

**91. EXERCISE** A preprogrammed workout on a treadmill consists of intervals walking at various rates and angles of incline. A 1% incline means 1 unit of vertical rise for every 100 units of horizontal run.

**a.** At what angle, with respect to the horizontal, is the treadmill bed when set at a 10% incline? Round to the nearest degree.

**b.** If the treadmill bed is 40 in-long, what is the vertical rise when set at an 8% incline?




**Evaluate each logarithm.**

**92.** $\log_8 64$

**93.** $\log_{125} 5$

**94.** $\log_2 32$

**95.** $\log_4 128$

**List all possible rational zeros of each function. Then determine which, if any, are zeros.**

**96.** $f(x) = x^3 - 4x^2 + x + 2$

**97.** $g(x) = x^3 + 6x^2 + 10x + 3$

**98.** $h(x) = x^4 - x^2 + x - 1$

**99.** $h(x) = 2x^3 + 3x^2 - 8x + 3$

**100.** $f(x) = 2x^4 + 3x^3 - 6x^2 - 11x - 3$

**101.** $g(x) = 4x^3 + x^2 + 8x + 2$

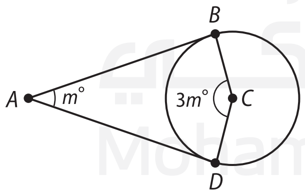
**102. NAVIGATION** A global positioning system (GPS) uses satellites to allow a user to determine his or her position on Earth. The system depends on satellite signals that are reflected to and from a hand-held transmitter. The time that the signal takes to reflect is used to determine the transmitter's position.
Radio waves travel through air at a speed of 299,792,458 m/s. Thus, $d(t) = 299{,}792{,}458t$ relates the time $t$ in seconds to the distance traveled $d(t)$ in meters.

**a.** Find the distance a radio wave will travel in 0.05, 0.2, 1.4, and 5.9 seconds.

**b.** If a signal from a GPS satellite is received at a transmitter in 0.08 second, how far from the transmitter is the satellite?

## Skills Review for Standardized Tests

**103. SAT/ACT** In the figure, $\overline{AB}$ and $\overline{AD}$ are tangents to circle $C$. What is the value of $m$?




**104.** Suppose $\theta$ is an angle in standard position with $\sin\theta > 0$. In which quadrant(s) could the terminal side of $\theta$ lie?

**A** I only

**B** I and II

**C** I and III

**D** I and IV

**105. REVIEW** Find the angular speed in radians per second of a point on a bicycle tire if it completes 2 revolutions in 3 seconds.

**F** $\frac{\pi}{3}$

**G** $\frac{\pi}{2}$

**H** $\frac{2\pi}{3}$

**J** $\frac{4\pi}{3}$

**106. REVIEW** Which angle has a tangent and cosine that are both negative?

**A** 110°

**B** 180°

**C** 210°

**D** 340°

EXPLORE 3-4

# Graphing Technology Lab
# Graphing the Sine Function Parametrically

## Objective

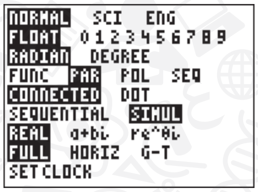
- Use a graphing calculator and parametric equations to graph the sine function and its inverse.

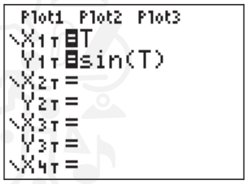
As functions of the real number system, you can graph trigonometric functions on the coordinate plane and apply the same graphical analysis that you did to functions in an earlier lesson. Parametric equations will be used to graph the sine function.

### Activity 1 Parametric Graph of $y = \sin x$

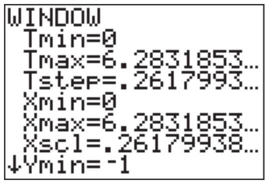
**Graph $x = t$, $y = \sin t$.**

**Step 1** Set the mode. In the MODE menu, select RADIAN, PAR, and SIMUL. This allows the equations to be graphed simultaneously. Next, enter the parametric equations. In parametric form, X,T,θ,n will use $t$ instead of $x$.

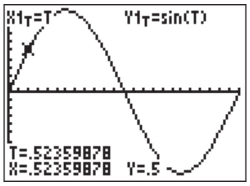



**Step 2** Set the $x$- and $t$-values to range from 0 to $2\pi$. Set Tstep and $x$-scale to $\frac{\pi}{12}$. Set $y$ to $[-1, 1]$ scl: 0.1. The calculator automatically converts to decimal form.



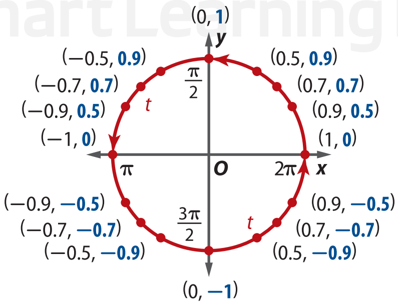

**Step 3** Graph the equations. Trace the function to identify points along the graph. Select TRACE and use the right arrow to move along the curve. Record the corresponding $x$- and $y$-values.

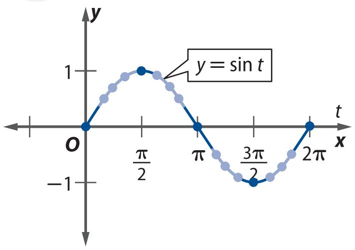



$[0, 2\pi]$ scl: $\frac{\pi}{12}$ by $[-1, 1]$ scl: 0.1
$t$: $[0, 2\pi]$; $t$step $\frac{\pi}{12}$

**Step 4** The table shows angle measures from 0° to 180°, or 0 to $\pi$, and the corresponding values for $\sin t$ on the unit circle. The figures below illustrate the relationship between the graph and the unit circle.

| Degrees | 0 | 30 | 45 | 60 | 90 | 120 | 135 | 150 | 180 |
|---|---|---|---|---|---|---|---|---|---|
| Radians | 0 | 0.52 | 0.79 | 1.05 | 1.571 | 2.094 | 2.356 | 2.618 | 3.14 |
| $y = \sin t$ | 0 | 0.5 | 0.707 | 0.866 | 1 | 0.866 | 0.707 | 0.5 | 0 |




**StudyTip**

**Decimal Equivalents** Below are the decimal equivalents of common trigonometric values.

$\frac{\sqrt{3}}{2} \approx 0.866$

$\frac{\sqrt{2}}{2} \approx 0.707$

## Exercises

**Graph each function on [0, 2π].**

**1.** $x = t, y = \cos t$

**2.** $x = t, y = \sin 2t$

**3.** $x = t, y = 3 \cos t$

**4.** $x = t, y = 4 \sin t$

**5.** $x = t, y = \cos (t + \pi)$

**6.** $x = t, y = 2 \sin \left(t - \frac{\pi}{4}\right)$

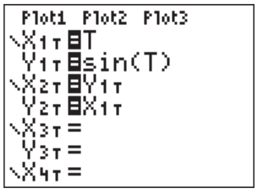
By definition, sin $t$ is the $y$-coordinate of the point $P(x, y)$ on the unit circle to which the real number $t$ on the number line gets wrapped. As shown in the diagram on the previous page, the graph of $y = \sin t$ follows the $y$-coordinate of the point determined by $t$ as it moves counterclockwise around the unit circle.

The graph of the sine function is called a *sine curve*. From Lesson 3-3, you know that the sine function is periodic with a period of $2\pi$. That is, the sine curve graphed from 0 to $2\pi$ would repeat every distance of $2\pi$ in either direction, positive or negative. Parametric equations can be used to graph the inverse of the sine function.

### Activity 2 Graph an Inverse

**Graph $x + t$, $y + \sin t$ and its inverse. Then determine a domain for which $y = \sin t$ is one-to-one.**

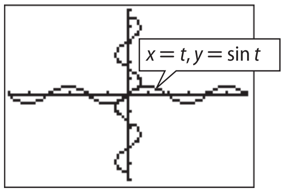
**Step 1** Inverses are found by switching $x$ and $y$. Enter the given equations as X1T and Y1T. To graph the inverse, set X2T = Y1T and Y2T = X1T. These are found in the VARS menu. Select Y-VARS, parametric, Y1T. Repeat for X1T.




**StudyTip**

**Tstep** If your graph appears to be pointed, you can change the tstep to a smaller value in order to get a smoother curve.

**Step 2** Graph the equations. Adjust the window so that both of the graphs can be seen, as shown. You may need to set the tstep to a smaller value in order to get a smooth curve.




$[-3\pi, 3\pi]$ scl: $\frac{\pi}{12}$ by $[-10, 10]$ scl: 2

$t$: $[-3\pi, 3\pi]$; $t$step $\frac{\pi}{12}$

**Step 3** Because the sine curve is periodic, there are an infinite number of domains for which the curve will pass the horizontal line test and be one-to-one. One such domain is $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.



## Exercises

**Graph each function and its inverse. Then determine a domain for which each function is one-to-one.**

**7.** $x = t, y = \cos 2t$

**8.** $x = t, y = -\sin t$

**9.** $x = t, y = 2 \cos t$

**10.** $x = t + \frac{\pi}{4}, y = \sin t$

**11.** $x = t, y = 2 \cos (t - \pi)$

**12.** $x = t - \frac{\pi}{6}, y = \sin t$

# LESSON 3-4 Graphing Sine and Cosine Functions

**Then**

- You analyzed graphs of functions.

**Now**

1. Graph transformations of the sine and cosine functions.
2. Use sinusoidal functions to solve problems.

**Why?**

- As you ride a Ferris wheel, the height that you are above the ground varies periodically as a function of time. You can model this behavior using a *sinusoidal function*.

**New Vocabulary**

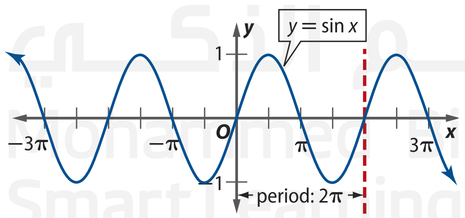
- sinusoid
- amplitude
- frequency
- phase shift
- vertical shift
- midline

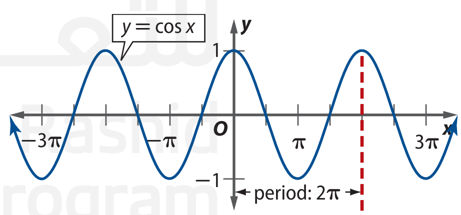
## 1 Transformations of Sine and Cosine Functions

As shown in Explore 4-4, the graph $y = \sin t$ follows the $y$-coordinate of the point determined by $t$ as it moves around the unit circle. Similarly, the graph of $y = \cos t$ follows the $x$-coordinate of this point. The graphs of these functions are periodic, repeating after a period of $2\pi$. The properties of the sine and cosine functions are summarized below.

### Key Concept Properties of the Sine and Cosine Functions

| Sine Function | Cosine Function |
|---|---|
| **Domain:** $(-\infty, \infty)$ **Range:** $[-1, 1]$ | **Domain:** $(-\infty, \infty)$ **Range:** $[-1, 1]$ |
| **$y$-intercept:** 0 | **$y$-intercept:** 1 |
| **$x$-intercepts:** $n\pi, n \in \mathbb{Z}$ | **$x$-intercepts:** $\frac{\pi}{2}n, n \in \mathbb{Z}$ |
| **Continuity:** continuous on $(-\infty, \infty)$ | **Continuity:** continuous on $(-\infty, \infty)$ |
| **Symmetry:** origin (odd function) | **Symmetry:** $y$-axis (even function) |
| **Extrema:** maximum of 1 at $x = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$; minimum of $-1$ at $x = \frac{3\pi}{2} + 2n\pi, n \in \mathbb{Z}$ | **Extrema:** maximum of 1 at $x = 2n\pi$, $n \in \mathbb{Z}$; minimum of $-1$ at $x = \pi + 2n\pi$, $n \in \mathbb{Z}$ |
| **End Behavior:** $\lim\limits_{x \to -\infty} \sin x$ and $\lim\limits_{x \to \infty} \sin x$ do not exist. | **End Behavior:** $\lim\limits_{x \to -\infty} \cos x$ and $\lim\limits_{x \to \infty} \cos x$ do not exist. |
| **Oscillation:** between $-1$ and 1 | **Oscillation:** between $-1$ and 1 |




The portion of each graph on $[0, 2\pi]$ represents one period or *cycle* of the function. Notice that the cosine graph is a horizontal translation of the sine graph. Any transformation of a sine function is called a **sinusoid**. The general form of the sinusoidal functions sine and cosine are

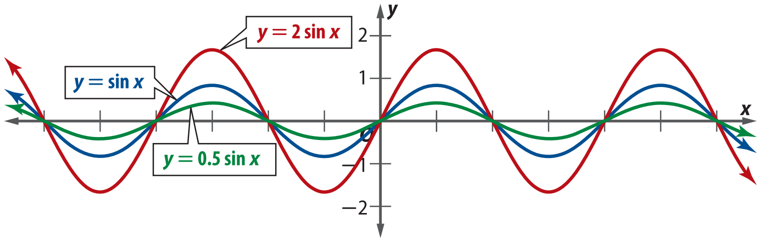
$$y = a \sin (bx + c) + d \qquad \text{and} \qquad y = a \cos (bx + c) + d$$

where $a$, $b$, $c$, and $d$ are constants and neither $a$ nor $b$ is 0.

Notice that the constant factor $a$ in $y = a \sin x$ and $y = a \cos x$ expands the graphs of $y = \sin x$ and $y = \cos x$ vertically if $|a| > 1$ and compresses them vertically if $|a| < 1$.

**StudyTip**

**Dilations and $x$-intercepts** Notice that a dilation of a sinusoidal function does not affect where the curve crosses the $x$-axis, at its $x$-intercepts.



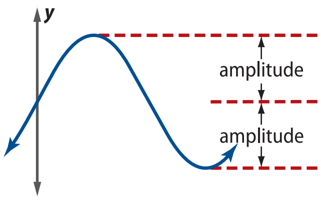

Vertical dilations affect the *amplitude* of sinusoidal functions.

## KeyConcept Amplitudes of Sine and Cosine Functions

**Words** The **amplitude** of a sinusoidal function is half the distance between the maximum and minimum values of the function or half the height of the wave.

**Symbols** For $y = a \sin (bx + c) + d$ and $y = a \cos (bx + c) + d$, amplitude $= |a|$.

**Model**




To graph a sinusoidal function of the form $y = a \sin x$ or $y = a \cos x$, plot the $x$-intercepts of the parent sine or cosine function and use the amplitude $|a|$ to plot the new maximum and minimum points. Then sketch the sine wave through these points.

## Example 1 Graph Vertical Dilations of Sinusoidal Functions

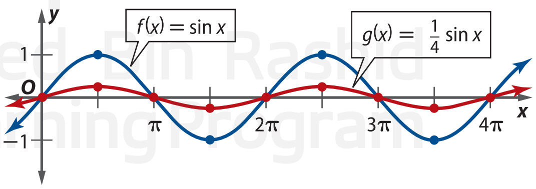
**Describe how the graphs of $f(x) = \sin x$ and $g(x) = \frac{1}{4} \sin x$ are related. Then find the amplitude of $g(x)$, and sketch two periods of both functions on the same coordinate axes.**

The graph of $g(x)$ is the graph of $f(x)$ compressed vertically. The amplitude of $g(x)$ is $\left|\frac{1}{4}\right|$ or $\frac{1}{4}$.

Create a table listing the coordinates of the $x$-intercepts and extrema for $f(x) = \sin x$ for one period on $[0, 2\pi]$. Then use the amplitude of $g(x)$ to find corresponding points on its graph.

| Function | $x$-intercept | Maximum | $x$-intercept | Minimum | $x$-intercept |
|---|---|---|---|---|---|
| $f(x) = \sin x$ | $(0, 0)$ | $\left(\frac{\pi}{2}, 1\right)$ | $(\pi, 0)$ | $\left(\frac{3\pi}{2}, -1\right)$ | $(2\pi, 0)$ |
| $g(x) = \frac{1}{4} \sin x$ | $(0, 0)$ | $\left(\frac{\pi}{2}, \frac{1}{4}\right)$ | $(\pi, 0)$ | $\left(\frac{3\pi}{2}, -\frac{1}{4}\right)$ | $(2\pi, 0)$ |

Sketch the curve through the indicated points for each function. Then repeat the pattern suggested by one period of each graph to complete a second period on $[2\pi, 4\pi]$. Extend each curve to the left and right to indicate that the curve continues in both directions.




**StudyTip**

**Radians Versus Degrees** You could rescale the $x$-axis in terms of degrees and produce sinusoidal graphs that look similar to those produced using radian measure. In calculus, however, you will encounter rules that depend on radian measure. So, in this book, we will graph all trigonometric functions in terms of radians.

### GuidedPractice

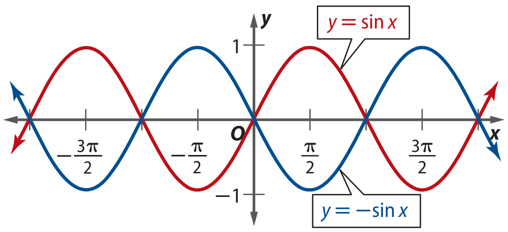
**Describe how the graphs of $f(x)$ and $g(x)$ are related. Then find the amplitude of $g(x)$, and sketch two periods of both functions on the same coordinate axes.**

**1A.** $f(x) = \cos x$
$g(x) = \frac{1}{3} \cos x$

**1B.** $f(x) = \sin x$
$g(x) = 5 \sin x$

**1C.** $f(x) = \cos x$
$g(x) = 2 \cos x$

If $a < 0$, the graph of the sinusoidal function is reflected in the $x$-axis.




## Example 2 Graph Reflections of Sinusoidal Functions

**Describe how the graphs of $f(x) = \cos x$ and $g(x) = -3 \cos x$ are related. Then find the amplitude of $g(x)$, and sketch two periods of both functions on the same coordinate axes.**

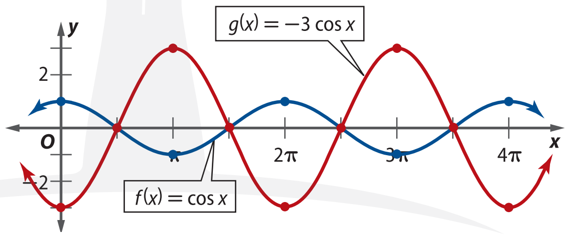
> **WatchOut!**
> **Amplitude** Notice that Example 2 does not state that the amplitude of $g(x) = -3 \cos x$ is $-3$. Amplitude is a height and is not directional.

The graph of $g(x)$ is the graph of $f(x)$ expanded vertically and then reflected in the $x$-axis. The amplitude of $g(x)$ is $|-3|$ or 3.

Create a table listing the coordinates of key points of $f(x) = \cos x$ for one period on $[0, 2\pi]$. Use the amplitude of $g(x)$ to find corresponding points on the graph of $y = 3 \cos x$. Then reflect these points in the $x$-axis to find corresponding points on the graph of $g(x)$.

| Function | Extremum | $x$-intercept | Extremum | $x$-intercept | Extremum |
|---|---|---|---|---|---|
| $f(x) = \cos x$ | $(0, 1)$ | $\left(\frac{\pi}{2}, 0\right)$ | $(\pi, -1)$ | $\left(\frac{3\pi}{2}, 0\right)$ | $(2\pi, 1)$ |
| $y = 3 \cos x$ | $(0, 3)$ | $\left(\frac{\pi}{2}, 0\right)$ | $(\pi, -3)$ | $\left(\frac{3\pi}{2}, 0\right)$ | $(2\pi, 3)$ |
| $g(x) = -3 \cos x$ | $(0, -3)$ | $\left(\frac{\pi}{2}, 0\right)$ | $(\pi, 3)$ | $\left(\frac{3\pi}{2}, 0\right)$ | $(2\pi, -3)$ |

Sketch the curve through the indicated points for each function. Then repeat the pattern suggested by one period of each graph to complete a second period on $[2\pi, 4\pi]$. Extend each curve to the left and right to indicate that the curve continues in both directions.




### GuidedPractice

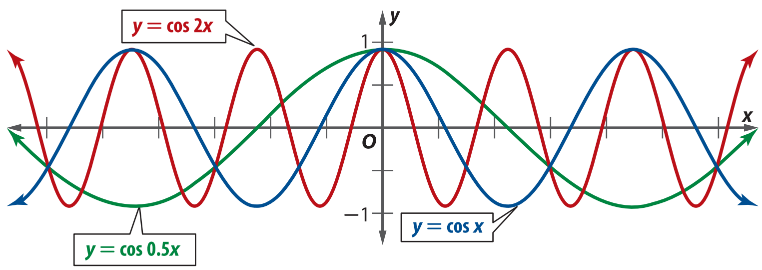
**Describe how the graphs of $f(x)$ and $g(x)$ are related. Then find the amplitude of $g(x)$, and sketch two periods of both functions on the same coordinate axes.**

**2A.** $f(x) = \cos x$
$g(x) = -\frac{1}{5} \cos x$

**2B.** $f(x) = \sin x$
$g(x) = -4 \sin x$



In Lesson 1-5, you learned that if $g(x) = f(bx)$, then $g(x)$ is the graph of $f(x)$ compressed horizontally if $|b| > 1$ and expanded horizontally if $|b| < 1$. Horizontal dilations affect the *period* of a sinusoidal function—the length of one full cycle.



**WatchOut!**

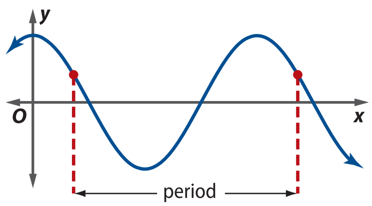
**Determining Period** When determining the period of a periodic function from its graph, remember that the period is the *smallest* distance that contains all values of the function.

## KeyConcept Periods of Sine and Cosine Functions

**Words** The period of a sinusoidal function is the distance between any two sets of repeating points on the graph of the function.

**Symbols** For $y = a \sin(bx + c) + d$ and $y = a \cos(bx + c) + d$, where $b \neq 0$, period $= \frac{2\pi}{|b|}$.

**Model**




To graph a sinusoidal function of the form $y = \sin bx$ or $y = \cos bx$, find the period of the function and successively add $\frac{\text{period}}{4}$ to the left endpoint of an interval with that length. Then use these values as the $x$-values for the key points on the graph.

## Example 3 Graph Horizontal Dilations of Sinusoidal Functions

**Describe how the graphs of $f(x) = \cos x$ and $g(x) = \cos \frac{x}{3}$ are related. Then find the period of $g(x)$, and sketch at least one period of both functions on the same coordinate axes.**

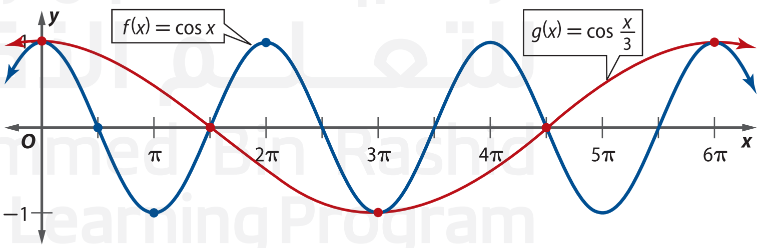
Because $\cos \frac{x}{3} = \cos \frac{1}{3}x$, the graph of $g(x)$ is the graph of $f(x)$ expanded horizontally.

The period of $g(x)$ is $\frac{2\pi}{\left|\frac{1}{3}\right|}$ or $6\pi$.

Because the period of $g(x)$ is $6\pi$, to find corresponding points on the graph of $g(x)$, change the $x$-coordinates of those key points on $f(x)$ so that they range from 0 to $6\pi$, increasing by increments of $\frac{6\pi}{4}$ or $\frac{3\pi}{2}$.

| Function | Maximum | $x$-intercept | Minimum | $x$-intercept | Maximum |
|---|---|---|---|---|---|
| $f(x) = \cos x$ | $(0, 1)$ | $\left(\frac{\pi}{2}, 0\right)$ | $(\pi, -1)$ | $\left(\frac{3\pi}{2}, 0\right)$ | $(2\pi, 1)$ |
| $g(x) = \cos \frac{x}{3}$ | $(0, 1)$ | $\left(\frac{3\pi}{2}, 0\right)$ | $(3\pi, -1)$ | $\left(\frac{9\pi}{2}, 0\right)$ | $(6\pi, 1)$ |

Sketch the curve through the indicated points for each function, continuing the patterns to complete one full cycle of each.




### GuidedPractice

**Describe how the graphs of $f(x)$ and $g(x)$ are related. Then find the period of $g(x)$, and sketch at least one period of each function on the same coordinate axes.**

**3A.** $f(x) = \cos x$
$g(x) = \cos \frac{x}{2}$

**3B.** $f(x) = \sin x$
$g(x) = \sin 3x$

**3C.** $f(x) = \cos x$
$g(x) = \cos \frac{1}{4}x$

Horizontal dilations also affect the *frequency* of sinusoidal functions.

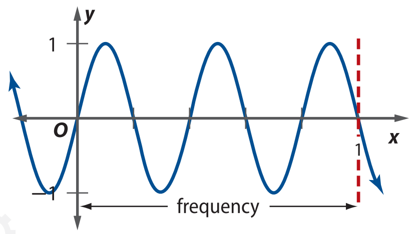
## KeyConcept Frequency of Sine and Cosine Functions

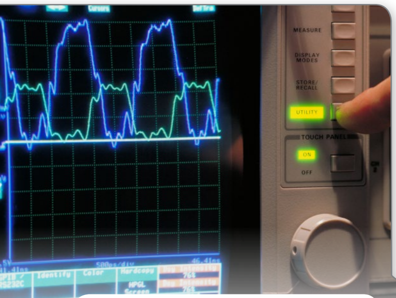
**Words** The **frequency** of a sinusoidal function is the number of cycles the function completes in a one unit interval. The frequency is the reciprocal of the period.

**Symbols** For $y = a \sin (bx + c) + d$ and $y = a \cos (bx + c) + d$,

frequency $= \dfrac{1}{\text{period}}$ or $\dfrac{|b|}{2\pi}$.

**Model**



### Real-WorldLink

In physics, frequency is measured in *hertz* or oscillations per second. For example, the number of sound waves passing a point A in one second would be the wave's frequency.

**Source:** *Science World*

Because the frequency of a sinusoidal function is the reciprocal of the period, it follows that the period of the function is the reciprocal of its frequency.

## Real-World Example 4 Use Frequency to Write a Sinusoidal Function

**MUSIC Musical notes are classified by frequency. In the equal tempered scale, middle C has a frequency of 262 hertz. Use this information and the information at the left to write an equation for a sine function that can be used to model the initial behavior of the sound wave associated with middle C having an amplitude of 0.2.**

The general form of the equation will be $y = a \sin bt$, where $t$ is the time in seconds. Because the amplitude is 0.2, $|a| = 0.2$. This means that $a = \pm 0.2$.

The period is the reciprocal of the frequency or $\frac{1}{262}$. Use this value to find $b$.

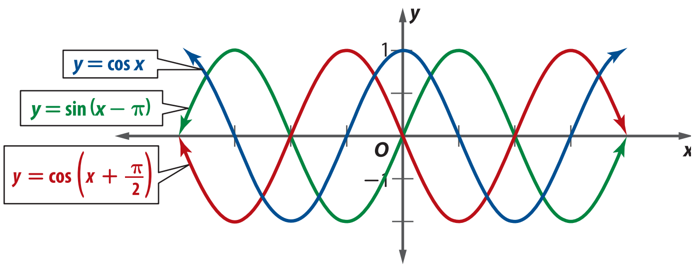
| | |
|---|---|
| $\text{period} = \dfrac{2\pi}{\lvert b \rvert}$ | Period formula |
| $\dfrac{1}{262} = \dfrac{2\pi}{\lvert b \rvert}$ | $\text{period} = \dfrac{1}{262}$ |
| $\lvert b \rvert = 2\pi(262)$ or $524\pi$ | Solve for $\lvert b \rvert$. |
| $b = \pm 524\pi$ | Solve for $b$. |

By arbitrarily choosing the positive values of $a$ and $b$, one sine function that models the initial behavior is $y = 0.2 \sin 524\pi t$.

### GuidedPractice

**4. MUSIC** In the same scale, the C above middle C has a frequency of 524 hertz. Write an equation for a sine function that can be used to model the initial behavior of the sound wave associated with this C having an amplitude of 0.1.

A *phase* of a sinusoid is the position of a wave relative to some reference point. A horizontal translation of a sinusoidal function results in a *phase shift*. Recall from Lesson 1-5 that the graph of $y = f(x + c)$ is the graph of $y = f(x)$ translated or *shifted* $|c|$ units left if $c > 0$ and $|c|$ units right if $c < 0$.

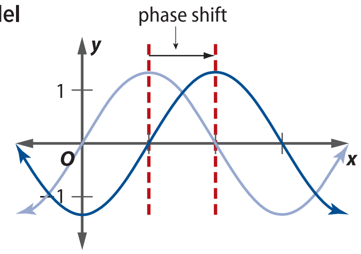


## KeyConcept Phase Shift of Sine and Cosine Functions

**Words** The **phase shift** of a sinusoidal function is the difference between the horizontal position of the function and that of an otherwise similar sinusoidal function.

**Symbols** For $y = a \sin (bx + c) + d$ and $y = a \cos (bx + c) + d$, where $b \neq 0$, phase shift $= -\frac{c}{|b|}$.

**Model**




You will verify the formula for phase shift in Exercise 44.

**StudyTip**

**Alternative Form** The general forms of the sinusoidal functions can also be expressed as $y = a \sin b(x - h) + k$ and $y = a \cos b(x - h) + k$. In these forms, each sinusoid has a phase shift of $h$ and a vertical translation of $k$ in comparison to the graphs of $y = a \sin bx$ and $y = a \cos bx$.

To graph the phase shift of a sinusoidal function of the form $y = a \sin (bx + c) + d$ or $y = a \cos (bx + c) + d$, first determine the endpoints of an interval that corresponds to one cycle of the graph by adding $-\frac{c}{b}$ to each endpoint on the interval $[0, 2\pi]$ of the parent function.

### Example 5 Graph Horizontal Translations of Sinusoidal Functions

**State the amplitude, period, frequency, and phase shift of $y = \sin\left(3x - \frac{\pi}{2}\right)$. Then graph two periods of the function.**

In this function, $a = 1$, $b = 3$, and $c = -\frac{\pi}{2}$.

Amplitude: $|a| = |1|$ or 1

Period: $\frac{2\pi}{|b|} = \frac{2\pi}{|3|}$ or $\frac{2\pi}{3}$

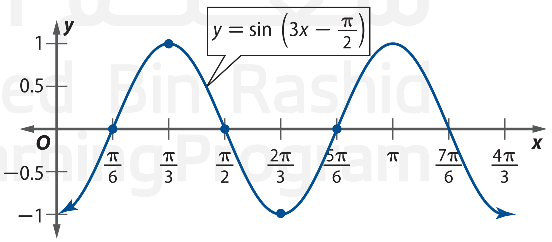
Frequency: $\frac{|b|}{2\pi} = \frac{|3|}{2\pi}$ or $\frac{3}{2\pi}$

Phase shift: $-\frac{c}{|b|} = -\frac{-\frac{\pi}{2}}{|3|}$ or $\frac{\pi}{6}$

To graph $y = \sin\left(3x - \frac{\pi}{2}\right)$, consider the graph of $y = \sin 3x$. The period of this function is $\frac{2\pi}{3}$. Create a table listing the coordinates of key points of $y = \sin 3x$ on the interval $\left[0, \frac{2\pi}{3}\right]$. To account for a phase shift of $\frac{\pi}{6}$, add $\frac{\pi}{6}$ to the $x$-values of each of the key points for the graph of $y = \sin 3x$.

| Function | $x$-intercept | Maximum | $x$-intercept | Minimum | $x$-intercept |
|---|---|---|---|---|---|
| $y = \sin 3x$ | $(0, 0)$ | $\left(\frac{\pi}{6}, 1\right)$ | $\left(\frac{\pi}{3}, 0\right)$ | $\left(\frac{\pi}{2}, -1\right)$ | $\left(\frac{2\pi}{3}, 0\right)$ |
| $y = \sin\left(3x - \frac{\pi}{2}\right)$ | $\left(\frac{\pi}{6}, 0\right)$ | $\left(\frac{\pi}{3}, 1\right)$ | $\left(\frac{\pi}{2}, 0\right)$ | $\left(\frac{2\pi}{3}, -1\right)$ | $\left(\frac{5\pi}{6}, 0\right)$ |

Sketch the graph of $y = \sin\left(3x - \frac{\pi}{2}\right)$ through these points, continuing the pattern to complete two cycles.




**GuidedPractice**

**State the amplitude, period, frequency, and phase shift of each function. Then graph two periods of the function.**

**5A.** $y = \cos\left(\frac{x}{2} + \frac{\pi}{4}\right)$

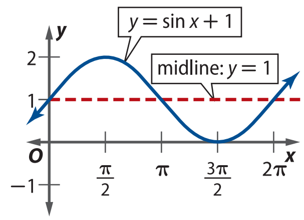
**5B.** $y = 3 \sin\left(2x - \frac{\pi}{3}\right)$

**StudyTip**

**Notation** $\sin(x + d) \neq \sin x + d$ The first expression indicates a phase shift, while the second expression indicates a vertical shift.

The final way to transform the graph of a sinusoidal function is through a vertical translation or **vertical shift**. Recall from Lesson 1-5 that the graph of $y = f(x) + d$ is the graph of $y = f(x)$ translated or *shifted* $|d|$ units up if $d > 0$ and $|d|$ units down if $d < 0$. The vertical shift is the average of the maximum and minimum values of the function.

The parent functions $y = \sin x$ and $y = \cos x$ oscillate about the $x$-axis. After a vertical shift, a new horizontal axis known as the **midline** becomes the reference line or equilibrium point about which the graph oscillates. For example, the midline of $y = \sin x + 1$ is $y = 1$, as shown.




In general, the midline for the graphs of $y = a \sin(bx + c) + d$ and $y = a \cos(bx + c) + d$ is $y = d$.

## Example 6 Graph Vertical Translations of Sinusoidal Functions

**State the amplitude, period, frequency, phase shift, and vertical shift of $y = \sin(x + 2\pi) - 1$. Then graph two periods of the function.**

In this function, $a = 1$, $b = 1$, $c = 2\pi$, and $d = -1$.

Amplitude: $|a| = |1|$ or 1

Period: $\frac{2\pi}{|b|} = \frac{2\pi}{|1|}$ or $2\pi$

Frequency: $\frac{|b|}{2\pi} = \frac{|1|}{2\pi}$ or $\frac{1}{2\pi}$

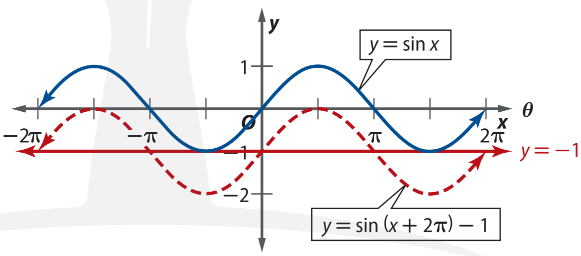
Phase shift: $-\frac{c}{|b|} = -\frac{2\pi}{|1|} = -2\pi$

Vertical shift: $d$ or $-1$

Midline: $y = d$ or $y = -1$

First, graph the midline $y = -1$. Then graph $y = \sin x$ shifted $2\pi$ units to the left and 1 unit down.

Notice that this transformation is equivalent to a translation 1 unit down because the phase shift was one period to the left.




**GuidedPractice**

**State the amplitude, period, frequency, phase shift, and vertical shift of each function. Then graph two periods of the function.**

**6A.** $y = 2 \cos x + 1$

**6B.** $y = \frac{1}{2} \sin\left(\frac{x}{4} - \frac{\pi}{2}\right) - 3$



The characteristics of transformations of the parent functions $y = \sin x$ and $y = \cos x$ are summarized below.

**TechnologyTip**

**Zoom Trig** When graphing a trigonometric function using your graphing calculator, be sure you are in radian mode and use the ZTrig selection under the zoom feature to change your viewing window from the standard window to a more appropriate window of $[-2\pi, 2\pi]$ scl: $\pi/2$ by $[-4, 4]$ scl: 1.

**ConceptSummary** Graphs of Sinusoidal Functions

The graphs of $y = a \sin(bx + c) + d$ and $y = a \cos(bx + c) + d$, where $a \neq 0$ and $b \neq 0$, have the following characteristics.

Amplitude: $|a|$

Period: $\frac{2\pi}{|b|}$

Frequency: $\frac{|b|}{2\pi}$ or $\frac{1}{\text{Period}}$

Phase shift: $-\frac{c}{|b|}$

Vertical shift: $d$

Midline: $y = d$

## 2 Applications of Sinusoidal Functions

Many real-world situations that exhibit periodic behavior over time can be modeled by transformations of $y = \sin x$ or $y = \cos x$.

### Real-WorldLink

The table shows the number of daylight hours on the 15th of each month in New York City.

| Month | Hours of Daylight |
|---|---|
| January | 9.58 |
| February | 10.67 |
| March | 11.9 |
| April | 13.3 |
| May | 14.43 |
| June | 15.07 |
| July | 14.8 |
| August | 13.8 |
| September | 12.48 |
| October | 11.15 |
| November | 9.9 |
| December | 9.27 |

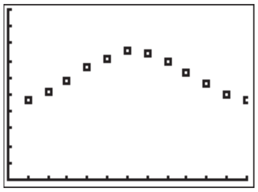
**Source:** U.S. Naval Observatory

### Real-World Example 7 Modeling Data Using a Sinusoidal Function

**METEOROLOGY Use the information at the left to write a sinusoidal function that models the number of hours of daylight for New York City as a function of time $x$, where $x = 1$ represents January 15, $x = 2$ represents February 15, and so on. Then use your model to estimate the number of hours of daylight on September 30 in New York City.**

**Step 1** Make a scatter plot of the data and choose a model.

The graph appears wave-like, so you can use a sinusoidal function of the form $y = a \sin (bx + c) + d$ or $y = a \cos (bx + c) + d$ to model the data. We will choose to use $y = a \cos (bx + c) + d$ to model the data.



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**Step 2** Find the maximum $M$ and minimum $m$ values of the data, and use these values to find $a$, $b$, $c$, and $d$.

The maximum and minimum hours of daylight are 15.07 and 9.27, respectively. The amplitude $a$ is half of the distance between the extrema.

$$a = \frac{1}{2}(M - m) = \frac{1}{2}(15.07 - 9.27) \text{ or } 2.9$$

The vertical shift $d$ is the average of the maximum and minimum data values.

$$d = \frac{1}{2}(M + m) = \frac{1}{2}(15.07 + 9.27) \text{ or } 12.17$$

A sinusoid completes half of a period in the time it takes to go from its maximum to its minimum value. One period is twice this time.

$$\text{Period} = 2(x_{max} - x_{min}) = 2(12 - 6) \text{ or } 12$$

$x_{max}$ = December 15 or month 12 and
$x_{min}$ = June 15 or month 6

Because the period equals $\frac{2\pi}{|b|}$, you can write $|b| = \frac{2\pi}{\text{Period}}$. Therefore, $|b| = \frac{2\pi}{12}$, or $\frac{\pi}{6}$.

The maximum data value occurs when $x = 6$. Since $y = \cos x$ attains its first maximum when $x = 0$, we must apply a phase shift of $6 - 0$ or 6 units. Use this value to find $c$.

$$\text{Phase shift} = -\frac{c}{|b|} \qquad \text{Phase shift formula}$$

$$6 = -\frac{c}{\frac{\pi}{6}} \qquad \text{Phase shift} = 6 \text{ and } |b| = \frac{\pi}{6}$$

$$c = -\pi \qquad \text{Solve for } c.$$

**Step 3** Write the function using the values for $a$, $b$, $c$, and $d$. Use $b = \frac{\pi}{6}$.

$y = 2.9 \cos \left(\frac{\pi}{6}x - \pi\right) + 12.17$ is one model for the hours of daylight

Graph the function and scatter plot in the same viewing window, as in Figure 3.4.1. To find the number of hours of daylight on September 30, evaluate the model for $x = 9.5$.

$y = 2.9 \cos \left(\frac{\pi}{6}(9.5) - \pi\right) + 12.17$ or about 11.42 hours of daylight

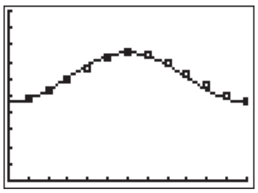

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**Figure 3.4.1**

### GuidedPractice

**METEOROLOGY** The average monthly temperatures for Seattle, Washington, are shown.

| Month | Jan | Feb | Mar | Apr | May | Jun | Jul | Aug | Sept | Oct | Nov | Dec |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Temp. (°F) | 41 | 44 | 47 | 50 | 56 | 61 | 65 | 66 | 61 | 54 | 46 | 42 |

**7A.** Write a function that models the monthly temperatures, using $x = 1$ to represent January.

**7B.** According to your model, what is Seattle's average monthly temperature in February?

## Exercises

**Describe how the graphs of $f(x)$ and $g(x)$ are related. Then find the amplitude of $g(x)$, and sketch two periods of both functions on the same coordinate axes.** (Examples 1 and 2)

1. $f(x) = \sin x$
   $g(x) = \frac{1}{2} \sin x$
2. $f(x) = \cos x$
   $g(x) = -\frac{1}{3} \cos x$
3. $f(x) = \cos x$
   $g(x) = 6 \cos x$
4. $f(x) = \sin x$
   $g(x) = -8 \sin x$

**Describe how the graphs of $f(x)$ and $g(x)$ are related. Then find the period of $g(x)$, and sketch at least one period of both functions on the same coordinate axes.** (Example 3)

5. $f(x) = \sin x$
   $g(x) = \sin 4x$
6. $f(x) = \cos x$
   $g(x) = \cos 2x$
7. $f(x) = \cos x$
   $g(x) = \cos \frac{1}{5} x$
8. $f(x) = \sin x$
   $g(x) = \sin \frac{1}{4} x$

9. **VOICES** The contralto vocal type includes the deepest female singing voice. Some contraltos can sing as low as the E below middle C (E3), which has a frequency of 165 hertz. Write an equation for a sine function that models the initial behavior of the sound wave associated with E3 having an amplitude of 0.15. (Example 4)

**Write a sine function that can be used to model the initial behavior of a sound wave with the frequency and amplitude given.** (Example 4)

10. $f = 440,\ a = 0.3$
11. $f = 932,\ a = 0.25$
12. $f = 1245,\ a = 0.12$
13. $f = 623,\ a = 0.2$

**State the amplitude, period, frequency, phase shift, and vertical shift of each function. Then graph two periods of the function.** (Examples 5 and 6)

14. $y = 3 \sin \left(x - \frac{\pi}{4}\right)$
15. $y = \cos \left(\frac{x}{3} + \frac{\pi}{2}\right)$
16. $y = 0.25 \cos x + 3$
17. $y = \sin 3x - 2$
18. $y = \cos \left(x - \frac{3\pi}{2}\right) - 1$
19. $y = \sin \left(x + \frac{5\pi}{6}\right) + 4$

20. **VACATIONS** The average number of reservations $R$ that a vacation resort has at the beginning of each month is shown. (Example 7)

| Month | $R$ | Month | $R$ |
|---|---|---|---|
| Jan | 200 | May | 121 |
| Feb | 173 | Jun | 175 |
| Mar | 113 | Jul | 198 |
| Apr | 87 | Aug | 168 |

   a. Write an equation of a sinusoidal function that models the average number of reservations using $x = 1$ to represent January.
   b. According to your model, approximately how many reservations can the resort anticipate in November?

21. **TIDES** The table shown below provides data for the first high and low tides of the day for a certain bay during one day in June. (Example 7)

| Tide | Height (m) | Time |
|---|---|---|
| first high tide | 12.95 | 4:25 A.M. |
| first low tide | 2.02 | 10:55 A.M. |

   a. Determine the amplitude, period, phase shift, and vertical shift of a sinusoidal function that models the height of the tide. Let $x$ represent the number of hours that the high or low tide occurred after midnight.
   b. Write a sinusoidal function that models the data.
   c. According to your model, what was the height of the tide at 8:45 P.M. that night?

22. **METEOROLOGY** The average monthly temperatures for Boston, Massachusetts are shown. (Example 7)

| Month | Temp. (°F) | Month | Temp. (°F) |
|---|---|---|---|
| Jan | 29 | Jul | 74 |
| Feb | 30 | Aug | 72 |
| Mar | 39 | Sept | 65 |
| Apr | 48 | Oct | 55 |
| May | 58 | Nov | 45 |
| Jun | 68 | Dec | 34 |

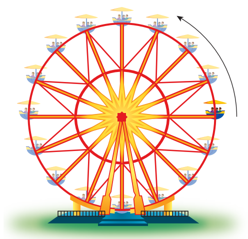
   a. Determine the amplitude, period, phase shift, and vertical shift of a sinusoidal function that models the monthly temperatures using $x = 1$ to represent January.
   b. Write an equation of a sinusoidal function that models the monthly temperatures.
   c. According to your model, what is Boston's average temperature in August?

**GRAPHING CALCULATOR** Find the values of $x$ in the interval $-\pi < x < \pi$ that make each equation or inequality true. (*Hint:* Use the intersection function.)

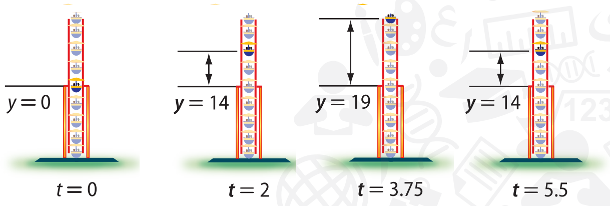
23. $-\sin x = \cos x$
24. $\sin x - \cos x = 1$
25. $\sin x + \cos x = 0$
26. $\cos x \leq \sin x$
27. $\sin x \cos x > 1$
28. $\sin x \cos x \leq 0$

29. **CAROUSELS** A wooden horse on a carousel moves up and down as the carousel spins. When the ride ends, the horse usually stops in a vertical position different from where it started. The position $y$ of the horse after $t$ seconds can be modeled by $y = 1.5 \sin (2t + c)$, where the phase shift $c$ must be continuously adjusted to compensate for the different starting positions. If during one ride the horse reached a maximum height after $\frac{7\pi}{12}$ seconds, find the equation that models the horse's position.

**30.** **AMUSEMENT PARKS** The position $y$ in feet of a passenger cart relative to the center of a Ferris wheel over $t$ seconds is shown below.



Side view of Ferris wheel over time interval [0, 5.5]




**a.** Find the time $t$ that it takes for the cart to return to $y = 0$ during its initial spin.

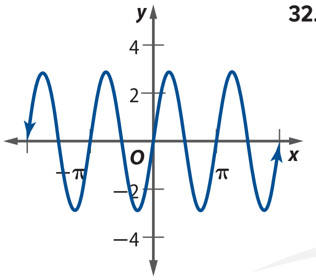
**b.** Find the period of the Ferris wheel.

**c.** Sketch the graph representing the position of the passenger cart over one period.

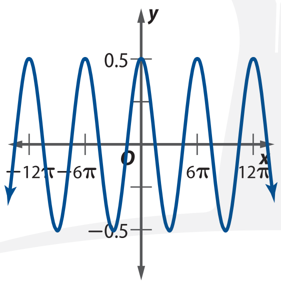
**d.** Write a sinusoidal function that models the position of the passenger cart as a function of time $t$.

**Write an equation that corresponds to each graph.**

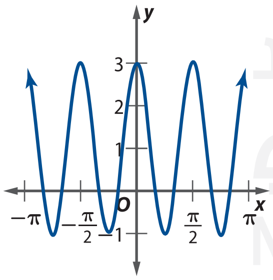
**31.**



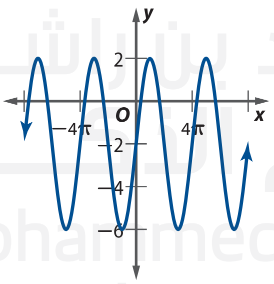

**32.**




**33.**




**34.**




**Write a sinusoidal function with the given period and amplitude that passes through the given point.**

**35.** period: $\pi$; amplitude: 5; point: $\left(\frac{\pi}{6}, \frac{5}{2}\right)$

**36.** period: $4\pi$; amplitude: 2; point: $(\pi, 2)$

**37.** period: $\frac{\pi}{2}$; amplitude: 1.5; point: $\left(\frac{\pi}{2}, \frac{3}{2}\right)$

**38.** period: $3\pi$; amplitude: 0.5; point: $\left(\pi, \frac{\sqrt{3}}{4}\right)$

**39.** **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the change in the graph of a sinusoidal function of the form $y = \sin x$ or $y = \cos x$ when multiplied by a polynomial function.

**a.** **GRAPHICAL** Use a graphing calculator to sketch the graphs of $y = 2x$, $y = -2x$, and $y = 2x \cos x$ on the same coordinate plane, on the interval $[-20, 20]$.

**b.** **VERBAL** Describe the behavior of the graph of $y = 2x \cos x$ in relation to the graphs of $y = 2x$ and $y = -2x$.

**c.** **GRAPHICAL** Use a graphing calculator to sketch the graphs of $y = x^2$, $y = -x^2$, and $y = x^2 \sin x$ on the same coordinate plane, on the interval $[-20, 20]$.

**d.** **VERBAL** Describe the behavior of the graph of $y = x^2 \sin x$ in relation to the graphs of $y = x^2$ and $y = -x^2$.

**e.** **ANALYTICAL** Make a conjecture as to the behavior of the graph of a sinusoidal function of the form $y = \sin x$ or $y = \cos x$ when multiplied by polynomial function of the form $y = f(x)$.

## H.O.T. Problems Use Higher-Order Thinking Skills

**40.** **CHALLENGE** Without graphing, find the exact coordinates of the first maximum point to the right of the $y$-axis for $y = 4 \sin\left(\frac{2}{3}x - \frac{\pi}{9}\right)$.

**REASONING** **Determine whether each statement is *true* or *false*. Explain your reasoning.**

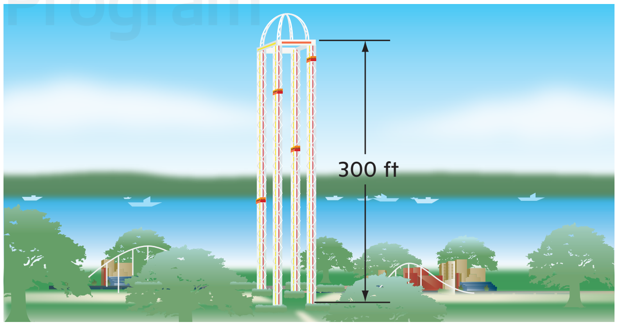
**41.** Every sine function of the form $y = a \sin(bx + c) + d$ can also be written as a cosine function of the form $y = a \cos(bx + c) + d$.

**42.** The period of $f(x) = \cos 8x$ is equal to four times the period of $g(x) = \cos 2x$.

**43.** **CHALLENGE** How many zeros does $y = \cos 1500x$ have on the interval $0 \le x \le 2\pi$?

**44.** **PROOF** Prove the phase shift formula.

**45.** **WRITING IN MATH** The Power Tower ride in Sandusky, Ohio, is shown below. Along the side of each tower is a string of lights that send a continuous pulse of light up and down each tower at a constant rate. Explain why the distance $d$ of this light from the ground over time $t$ cannot be represented by a sinusoidal function.



## Spiral Review

**The given point lies on the terminal side of an angle $\theta$ in standard position. Find the values of the six trigonometric functions of $\theta$.**

**46.** $(-4, 4)$

**47.** $(8, -2)$

**48.** $(-5, -9)$

**49.** $(4, 5)$

**Write each degree measure in radians as a multiple of $\pi$ and each radian measure in degrees.**

**50.** $25°$

**51.** $-420°$

**52.** $-\frac{\pi}{4}$

**53.** $\frac{8\pi}{3}$

**54.** **SCIENCE** Radiocarbon dating is a method of estimating the age of an organic material by calculating the amount of carbon-14 present in the material. The age of a material can be calculated using $A = t \cdot \frac{\ln R}{-0.693}$, where $A$ is the age of the object in years, $t$ is the half-life of carbon-14 or 5700 years, and $R$ is the ratio of the amount of carbon-14 in the sample to the amount of carbon-14 in living tissue.

**a.** A sample of organic material contains 0.000076 gram of carbon-14. A living sample of the same material contains 0.00038 gram. About how old is the sample?

**b.** A specific sample is at least 20,000 years old. What is the maximum percent of carbon-14 remaining in the sample?

**State the number of possible real zeros and turning points of each function. Then determine all of the real zeros by factoring.**

**55.** $f(x) = x^3 + 2x^2 - 8x$

**56.** $f(x) = x^4 - 10x^2 + 9$

**57.** $f(x) = x^5 + 2x^4 - 4x^3 - 8x^2$

**58.** $f(x) = x^4 - 1$

**Determine whether $f$ has an inverse function. If it does, find the inverse function and state any restrictions on its domain.**

**59.** $f(x) = -x - 2$

**60.** $f(x) = \frac{1}{x + 4}$

**61.** $f(x) = (x - 3)^2 - 7$

**62.** $f(x) = \frac{1}{(x - 1)^2}$

## Skills Review for Standardized Tests

**63.** **SAT/ACT** If $x + y = 90°$ and $x$ and $y$ are both nonnegative angles, what is equal to $\frac{\cos x}{\sin y}$?

**A** 0

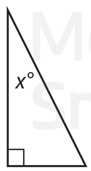
**B** $\frac{1}{2}$

**C** 1

**D** 1.5

**E** Cannot be determined from the information given.

**64.** **REVIEW** If $\tan x = \frac{10}{24}$ in the figure below, what are $\sin x$ and $\cos x$?



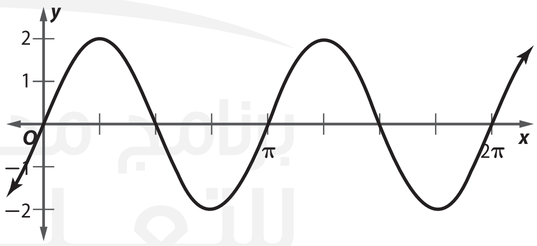

**F** $\sin x = \frac{26}{10}$ and $\cos x = \frac{24}{26}$

**G** $\sin x = \frac{10}{26}$ and $\cos x = \frac{24}{26}$

**H** $\sin x = \frac{26}{10}$ and $\cos x = \frac{26}{24}$

**J** $\sin x = \frac{10}{26}$ and $\cos x = \frac{26}{24}$

**65.** Identify the equation represented by the graph.




**A** $y = \frac{1}{2}\sin 4x$

**B** $y = \frac{1}{4}\sin 2x$

**C** $y = 2\sin 2x$

**D** $y = 4\sin\frac{1}{2}x$

**66.** **REVIEW** If $\cos\theta = \frac{8}{17}$ and the terminal side of the angle is in Quadrant IV, what is the exact value of $\sin\theta$?

**F** $-\frac{15}{8}$

**G** $-\frac{17}{15}$

**H** $-\frac{15}{17}$

**J** $-\frac{8}{15}$

# EXTEND 3-4 Graphing Technology Lab
# Sums and Differences of Sinusoids

## Objective

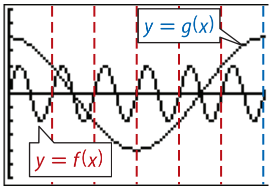
- Graph and examine the periods of sums and differences of sinusoids.

The graphs of the sums and differences of two sinusoids will often have different periods than the graphs of the original functions.

## Activity 1 Sum of Sinusoids

**Determine a common interval on which both $f(x) = 2 \sin 3x$ and $g(x) = 4 \cos \frac{x}{2}$ complete a whole number of cycles. Then graph $h(x) = f(x) + g(x)$, and identify the period of the function.**

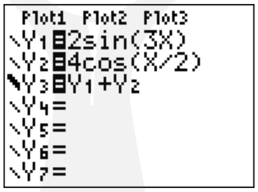
**Step 1** Enter $f(x)$ for Y1 and $g(x)$ for Y2. Then adjust the window until each graph completes one or more whole cycles on the same interval. One interval on which this occurs is $[0, 4\pi]$. On this interval, $g(x)$ completes one whole cycle and $f(x)$ completes six whole cycles.

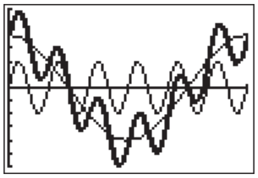



$[0, 4\pi]$ scl: $\pi$ by $[-6, 6]$ scl: 1

**Step 2** To graph $h(x)$ as Y3, under the VARS menu, select Y-VARS, function, and Y1 to enter Y1. Then press + and select Y-VARS, function, and Y2 to enter Y2.

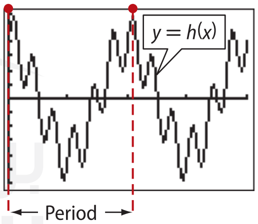
**Step 3** Graph $f(x)$, $g(x)$ and $h(x)$ on the same screen. To make the graph of $h(x)$ stand out, scroll to the left of the equals sign next to Y3, and press ENTER. Then graph the functions using the same window as above.



$[0, 4\pi]$ scl: $\pi$ by $[-6, 6]$ scl: 1

**Step 4** By adjusting the x-axis from $[0, 4\pi]$ to $[0, 8\pi]$ to observe the full pattern of $h(x)$, we can see that the period of the sum of the two sinusoids is $4\pi$.




$[0, 8\pi]$ scl: $2\pi$ by $[-6, 6]$ scl: 1

**TechnologyTip**

**Hiding Graphs** Scroll to the equals sign and select enter to make a graph disappear.



## Exercises

**Determine a common interval on which both $f(x)$ and $g(x)$ complete a whole number of cycles. Then graph $a(x) = f(x) + g(x)$ and $b(x) = f(x) - g(x)$, and identify the period of the function.**

1. $f(x) = 4 \sin 2x$
   $g(x) = -2 \cos 3x$

2. $f(x) = \sin 8x$
   $g(x) = \cos 6x$

3. $f(x) = 3 \sin (x - \pi)$
   $g(x) = -2 \cos 2x$

4. $f(x) = \frac{1}{2} \sin 4x$
   $g(x) = 2 \sin \left(x - \frac{\pi}{2}\right)$

5. $f(x) = \frac{1}{4} \cos \frac{x}{2}$
   $g(x) = -2 \cos \left(x - \frac{\pi}{2}\right)$

6. $f(x) = -\frac{1}{2} \sin 2x$
   $g(x) = 3 \cos 2x$

7. **MAKE A CONJECTURE** Explain how you can use the periods of two sinusoids to find the period of the sum or difference of the two sinusoids.

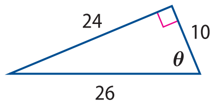
CHAPTER 3

# Mid-Chapter Quiz

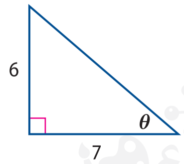
## Lessons 3-1 through 3-4

**Find the exact values of the six trigonometric functions of $\theta$.** (Lesson 3-1)

**1.**

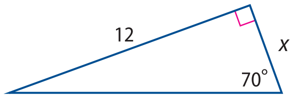



**2.**

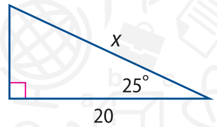



**Find the value of $x$. Round to the nearest tenth if necessary.** (Lesson. 3-1)

**3.**




**4.**



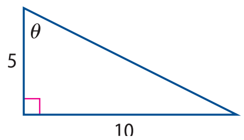

**5.** **SHADOWS** A pine tree casts a shadow that is 7.9 ft long when the Sun is 80° above the horizon. (Lesson 3-1)

**a.** Find the height of the tree.

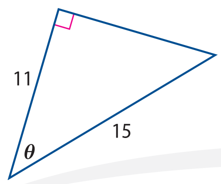
**b.** Later that same day, a person 6 ft tall casts a shadow 6.7 ft long. At what angle is the Sun above the horizon?

**Find the measure of angle $\theta$. Round to the nearest degree if necessary.** (Lesson 3-1)

**6.**




**7.**



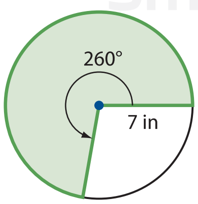

**8.** Write $\frac{2\pi}{9}$ in degrees. (Lesson 3-2)

**Identify all angles that are coterminal with the given angle. Then find and draw one positive and one negative angle coterminal with the given angle.** (Lesson 3-2)

**9.** $\frac{3\pi}{10}$

**10.** $-22°$

**11.** **MULTIPLE CHOICE** Find the approximate area of the shaded region. (Lesson 3-2)




**A** 12.2 in$^2$

**B** 42.8 in$^2$

**C** 85.5 in$^2$

**D** 111.2 in$^2$

**12.** **TRAVEL** A car is traveling at a speed of 55 mi/h on tires that measure 2.6 ft in diameter. Find the approximate angular speed of the tires in radians per minute. (Lesson 3-2)

**Sketch each angle. Then find its reference angle.** (Lesson 3-3)

**13.** 175°

**14.** $\frac{21\pi}{13}$

**Find the exact value of each expression. If undefined, write *undefined*.** (Lesson 3-3)

**15.** $\cos 315°$

**16.** $\sec \frac{3\pi}{2}$

**17.** $\sin \frac{5\pi}{3}$

**18.** $\tan \frac{5\pi}{6}$

**Find the exact values of the five remaining trigonometric functions of $\theta$.** (Lesson 3-3)

**19.** $\cos \theta = -\frac{2}{5}$, where $\sin \theta < 0$ and $\tan \theta > 0$

**20.** $\cot \theta = \frac{4}{3}$, where $\cos \theta > 0$ and $\sin \theta > 0$

**State the amplitude, period, frequency, phase shift, and vertical shift of each function. Then graph two full periods of the function.** (Lesson 3-4)

**21.** $y = -3 \sin \left(x - \frac{3\pi}{2}\right)$

**22.** $y = 5 \cos 2x - 2$

**23.** **MULTIPLE CHOICE** Which of the functions has the same graph as $y = 3 \sin (x - \pi)$? (Lesson 3-4)

**F** $y = 3 \sin (x + \pi)$

**G** $y = 3 \cos \left(x - \frac{\pi}{2}\right)$

**H** $y = -3 \sin (x - \pi)$

**J** $y = -3 \cos \left(x + \frac{\pi}{2}\right)$

**24.** **SPRING** The motion of an object attached to a spring oscillating across its original position of rest can be modeled by $x(t) = A \cos \omega t$, where $A$ is the initial displacement of the object from its resting position, $\omega$ is a constant dependent on the spring and the mass of the object attached to the spring, and $t$ is time measured in seconds. (Lesson 3-4)

**a.** Draw a graph for the motion of an object attached to a spring and displaced 4 cm, where $\omega = 3$.

**b.** How long will it take for the object to return to its initial position for the first time?

**c.** The constant $\omega$ is equal to $\sqrt{\frac{k}{m}}$, where $k$ is the spring constant, and $m$ is the mass of the object. How would increasing the mass of an object affect the period of its oscillations? Explain your reasoning.

**25.** **BUOY** The height above sea level in feet of a signal buoy's transmitter is modeled by $h = a \sin bt + \frac{11}{2}$. In rough waters, the height cycles between 1 and 10 ft, with 4 seconds between cycles. Find the values of $a$ and $b$.

# LESSON 3-5 Graphing Other Trigonometric Functions

## :: Then

- You analyzed graphs of trigonometric functions.

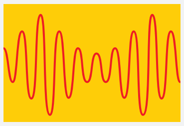
## :: Now

- **1** Graph tangent and reciprocal trigonometric functions.
- **2** Graph damped trigonometric functions.

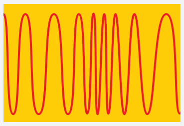
## :: Why?

- There are two types of radio transmissions known as amplitude modulation (AM) and frequency modulation (FM). When sound is transmitted by an AM radio station, the amplitude of a sinusoidal wave called the *carrier wave* is changed to produce sound. The transmission of an FM signal results in a change in the frequency of the carrier wave. You will learn more about the graphs of these waves, known as *damped waves*, in this lesson.

AM signal



FM signal



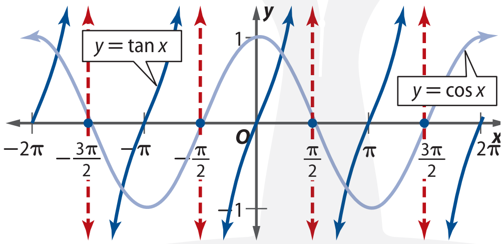
### NewVocabulary

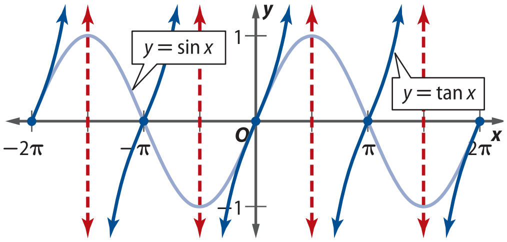
damped trigonometric function
damping factor
damped oscillation
damped wave
damped harmonic motion

## 1 Tangent and Reciprocal Functions

In Lesson 3-4, you graphed the sine and cosine functions on the coordinate plane. You can use the same techniques to graph the tangent function and the reciprocal trigonometric functions—cotangent, secant, and cosecant.

Since $\tan x = \frac{\sin x}{\cos x}$, the tangent function is undefined when $\cos x = 0$. Therefore, the tangent function has a *vertical asymptote* whenever $\cos x = 0$. Similarly, the tangent and sine functions each have zeros at integer multiples of $\pi$ because $\tan x = 0$ when $\sin x = 0$.

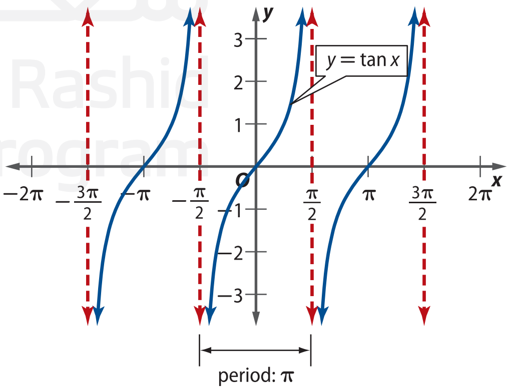



The properties of the tangent function are summarized below.

### KeyConcept Properties of the Tangent Function

| | |
|---|---|
| **Domain:** | $x \in \mathbb{R}, x \neq \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$ |
| **Range:** | $(-\infty, \infty)$ |
| ***x*-intercepts:** | $n\pi, n \in \mathbb{Z}$ |
| ***y*-intercept:** | 0 |
| **Continuity:** | infinite discontinuity at $x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$ |
| **Asymptotes:** | $x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$ |
| **Symmetry:** | origin (odd function) |
| **Extrema:** | none |
| **End Behavior:** | $\lim_{x \to -\infty} \tan x$ and $\lim_{x \to \infty} \tan x$ do not exist. The function oscillates between $-\infty$ and $\infty$. |



**StudyTip**

**Amplitude** The term *amplitude* does not apply to the tangent or cotangent functions because the heights of these functions are infinite.

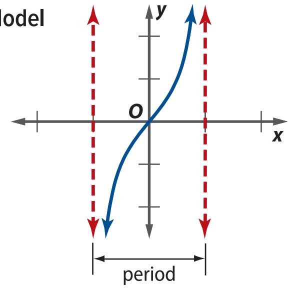
The general form of the tangent function, which is similar to that of the sinusoidal functions, is $y = a \tan (bx + c) + d$, where $a$ produces a vertical stretch or compression, $b$ affects the period, $c$ produces a phase shift, $d$ produces a vertical shift and neither $a$ or $b$ are 0.

## KeyConcept Period of the Tangent Function

**Words** The *period* of a tangent function is the distance between any two consecutive vertical asymptotes.

**Symbols** For $y = a \tan (bx + c)$, where $b \neq 0$, period $= \frac{\pi}{|b|}$.

**Model**




Two consecutive vertical asymptotes for $y = \tan x$ are $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$. You can find two consecutive vertical asymptotes for any tangent function of the form $y = a \tan (bx + c) + d$ by solving the equations $bx + c = -\frac{\pi}{2}$ and $bx + c = \frac{\pi}{2}$.

You can sketch the graph of a tangent function by plotting the vertical asymptotes, $x$-intercepts, and points between the asymptotes and $x$-intercepts.

## Example 1 Graph Horizontal Dilations of the Tangent Function

**Locate the vertical asymptotes, and sketch the graph of $y = \tan 2x$.**

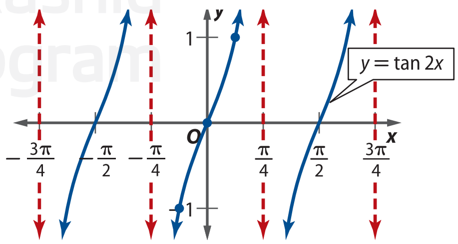
The graph of $y = \tan 2x$ is the graph of $y = \tan x$ compressed horizontally. The period is $\frac{\pi}{|2|}$ or $\frac{\pi}{2}$. Find two consecutive vertical asymptotes.

| | | |
|---|---|---|
| $bx + c = -\frac{\pi}{2}$ | Tangent asymptote equations | $bx + c = \frac{\pi}{2}$ |
| $2x + 0 = -\frac{\pi}{2}$ | $b = 2, c = 0$ | $2x + 0 = \frac{\pi}{2}$ |
| $x = -\frac{\pi}{4}$ | Simplify. | $x = \frac{\pi}{4}$ |

Create a table listing key points, including the $x$-intercept, that are located between the two vertical asymptotes at $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$.

| Function | Vertical Asymptote | Intermediate Point | $x$-intercept | Intermediate Point | Vertical Asymptote |
|---|---|---|---|---|---|
| $y = \tan x$ | $x = -\frac{\pi}{2}$ | $\left(-\frac{\pi}{4}, -1\right)$ | $(0, 0)$ | $\left(\frac{\pi}{4}, 1\right)$ | $x = \frac{\pi}{2}$ |
| $y = \tan 2x$ | $x = -\frac{\pi}{4}$ | $\left(-\frac{\pi}{8}, -1\right)$ | $(0, 0)$ | $\left(\frac{\pi}{8}, 1\right)$ | $x = \frac{\pi}{4}$ |

Sketch the curve through the indicated key points for the function. Then sketch one cycle to the left on the interval $\left(-\frac{3\pi}{4}, -\frac{\pi}{4}\right)$ and one cycle to the right on the interval $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$.




**GuidedPractice**

**Locate the vertical asymptotes, and sketch the graph of each function.**

**1A.** $y = \tan 4x$

**1B.** $y = \tan \frac{x}{2}$

## Example 2 Graph Reflections and Translations of the Tangent Function

**Locate the vertical asymptotes, and sketch the graph of each function.**

**a.** $y = -\tan \frac{x}{2}$

The graph of $y = -\tan \frac{x}{2}$ is the graph of $y = \tan x$ expanded horizontally and then reflected in the $x$-axis. The period is $\frac{\pi}{\left|\frac{1}{2}\right|}$ or $2\pi$. Find two consecutive vertical asymptotes.

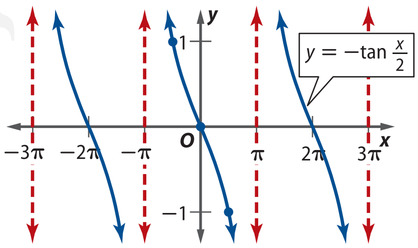
$\frac{x}{2} + 0 = -\frac{\pi}{2}$    $b = \frac{1}{2}, c = 0$    $\frac{x}{2} + 0 = \frac{\pi}{2}$

$x = 2\left(-\frac{\pi}{2}\right)$ or $-\pi$    Simplify.    $x = 2\left(\frac{\pi}{2}\right)$ or $\pi$

Create a table listing key points, including the x-intercept, that are located between the two vertical asymptotes at $x = -\pi$ and $x = \pi$.

| Function | Vertical Asymptote | Intermediate Point | $x$-intercept | Intermediate Point | Vertical Asymptote |
|---|---|---|---|---|---|
| $y = \tan x$ | $x = -\frac{\pi}{2}$ | $\left(-\frac{\pi}{4}, -1\right)$ | $(0, 0)$ | $\left(\frac{\pi}{4}, 1\right)$ | $x = \frac{\pi}{2}$ |
| $y = -\tan \frac{x}{2}$ | $x = -\pi$ | $\left(-\frac{\pi}{2}, 1\right)$ | $(0, 0)$ | $\left(\frac{\pi}{2}, -1\right)$ | $x = \pi$ |

Sketch the curve through the indicated key points for the function. Then repeat the pattern for one cycle to the left and right of the first curve.




**b.** $y = \tan\left(x - \frac{3\pi}{2}\right)$

The graph of $y = \tan\left(x - \frac{3\pi}{2}\right)$ is the graph of $y = \tan x$ shifted $\frac{3\pi}{2}$ units to the right. The period is $\frac{\pi}{|1|}$ or $\pi$. Find two consecutive vertical asymptotes.

$x - \frac{3\pi}{2} = -\frac{\pi}{2}$    $b = 1, c = -\frac{3\pi}{2}$    $x - \frac{3\pi}{2} = \frac{\pi}{2}$

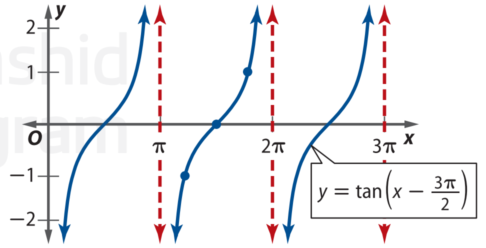
$x = -\frac{\pi}{2} + \frac{3\pi}{2}$ or $\pi$    Simplify.    $x = \frac{\pi}{2} + \frac{3\pi}{2}$ or $2\pi$

| Function | Vertical Asymptote | Intermediate Point | $x$-intercept | Intermediate Point | Vertical Asymptote |
|---|---|---|---|---|---|
| $y = \tan x$ | $x = -\frac{\pi}{2}$ | $\left(-\frac{\pi}{4}, -1\right)$ | $(0, 0)$ | $\left(\frac{\pi}{4}, 1\right)$ | $x = \frac{\pi}{2}$ |
| $y = \tan\left(x - \frac{3\pi}{2}\right)$ | $x = \pi$ | $\left(\frac{5\pi}{4}, -1\right)$ | $\left(\frac{3\pi}{2}, 0\right)$ | $\left(\frac{7\pi}{4}, 1\right)$ | $x = 2\pi$ |

**StudyTip**

**Alternate Method** When graphing a function with only a horizontal translation $c$, you can find the key points by adding $c$ to each of the $x$-coordinates of the key points of the parent function.

Sketch the curve through the indicated key points for the function. Then sketch one cycle to the left and right of the first curve.




### GuidedPractice

**2A.** $y = \tan\left(2x + \frac{\pi}{2}\right)$

**2B.** $y = -\tan\left(x - \frac{\pi}{6}\right)$

The cotangent function is the reciprocal of the tangent function, and is defined as $\cot x = \frac{\cos x}{\sin x}$. Like the tangent function, the period of a cotangent function of the form $y = a \cot (bx + c) + d$ can be found by calculating $\frac{\pi}{|b|}$. Two consecutive vertical asymptotes can be found by solving the equations $bx + c = 0$ and $bx + c = \pi$. The properties of the cotangent function are summarized below.

## KeyConcept Properties of the Cotangent Function

**Domain:** $x \in \mathbb{R}, x \neq n\pi, n \in \mathbb{Z}$

**Range:** $(-\infty, \infty)$

***x*-intercepts:** $\frac{\pi}{2} + n\pi, n \in \mathbb{Z}$

***y*-intercept:** none

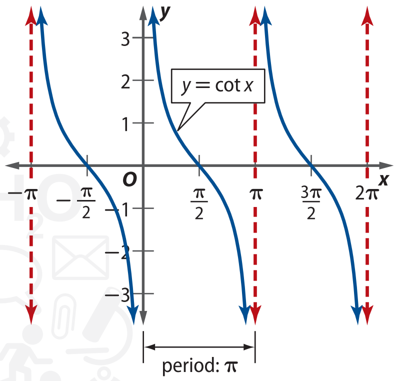
**Continuity:** infinite discontinuity at $x = n\pi, n \in \mathbb{Z}$

**Asymptotes:** $x = n\pi, n \in \mathbb{Z}$

**Symmetry:** origin (odd function)

**Extrema:** none

**End Behavior:** $\lim\limits_{x \to -\infty} \cot x$ and $\lim\limits_{x \to \infty} \cot x$ do not exist. The function oscillates between $-\infty$ and $\infty$.




You can sketch the graph of a cotangent function using the same techniques that you used to sketch the graph of a tangent function.

## Example 3 Sketch the Graph of a Cotangent Function

**Locate the vertical asymptotes, and sketch the graph of $y = \cot \frac{x}{3}$.**

The graph of $y = \cot \frac{x}{3}$ is the graph of $y = \cot x$ expanded horizontally. The period is $\frac{\pi}{\left|\frac{1}{3}\right|}$ or $3\pi$. Find two consecutive vertical asymptotes by solving $bx + c = 0$ and $bx + c = \pi$.

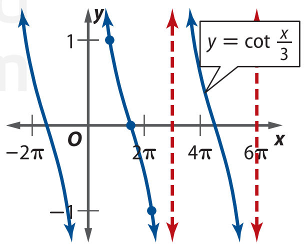
$\frac{x}{3} + 0 = 0$     $b = \frac{1}{3}, c = 0$     $\frac{x}{3} + 0 = \pi$

$x = 3(0)$ or $0$     Simplify.     $x = 3(\pi)$ or $3\pi$

Create a table listing key points, including the $x$-intercept, that are located between the two vertical asymptotes at $x = 0$ and $x = 3\pi$.

| Function | Vertical Asymptote | Intermediate Point | *x*-intercept | Intermediate Point | Vertical Asymptote |
|---|---|---|---|---|---|
| $y = \cot x$ | $x = 0$ | $\left(\frac{\pi}{4}, 1\right)$ | $\left(\frac{\pi}{2}, 0\right)$ | $\left(\frac{3\pi}{4}, -1\right)$ | $x = \pi$ |
| $y = \cot \frac{x}{3}$ | $x = 0$ | $\left(\frac{3\pi}{4}, 1\right)$ | $\left(\frac{3\pi}{2}, 0\right)$ | $\left(\frac{9\pi}{4}, -1\right)$ | $x = 3\pi$ |

Following the same guidelines that you used for the tangent function, sketch the curve through the indicated key points that you found. Then sketch one cycle to the left and right of the first curve.




### TechnologyTip

**Graphing a Cotangent Function** When using a calculator to graph a cotangent function, enter the reciprocal of tangent, $y = \frac{1}{\tan x}$. Graphing calculators may produce solid lines where the asymptotes occur. Setting the mode to DOT will eliminate the line.

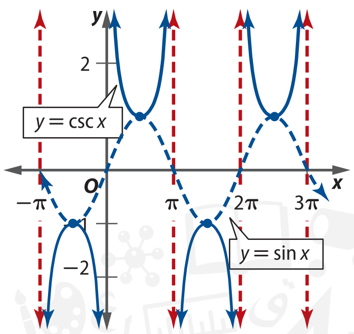
### GuidedPractice

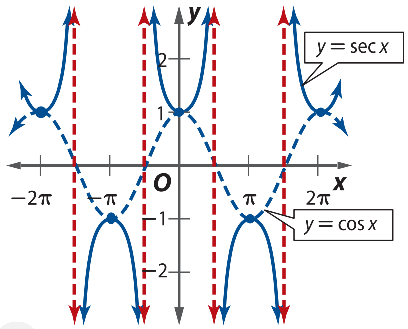
**Locate the vertical asymptotes, and sketch the graph of each function.**

**3A.** $y = -\cot 3x$

**3B.** $y = 3 \cot \frac{x}{2}$

The reciprocals of the sine and cosine functions are defined as $\csc x = \frac{1}{\sin x}$ and $\sec x = \frac{1}{\cos x}$, as shown below.




The cosecant function has asymptotes when $\sin x = 0$, which occurs at integer multiples of $\pi$. Likewise, the secant function has asymptotes when $\cos x = 0$, located at odd multiples of $\frac{\pi}{2}$. Notice also that the graph of $y = \csc x$ has a relative minimum at each maximum point on the sine curve, and a relative maximum at each minimum point on the sine curve. The same is true for the graphs of $y = \sec x$ and $y = \cos x$.

The properties of the cosecant and secant functions are summarized below.

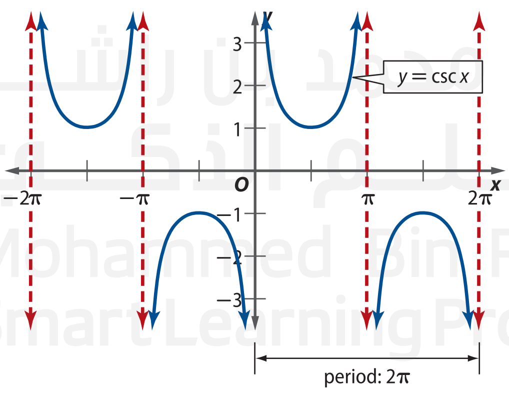
**TechnologyTip**

**Graphing** Graphing the cosecant and secant functions on a calculator is similar to graphing the cotangent function. Enter the reciprocals of the sine and cosine functions.

## KeyConcept Properties of the Cosecant and Secant Functions

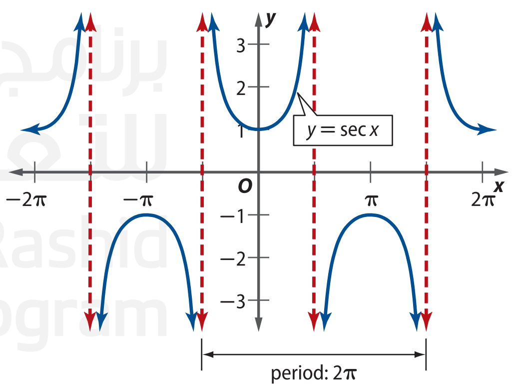
### Cosecant Function

| | |
|---|---|
| **Domain:** | $x \in \mathbb{R}, x \neq n\pi, n \in \mathbb{Z}$ |
| **Range:** | $(-\infty, -1]$ and $[1, \infty)$ |
| ***x*-intercepts:** | none |
| ***y*-intercept:** | none |
| **Continuity:** | infinite discontinuity at $x = n\pi, n \in \mathbb{Z}$ |
| **Asymptotes:** | $x = n\pi, n \in \mathbb{Z}$ |
| **Symmetry:** | origin (odd function) |
| **End Behavior:** | $\lim_{x \to -\infty} \csc x$ and $\lim_{x \to \infty} \csc x$ do not exist. The function oscillates between $-\infty$ and $\infty$. |




### Secant Function

| | |
|---|---|
| **Domain:** | $x \in \mathbb{R}, x \neq \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$ |
| **Range:** | $(-\infty, -1]$ and $[1, \infty)$ |
| ***x*-intercepts:** | none |
| ***y*-intercept:** | 1 |
| **Continuity:** | infinite discontinuity at $x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$ |
| **Asymptotes:** | $x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$ |
| **Symmetry:** | *y*-axis (even function) |
| **Behavior:** | $\lim_{x \to -\infty} \sec x$ and $\lim_{x \to \infty} \sec x$ do not exist. The function oscillates between $-\infty$ and $\infty$. |




Like the sinusoidal functions, the period of a secant function of the form $y = a \sec (bx + c) + d$ or cosecant function of the form $y = a \csc (bx + c) + d$ can be found by calculating $\frac{2\pi}{|b|}$. Two vertical asymptotes for the secant function can be found by solving the equations $bx + c = -\frac{\pi}{2}$ and $bx + c = \frac{3\pi}{2}$ and two vertical asymptotes for the cosecant function can be found by solving $bx + c = -\pi$ and $bx + c = \pi$.

To sketch the graph of a cosecant or secant function, locate the asymptotes of the function and find the corresponding relative maximum and minimums points.

## Example 4 Sketch Graphs of Cosecant and Secant Functions

**Locate the vertical asymptotes, and sketch the graph of each function.**

**a.** $y = \csc\left(x + \frac{\pi}{2}\right)$

The graph of $y = \csc\left(x + \frac{\pi}{2}\right)$ is the graph of $y = \csc x$ shifted $\frac{\pi}{2}$ units to the left. The period is $\frac{2\pi}{|1|}$ or $2\pi$. Two vertical asymptotes occur when $bx + c = -\pi$ and $bx + c = \pi$. Therefore, two asymptotes are $x + \frac{\pi}{2} = -\pi$ or $x = -\frac{3\pi}{2}$ and $x + \frac{\pi}{2} = \pi$ or $x = \frac{\pi}{2}$.

Create a table listing key points, including the relative maximum and minimum, that are located between the two vertical asymptotes at $x = -\frac{3\pi}{2}$ and $x = \frac{\pi}{2}$.

| Function | Vertical Asymptote | Relative Maximum | Vertical Asymptote | Relative Minimum | Vertical Asymptote |
|---|---|---|---|---|---|
| $y = \csc x$ | $x = -\pi$ | $\left(-\frac{\pi}{2}, -1\right)$ | $x = 0$ | $\left(\frac{\pi}{2}, 1\right)$ | $x = \pi$ |
| $y = \csc\left(x + \frac{\pi}{2}\right)$ | $x = -\frac{3\pi}{2}$ | $(-\pi, -1)$ | $x = -\frac{\pi}{2}$ | $(0, 1)$ | $x = \frac{\pi}{2}$ |

Sketch the curve through the indicated key points for the function. Then sketch one cycle to the left and right. The graph is shown in Figure 4.5.1 below.

**StudyTip**

**Finding Asymptotes and Key Points** You can use the periodic nature of trigonometric graphs to help find asymptotes and key points. In Example 4a, notice that the vertical asymptote $x = -\frac{\pi}{2}$ is equidistant from the calculated asymptotes, $x = -\frac{3\pi}{2}$ and $x = \frac{\pi}{2}$.

**b.** $y = \sec \frac{x}{4}$

The graph of $y = \sec \frac{x}{4}$ is the graph of $y = \sec x$ expanded horizontally. The period is $\frac{2\pi}{\left|\frac{1}{4}\right|}$ or $8\pi$. Two vertical asymptotes occur when $bx + c = -\frac{\pi}{2}$ and $bx + c = \frac{3\pi}{2}$. Therefore, two asymptotes are $\frac{x}{4} + 0 = -\frac{\pi}{2}$ or $x = -2\pi$ and $\frac{x}{4} + 0 = \frac{3\pi}{2}$ or $x = 6\pi$.

Create a table listing key points that are located between the asymptotes at $x = -2\pi$ and $x = 6\pi$.

| Function | Vertical Asymptote | Relative Minimum | Vertical Asymptote | Relative Maximum | Vertical Asymptote |
|---|---|---|---|---|---|
| $y = \sec x$ | $x = -\frac{\pi}{2}$ | $(0, 1)$ | $x = \frac{\pi}{2}$ | $(\pi, -1)$ | $x = \frac{3\pi}{2}$ |
| $y = \sec \frac{x}{4}$ | $x = -2\pi$ | $(0, 1)$ | $x = 2\pi$ | $(4\pi, -1)$ | $x = 6\pi$ |

Sketch the curve through the indicated key points for the function. Then sketch one cycle to the left and right. The graph is shown in Figure 4.5.2 below.

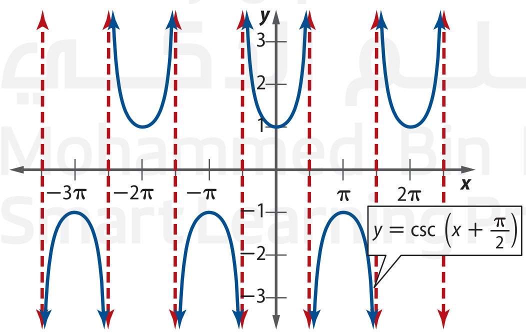



Figure 3.5.1

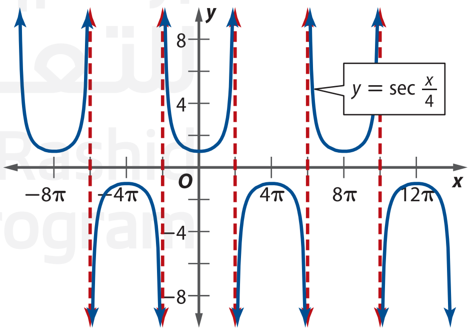



Figure 3.5.2

### GuidedPractice

**4A.** $y = \csc 2x$

**4B.** $y = \sec(x + \pi)$

## 2 Damped Trigonometric Functions

When a sinusoidal function is multiplied by another function $f(x)$, the graph of their product oscillates between the graphs of $y = f(x)$ and $y = -f(x)$. When this product reduces the amplitude of the wave of the original sinusoid, it is called **damped oscillation**, and the product of the two functions is known as a **damped trigonometric function**. This change in oscillation can be seen in Figures 3.5.3 and 3.5.4 for the graphs of $y = \sin x$ and $y = 2x \sin x$.

> **StudyTip**
>
> **Damped Functions** Trigonometric functions that are multiplied by constants do not experience damping. The constant affects the amplitude of the function.

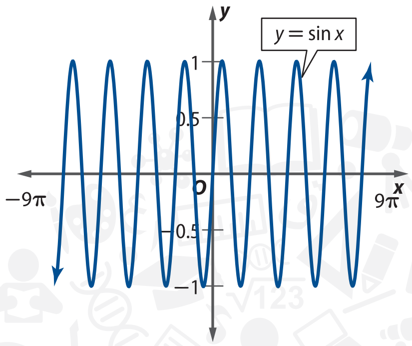



**Figure 3.5.3**

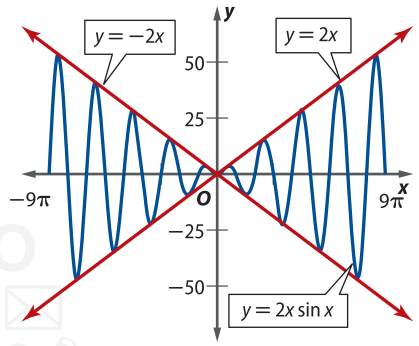



**Figure 3.5.4**

A damped trigonometric function is of the form $y = f(x) \sin bx$ or $y = f(x) \cos bx$, where $f(x)$ is the **damping factor**.

Damped oscillation occurs as $x$ approaches $\pm\infty$ or as $x$ approaches 0 from both directions.

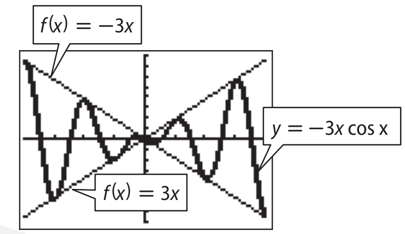
### Example 5 Sketch Damped Trigonometric Functions

**Identify the damping factor $f(x)$ of each function. Then use a graphing calculator to sketch the graphs of $f(x)$, $-f(x)$, and the given function in the same viewing window. Describe the behavior of the graph.**

**a.** $y = -3x \cos x$

The function $y = -3x \cos x$ is the product of the functions $y = -3x$ and $y = \cos x$, so $f(x) = -3x$.

The amplitude of the function is decreasing as $x$ approaches 0 from both directions.

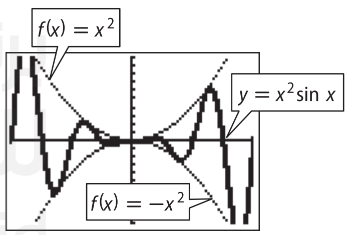



$[-4\pi, 4\pi]$ scl: $\pi$ by $[-40, 40]$ scl: 5

**b.** $y = x^2 \sin x$

The function $y = x^2 \sin x$ is the product of the functions $y = x^2$ and $y = \sin x$. Therefore, the damping factor is $f(x) = x^2$.

The amplitude of the function is decreasing as $x$ approaches 0 from both directions.



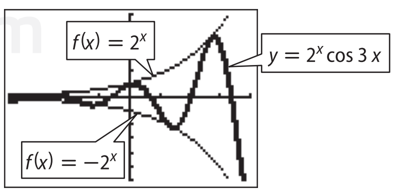

$[-4\pi, 4\pi]$ scl: $\pi$ by $[-100, 100]$ scl: 10

> **Math HistoryLink**
>
> **Cathleen Synge Morawetz (1923– )**
> A Canadian, Morawetz studied the scattering of sound and magnetic waves and later proved results relating to the nonlinear wave equation.

**c.** $y = 2^x \cos 3x$

The function $y = 2^x \cos 3x$ is the product of the functions $y = 2^x$ and $y = \cos 3x$, so $f(x) = 2^x$.

The amplitude of the function is decreasing as $x$ approaches $-\infty$.




$[-\pi, \pi]$ scl: $\frac{\pi}{4}$ by $[-6, 6]$ scl: 1

**GuidedPractice**

**5A.** $y = 5x \sin x$

**5B.** $y = \frac{1}{x} \cos x$

**5C.** $y = 3^x \sin x$

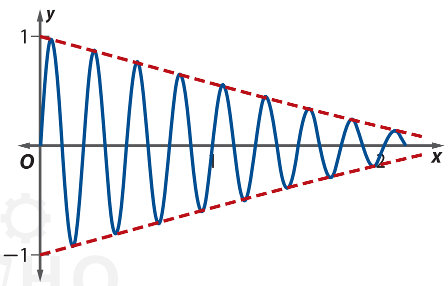
When the amplitude of the motion of an object decreases with time due to friction, the motion is called *damped harmonic motion*.

## KeyConcept Damped Harmonic Motion

**Words** An object is in **damped harmonic motion** when the amplitude is determined by the function $a(t) = ke^{-ct}$.

**Symbols** For $y = ke^{-ct} \sin \omega t$ and $y = ke^{-ct} \cos \omega t$, where $c > 0$, $k$ is the displacement, $c$ is the damping constant, $t$ is time, and $\omega$ is the period.

**Model**




The greater the damping constant $c$, the faster the amplitude approaches 0. The magnitude of $c$ depends on the size of the object and the material of which it is composed.

### Real-WorldLink

Each string on a guitar is stretched to a particular length and tautness. These aspects, along with the weight and type of string, cause it to vibrate with a characteristic frequency or pitch called its fundamental frequency, producing the note we hear.

**Source:** *How Stuff Works*

## Real-World Example 6 Damped Harmonic Motion

**MUSIC A guitar string is plucked at a distance of 0.8 cm above its rest position and then released, causing a vibration. The damping constant for the string is 2.1, and the note produced has a frequency of 175 cycles per second.**

**a. Write a trigonometric function that models the motion of the string.**

The maximum displacement of the string occurs when $t = 0$, so $y = ke^{-ct} \cos \omega t$ can be used to model the motion of the string because the graph of $y = \cos t$ has a $y$-intercept other than 0.

The maximum displacement occurs when the string is plucked 0.8 cm. The total displacement is the maximum displacement $M$ minus the minimum displacement $m$, so

$$k = M - m = 0.8 - 0 \text{ or } 0.8 \text{ cm.}$$

You can use the value of the frequency to find $\omega$.

$\dfrac{|\omega|}{2\pi} = 175$ $\quad$ $\dfrac{|\omega|}{2\pi} =$ frequency

$|\omega| = 350\pi$ $\quad$ Multiply each side by $2\pi$.

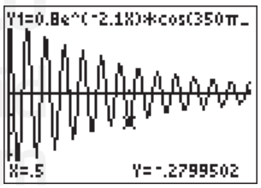
Write a function using the values of $k$, $\omega$, and $c$.

$y = 0.8e^{-2.1t} \cos 350\pi t$ is one model that describes the motion of the string.

**b. Determine the amount of time $t$ that it takes the string to be damped so that $-0.28 \le y \le 0.28$.**

Use a graphing calculator to determine the value of $t$ when the graph of $y = 0.8e^{-2.1t} \cos 350\pi t$ is oscillating between $y = -0.28$ and $y = 0.28$.

From the graph, you can see that it takes approximately 0.5 second for the graph of $y = 0.8e^{-2.1t} \cos 350\pi t$ to oscillate within the interval $-0.28 \le y \le 0.28$.




[0, 1] scl: 0.5 by [−0.75, 0.75] scl: 0.25

### GuidedPractice

6. **MUSIC** Suppose another string on the guitar was plucked 0.5 cm above its rest position with a frequency of 98 cycles per second and a damping constant of 1.7.

   **A.** Write a trigonometric function that models the motion of the string $y$ as a function of time $t$.

   **B.** Determine the time $t$ that it takes the string to be damped so that $-0.15 \le y \le 0.15$.

## Exercises

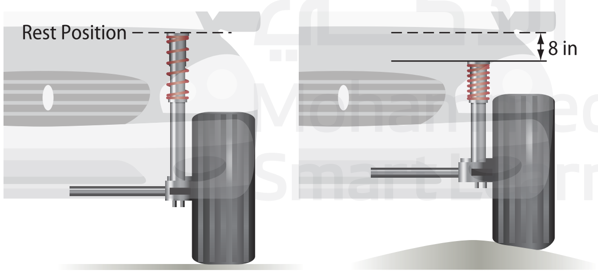
**Locate the vertical asymptotes, and sketch the graph of each function.** (Examples 1–4)

1. $y = 2 \tan x$
2. $y = \tan\left(x + \frac{\pi}{4}\right)$
3. $y = \cot\left(x - \frac{\pi}{6}\right)$
4. $y = -3 \tan \frac{x}{3}$
5. $y = -\frac{1}{4} \cot x$
6. $y = -\tan 3x$
7. $y = -2 \tan (6x - \pi)$
8. $y = \cot \frac{x}{2}$
9. $y = \frac{1}{5} \csc 2x$
10. $y = \csc\left(4x + \frac{7\pi}{6}\right)$
11. $y = \sec (x + \pi)$
12. $y = -2 \csc 3x$
13. $y = 4 \sec\left(x - \frac{3\pi}{4}\right)$
14. $y = \sec\left(\frac{x}{5} + \frac{\pi}{5}\right)$
15. $y = \frac{3}{2} \csc\left(x - \frac{2\pi}{3}\right)$
16. $y = -\sec \frac{x}{8}$

**Identify the damping factor $f(x)$ of each function. Then use a graphing calculator to sketch the graphs of $f(x)$, $-f(x)$, and the given function in the same viewing window. Describe the behavior of the graph.** (Example 5)

17. $y = \frac{3}{5} x \sin x$
18. $y = 4x \cos x$
19. $y = 2x^2 \cos x$
20. $y = \frac{x^3}{2} \sin x$
21. $y = \frac{1}{3} x \sin 2x$
22. $y = (x - 2)^2 \sin x$
23. $y = e^{0.5x} \cos x$
24. $y = 3^x \sin x$
25. $y = |x| \cos 3x$
26. $y = \ln x \cos x$

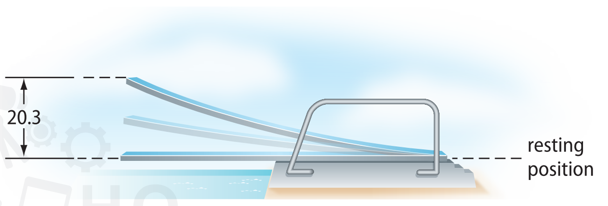
27. **MECHANICS** When the car shown below hit a bump in the road, the shock absorber was compressed 8 in, released, and then began to vibrate in damped harmonic motion with a frequency of 2.5 cycles per second. The damping constant for the shock absorber is 3. (Example 6)




a. Write a trigonometric function that models the displacement of the shock absorber $y$ as a function of time $t$. Let $t = 0$ be the instant the shock absorber is released.

b. Determine the amount of time $t$ that it takes for the amplitude of the vibration to decrease to 4 in.

28. **DIVING** The end of a diving board is 20.3 cm above its resting position at the moment a diver leaves the board. Two seconds later, the board has moved down and up 12 times. The damping constant for the board is 0.901. (Example 6)



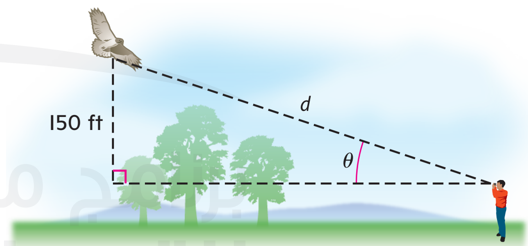

a. Write a trigonometric function that models the motion of the diving board $y$ as a function of time $t$.

b. Determine the amount of time $t$ that it takes the diving board to be damped so that $-0.5 \le y \le 0.5$.

**Locate the vertical asymptotes, and sketch the graph of each function.**

29. $y = \sec x + 3$
30. $y = \sec\left(x - \frac{\pi}{2}\right) + 4$
31. $y = \csc \frac{x}{3} - 2$
32. $y = \csc\left(3x + \frac{\pi}{6}\right) + 3$
33. $y = \cot (2x + \pi) - 3$
34. $y = \cot\left(\frac{x}{2} + \frac{\pi}{2}\right) - 1$

35. **PHOTOGRAPHY** Saeed is taking pictures of a hawk that is flying 150 ft above him. The hawk will eventually fly directly over Saeed. Let $d$ be the distance Saeed is from the hawk and $\theta$ be the angle of elevation to the hawk from Saeed's camera.




a. Write $d$ as a function of $\theta$.

b. Graph the function on the interval $0 < \theta < \pi$.

c. Approximately how far away is the hawk from Saeed when the angle of elevation is 45°?

36. **DISTANCE** A spider is slowly climbing up a wall. Hiyam is standing 6 ft away from the wall watching the spider. Let $d$ be the distance Hiyam is from the spider and $\theta$ be the angle of elevation to the spider from Hiyam.

a. Write $d$ as a function of $\theta$.

b. Graph the function on the interval $0 < \theta < \frac{\pi}{2}$.

c. Approximately how far away is the spider from Hiyam when the angle of elevation is 32°?

**GRAPHING CALCULATOR** **Find the values of $\theta$ on the interval $-\pi < \theta < \pi$ that make each equation true.**

**37.** $\cot \theta = 2 \sec \theta$

**38.** $\sin \theta = \cot \theta$

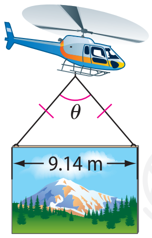
**39.** $4 \cos \theta = \csc \theta$

**40.** $\tan \frac{\theta}{2} = \sin \theta$

**41.** $\csc \theta = \sec \theta$

**42.** $\tan \theta = \sec \frac{\theta}{2}$

**43.** **TENSION** A helicopter is delivering a large mural that is to be displayed in the center of town. Two ropes are used to attach the mural to the helicopter, as shown. The tension $T$ on each rope is equal to half the downward force times $\sec \frac{\theta}{2}$.




**a.** The downward force in newtons equals the mass of the mural times gravity, which is 9.8 newtons per kilogram. If the mass of the mural is 544 kilograms, find the downward force.

**b.** Write an equation that represents the tension $T$ on each rope.

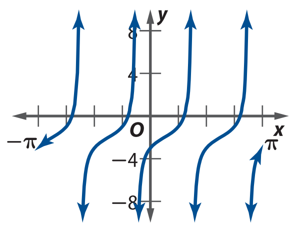
**c.** Graph the equation from part **b** on the interval $[0, 180°]$.

**d.** Suppose the mural is 9.14 m long and the ideal angle $\theta$ for tension purposes is a right angle. Determine how much rope is needed to transport the mural and the tension that is being applied to each rope.

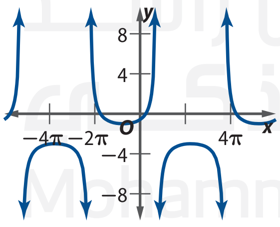
**e.** Suppose you have 12.2 m of rope to use to transport the mural. Find $\theta$ and the tension that is being applied to each rope.

**Match each function with its graph.**

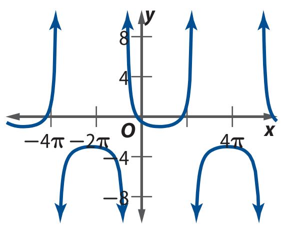
**a.**



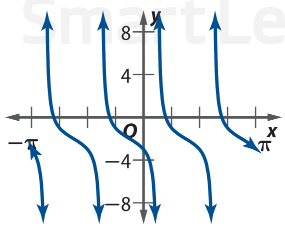

**b.**




**c.**




**d.**




**44.** $y = \csc\left(\frac{x}{3} + \frac{\pi}{4}\right) - 2$

**45.** $y = \sec\left(\frac{x}{3} + \frac{\pi}{4}\right) - 2$

**46.** $y = \cot\left(2x - \frac{\pi}{4}\right) - 2$

**47.** $y = \tan\left(2x - \frac{\pi}{4}\right) - 2$

**GRAPHING CALCULATOR** **Graph each pair of functions on the same screen and make a conjecture as to whether they are equivalent for all real numbers. Then use the properties of the functions to verify each conjecture.**

**48.** $f(x) = \sec x \cos x$; $g(x) = 1$

**49.** $f(x) = \sec^2 x$; $g(x) = \tan^2 x + 1$

**50.** $f(x) = \cos x \csc x$; $g(x) = \cot x$

**51.** $f(x) = \dfrac{1}{\sec\left(x - \frac{\pi}{2}\right)}$; $g(x) = \sin x$

**Write an equation for the given function given the period, phase shift (ps), and vertical shift (vs).**

**52.** function: sec; period: $3\pi$; ps: 0; vs: 2

**53.** function: tan; period: $\frac{\pi}{2}$; ps: $\frac{\pi}{4}$; vs: $-1$

**54.** function: csc; period: $\frac{\pi}{4}$; ps: $-\pi$; vs: 0

**55.** function: cot; period: $3\pi$; ps: $\frac{\pi}{2}$; vs: 4

**56.** function: csc; period: $\frac{\pi}{3}$; ps: $-\frac{\pi}{2}$; vs: $-3$

## H.O.T. Problems Use Higher-Order Thinking Skills

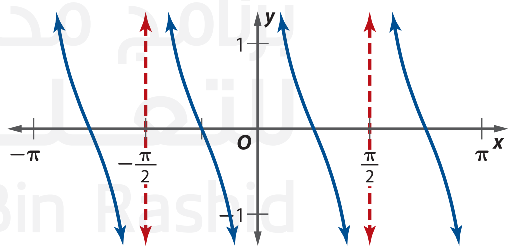
**57.** **PROOF** Verify that the $y$-intercept for the graph of any function of the form $y = ke^{-ct} \cos \omega t$ is $k$.

**REASONING** **Determine whether each statement is *true* or *false*. Explain your reasoning.**

**58.** If $b \neq 0$, then $y = a + b \sec x$ has extrema of $\pm(a + b)$.

**59.** If $x = \theta$ is an asymptote of $y = \csc x$, then $x = \theta$ is also an asymptote of $y = \cot x$.

**60.** **ERROR ANALYSIS** Hana and Huda are studying the graph shown. Hana thinks that it is the graph of $y = -\frac{1}{3} \tan 2x$, and Huda thinks that it is the graph of $y = \frac{1}{3} \cot 2x$. Is either of them correct? Explain your reasoning.




**61.** **CHALLENGE** Write a cosecant function and a cotangent function that have the same graphs as $y = \sec x$ and $y = \tan x$ respectively. Check your answers by graphing.

**62.** **WRITING IN MATH** A damped trigonometric function oscillates between the positive and negative graphs of the damping factor. Explain why a damped trigonometric function oscillates between the positive and negative graphs of the damping factor and why the amplitude of the function depends on the damping factor.

## Spiral Review

**State the amplitude, period, frequency, phase shift, and vertical shift of each function. Then graph two periods of the function.**

**63.** $y = 3\sin\left(2x - \frac{\pi}{3}\right) + 10$

**64.** $y = 2\cos\left(3x + \frac{3\pi}{4}\right) - 6$

**65.** $y = \frac{1}{2}\cos(4x - \pi) + 1$

**Find the exact values of the five remaining trigonometric functions of $\theta$.**

**66.** $\sin\theta = \frac{4}{5}$, $\cos\theta > 0$

**67.** $\cos\theta = \frac{6\sqrt{37}}{37}$, $\sin\theta > 0$

**68.** $\tan\theta = \frac{24}{7}$, $\sin\theta > 0$

**69. POPULATION** The population of a city 10 years ago was 45,600. Since then, the population has increased at a steady rate each year. If the population is currently 64,800, find the annual rate of growth for this city.

**70. MEDICINE** The half-life of a radioactive substance is the amount of time it takes for half of the atoms of the substance to disintegrate. Nuclear medicine technologists use the iodine isotope I-131, with a half-life of 8 days, to check a patient's thyroid function. After ingesting a tablet containing the iodine, the isotopes collect in the patient's thyroid, and a special camera is used to view its function. Suppose a patient ingests a tablet containing 9 microcuries of I-131. To the nearest hour, how long will it be until there are only 2.8 microcuries in the patient's thyroid?

**Factor each polynomial completely using the given factor and long division.**

**71.** $x^3 + 2x^2 - x - 2$; $x - 1$

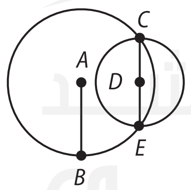
**72.** $x^3 + x^2 - 16x - 16$; $x + 4$

**73.** $x^3 - x^2 - 10x - 8$; $x + 1$

**74. EXERCISE** The American College of Sports Medicine recommends that healthy adults exercise at a target level of 60% to 90% of their maximum heart rates. You can estimate your maximum heart rate by subtracting your age from 220. Write a compound inequality that models age $a$ and target heart rate $r$.

## Skills Review for Standardized Tests

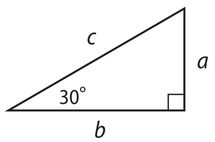
**75. SAT/ACT** In the figure, $A$ and $D$ are the centers of the two circles, which intersect at points $C$ and $E$. $\overline{CE}$ is a diameter of circle $D$. If $AB = CE = 10$, what is $AD$?




**A** 5 **C** $5\sqrt{3}$ **E** $10\sqrt{3}$

**B** $5\sqrt{2}$ **D** $10\sqrt{2}$

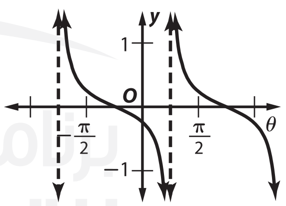
**76. REVIEW** Refer to the figure below. If $c = 14$, find the value of $b$.




**F** $\frac{\sqrt{3}}{2}$ **H** 7

**G** $14\sqrt{3}$ **J** $7\sqrt{3}$

**77.** Which equation is represented by the graph?




**A** $y = \cot\left(\theta + \frac{\pi}{4}\right)$

**B** $y = \cot\left(\theta - \frac{\pi}{4}\right)$

**C** $y = \tan\left(\theta + \frac{\pi}{4}\right)$

**D** $y = \tan\left(\theta - \frac{\pi}{4}\right)$

**78. REVIEW** If $\sin\theta = -\frac{1}{2}$ and $\pi < \theta < \frac{3\pi}{2}$, then $\theta = ?$

**F** $\frac{13\pi}{12}$ **H** $\frac{5\pi}{4}$

**G** $\frac{7\pi}{6}$ **J** $\frac{4\pi}{3}$

LESSON 3-6

# Inverse Trigonometric Functions

**Then**

- You found and graphed the inverses of relations and functions.

**Now**

1. Evaluate and graph inverse trigonometric functions.
2. Find compositions of trigonometric functions.

**Why?**

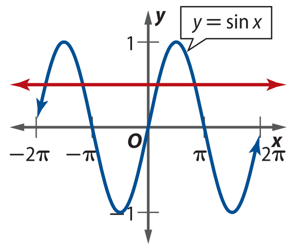
- Inverse trigonometric functions can be used to model the changing horizontal angle of rotation needed for a television camera to follow the motion of a drag-racing vehicle.

**NewVocabulary**

arcsine function
arccosine function
arctangent function

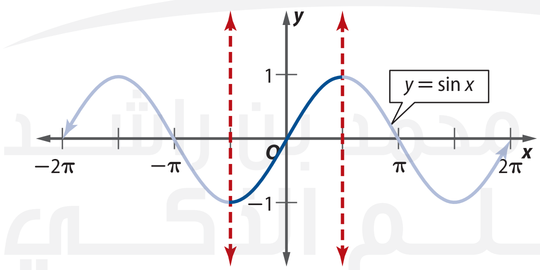
## 1 Inverse Trigonometric Functions

In Lesson 1-7, you learned that a function has an inverse function if and only if it is *one-to-one*, meaning that each $y$-value of the function can be matched with no more than one $x$-value. Because the sine function fails the horizontal line test, it is not one-to-one.

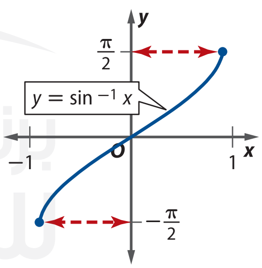



If, however, we restrict the domain of the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the restricted function *is* one-to-one and takes on all possible range values $[-1, 1]$ of the unrestricted function. It is on this restricted domain that $y = \sin x$ has an inverse function called the *inverse sine function* $y = \sin^{-1} x$. The graph of $y = \sin^{-1} x$ is found by reflecting the graph of the restricted sine function in the line $y = x$.

**Restricted Sine Function**




**Inverse Sine Function**






Notice that the domain of $y = \sin^{-1} x$ is $[-1, 1]$, and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Because angles and arcs given on the unit circle have equivalent radian measures, the inverse sine function is sometimes referred to as the **arcsine function** $y = \arcsin x$.

In Lesson 3-1, you used the inverse relationship between the sine and inverse sine functions to find acute angle measures. From the graphs above, you can see that in general,

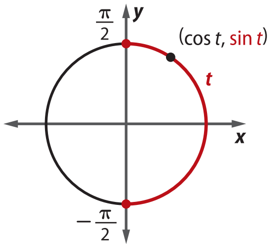
$$y = \sin^{-1} x \text{ or } y = \arcsin x \text{ iff } \sin y = x, \text{ when } -1 \le x \le 1 \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

iff means if and only if.

This means that $\sin^{-1} x$ or $\arcsin x$ can be interpreted as *the angle (or arc) between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ *with a sine of* $x$. For example, $\sin^{-1} 0.5$ is the angle with a sine of 0.5.

Recall that $\sin t$ is the $y$-coordinate of the point on the unit circle corresponding to the angle or arc length $t$. Because the range of the inverse sine function is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the possible angle measures of the inverse sine function are located on the right half of the unit circle, as shown.

**Inverse Sine Values**

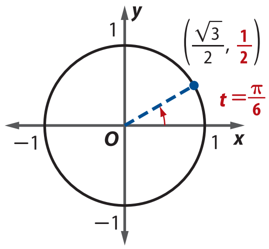



You can use the unit circle to find the exact value of some expressions involving $\sin^{-1} x$ or $\arcsin x$.

## Example 1 Evaluate Inverse Sine Functions

**Find the exact value of each expression, if it exists.**

**a.** $\sin^{-1} \frac{1}{2}$

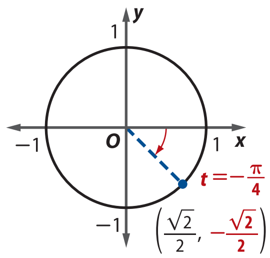



Find a point on the unit circle on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with a $y$-coordinate of $\frac{1}{2}$. When $t = \frac{\pi}{6}$, $\sin t = \frac{1}{2}$.

Therefore, $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$.

**CHECK** If $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$, then $\sin \frac{\pi}{6} = \frac{1}{2}$. ✓

**b.** $\arcsin \left(-\frac{\sqrt{2}}{2}\right)$




Find a point on the unit circle on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with a $y$-coordinate of $-\frac{\sqrt{2}}{2}$. When $t = -\frac{\pi}{4}$, $\sin t = -\frac{\sqrt{2}}{2}$.

Therefore, $\arcsin \left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

**CHECK** If $\arcsin \left(-\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$, then $\sin \left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$. ✓

**c.** $\sin^{-1} 3$

Because the domain of the inverse sine function is $[-1, 1]$ and $3 > 1$, there is no angle with a sine of 3. Therefore, the value of $\sin^{-1} 3$ does not exist.

### GuidedPractice

**1A.** $\arcsin \left(\frac{\sqrt{3}}{2}\right)$ **1B.** $\sin^{-1} (-2\pi)$ **1C.** $\arcsin (-1)$

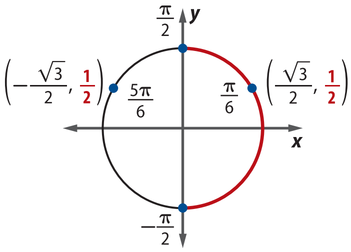
**TechnologyTip**

**Evaluate $\sin^{-1}$** You can also use a graphing calculator to find the angle that has a sine of $\frac{1}{2}$.

```
sin⁻¹(0.5)
        .5235987756
π/6
        .5235987756
```

Make sure you select RADIAN on the MODE feature of your graphing calculator.

Notice in Example 1a that while $\sin \frac{5\pi}{6}$ is also $\frac{1}{2}$, $\frac{5\pi}{6}$ is not in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, $\sin^{-1} \frac{1}{2} \neq \frac{5\pi}{6}$.

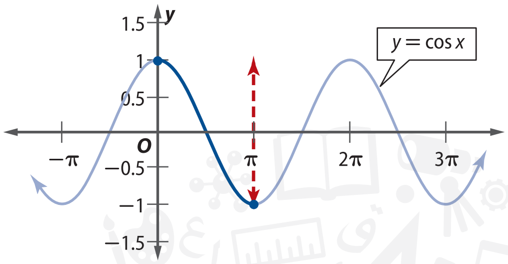


**StudyTip**

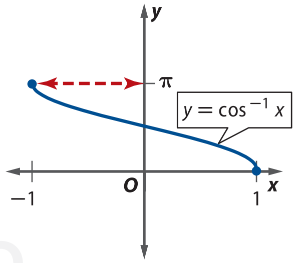
**Principal Values** Trigonometric functions with restricted domains are sometimes indicated with capital letters. For example, $y = \text{Sin } x$ represents the function $y = \sin x$, where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. The values in these restricted domains are often called *principal values*.

When restricted to a domain of $[0, \pi]$, the cosine function is one-to-one and takes on all of its possible range values on $[-1, 1]$. It is on this restricted domain that the cosine function has an inverse function, called the *inverse cosine function* $y = \cos^{-1} x$ or **arccosine function** $y = \arccos x$. The graph of $y = \cos^{-1} x$ is found by reflecting the graph of the restricted cosine function in the line $y = x$.

**Restricted Cosine Function**

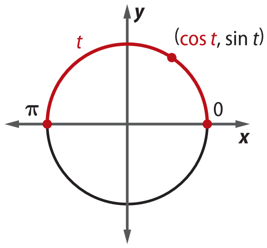



**Inverse Cosine Function**




Recall that $\cos t$ is the $x$-coordinate of the point on the unit circle corresponding to the angle or arc length $t$. Because the range of $y = \cos^{-1} x$ is restricted to $[0, \pi]$, the values of an inverse cosine function are located on the upper half of the unit circle.

**Inverse Cosine Values**

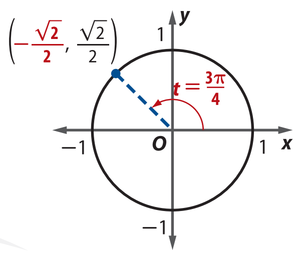



## Example 2 Evaluate Inverse Cosine Functions

**Find the exact value of each expression, if it exists.**

**a.** $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Find a point on the unit circle in the interval $[0, \pi]$ with an $x$-coordinate of $-\frac{\sqrt{2}}{2}$. When $t = \frac{3\pi}{4}$, $\cos t = -\frac{\sqrt{2}}{2}$.




Therefore, $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

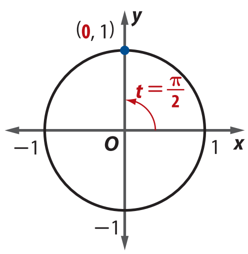
**CHECK** If $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$, then $\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$. ✔

**b.** $\arccos(-2)$

Since the domain of the cosine function is $[-1, 1]$ and $-2 < -1$, there is no angle with a cosine of $-2$. Therefore, the value of $\arccos(-2)$ does not exist.

**c.** $\cos^{-1} 0$

Find a point on the unit circle in the interval $[0, \pi]$ with an $x$-coordinate of 0. When $t = \frac{\pi}{2}$, $\cos t = 0$.




Therefore, $\cos^{-1} 0 = \frac{\pi}{2}$.

**CHECK** If $\cos^{-1} 0 = \frac{\pi}{2}$, then $\cos \frac{\pi}{2} = 0$. ✔

**GuidedPractice**

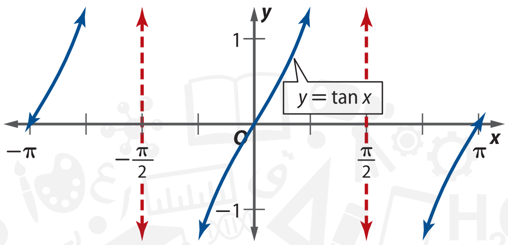
**2A.** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **2B.** $\arccos 2.5$ **2C.** $\cos^{-1}\left(-\frac{1}{2}\right)$

**StudyTip**

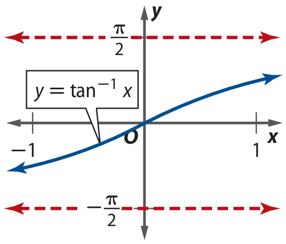
**End Behavior of Inverse Tangent** Notice that when the graph of the restricted tangent function is reflected in the line $y = x$, the vertical asymptotes at $x = \pm\frac{\pi}{2}$ become the horizontal asymptotes $y = \pm\frac{\pi}{2}$ of the inverse tangent function. Therefore, $\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$ and $\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$.

When restricted to a domain of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the tangent function is one-to-one. It is on this restricted domain that the tangent function has an inverse function called the *inverse tangent function* $y = \tan^{-1} x$ or **arctangent function** $y = \arctan x$. The graph of $y = \tan^{-1} x$ is found by reflecting the graph of the restricted tangent function in the line $y = x$. Notice that unlike the sine and cosine functions, the domain of the inverse tangent function is $(-\infty, \infty)$.

**Restricted Tangent Function**

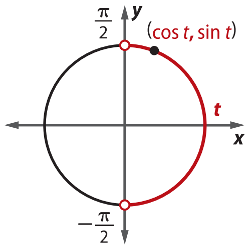



**Inverse Tangent Function**




You can also use the unit circle to find the value of an inverse tangent expression. On the unit circle, $\tan t = \frac{\sin t}{\cos t}$ or $\frac{y}{x}$. The values of $y = \tan^{-1} x$ will be located on the right half of the unit circle, not including $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, because the tangent function is undefined at those points.

**Inverse Tangent Values**




**TechnologyTip**

**Evaluate $\tan^{-1}$** You can also use a graphing calculator to find the angle that has a tangent of $\sqrt{3}$.

```
tan⁻¹(√(3))
          1.047197551
π/3
          1.047197551
```

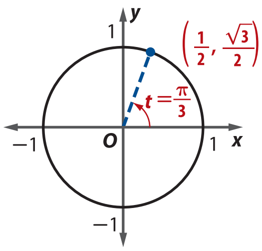
Make sure you select RADIAN on the MODE feature of your graphing calculator.

## Example 3 Evaluate Inverse Tangent Functions

**Find the exact value of each expression, if it exists.**

**a.** $\tan^{-1} \sqrt{3}$

Find a point $(x, y)$ on the unit circle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\frac{y}{x} = \sqrt{3}$. When $t = \frac{\pi}{3}$, $\tan t = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$ or $\sqrt{3}$. Therefore, $\tan^{-1} \sqrt{3} = \frac{\pi}{3}$.

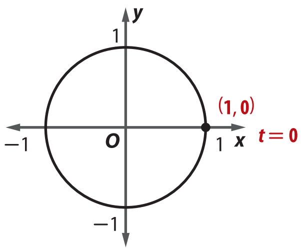



**CHECK** If $\tan^{-1} \sqrt{3} = \frac{\pi}{3}$, then $\tan \frac{\pi}{3} = \sqrt{3}$. ✓

**b.** $\arctan 0$

Find a point $(x, y)$ on the unit circle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\frac{y}{x} = 0$. When $t = 0$, $\tan t = \frac{0}{1}$ or 0.

Therefore, $\arctan 0 = 0$.




**CHECK** If $\arctan 0 = 0$, then $\tan 0 = 0$. ✓

**GuidedPractice**

**3A.** $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

**3B.** $\tan^{-1}(-1)$

While inverse functions for secant, cosecant, and cotangent do exist, these functions are rarely used in computations because the inverse functions for their reciprocals exist. Also, deciding how to restrict the domains of secant, cosecant, and cotangent to obtain arcsecant, arccosecant, and arccotangent is not as apparent. You will explore these functions in Exercise 66.

The three most common inverse trigonometric functions are summarized below.

## KeyConcept Inverse Trigonometric Functions

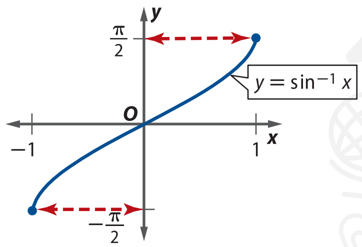
**Inverse Sine of $x$**

**Words** The angle (or arc) between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with a sine of $x$.

**Symbols** $y = \sin^{-1} x$ if and only if $\sin y = x$, for $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

**Domain:** $[-1, 1]$

**Range:** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



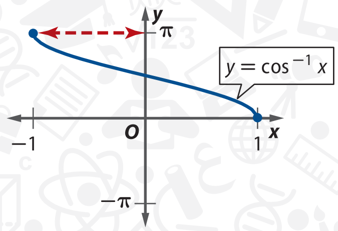

**Inverse Cosine of $x$**

**Words** The angle (or arc) between 0 and $\pi$ with a cosine of $x$.

**Symbols** $y = \cos^{-1} x$ if and only if $\cos y = x$, for $-1 \le x \le 1$ and $0 \le y \le \pi$.

**Domain:** $[-1, 1]$

**Range:** $[0, \pi]$



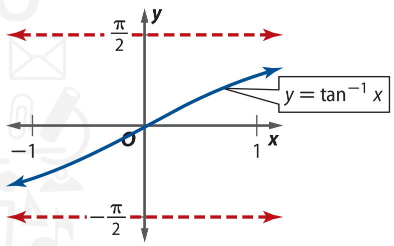

**Inverse Tangent of $x$**

**Words** The angle (or arc) between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with a tangent of $x$.

**Symbols** $y = \tan^{-1} x$ if and only if $\tan y = x$, for $-\infty < x < \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

**Domain:** $(-\infty, \infty)$

**Range:** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$




You can sketch the graph of one of the inverse trigonometric functions shown above by rewriting the function in the form $\sin y = x$, $\cos y = x$, or $\tan y = x$, assigning values to $y$ and making a table of values, and then plotting the points and connecting the points with a smooth curve.

## Example 4 Sketch Graphs of Inverse Trigonometric Functions

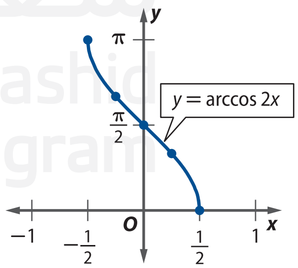
**Sketch the graph of $y = \arccos 2x$.**

By definition, $y = \arccos 2x$ and $\cos y = 2x$ are equivalent on $0 \le y \le \pi$, so their graphs are the same. Rewrite $\cos y = 2x$ as $x = \frac{1}{2} \cos y$ and assign values to $y$ on the interval $[0, \pi]$ to make a table of values.

**WatchOut!**
Remember that $\pi \approx 3.14$ radians or 180°.

| $y$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{6}$ | $\frac{\pi}{2}$ | $\frac{5\pi}{6}$ | $\frac{3\pi}{4}$ | $\pi$ |
|---|---|---|---|---|---|---|---|
| $x = \frac{1}{2} \cos y$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{4}$ | $\frac{\sqrt{3}}{4}$ | 0 | $-\frac{\sqrt{3}}{4}$ | $-\frac{\sqrt{2}}{4}$ | $-\frac{1}{2}$ |

Then plot the points $(x, y)$ and connect them with a smooth curve. Notice that this curve has endpoints at $\left(-\frac{1}{2}, \pi\right)$ and $\left(\frac{1}{2}, 0\right)$, indicating that the entire graph of $y = \arccos 2x$ is shown.

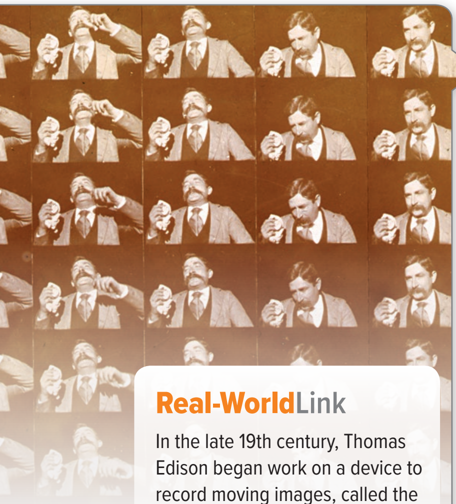



### GuidedPractice

**Sketch the graph of each function.**

**4A.** $y = \arcsin 3x$

**4B.** $y = \tan^{-1} 2x$



**Real-WorldLink**

In the late 19th century, Thomas Edison began work on a device to record moving images, called the kinetoscope, which would later become the film projector. The earliest copyrighted motion picture is a film of one of Edison's employees sneezing.

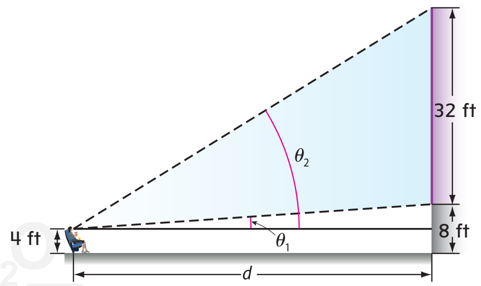
**Source:** The Library of Congress

## Real-World Example 5 Use an Inverse Trigonometric Function

**MOVIES In a movie theater, a person's viewing angle for watching a movie changes depending on where he or she sits in the theater.**

**a. Write a function modeling the viewing angle $\theta$ for a person in the theater whose eye-level when sitting is 4 ft above ground.**

Draw a diagram to find the measure of the viewing angle. Let $\theta_1$ represent the angle formed from eye-level to the bottom of the screen, and let $\theta_2$ represent the angle formed from eye-level to the top of the screen.




So, the viewing angle is $\theta = \theta_2 - \theta_1$. You can use the tangent function to find $\theta_1$ and $\theta_2$. Because the eye-level of the person when seated is 4 ft above the floor, the distance opposite $\theta_1$ is 8 - 4 ft or 4 ft long.

$\tan \theta_1 = \frac{4}{d}$ opp = 4 and adj = $d$

$\theta_1 = \tan^{-1} \frac{4}{d}$ Inverse tangent function

The distance opposite $\theta_2$ is $(32 + 8) - 4$ ft or 36 ft.

$\tan \theta_2 = \frac{36}{d}$ opp = 36 and adj = $d$

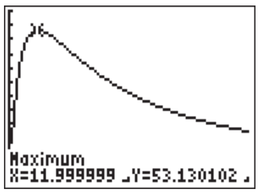
$\theta_2 = \tan^{-1} \frac{36}{d}$ Inverse tangent function

So, the viewing angle can be modeled by $\theta = \tan^{-1}\frac{36}{d} - \tan^{-1}\frac{4}{d}$.

**b. Determine the distance that corresponds to the maximum viewing angle.**

The distance at which the maximum viewing angle occurs is the maximum point on the graph. You can use a graphing calculator to find this point.

From the graph, you can see that the maximum viewing angle occurs approximately 12 ft from the screen.

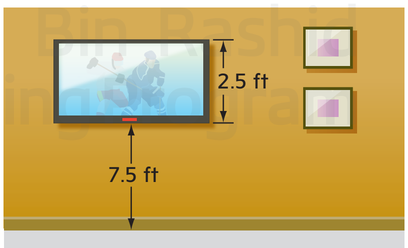



[0, 100] scl: 10 by [0, 60] scl: 5



**GuidedPractice**

**5. TELEVISION** Ahmed has purchased a new flat-screen television. So that his family will be able to see, he has decided to hang the television on the wall as shown.




**A.** Write a function modeling the distance $d$ of the maximum viewing angle $\theta$ for Ahmed if his eye level when sitting is 3 ft above ground.

**B.** Determine the distance that corresponds to the maximum viewing angle.

## 2 Compositions of Trigonometric Functions

In Lesson 1-7, you learned that if $x$ is in the domain of $f(x)$ and $f^{-1}(x)$, then

$$f[f^{-1}(x)] = x \qquad \text{and} \qquad f^{-1}[f(x)] = x.$$

Because the domains of the trigonometric functions are restricted to obtain the inverse trigonometric functions, the properties do not apply for all values of $x$.

For example, while $\sin x$ is defined for all $x$, the domain of $\sin^{-1} x$ is $[-1, 1]$. Therefore, $\sin(\sin^{-1} x) = x$ is only true when $-1 \leq x \leq 1$. A different restriction applies for the composition $\sin^{-1}(\sin x)$. Because the domain of $\sin x$ is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\sin^{-1}(\sin x) = x$ is only true when $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

These domain restrictions are summarized below.

### KeyConcept Domain of Compositions of Trigonometric Functions

| $f[f^{-1}(x)] = x$ | $f^{-1}[f(x)] = x$ |
|---|---|
| If $-1 \leq x \leq 1$, then $\sin(\sin^{-1} x) = x$. | If $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, then $\sin^{-1}(\sin x) = x$. |
| If $-1 \leq x \leq 1$, then $\cos(\cos^{-1} x) = x$. | If $0 \leq x \leq \pi$, then $\cos^{-1}(\cos x) = x$. |
| If $-\infty < x < \infty$, then $\tan(\tan^{-1} x) = x$. | If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then $\tan^{-1}(\tan x) = x$. |

### Example 6 Use Inverse Trigonometric Properties

**Find the exact value of each expression, if it exists.**

**a.** $\sin\left[\sin^{-1}\left(-\frac{1}{4}\right)\right]$

The inverse property applies because $-\frac{1}{4}$ lies on the interval $[-1, 1]$.

Therefore, $\sin\left[\sin^{-1}\left(-\frac{1}{4}\right)\right] = -\frac{1}{4}$.

**b.** $\arctan\left(\tan\frac{\pi}{2}\right)$

Because $\tan x$ is not defined when $x = \frac{\pi}{2}$, $\arctan\left(\tan\frac{\pi}{2}\right)$ does not exist.

**c.** $\arcsin\left(\sin\frac{7\pi}{4}\right)$

Notice that the angle $\frac{7\pi}{4}$ does not lie on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. However, $\frac{7\pi}{4}$ is coterminal with $\frac{7\pi}{4} - 2\pi$ or $-\frac{\pi}{4}$, which is on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\arcsin\left(\sin\frac{7\pi}{4}\right) = \arcsin\left[\sin\left(-\frac{\pi}{4}\right)\right]$ $\quad$ $\sin\frac{7\pi}{4} = \sin\left(-\frac{\pi}{4}\right)$

$= -\frac{\pi}{4}$ $\quad$ Since $-\frac{\pi}{2} \leq -\frac{\pi}{4} \leq \frac{\pi}{2}$, $\arcsin(\sin x) = x$.

Therefore, $\arcsin\left(\sin\frac{7\pi}{4}\right) = -\frac{\pi}{4}$.

**WatchOut!**

**Compositions and Inverses** When computing $f^{-1}[f(x)]$ with trigonometric functions, the domain appears to be $(-\infty, \infty)$. However, because the ranges of the inverse functions are restricted, coterminal angles must sometimes be found.

**GuidedPractice**

**6A.** $\tan\left(\tan^{-1}\frac{\pi}{3}\right)$

**6B.** $\cos^{-1}\left(\cos\frac{3\pi}{4}\right)$

**6C.** $\arcsin\left(\sin\frac{2\pi}{3}\right)$

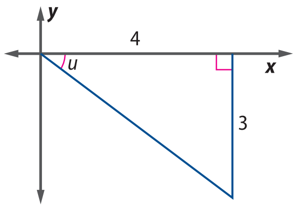
You can also evaluate the composition of two different inverse trigonometric functions.

## Example 7 Evaluate Compositions of Trigonometric Functions

**Find the exact value of $\cos\left[\tan^{-1}\left(-\frac{3}{4}\right)\right]$.**

To simplify the expression, let $u = \tan^{-1}\left(-\frac{3}{4}\right)$, so $\tan u = -\frac{3}{4}$.

Because the tangent function is negative in Quadrants II and IV, and the domain of the inverse tangent function is restricted to Quadrants I and IV, $u$ must lie in Quadrant IV.




Using the Pythagorean Theorem, you can find that the length of the hypotenuse is 5. Now, solve for $\cos u$.

$\cos u = \frac{\text{adj}}{\text{hyp}}$ Cosine function

$= \frac{4}{5}$ adj = 4 and hyp = 5

So, $\cos\left[\tan^{-1}\left(-\frac{3}{4}\right)\right] = \frac{4}{5}$.

### GuidedPractice

**Find the exact value of each expression.**

**7A.** $\cos^{-1}\left(\sin\frac{\pi}{3}\right)$

**7B.** $\sin\left(\arctan\frac{5}{12}\right)$

Sometimes the composition of two trigonometric functions reduces to an algebraic expression that does not involve *any* trigonometric expressions.

**StudyTip**

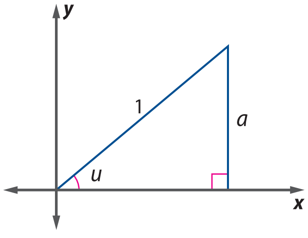
**Decomposing Algebraic Functions** The technique used to convert a trigonometric expression into an algebraic expression can be reversed. Decomposing an algebraic function as the composition of two trigonometric functions is a technique used frequently in calculus.

## Example 8 Evaluate Compositions of Trigonometric Functions

**Write tan (arcsin $a$) as an algebraic expression of $a$ that does not involve trigonometric functions.**

Let $u = \arcsin a$, so $\sin u = a$.

Because the domain of the inverse sine function is restricted to Quadrants I and IV, $u$ must lie in Quadrant I or IV. The solution is similar for each quadrant, so we will solve for Quadrant I.




From the Pythagorean Theorem, you can find that the length of the side adjacent to $u$ is $\sqrt{1 - a^2}$. Now, solve for $\tan u$.

$\tan u = \frac{\text{opp}}{\text{adj}}$ Tangent function

$= \frac{a}{\sqrt{1-a^2}}$ or $\frac{a\sqrt{1-a^2}}{1-a^2}$ opp = $a$ and adj = $\sqrt{1-a^2}$

So, $\tan(\arcsin a) = \frac{a\sqrt{1-a^2}}{1-a^2}$.

### GuidedPractice

**Write each expression as an algebraic expression of $x$ that does not involve trigonometric functions.**

**8A.** $\sin(\arccos x)$

**8B.** $\cot[\sin^{-1} x]$

# Exercises

**Find the exact value of each expression, if it exists.** (Examples 1–3)

**1.** $\sin^{-1} 0$

**2.** $\arcsin \frac{\sqrt{3}}{2}$

**3.** $\arcsin \frac{\sqrt{2}}{2}$

**4.** $\sin^{-1} \frac{1}{2}$

**5.** $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

**6.** $\arccos 0$

**7.** $\cos^{-1} \frac{\sqrt{2}}{2}$

**8.** $\arccos (-1)$

**9.** $\arccos \frac{\sqrt{3}}{2}$

**10.** $\cos^{-1} \frac{1}{2}$

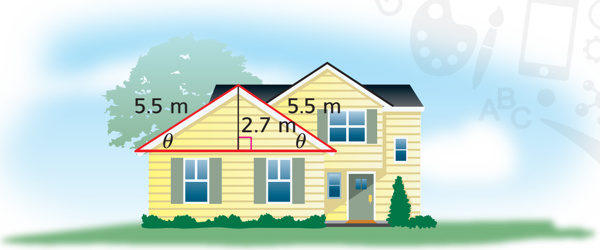
**11.** $\arctan 1$

**12.** $\arctan (-\sqrt{3})$

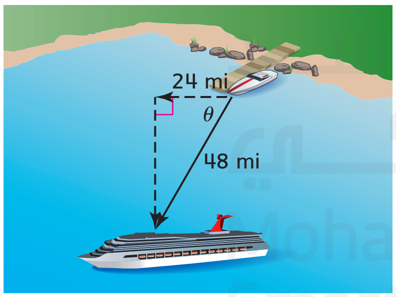
**13.** $\tan^{-1} \frac{\sqrt{3}}{3}$

**14.** $\tan^{-1} 0$

**15.** **ARCHITECTURE** The support for a roof is shaped like two right triangles, as shown below. Find $\theta$. (Example 3)




**16.** **RESCUE** A cruise ship sailed due west 24 mi before turning south. When the cruise ship became disabled and the crew radioed for help, the rescue boat found that the fastest route covered a distance of 48 mi. Find the angle $\theta$ at which the rescue boat should travel to aid the cruise ship. (Example 3)




**Sketch the graph of each function.** (Example 4)

**17.** $y = \arcsin x$

**18.** $y = \sin^{-1} 2x$

**19.** $y = \sin^{-1} (x + 3)$

**20.** $y = \arcsin x - 3$

**21.** $y = \arccos x$

**22.** $y = \cos^{-1} 3x$

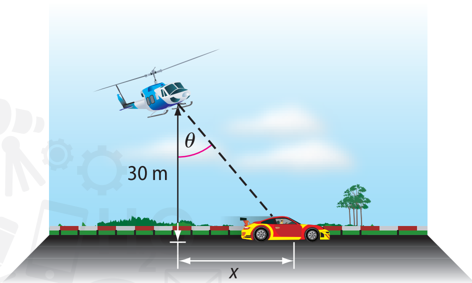
**23.** $y = \arctan x$

**24.** $y = \tan^{-1} 3x$

**25.** $y = \tan^{-1} (x + 1)$

**26.** $y = \arctan x - 1$

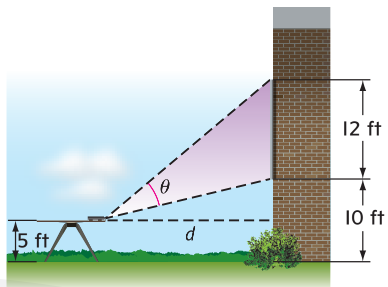
**27.** **DRAG RACE** A television camera is filming a drag race. The camera rotates as the vehicles move past it. The camera is 30 m away from the track. Consider $\theta$ and $x$ as shown in the figure. (Example 5)




**a.** Write $\theta$ as a function of $x$.

**b.** Find $\theta$ when $x = 6$ m and $x = 14$ m.

**28.** **SPORTS** Salem and Rashid want to project a pro soccer game on the side of their apartment building. They have placed a projector on a table that stands 5 ft above the ground and have hung a 12 ft-tall screen that is 10 ft above the ground. (Example 5)




**a.** Write a function expressing $\theta$ in terms of distance $d$.

**b.** Use a graphing calculator to determine the distance for the maximum projecting angle.

**Find the exact value of each expression, if it exists.** (Examples 6 and 7)

**29.** $\sin\left(\sin^{-1} \frac{3}{4}\right)$

**30.** $\sin^{-1}\left(\sin \frac{\pi}{2}\right)$

**31.** $\cos\left(\cos^{-1} \frac{2}{9}\right)$

**32.** $\cos^{-1} (\cos \pi)$

**33.** $\tan\left(\tan^{-1} \frac{\pi}{4}\right)$

**34.** $\tan^{-1}\left(\tan \frac{\pi}{3}\right)$

**35.** $\cos (\tan^{-1} 1)$

**36.** $\sin^{-1}\left(\cos \frac{\pi}{2}\right)$

**37.** $\sin\left(2 \cos^{-1} \frac{\sqrt{2}}{2}\right)$

**38.** $\sin (\tan^{-1} 1 - \sin^{-1} 1)$

**39.** $\cos (\tan^{-1} 1 - \sin^{-1} 1)$

**40.** $\cos\left(\cos^{-1} 0 + \sin^{-1} \frac{1}{2}\right)$

**Write each trigonometric expression as an algebraic expression of $x$.** (Example 8)

**41.** $\tan(\arccos x)$

**42.** $\csc(\cos^{-1} x)$

**43.** $\sin(\cos^{-1} x)$

**44.** $\cos(\arcsin x)$

**45.** $\csc(\sin^{-1} x)$

**46.** $\sec(\arcsin x)$

**47.** $\cot(\arccos x)$

**48.** $\cot(\arcsin x)$

**Describe how the graphs of $g(x)$ and $f(x)$ are related.**

**49.** $f(x) = \sin^{-1} x$ and $g(x) = \sin^{-1}(x - 1) - 2$

**50.** $f(x) = \arctan x$ and $g(x) = \arctan 0.5x - 3$

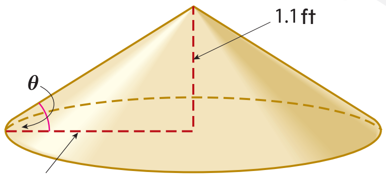
**51.** $f(x) = \cos^{-1} x$ and $g(x) = 3(\cos^{-1} x - 2)$

**52.** $f(x) = \arcsin x$ and $g(x) = \frac{1}{2}\arcsin(x + 2)$

**53.** $f(x) = \arccos x$ and $g(x) = 5 + \arccos 2x$

**54.** $f(x) = \tan^{-1} x$ and $g(x) = \tan^{-1} 3x - 4$

**55.** **SAND** When piling sand, the angle formed between the pile and the ground remains fairly consistent and is called the *angle of repose*. Suppose Fatheya creates a pile of sand at the beach that is 3 ft in diameter and 1.1 ft high.




**a.** What is the angle of repose?

**b.** If the angle of repose remains constant, how many feet in diameter would a pile need to be to reach a height of 4 ft?

**Give the domain and range of each composite function. Then use your graphing calculator to sketch its graph.**

**56.** $y = \cos(\tan^{-1} x)$

**57.** $y = \sin(\cos^{-1} x)$

**58.** $y = \arctan(\sin x)$

**59.** $y = \sin^{-1}(\cos x)$

**60.** $y = \cos(\arcsin x)$

**61.** $y = \tan(\arccos x)$

**62.** **INVERSES** The arcsecant function is graphed by restricting the domain of the secant function to the intervals $\left[0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right]$, and the arccosecant function is graphed by restricting the domain of the cosecant function to the intervals $\left[-\frac{\pi}{2}, 0\right)$ and $\left(0, \frac{\pi}{2}\right]$.

**a.** State the domain and range of each function.

**b.** Sketch the graph of each function.

**c.** Explain why a restriction on the domain of the secant and cosecant functions is necessary in order to graph the inverse functions.

**Write each algebraic expression as a trigonometric function of an inverse trigonometric function of $x$.**

**63.** $\dfrac{x}{\sqrt{1 - x^2}}$

**64.** $\dfrac{\sqrt{1 - x^2}}{x}$

**65.** **MULTIPLE REPRESENTATIONS** In this problem, you will explore the graphs of compositions of trigonometric functions.

**a.** **ANALYTICAL** Consider $f(x) = \sin x$ and $f^{-1}(x) = \arcsin x$. Describe the domain and range of $f \circ f^{-1}$ and $f^{-1} \circ f$.

**b.** **GRAPHICAL** Create a table of several values for each composite function on the interval $[-2, 2]$. Then use the table to sketch the graphs of $f \circ f^{-1}$ and $f^{-1} \circ f$. Use a graphing calculator to check your graphs.

**c.** **ANALYTICAL** Consider $g(x) = \cos x$ and $g^{-1}(x) = \arccos x$. Describe the domain and range of $g \circ g^{-1}$ and $g^{-1} \circ g$ and make a conjecture as to what the graphs of $g \circ g^{-1}$ and $g^{-1} \circ g$ will look like. Explain your reasoning.

**d.** **GRAPHICAL** Sketch the graphs of $g \circ g^{-1}$ and $g^{-1} \circ g$. Use a graphing calculator to check your graphs.

**e.** **VERBAL** Make a conjecture as to what the graphs of the two possible compositions of the tangent and arctangent functions will look like. Explain your reasoning. Then check your conjecture using a graphing calculator.

## H.O.T. Problems Use Higher-Order Thinking Skills

**66.** **ERROR ANALYSIS** Ahmed and Ali are discussing inverse trigonometric functions. Because $\tan x = \frac{\sin x}{\cos x}$, Ahmed conjectures that $\tan^{-1} x = \frac{\sin^{-1} x}{\cos^{-1} x}$. Ali disagrees. Is either of them correct? Explain.

**67.** **CHALLENGE** Use the graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ to find the value of $\sin^{-1} x + \cos^{-1} x$ on the interval $[-1, 1]$. Explain your reasoning.

**68.** **REASONING** Determine whether the following statement is *true* or *false*: If $\cos \frac{7\pi}{4} = \frac{\sqrt{2}}{2}$, then $\cos^{-1} \frac{\sqrt{2}}{2} = \frac{7\pi}{4}$. Explain your reasoning.

**REASONING** **Determine whether each function is *odd*, *even*, or *neither*. Justify your answer.**

**69.** $y = \sin^{-1} x$

**70.** $y = \cos^{-1} x$

**71.** $y = \tan^{-1} x$

**72.** **WRITING IN MATH** Explain how the restrictions on the sine, cosine, and tangent functions dictate the domain and range of their inverse functions.

## Spiral Review

**Locate the vertical asymptotes, and sketch the graph of each function.**

**73.** $y = 3 \tan \theta$

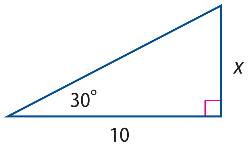
**74.** $y = \cot 5\theta$

**75.** $y = 3 \csc \frac{1}{2}\theta$

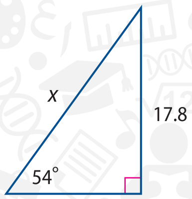
**76.** **WAVES** A leaf floats on the water bobbing up and down. The distance between its highest and lowest points is 4 cm. It moves from its highest point down to its lowest point and back to its highest point every 10 seconds. Write a cosine function that models the movement of the leaf in relationship to the equilibrium point.

**Find the value of $x$. Round to the nearest tenth, if necessary.**

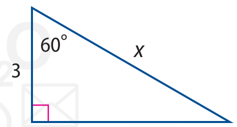
**77.**




**78.**




**79.**




**For each pair of functions, find $[f \circ g](x)$, $[g \circ f](x)$, and $[f \circ g](4)$.**

**80.** $f(x) = x^2 + 3x - 6$
$g(x) = 4x + 1$

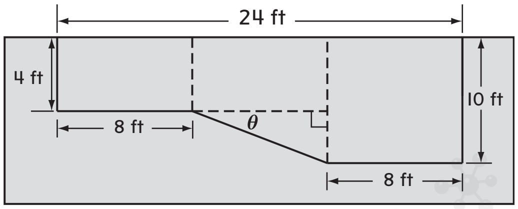
**81.** $f(x) = 6 - 5x$
$g(x) = \frac{1}{x}$

**82.** $f(x) = \sqrt{x + 3}$
$g(x) = x^2 + 1$

**83.** **EDUCATION** Tarek has answered 11 of his last 20 daily quiz questions correctly. His baseball coach told him that he must raise his average to at least 70% if he wants to play in the season opener. Tarek vows to study diligently and answer all of the daily quiz questions correctly in the future. How many consecutive daily quiz questions must he answer correctly to raise his average to 70%?

## Skills Review for Standardized Tests

**84. SAT/ACT** To the nearest degree, what is the angle of depression $\theta$ between the shallow end and the deep end of the swimming pool?




Side View of Swimming Pool

**A** 25° **C** 41° **E** 73°

**B** 37° **D** 53°

**85.** Which of the following represents the exact value of $\sin\left(\tan^{-1}\frac{1}{2}\right)$?

**F** $-\frac{2\sqrt{5}}{5}$

**G** $-\frac{\sqrt{5}}{5}$

**H** $\frac{\sqrt{5}}{5}$

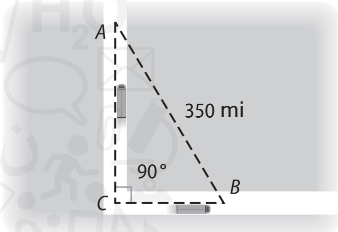
**J** $\frac{2\sqrt{5}}{5}$

**86. REVIEW** The hypotenuse of a right triangle is 67 cm. If one of the angles has a measure of 47°, what is the length of the shortest leg of the triangle?

**A** 45.7 cm **C** 62.5 cm

**B** 49.0 cm **D** 71.8 cm

**87. REVIEW** Two trucks, *A* and *B*, start from the intersection *C* of two straight roads at the same time. Truck *A* is traveling twice as fast as truck *B* and after 4 hours, the two trucks are 350 mi apart. Find the approximate speed of truck *B* in mi/h.




**F** 39 **H** 51

**G** 44 **J** 78

CHAPTER 3

# Study Guide and Review

## Chapter Summary

### KeyConcepts

**Right Triangle Trigonometry** (Lesson 3-1)

$\sin \theta = \frac{\text{opp}}{\text{hyp}}$ $\quad \cos \theta = \frac{\text{adj}}{\text{hyp}}$ $\quad \tan \theta = \frac{\text{opp}}{\text{adj}}$

$\csc \theta = \frac{\text{hyp}}{\text{opp}}$ $\quad \sec \theta = \frac{\text{hyp}}{\text{adj}}$ $\quad \cot \theta = \frac{\text{adj}}{\text{opp}}$

**Degrees and Radians** (Lesson 3-2)

- To convert from degrees to radians, multiply by $\frac{\pi \text{ radians}}{180°}$.
- To convert from radians to degrees, multiply by $\frac{180°}{\pi \text{ radians}}$.
- Linear speed: $v = \frac{s}{t}$, where $s$ is the arc length traveled during time $t$
- Angular speed: $\omega = \frac{\theta}{t}$, where $\theta$ is the angle of rotation (in radians) moved during time $t$

**Trigonometric Functions on the Unit Circle** (Lesson 3-3)

- For an angle $\theta$ in radians containing $(x, y)$, $\cos \theta = \frac{x}{r}$, $\sin \theta = \frac{y}{r}$, and $\tan \theta = \frac{y}{x}$, where $r = \sqrt{x^2 + y^2}$.
- For an angle $t$ containing $(x, y)$ on the unit circle, $\cos \theta = x$, $\sin \theta = y$, and $\tan \theta = \frac{y}{x}$.

**Graphing Sine and Cosine Functions** (Lesson 3-4)

- A sinusoidal function is of the form $y = a \sin (bx + c) + d$ or $y = a \cos (bx + c) + d$, where amplitude $= |a|$, period $= \frac{2\pi}{|b|}$, frequency $= \frac{|b|}{2\pi}$, phase shift $= -\frac{c}{|b|}$, and vertical shift $= d$.

**Graphing Other Trigonometric Functions** (Lesson 3-5)

- A damped trigonometric function is of the form $y = f(x) \sin bx$ or $y = f(x) \cos bx$, where $f(x)$ is the damping factor.

**Inverse Trigonometric Functions** (Lesson 3-6)

- $y = \sin^{-1} x$ iff $\sin y = x$, for $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
- $y = \cos^{-1} x$ iff $\cos y = x$, for $-1 \le x \le 1$ and $0 \le y \le \pi$.
- $y = \tan^{-1} x$ iff $\tan y = x$, for $-\infty < x < \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

**The Law of Sines and the Law of Cosines** (Lesson 3-7)

Let $\triangle ABC$ be any triangle.

- The Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
- The Law of Cosines: $a^2 = b^2 + c^2 - 2bc \cos A$
  $b^2 = a^2 + c^2 - 2ac \cos B$
  $c^2 = a^2 + b^2 - 2ab \cos C$

### KeyVocabulary

| | |
|---|---|
| amplitude | oblique triangles |
| angle of depression | period |
| angle of elevation | periodic function |
| angular speed | phase shift |
| circular function | quadrantal angle |
| cosecant | radian |
| cosine | reciprocal function |
| cotangent | reference angle |
| coterminal angles | secant |
| damped trigonometric function | sector |
| damped wave | sine |
| damping factor | sinusoid |
| frequency | standard position |
| initial side | tangent |
| inverse trigonometric function | terminal side |
| Law of Cosines | trigonometric functions |
| Law of Sines | trigonometric ratios |
| linear speed | unit circle |
| midline | vertical shift |

### VocabularyCheck

**State whether each sentence is *true* or *false*. If *false*, replace the underlined term to make a true sentence.**

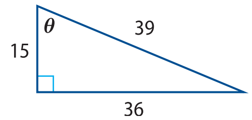
1. The <u>sine</u> of an acute angle in a right triangle is the ratio of the lengths of its opposite leg to the hypotenuse.
2. The <u>secant</u> ratio is the reciprocal of the sine ratio.
3. An <u>angle of elevation</u> is the angle formed by a horizontal line and an observer's line of sight to an object below the line.
4. The <u>radian</u> measure of an angle is equal to the ratio of the length of its intercepted arc to the radius.
5. The rate at which an object moves along a circular path is called its <u>linear speed</u>.
6. 0°, $\pi$, and $-\frac{\pi}{2}$ are examples of <u>reference angles</u>.
7. The <u>period</u> of the graph of $y = 4 \sin 3x$ is 4.
8. For $f(x) = \cos bx$, as $b$ increases, the <u>frequency</u> decreases.
9. The range of the <u>arcsine</u> function is $[0, \pi]$.
10. The <u>Law of Sines</u> can be used to determine unknown side lengths or angle measures of some triangles.

# Lesson-by-Lesson Review

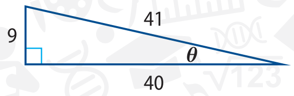
## 3-1 Right Triangle Trigonometry

Find the exact values of the six trigonometric functions of $\theta$.

**11.**

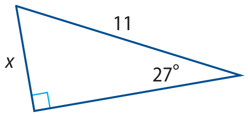



**12.**

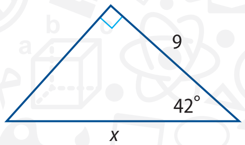



Find the value of $x$. Round to the nearest tenth, if necessary.

**13.**

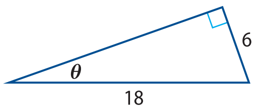



**14.**

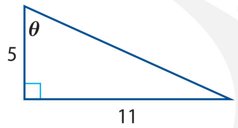



Find the measure of angle $\theta$. Round to the nearest degree, if necessary.

**15.**

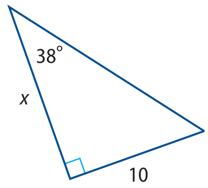



**16.**




### Example 1

Find the value of $x$. Round to the nearest tenth, if necessary.




| | |
|---|---|
| $\tan\theta = \dfrac{\text{opp}}{\text{adj}}$ | Tangent function |
| $\tan 38° = \dfrac{10}{x}$ | $\theta = 38°$, opp $= 10$, and adj $= x$ |
| $x \tan 38° = 10$ | Multiply each side by $x$. |
| $x = \dfrac{10}{\tan 38°}$ | Divide each side by tan 38°. |
| $x \approx 12.8$ | Use a calculator. |

## 3-2 Degrees and Radians

Write each degree measure in radians as a multiple of $\pi$ and each radian measure in degrees.

**17.** 135°

**18.** 450°

**19.** $\frac{7\pi}{4}$

**20.** $\frac{13\pi}{10}$

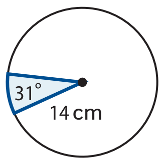
Identify all angles coterminal with the given angle. Then find and draw one positive and one negative angle coterminal with the given angle.

**21.** 342°

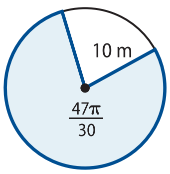
**22.** $-\frac{\pi}{6}$

Find the area of each sector.

**23.**




**24.**



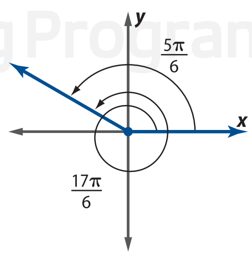

### Example 2

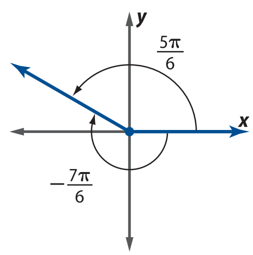
Identify all angles coterminal with $\frac{5\pi}{12}$. Then find and draw one positive and one negative coterminal angle.

All angles measuring $\frac{5\pi}{12} + 2n\pi$ are coterminal with a $\frac{5\pi}{12}$ angle.

Let $n = 1$ and $-1$.

$\frac{5\pi}{6} + 2\pi(1) = \frac{17\pi}{6}$ $\qquad$ $\frac{5\pi}{6} - 2\pi(-1) = -\frac{7\pi}{6}$



## 3-3 Trigonometric Functions on the Unit Circle

Sketch each angle. Then find its reference angle.

**25.** $240°$

**26.** $75°$

**27.** $-\frac{3\pi}{4}$

**28.** $\frac{11\pi}{18}$

Find the exact values of the five remaining trigonometric functions of $\theta$.

**29.** $\cos\theta = \frac{2}{5}$, where $\sin\theta > 0$ and $\tan\theta > 0$

**30.** $\tan\theta = -\frac{3}{4}$, where $\sin\theta > 0$ and $\cos\theta < 0$

**31.** $\sin\theta = -\frac{5}{13}$, where $\cos\theta > 0$ and $\cot\theta < 0$

**32.** $\cot\theta = \frac{2}{3}$, where $\sin\theta < 0$ and $\tan\theta > 0$

Find the exact value of each expression. If undefined, write *undefined*.

**33.** $\sin 180°$

**34.** $\cot\frac{11\pi}{6}$

**35.** $\sec 450°$

**36.** $\cos\left(-\frac{19\pi}{6}\right)$

### Example 3

**Let $\cos\theta = \frac{5}{13}$, where $\sin\theta < 0$. Find the exact values of the five remaining trigonometric functions of $\theta$.**

Since $\cos\theta$ is positive and $\sin\theta$ is negative, $\theta$ lies in Quadrant IV. This means that the $x$-coordinate of a point on the terminal side of $\theta$ is positive and the $y$-coordinate is negative.

Since $\cos\theta = \frac{x}{r} = \frac{5}{13}$, use $x = 5$ and $r = 13$ to find $y$.

$y = \sqrt{r^2 - x^2}$ Pythagorean Theorem

$= \sqrt{169 - 25}$ or 12 $r = 13$ and $x = 5$

$\sin\theta = \frac{y}{r}$ or $\frac{12}{13}$ $\tan\theta = \frac{y}{x}$ or $\frac{12}{5}$ $\sec\theta = \frac{r}{x}$ or $\frac{13}{5}$

$\csc\theta = \frac{r}{y}$ or $\frac{13}{12}$ $\cot\theta = \frac{x}{y}$ or $\frac{5}{12}$

## 3-4 Graphing Sine and Cosine Functions

Describe how the graphs of $f(x)$ and $g(x)$ are related. Then find the amplitude and period of $g(x)$, and sketch at least one period of both functions on the same coordinate axes.

**37.** $f(x) = \sin x$
$g(x) = 5\sin x$

**38.** $f(x) = \cos x$
$g(x) = \cos 2x$

**39.** $f(x) = \sin x$
$g(x) = \frac{1}{2}\sin x$

**40.** $f(x) = \cos x$
$g(x) = -\cos\frac{1}{3}x$

State the amplitude, period, frequency, phase shift, and vertical shift of each function. Then graph two periods of the function.

**41.** $y = 2\cos(x - \pi)$

**42.** $y = -\sin 2x + 1$

**43.** $y = \frac{1}{2}\cos\left(x + \frac{\pi}{2}\right)$

**44.** $y = 3\sin\left(x + \frac{2\pi}{3}\right)$

### Example 4

**State the amplitude, period, frequency, phase shift, and vertical shift of $y = 4\sin\left(x - \frac{\pi}{2}\right) - 4$. Then graph two periods of the function.**

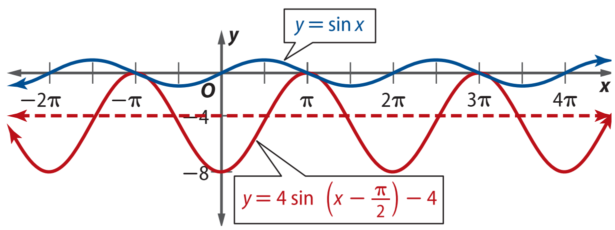
In this function, $a = 4$, $b = 1$, $c = -\frac{\pi}{2}$, and $d = -4$.

Amplitude: $|a| = |4|$ or 4 Period: $\frac{2\pi}{|b|} = \frac{2\pi}{|1|}$ or $2\pi$

Frequency: $\frac{|b|}{2\pi} = \frac{|1|}{2\pi}$ or $\frac{1}{2\pi}$ Vertical shift: $d$ or $-4$

Phase shift: $-\frac{c}{|b|} = -\frac{-\frac{\pi}{2}}{|1|}$ or $\frac{\pi}{2}$

First, graph the midline $y = -4$. Then graph $y = 4\sin x$ shifted $\frac{\pi}{2}$ units to the right and 4 units down.



## 3-5 Graphing Other Trigonometric Functions

**Locate the vertical asymptotes, and sketch the graph of each function.**

**45.** $y = 3 \tan x$

**46.** $y = \frac{1}{2} \tan \left(x - \frac{\pi}{2}\right)$

**47.** $y = \cot \left(x + \frac{\pi}{3}\right)$

**48.** $y = -\cot (x - \pi)$

**49.** $y = 2 \sec \left(\frac{x}{2}\right)$

**50.** $y = -\csc (2x)$

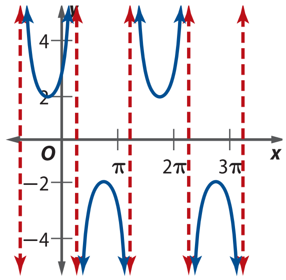
**51.** $y = \sec (x - \pi)$

**52.** $y = \frac{2}{3} \csc \left(x + \frac{\pi}{2}\right)$

### Example 5

**Locate the vertical asymptotes, and sketch the graph of $y = 2 \sec \left(x + \frac{\pi}{4}\right)$.**

Because the graph of $y = 2 \sec \left(x + \frac{\pi}{4}\right)$ is the graph of $y = 2 \sec x$ shifted to the left $\frac{\pi}{4}$ units, the vertical asymptotes for one period are located at $-\frac{3\pi}{4}$, $\frac{\pi}{4}$, and $\frac{5\pi}{4}$. Graph two cycles on the interval $\left[-\frac{3\pi}{4}, \frac{13\pi}{4}\right]$.




## 3-6 Inverse Trigonometric Functions

**Find the exact value of each expression, if it exists.**

**53.** $\sin^{-1} (-1)$

**54.** $\cos^{-1} \frac{\sqrt{3}}{2}$

**55.** $\tan^{-1} \left(-\frac{\sqrt{3}}{3}\right)$

**56.** arcsin 0

**57.** arctan $(-1)$

**58.** arccos $\frac{\sqrt{2}}{2}$

**59.** $\sin^{-1} \left[\sin \left(-\frac{\pi}{3}\right)\right]$

**60.** $\cos^{-1} [\cos (-3\pi)]$

### Example 6

**Find the exact value of arctan $-\sqrt{3}$.**

Find a point on the unit circle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with a tangent of $-\sqrt{3}$. When $t = -\frac{\pi}{3}$, $\tan t = -\sqrt{3}$. Therefore, arctan $-\sqrt{3} = -\frac{\pi}{3}$.

# Study Guide and Review *Continued*

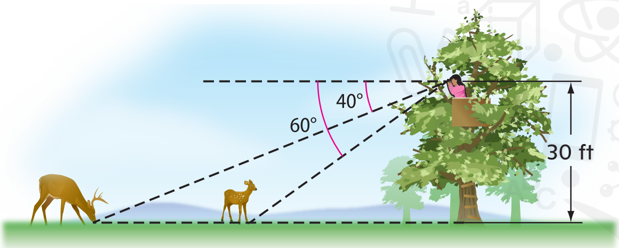
## Applications and Problem Solving

**67. CONSTRUCTION** A construction company is installing a three-foot-high wheelchair ramp onto a landing outside of an office. The angle of the ramp must be 4°. (Lesson 3-1)

**a.** What is the length of the ramp?

**b.** What is the slope of the ramp?

**68. NATURE** For a photography project, Mariam is photographing deer from a tree stand. From her sight 30 ft above the ground, she spots two deer in a straight line, as shown below. How much farther away is the second deer than the first? (Lesson 3-1)

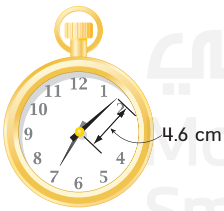



**69. FIGURE SKATING** An Olympic ice skater performs a routine in which she jumps in the air for 2.4 seconds while spinning 3 full revolutions. (Lesson 3-2)

**a.** Find the angular speed of the figure skater.

**b.** Express the angular speed of the figure skater in degrees per minute.

**70. TIMEPIECES** The length of the minute hand of a pocket watch is 4.6 cm. What is the area swept by the minute hand in 40 minutes? (Lesson 3-2)




**71. WORLD'S FAIR** The first Ferris wheel had a diameter of 250 ft and took 10 minutes to complete one full revolution. (Lesson 3-3)

**a.** How many degrees would the Ferris wheel rotate in 100 seconds?

**b.** How far has a person traveled if he or she has been on the Ferris wheel for 7 minutes?

**c.** How long would it take for a person to travel 200 ft?

**72. AIR CONDITIONING** An air-conditioning unit turns on and off to maintain the desired temperature. On one summer day, the air conditioner turns on at 8:30 A.M. when the temperature is 80° Fahrenheit and turns off at 8:55 A.M. when the temperature is 74°. (Lesson 3-4)

**a.** Find the amplitude and period if you were going to use a trigonometric function to model this change in temperature, assuming that the temperature cycle will continue.

**b.** Is it appropriate to model this situation with a trigonometric function? Explain your reasoning.

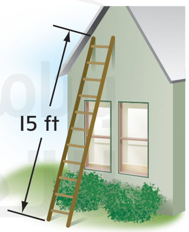
**73. TIDES** In Lewis Bay, the low tide is recorded as 2 ft at 4:30 A.M., and the high tide is recorded as 5.5 ft at 10:45 A.M. (Lesson 3-4)

**a.** Find the period for the trigonometric model.

**b.** At what time will the next high tide occur?

**74. MUSIC** When plucked, a bass string is displaced 1.5 in, and its damping factor is 1.9. It produces a note with a frequency of 90 cycles per second. Determine the amount of time it takes the string's motion to be dampened so that $-0.1 < y < 0.1$. (Lesson 3-5)

**75. PAINTING** A painter is using a 15 ft ladder to paint the side of a house. If the angle the ladder makes with the ground is less than 65°, it will slide out from under him. What is the greatest distance that the bottom of the ladder can be from the side of the house and still be safe for the painter? (Lesson 3-6)

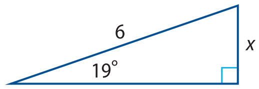


CHAPTER 3

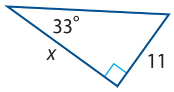
# Practice Test

Find the value of $x$. Round to the nearest tenth, if necessary.

**1.**

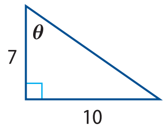



**2.**

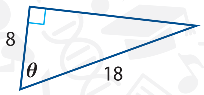



Find the measure of angle $\theta$. Round to the nearest degree, if necessary.

**3.**




**4.**




**5.** **MULTIPLE CHOICE** What is the linear speed of a point rotating at an angular speed of 36 radians per second at a distance of 12 cm from the center of the rotation?

**A** 420 cm/s

**B** 432 cm/s

**C** 439 cm/s

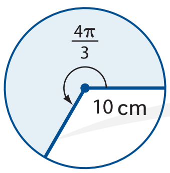
**D** 444 cm/s

Write each degree measure in radians as a multiple of $\pi$ and each radian measure in degrees.

**6.** 200°

**7.** $-\frac{8\pi}{3}$

**8.** Find the area of the sector of the circle shown.




Sketch each angle. Then find its reference angle.

**9.** 165°

**10.** $\frac{21\pi}{13}$

Find the exact value of each expression.

**11.** $\sec \frac{7\pi}{6}$

**12.** $\cos(-240°)$

**13.** **MULTIPLE CHOICE** An angle $\theta$ satisfies the following inequalities: $\csc\theta < 0$, $\cot\theta > 0$, and $\sec\theta < 0$. In which quadrant does $\theta$ lie?

**F** I

**G** II

**H** III

**J** IV

State the amplitude, period, frequency, phase shift, and vertical shift of each function. Then graph two periods of the function.

**14.** $y = 4\cos\frac{x}{2} - 5$

**15.** $y = -\sin\left(x + \frac{\pi}{2}\right)$

**16.** **TIDES** The table gives the approximate times that the high and low tides occurred in San Azalea Bay over a 2-day period.

| Tide | High 1 | Low 1 | High 2 | Low 2 |
|---|---|---|---|---|
| Day 1 | 2:35 A.M. | 8:51 A.M. | 3:04 P.M. | 9:19 P.M. |
| Day 2 | 3:30 A.M. | 9:48 A.M. | 3:55 P.M. | 10:20 P.M. |

**a.** The tides can be modeled with a trigonometric function. Approximately what is the period of this function?

**b.** The difference in height between the high and low tides is 7 ft. What is the amplitude of this function?

**c.** Write a function that models the tides where $t$ is measured in hours. Assume the function has no phase shift or vertical shift.

Locate the vertical asymptotes, and sketch the graph of each function.

**17.** $y = \tan\left(x + \frac{\pi}{4}\right)$

**18.** $y = \frac{1}{2}\sec 2x$

Find all solutions for the given triangle, if possible. If no solution exists, write *no solution*. Round side lengths to the nearest tenth and angle measurements to the nearest degree.

**19.** $a = 8, b = 16, A = 22°$

**20.** $a = 9, b = 7, A = 84°$

**21.** $a = 3, b = 5, c = 7$

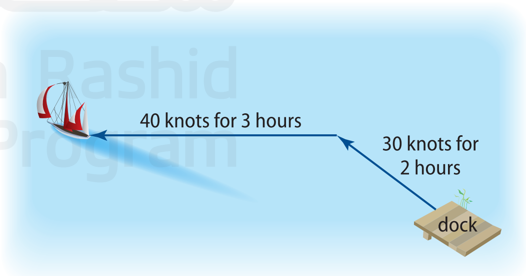
**22.** $a = 8, b = 10, C = 46°$

Find the exact value of each expression, if it exists.

**23.** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

**24.** $\sin^{-1}\left(-\frac{1}{2}\right)$

**25.** **NAVIGATION** A boat leaves a dock and travels 45° north of west averaging 30 knots for 2 hours. The boat then travels directly west averaging 40 knots for 3 hours.




**a.** How many nautical miles is the boat from the dock after 5 hours?

**b.** How many degrees south of east is the dock from the boat's present position?

CHAPTER 3

## Connect to AP Calculus

# Related Rates

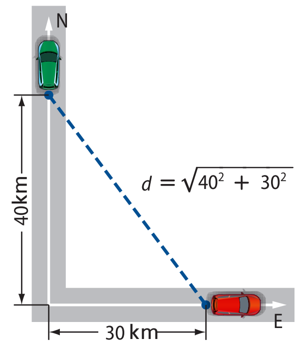
### Objective

- Model and solve related rates problems.

If air is being pumped into a balloon at a given rate, can we find the rate at which the volume of the balloon is expanding? How does the rate a company spends money on advertising affect the rate of its sales? *Related rates* problems occur when the rate of change for one variable can be found by *relating* that to rates of change for other variables.

Suppose two cars leave a point at the same time. One car is traveling 40 km/h due north, while the second car is traveling 30 km/h due east. How far apart are the two cars after 1 hour? 2 hours? 3 hours? We can use the formula $d = rt$ and the Pythagorean Theorem to solve for these values.

In this situation, we know the rates of change for each car. What if we want to know the rate at which the distance between the two cars is changing?




### Activity 1 Rate of Change

**Two cars leave a house at the same time. One car travels due north at 35 km/h, while the second car travels due east at 55 km/h. Approximate the rate at which the distance between the two cars is changing.**

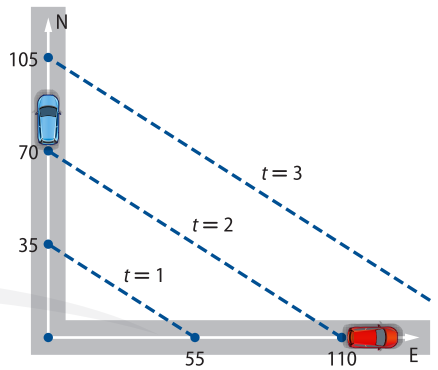
**Step 1** Make a sketch of the situation.

**Step 2** Write equations for the distance traveled by each car after $t$ hours.

**Step 3** Find the distance traveled by each car after 1, 2, 3, and 4 hours.

**Step 4** Use the Pythagorean Theorem to find the distance between the two cars at each point in time.

**Step 5** Find the average rate of change of the distance between the two cars for $1 \leq t \leq 2$, $2 \leq t \leq 3$, and $3 \leq t \leq 4$.



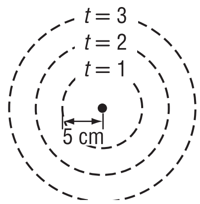

#### Analyze the Results

1. Make a scatter plot displaying the total distance between the two cars. Let time t be the independent variable and total distance $d$ be the dependent variable. Draw a line through the points.
2. What type of function does the graph seem to model? How is your conjecture supported by the values found in Step 5?
3. What would happen to the average rate of change of the distance between the two cars if one of the cars slowed down? sped up? Explain your reasoning.

The rate that the distance between the two cars is changing is *related* to the rates of the two cars. In calculus, problems involving related rates can be solved using *implicit differentiation*. However, before we can use advanced techniques of differentiation, we need to understand how the rates involved relate to one another. Therefore, the first step to solving any related rates problem should always be to model the situation with a sketch or graph and to write equations using the relevant values and variables.

## Activity 2 Model Related Rates

**A rock tossed into a still body of water creates a circular ripple that grows at a rate of 5 cm/s. Find the area of the circle after 3 seconds if the radius of the circle is 5 cm at $t = 1$.**




**Step 1** Make a sketch of the situation.

**Step 2** Write an equation for the radius $r$ of the circle after $t$ seconds.

**Step 3** Find the radius for $t = 3$, and then find the area.

### Analyze the Results

4. Find an equation for the area $A$ of the circle in terms of $t$.
5. Find the area of the circle for $t = 1, 2, 3, 4$, and 5 seconds.
6. Make a graph of the values. What type of function does the graph seem to model?

You can use the difference quotient to calculate the rate of change for the area of the circle at a certain point in time.

## Activity 3 Approximate Related Rate

**Approximate the rate of change for the area of the circle in Activity 2.**

**Step 1** Substitute the expression for the area of the circle into the difference quotient.

$$m = \frac{\pi[5(t + h)]^2 - \pi(5t)^2}{h}$$

**Step 2** Approximate the rate of change of the circle at 2 seconds. Let $h = 0.1$, 0.01, and 0.001.

**Step 3** Repeat Steps 1 and 2 for $t = 3$ seconds and $t = 4$ seconds.

**StudyTip**

**Difference Quotient** Recall that the difference quotient for calculating the slope of the line tangent to the graph of $f(x)$ at the point $(x, f(x))$ is $m = \frac{f(x + h) - f(x)}{h}$.

### Analyze the Results

7. What do the rates of change appear to approach for each value of $t$?
8. What happens to the rate of change of the area of the circle as the radius increases? Explain.
9. How does this approach differ from the approach you used in Activity 1 to find the rate of change for the distance between the two cars? Explain why this was necessary.

## Model and Apply

10. A 4 m ladder is leaning against a wall so that the base of the ladder is exactly 1.5 m from the base of the wall. If the bottom of the ladder starts to slide away from the wall at a rate of 0.6 m/s, how fast is the top of the ladder sliding down the wall?
    a. Sketch a model of the situation. Let $d$ be the distance from the top of the ladder to the ground and $m$ be the rate at which the top of the ladder is sliding down the wall.
    b. Write an expression for the distance from the base of the ladder to the wall after $t$ seconds.
    c. Find an equation for the distance $d$ from the top of the ladder to the ground in terms of $t$ by substituting the expression found in part $b$ into the Pythagorean Theorem.
    d. Use the Pythagorean Theorem to find the distance $d$ from the top of the ladder to the ground for $t = 0, 1, 2, 3, 3.5$, and 3.75.
    e. Make a graph of the values. What type of function does the graph seem to model?
    f. Use the difference quotient to approximate the rate of change $m$ for the distance from the top of the ladder to the ground at $t = 2$. Let $h = 0.1$, 0.01, and 0.001. As $h$ approaches 0, what do the values for $m$ appear to approach?