Knockoffs for Trading US Equities

BSc Project Presentation

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Overview of Presentation

1. Introduction
2. False Discovery Rate
3. Linear Modelling
4. Construct knockoffs
5. Asset Selection
6. Portfolio Construction
7. Q&A

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Introduction

Question: How can we use statistical methods to select US equities capable of tracking the performance of a chosen Index?

Chosen indices: S&P 500, Russell 1000 and DJIA

US Equities: ~2000 US Equities traded on NYSE, AMEX, and Nasdaq exchanges

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False Discovery Rate

Benjamini and Hochberg [1] introduced the False Discovery Rate (FDR) in 1995.

	Declared non-significant	**Declared significant**	**Total**
True null hypotheses	U	V	m_0
Non-true null hypotheses	T	S	$m - m_0$
	$m - R$	R	m

Table 1: Testing m null hypotheses

Define: $Q = V/(V + S)$

V is the number of erroneously rejected hypotheses: $H_0\ is\ True, H_0\ rejected$
S is the number of correctly rejected hypotheses: $H_0\ is\ non - true, H_0\ rejected$

[1] Yoav Benjamini and Yosef Hochberg. "Controlling the False Discovery Rate: A Practical and Powerful Approach to Multiple Testing". In: Journal of the Royal Statistical Society. Series B (Methodological) 57.1 (1995), pp. 289–300. issn: 00359246. url: `http://www.jstor.org/stable/2346101`

False Discovery Rate

The False Discovery Rate is the expected proportion of incorrectly rejected null hypotheses

$$FDR = \mathbb{E}(Q) = \mathbb{E}(V/\underbrace{(V + S)}) = \mathbb{E}(V/R)$$

$$\mathrm{V} + \mathrm{S} = 0 \;\Rightarrow Q = 0$$

False Discovery Rate

The False Discovery Rate is the expected proportion of incorrectly rejected null hypotheses

$$FDR = \mathbb{E}(Q) = \mathbb{E}(V/(V + S)) = \mathbb{E}(V/R)$$

Consider testing $H_1, H_2, H_3, \dots, H_m$

FDR Controlling Procedure

1. Each $H_1, H_2, H_3, \dots, H_m$ has a corresponding p-value $P_1, P_2, P_3, \dots, P_m$
2. Order the p-values $P_{(1)} \leq P_{(2)} \leq P_{(3)} \leq \cdots \leq P_{(m)}$ such that $P_{(j)}$ corresponds to null $H_{(j)}$
3. Let k be the largest i for which $P_{(i)} \leq \frac{i}{m} q^*$
4. Reject all $H_{(i)}: i = 1,2,3, \dots, k$

False Discovery Rate

Theorem 1

For independent test statistics and for any configuration of false null hypotheses, the FDR controlling procedure controls the FDR at q^*.

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Linear Modelling

Barber and Candès [2] introduced the knockoff filter for statistical linear models.

Discover which features are truly associated with the response in a linear model

Guaranteed control of the FDR

Computation is cheap and construction does not require any new data

Method can work with a broad class of test statistics

[2] Rina Foygel Barber and Emmanuel J. Candès. “Controlling the false discovery rate via knockoffs”. In: The Annals of Statistics 43.5 (Oct. 2015). issn: 0090-5364. doi: 10.1214/15-aos1337. url: http://dx.doi.org/10.1214/15-AOS1337.

Linear Modelling

- Measuring returns from Friday to Friday, the first Friday of 2010 was on 2010-01-08 and the last Friday of 2019 was on 2019-12-27.
- Include only the largest 100 companies by market capitalisation

$$\boldsymbol{n} = 520$$
$$\boldsymbol{p} = 100$$

$$\overset{\boldsymbol{y}}{\begin{pmatrix} \textbf{Index} \\ \text{W1 Return} \\ \text{W2 Return} \\ \text{W3 Return} \\ \dots \\ \text{W}n\text{ Return} \end{pmatrix}} = \overset{\boldsymbol{X}}{\begin{pmatrix} \textbf{Equity } \boldsymbol{j} \\ \text{W1 Return} \\ \text{W2 Return} \\ \text{W3 Return} \\ \dots \\ \text{W}n\text{ Return} \end{pmatrix}} \overset{\boldsymbol{\beta}}{\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \dots \\ \beta_p \end{pmatrix}} + \overset{\boldsymbol{z}}{\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \dots \\ z_n \end{pmatrix}}$$

Linear Modelling

Data Pre-processing

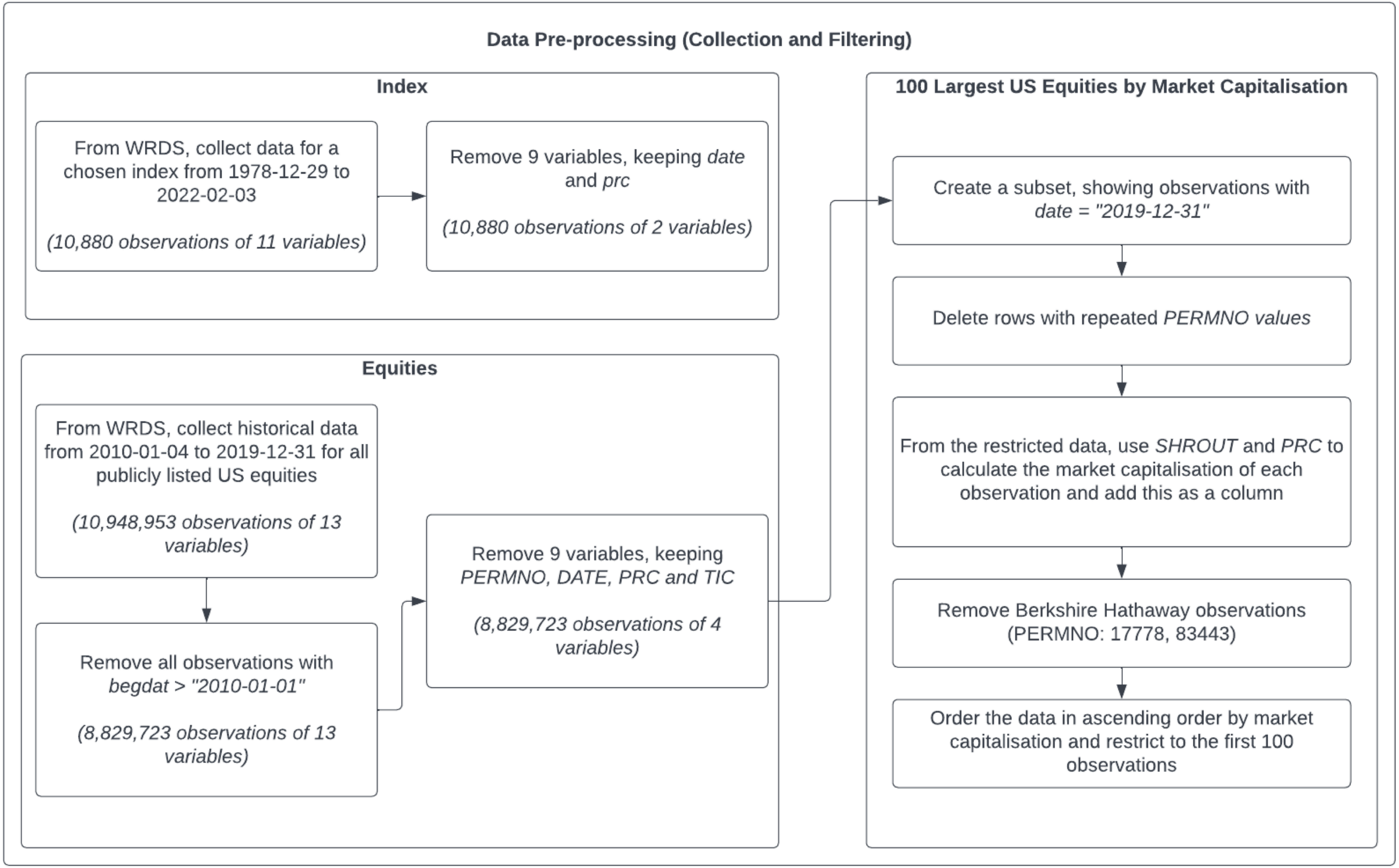
1. **PERMNO**
2. **BEGDAT**
3. **SHROUT**

Index dataframe

119,680 to 21,760 elements

Equities dataframe

142,336,389 to 35,318,892 elements


Data Pre-processing (Collection and Filtering)
Index
From WRDS, collect data for a chosen index from 1978-12-29 to 2022-02-03
(10,880 observations of 11 variables)
Remove 9 variables, keeping date and prc
(10,880 observations of 2 variables)
Equities
From WRDS, collect historical data from 2010-01-04 to 2019-12-31 for all publicly listed US equities
(10,948,953 observations of 13 variables)
Remove all observations with begdat > "2010-01-01"
(8,829,723 observations of 13 variables)
Remove 9 variables, keeping PERMNO, DATE, PRC and TIC
(8,829,723 observations of 4 variables)
100 Largest US Equities by Market Capitalisation
Create a subset, showing observations with date = "2019-12-31"
Delete rows with repeated PERMNO values
From the restricted data, use SHROUT and PRC to calculate the market capitalisation of each observation and add this as a column
Remove Berkshire Hathaway observations (PERMNO: 17778, 83443)
Order the data in ascending order by market capitalisation and restrict to the first 100 observations

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The Knockoff Filter

Construct knockoffs

For each $\boldsymbol{X_j}$ in our design matrix, we construct a knockoff feature $\widetilde{\boldsymbol{X_j}}$

Calculate $\boldsymbol{\Sigma} = \boldsymbol{X}^T\boldsymbol{X}$, after normalising we want:

$$\widetilde{\boldsymbol{X}}^{\boldsymbol{T}}\widetilde{\boldsymbol{X}} = \boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{\Sigma}$$

$$\boldsymbol{X}^{\boldsymbol{T}}\widetilde{\boldsymbol{X}} = \boldsymbol{\Sigma} \; - \mathrm{diag}(\boldsymbol{s})$$

Comparing a feature to its knockoff:

$$\boldsymbol{X}_{\boldsymbol{j}}^{\boldsymbol{T}}\widetilde{\boldsymbol{X}_{\boldsymbol{j}}} = \Sigma_{\mathrm{jj}} \; - \mathrm{s}_{\mathrm{j}} = 1 \; - \mathrm{s}_{\mathrm{j}}$$

The Knockoff Filter

Construct knockoffs

Construction Strategy

$$\tilde{\boldsymbol{X}} = \boldsymbol{X}(\boldsymbol{I} \ - \boldsymbol{\Sigma}^{-1}\text{diag}(\mathbf{s})) + \tilde{\boldsymbol{U}}\boldsymbol{C}$$

$\tilde{\boldsymbol{U}}$ is an $n \times p$ orthonormal matrix that is orthogonal to the span of the features $\boldsymbol{X}$

$$\boldsymbol{C}^{\mathrm{T}}\mathbf{C} = \mathbf{2}\text{diag}(\mathbf{s}) - \text{diag}(\mathbf{s})\ \boldsymbol{\Sigma}^{-1}\text{diag}(\mathbf{s})$$

SDP knockoffs

Minimise: $\Sigma(1 - s_j)$

Subject to: $0 \leq s_j \leq 1$

$\text{diag}(\mathbf{s}) \preccurlyeq 2\boldsymbol{\Sigma}$

$\Leftrightarrow$

$\boldsymbol{E}_{\boldsymbol{i}_{(\boldsymbol{r},\boldsymbol{c})}} = -1, 0 \text{ otherwise}$

Maximise: $tr\big(\boldsymbol{I}\ \text{diag}(\mathbf{s})\big)$

Subject to: $\Sigma s_i E_i \succcurlyeq -2\Sigma$

$tr\big(-E_i\text{diag}(\mathbf{s})\big) \leq 1$

$tr\big(E_i\ \text{diag}(\mathbf{s})\big) \leq 1$

The Knockoff Filter

Construct knockoffs

Construction Strategy

$$\tilde{\boldsymbol{X}} = \boldsymbol{X}(\boldsymbol{I} - \boldsymbol{\Sigma}^{-1}\text{diag}(\mathbf{s})) + \tilde{\boldsymbol{U}}\boldsymbol{C}$$

$\tilde{\boldsymbol{U}}$ is an $n \times p$ orthonormal matrix that is orthogonal to the span of the features $\boldsymbol{X}$

$$\boldsymbol{C}^{\mathrm{T}}\mathbf{C} = \mathbf{2}\text{diag}(\mathbf{s}) - \text{diag}(\mathbf{s})\,\boldsymbol{\Sigma}^{-1}\text{diag}(\mathbf{s})$$

Equi-correlated knockoffs

Maximise: $\boldsymbol{s}$

Subject to: $2\Sigma \succcurlyeq \text{diag}(\mathbf{s})$

$\boldsymbol{s} \geq \mathbf{0}$

Consider the constraint

$$2\Sigma \succcurlyeq \text{diag}(\mathbf{s}) \Leftrightarrow \lambda_{min}\big(2\Sigma - \text{diag}(\mathbf{s})\big) \geq \mathbf{0}$$
$$\Leftrightarrow 2\lambda_{min}(\Sigma) \geq s_j$$

Obvious solution: $s_j = 2\lambda_{min}(\Sigma)$

The Knockoff Filter

Calculate statistics for each pair of original and knockoff variables

We now wish to introduce the statistics W_j for each β_j

- Large positive values are evidence against the null hypothesis $H_0\colon\ \beta_j = 0$

Consider Tibshirani's Lasso model [3] as a method for constructing coefficient estimates

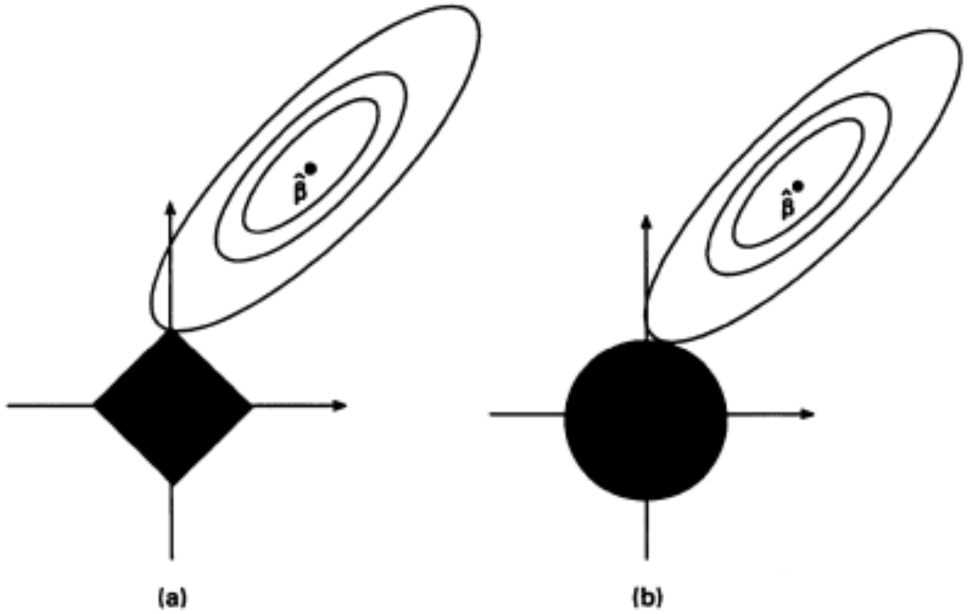
$$\widehat{\boldsymbol{\beta}}(\lambda) = argmin_{\boldsymbol{b}}\left\{\frac{1}{2}\underbrace{||\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b}||_2^{\boldsymbol{2}}}_{\text{OLS Estimate}} + \boldsymbol{\lambda}\overbrace{||\boldsymbol{b}||_1}^{\text{Constraint: } \Sigma_j \beta_j \leq t}\right\}$$

[3] Robert Tibshirani. "Regression Shrinkage and Selection via the Lasso". In: Journal of the Royal Statistical Society. Series B (Methodological) 58.1 (1996), pp. 267–288. issn: 00359246. url: `http://www.jstor.org/stable/2346178` (visited on 04/17/2022)

The Knockoff Filter

Calculate statistics for each pair of original and knockoff variables

$$\hat{\boldsymbol{\beta}}(\lambda) = argmin_{\boldsymbol{b}}\left\{\frac{1}{2}\underbrace{||\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b}||_2^{\mathbf{2}}}_{\text{OLS Estimate}} + \overbrace{\boldsymbol{\lambda}||\boldsymbol{b}||_1}^{\text{Constraint: } \Sigma_j \beta_j \leq t}\right\}$$



Comparison between Lasso (a) and ridge (b) regression

[3] Robert Tibshirani. “Regression Shrinkage and Selection via the Lasso”. In: Journal of the Royal Statistical Society. Series B (Methodological) 58.1 (1996), pp. 267–288. issn: 00359246.
url: `http://www.jstor.org/stable/2346178` (visited on 04/17/2022)

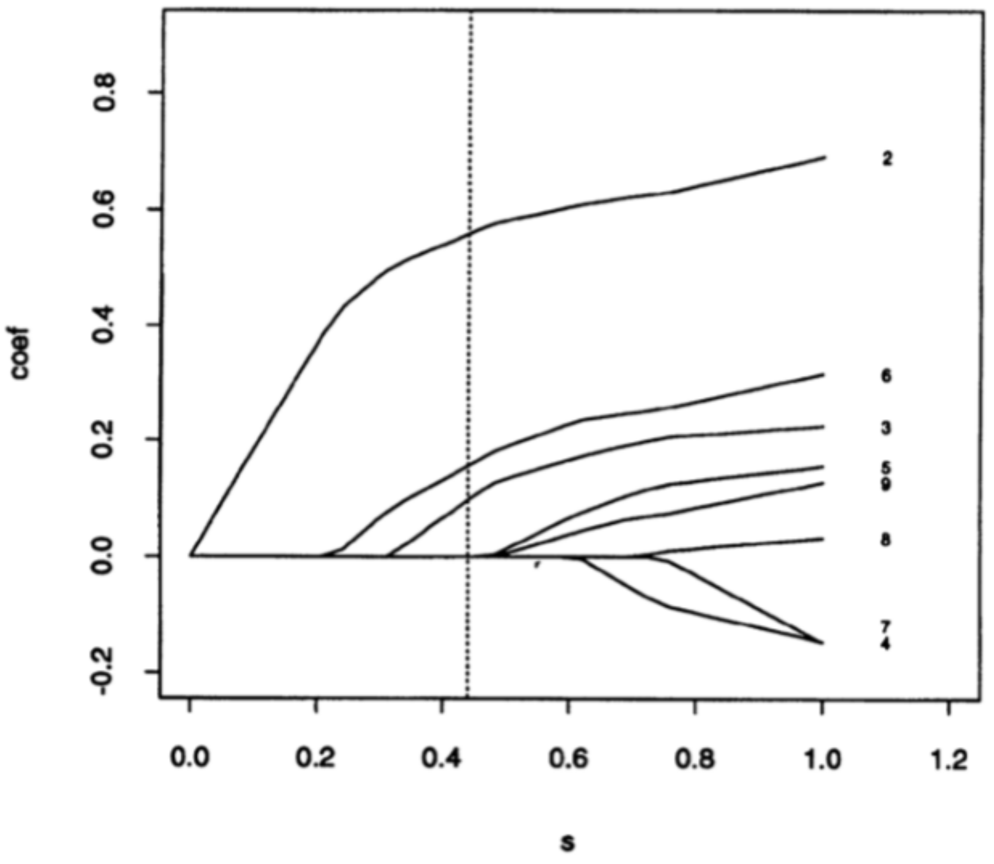
The Knockoff Filter

Calculate statistics for each pair of original and knockoff variables

$$Test\ statistic\ for\ feature\ j = \sup\{\lambda : \hat{\beta}_j(\lambda) \neq 0\}$$

1. Apply the Lasso model to the augmented matrix $[\boldsymbol{X}, \widetilde{\boldsymbol{X}}]$
2. Construct a vector of test statistics $(Z_1, Z_2, \dots, Z_p, \widetilde{Z}_1, \widetilde{Z}_2, \dots, \widetilde{Z}_p)$
3. For each j, define W_j

$$W_j = Z_j \vee \widetilde{Z}_j \cdot \begin{cases} +1: Z_j > \widetilde{Z}_j \\ -1: Z_j < \widetilde{Z}_j \\ 0: Z_j = \widetilde{Z}_j \end{cases}$$



Lasso shrinkage of coefficients, $s = 1/\lambda$ *and* $coef = |\beta_j|$

The Knockoff Filter

Calculating a Threshold for the Statistics

We wish to select large positive W_j such that $W_j \geq t$ for some $t > 0$

Let $W = \{|W_j| : j = 1, \ldots, p\} / \{0\}$

For some target FDR q, define the following data-dependent threshold

$$T = \min\left\{t \in W : \frac{\#\{j : W_j \leq -t\}}{\#\{j : W_j \geq t\} \vee 1} \leq q\right\}$$

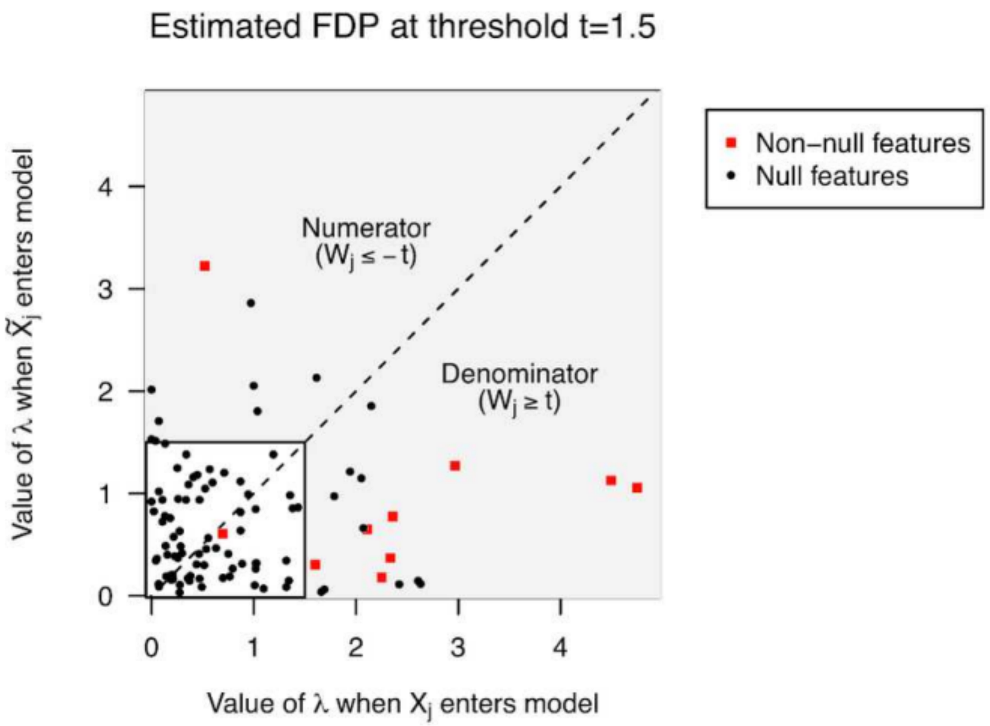
The Knockoff Filter

Calculating a Threshold for the Statistics

Data Dependent Threshold:

$$T = \min\left\{t \in W: \frac{\#\{j: W_j \leq -t\}}{\#\{j: W_j \geq t\} \vee 1} \leq q\right\}$$

- 9 features above the diagonal and 18 below: 9/18 estimates the FDR
- 8 true discoveries out of 18 selected features: 8/18 is the true FDR



Visualisation of the knockoff procedure, black points correspond to $\beta_j = 0$ and red points correspond to $\beta_j \neq 0$.

The Knockoff Filter

Theorems from Knockoffs

Define: $\hat{S} = \{j : W_j \geq T\}$

Theorem 2

The knockoff procedure controls a quantity nearly equal to the FDR in feature selection.

More Specifically:

$$\mathbb{E}\left[\# \frac{\{j\colon \beta_j = 0 \; and \; j \in \hat{S}\}}{\#\{j\colon j \in \hat{S}\} + q^{-1}}\right] \leq q$$

The Knockoff Filter

Theorems from Knockoffs

The knockoff+ procedure:

$$T' = \min\left\{t \in \mathcal{W}: \frac{1 + \#\{j: W_j \le -t\}}{\#\{j: W_j \ge t\} \vee 1} \le q\right\}$$

Define: $\hat{S} = \{j : W_j \ge T'\}$

Theorem 3

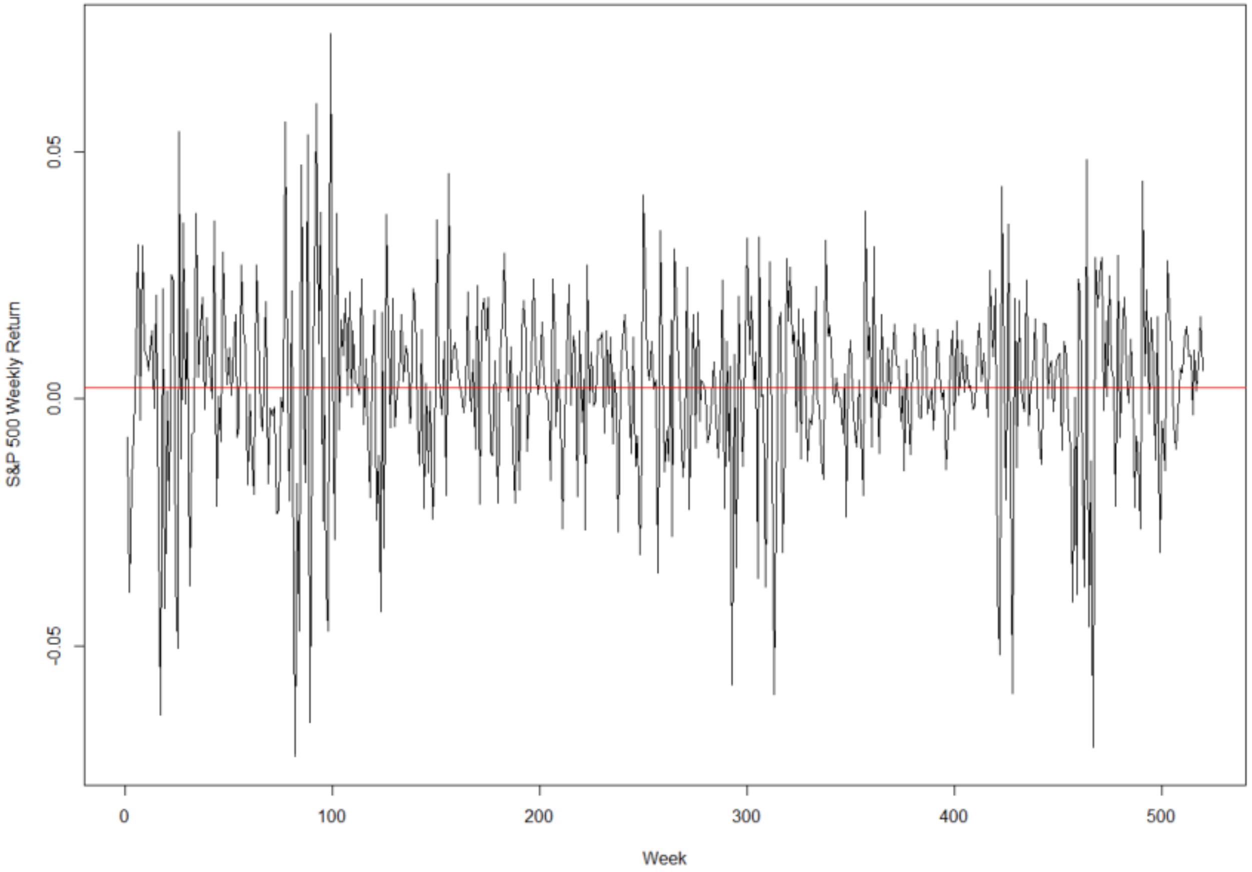
The knockoff+ procedure controls the FDR exactly in feature selection.

More Specifically:

$$\mathbb{E}\left[\# \frac{\{j: \beta_j = 0 \; and \; j \in \hat{S}\}}{\#\{j: j \in \hat{S}\} \vee 1}\right] \le q$$

Overview of Presentation

Asset Selection



Return series of the S&P 500 Index from 2010-01-15 to 2019-12-27

Mean weekly returns

0.002190

Observed variance

0.0003734

Asset Selection

- Target FDR of 0.05
- ‘knockoff’ package used in R
- 42 selected US equities

AAPL	MSFT	AMZN	JPM	MA	XOM	UNH
DIS	PFE	CMCSA	CSCO	PEP	C	ORCL
ADBE	NVDA	TMO	RTC	HON	TXN	DHR
SBUX	CVS	MO	USB	LOW	BKNG	MS
CAT	GS	MDLZ	FISV	ANTM	TFC	PROV
ISRG	PCS	SPGI	BSX	SCHQ	ITW	ECL

S&P 500 Knockoff Portfolio constituents

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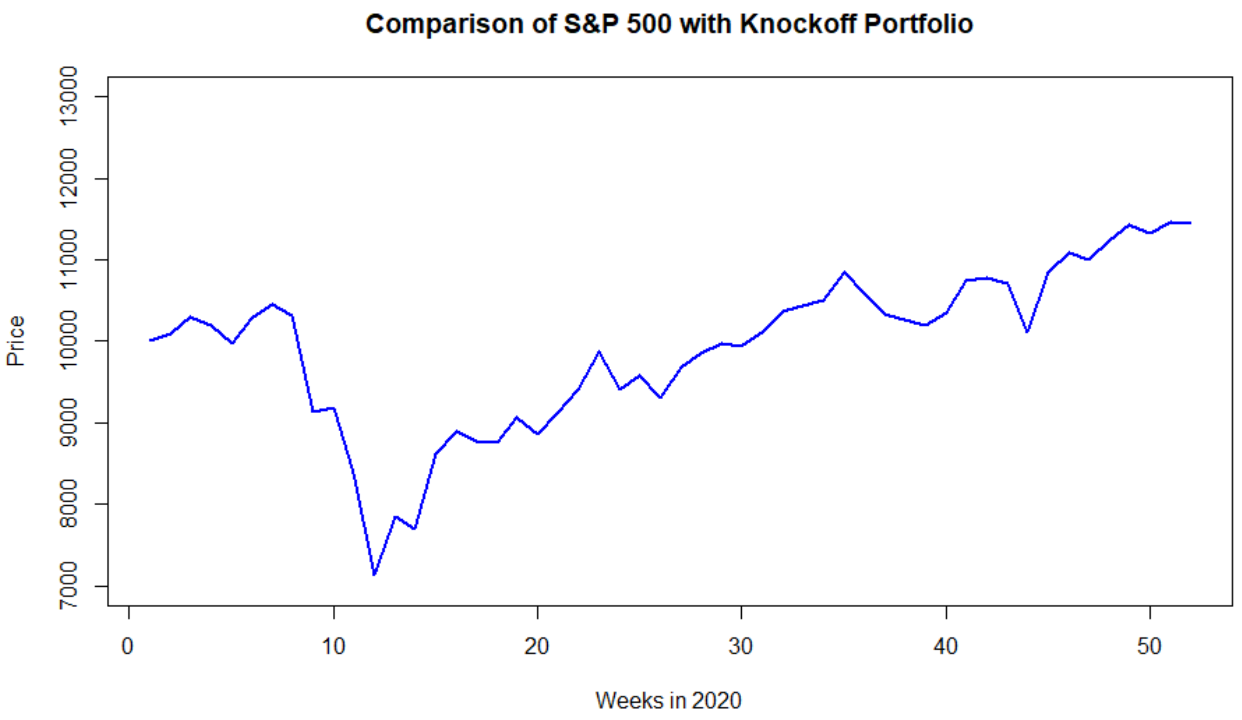
Portfolio Construction

Weighting by Market Capitalisation

Replicate Index weightings

What would happen if I invested $10,000 on the first Friday of January 2020?



Portfolio Construction

Weighting by Market Capitalisation

Replicate Index weightings

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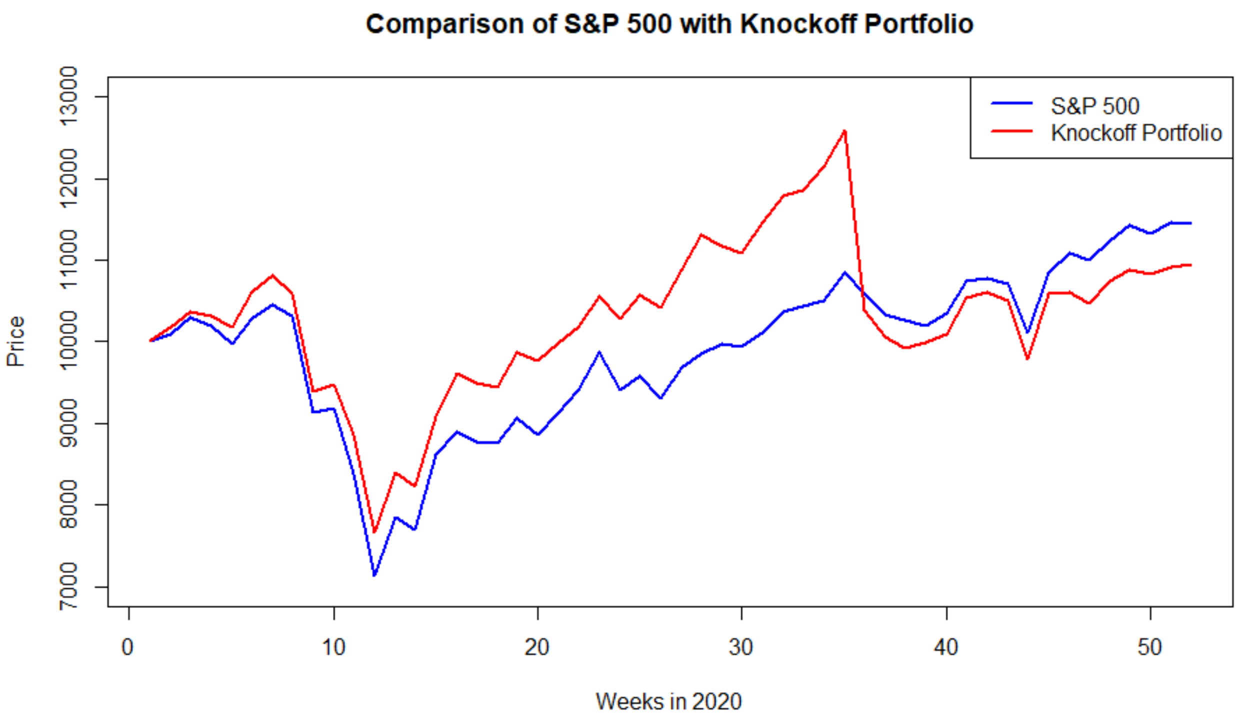

Portfolio Construction

Weighting by Market Capitalisation

Replicate Index weightings

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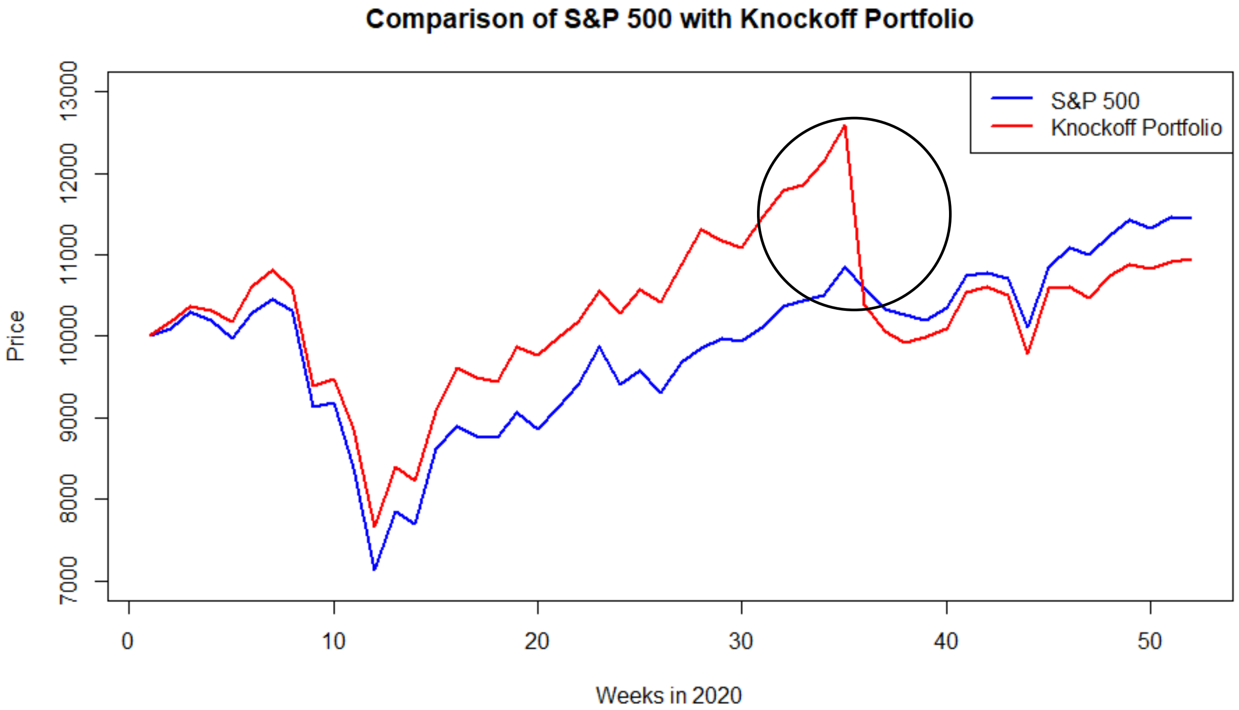


Portfolio Construction

Modern Portfolio Theory

MPT assumes that investors are risk averse

Portfolio return variance: $\boldsymbol{w}^T \boldsymbol{\Sigma} \mathbf{w}$

Expected portfolio returns: $\boldsymbol{R}^T \boldsymbol{w}$

Covariance matrix of returns

Minimise: $\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}$

Subject to: $\boldsymbol{R}^T \boldsymbol{w} = \mu^*$ → Target return

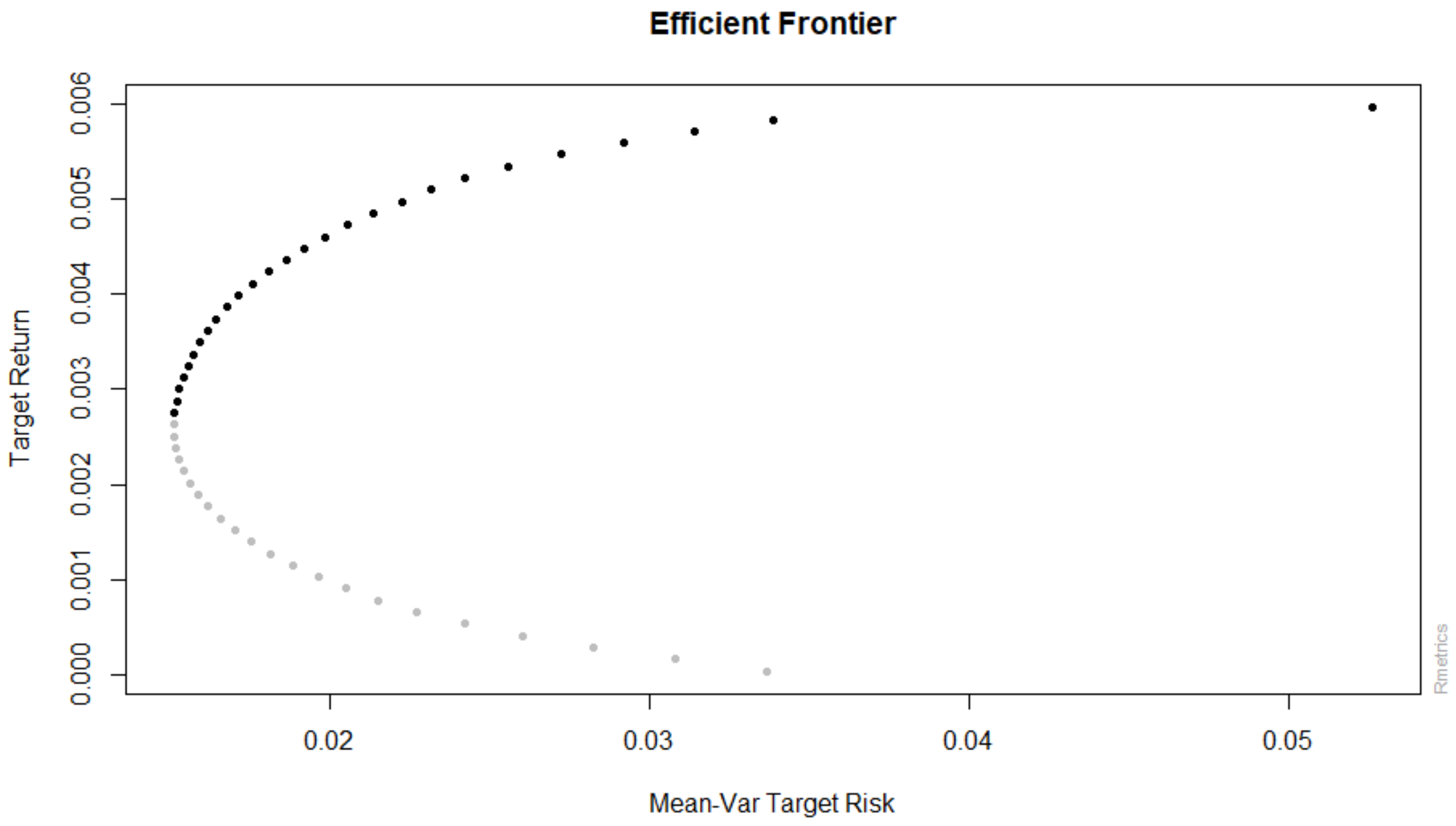
$\Sigma w_i = 1$ → Sum of weights must be 1

$\boldsymbol{w} \geq \mathbf{0}$ → No short selling

Vector of expected returns

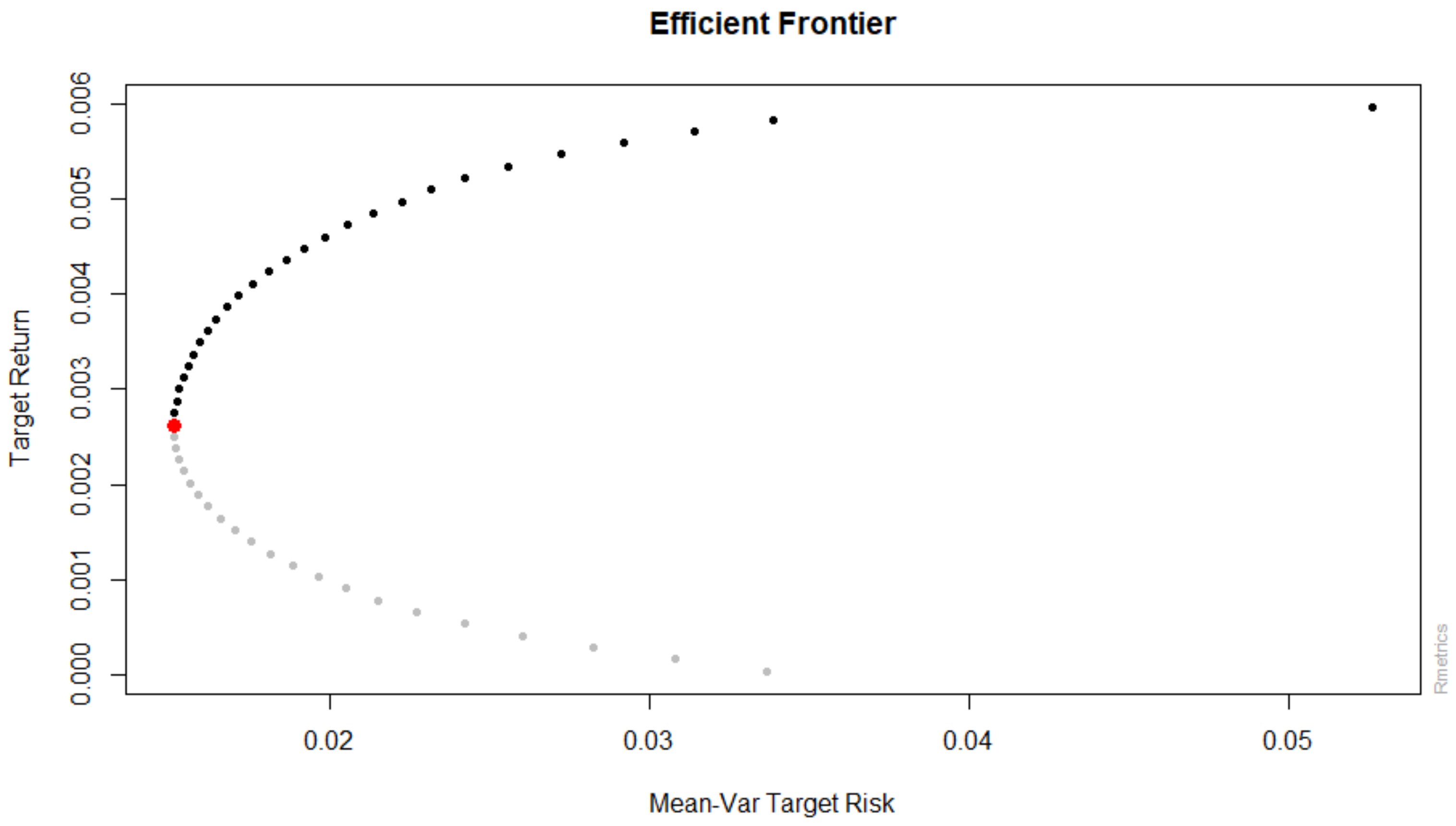
Portfolio Construction

Efficient Frontier



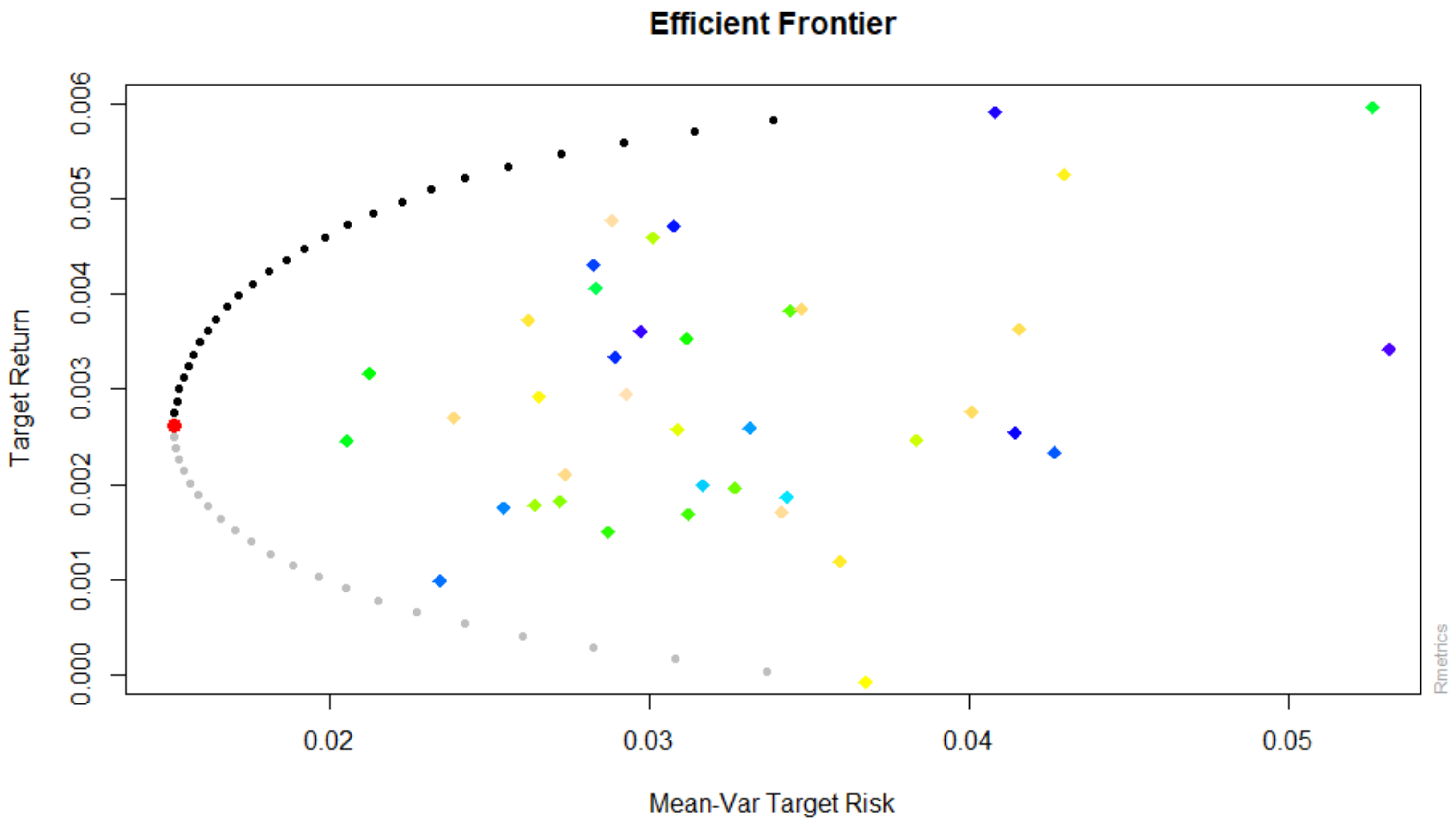
Portfolio Construction

Efficient Frontier with Global Minimum Variance



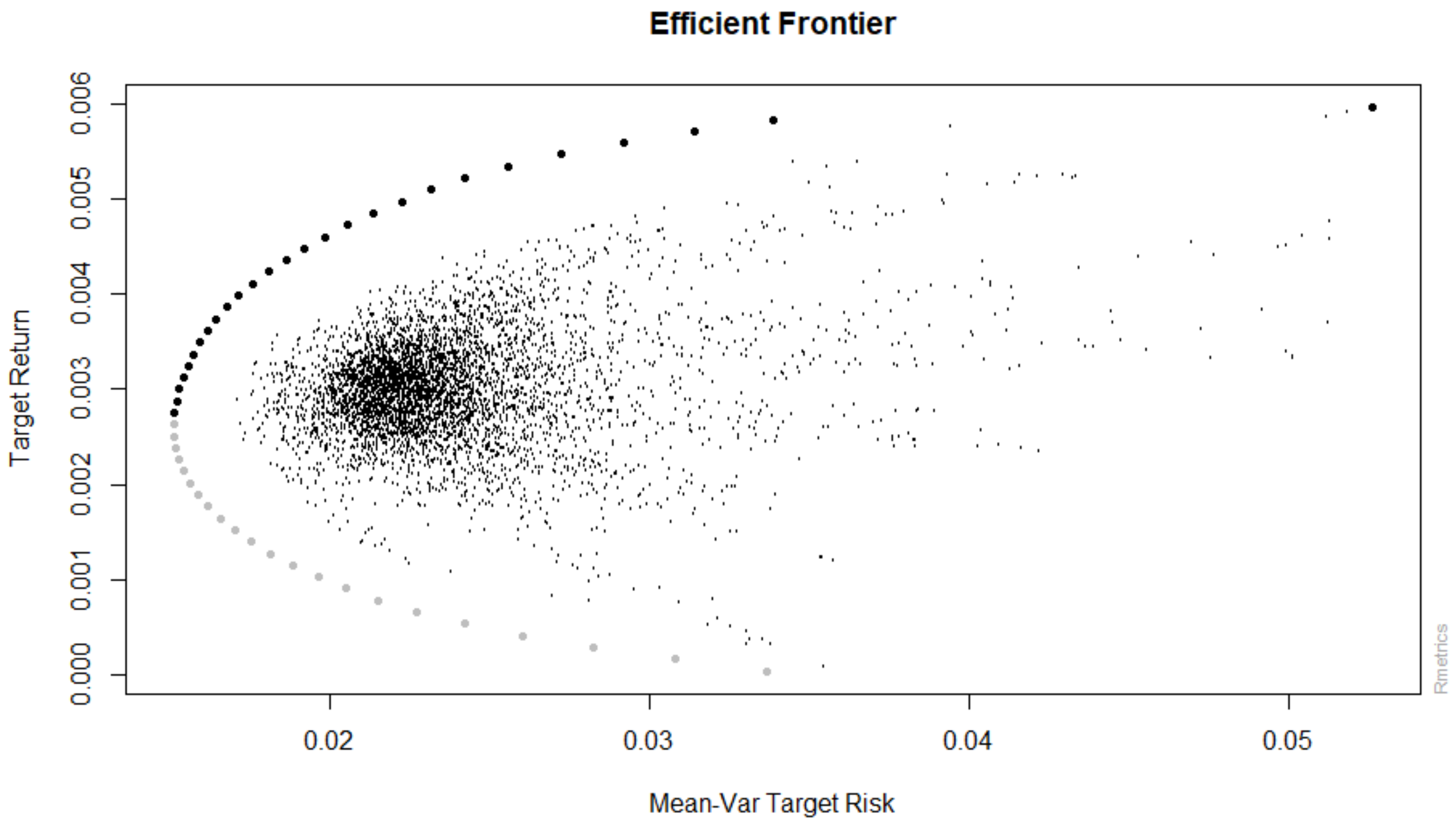
Portfolio Construction

Efficient Frontier with Global Minimum Variance and Risk/Return for Each Asset



Portfolio Construction

Efficient Frontier with Monte Carlo Portfolios



Questions?