Assignment 1
COMP3670

August 22, 2020

name: Xuecheng Zhang
UID:u6284513

5/8/2020

Exercise 1 Solving Linear Systems (5+5 credits)
Find the set S of all solutions x of the following inhomogenous linear systems Ax = b, where A and b are defined as follows. Write the solution space S in parametric form.

(a)

Solve for $Ax = b$

$$\left[\begin{array}{ccc|c} 2 & -2 & -5 & -4 \\ 1 & -1 & 3 & 9 \\ 3 & -3 & -2 & -5 \end{array}\right]$$

swap r_1 with r_2

$$\xrightarrow{} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 9 \\ 2 & -2 & -5 & -4 \\ 3 & -3 & -2 & -5 \end{array}\right]$$

$$\xrightarrow[r_3 = r_2 - 3r_1]{r_2 = r_3 - 2r_1} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 9 \\ 0 & 0 & -11 & -22 \\ 0 & 0 & -11 & -22 \end{array}\right]$$

$$\xrightarrow[r_2 = r_2/-11]{r_3 = r_3 - r_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{r_1 = r_1 - 3r_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

Solve for $Ax = 0$

$$\left[\begin{array}{ccc|c} 2 & -2 & -5 & 0 \\ 1 & -1 & 3 & 0 \\ 3 & -3 & -2 & 0 \end{array}\right]$$

swap r_1 with r_2

$$\xrightarrow{} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 2 & -2 & -5 & 0 \\ 3 & -3 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[r_3 = r_2 - 3r_1]{r_2 = r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 0 & -11 & 0 \\ 0 & 0 & -11 & 0 \end{array}\right]$$

$$\xrightarrow[r_2 = r_2/-11]{r_3 = r_3 - r_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{r_1 = r_1 - 3r_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \lambda \in \mathbb{R}$$

∴ combine this two solutions:

the general solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \lambda \in R$$

The solution Space S $= \left\{ \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ \lambda \in \mathbb{R} \right\}$

(b)

$$\left[\begin{array}{ccc|c} 2 & -3 & -10 & 4 \\ -4 & 2 & 3 & 0 \\ 10 & -3 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 2 & -3 & -10 & 4 \\ 0 & -4 & -17 & 8 \\ 0 & 12 & 51 & -19 \end{array}\right] \quad \begin{array}{l} r_2 = r_2 + 2r_2 \\ r_3 = r_3 - 5r_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & -10 & 4 \\ 0 & -4 & -17 & 8 \\ 0 & 0 & 0 & 5 \end{array}\right] \quad r_3 = r_3 + 3r_2$$

$$\left[\begin{array}{ccc|c} 2 & -3 & -10 & 4 \\ 0 & -4 & -17 & 8 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad r_3 = r_3/5$$

$$\left[\begin{array}{ccc|c} 2 & -3 & -10 & 0 \\ 0 & -4 & -17 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} r_1 = r_1 - 4r_3 \\ r_2 = r_2 - 8r_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & -10 & 0 \\ 0 & 1 & \frac{17}{4} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad r_2 = r_2/-4$$

$$\left[\begin{array}{ccc|c} 2 & 0 & \frac{11}{4} & 0 \\ 0 & 1 & \frac{17}{4} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad r_1 = r_1 + 3r_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{11}{8} & 0 \\ 0 & 1 & \frac{17}{4} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad r_1 = r_1/2.$$

The system has no solutions and the Space S $= \varnothing$

Exercise 2 For what values of λ does the inverse of the following matrix exist?
When the inverse of matrix exists, the determinant of matrix cannot be 0.

$$det = \begin{vmatrix} \lambda & 1 & 2 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 3\lambda + 1 - 6 - \lambda = 2\lambda - 5 = 0$$

If $\lambda \neq \frac{5}{2}$, the inverse of matrix exists.

Exercise 3 Which of the following sets are subspaces of $\mathbb{R}^3$. Prove your answer. (That is, if it is a subspace, you must demonstrate the subspace axioms are satisfied, and if it is not a subspace, you must show which axiom fails.)
(a) A = (x, y, 1) : x, y, $\in$ R
(b) B = (x, y, z) : x + 4y - 3z = t, where t is some real number. (Your answer may depend on the value of t.)
(c) C = (x, y, z) : x $\geq$ 0, y $\geq$ 0, z $\geq$ 0
(d) D = (x, y, z) : x, y $\in \mathbb{R}$, z $\in \mathbb{Q}$.

(a) It is not subspace of $\mathbb{R}^3$. Since 0 is not in A, A is not the subspace of $\mathbb{R}^3$
(b) If t$\neq$0, then 0 $\notin$ B and it implies that B is not a subspace of $\mathbb{R}^3$.
if t = 0, then 0 $\in$ B and B $\subseteq \mathbb{R}^3$.
Also, assume $B_1(x_1,y_1,z_1) \in$ B, $B_2(x_2,y_2,z_2) \in$ B, We get x_1 + $4y_1$ - $3z_1$ + x_2 + $4y_2$ - $3z_2$ = (x_1 + x_2) + 4(y_1 + y_2) - 3(z_1 + z_2) $\in$ B, i.e. B_1 + B_2 $\in$ B
Assume $\lambda \in \mathbb{R}$, (X,Y,Z) $\in$ S and we can obtain that X + 4Y - 3Z = 0. λ X + 4λ Y - 3λ Z = λ (X + 4Y - 3Z). i.e. λ(X,Y,Z) $\in$ S.
Therefore, we can conclude that when t$\neq$0, B is not a subspace of $\mathbb{R}^3$ and when t = 0, B is a subspace of $\mathbb{R}^3$.

(c) It is not subspace of $\mathbb{R}^3$. We assume that C_1 (x_c,y_c,z_c) $\in$ C, C_2 λ(x_c,y_c,z_c) (λ = -1). -$x_c \leq 0$; -$y_c \leq 0$;-$z_c \leq 0$; $C_2 \notin$ C. Therefore, C is not subspace of $\mathbb{R}^3$.

(d) It is not subspace of $\mathbb{R}^3$. Here is a counterexample, D_1 (2,3,4), λ = $\sqrt{3}$, D_2 = λ D_1 = (2$\sqrt{3}$,3$\sqrt{3}$,4$\sqrt{3}$), as 4$\sqrt{3} \notin \mathbb{Q}$. Therefore, D is not a subspace of $\mathbb{R}^3$.

Exercise 4 Let v1, v2, v3 be vectors in $\mathbb{R}^2$. Prove that the set v1, v2, v3 is linearly dependant.
Assume v1,v2,v3 is linearly independent. From the definition of the basis of a vector space, v1,v2 can form a basis in $\mathbb{R}^2$ which means r3 can be written in terms of v1 and v2. It implies that v1,v2,v3 set is linearly dependent which draws the contradiction. Therefore, v1,v2,v3 is linearly dependent.

Exercise 5
(a) Is the following function $< .,. >$ defined for all x = $[x1, x2]^T \in \mathbb{R}^2$ and y = $[y_1, y_2]^T \in R^2$ as <x,y> = $y_1(x_1 - x_2) + y_2(x_2 - x_1)$

It is not an inner mapping.
For bilinear mapping:

$$\begin{aligned}
\Omega(\lambda x+\phi y,z) &= z_1(\lambda x_1+\phi y_1-\lambda x_2-\phi y_2)+z_2(\lambda x_2+\phi y_2-\lambda x_1-\phi y_1) && (1)\\
&= \lambda(z_1x_1-z_1x_2+z_2x_2-z_2x_1)+\phi(y_1z_1-y_2z_1+y_2z_2-y_1z_2) && (2)\\
&= \lambda\Omega(x,z)+\phi\Omega(y,z) && (3)
\end{aligned}$$

$$\begin{aligned}
\lambda\Omega(x,y)+\phi\Omega(x,z) &= \lambda[y_1(x_1-x_2)+y_2(x_2-x_1)]+\phi[z_1(x_1-x_2)+z_2(x_2-x_1)] && (4)\\
&= (x_1-x_2)(\lambda y_1+\phi z_1)+(x_2-x_1)(\lambda y_2+\phi z_2) && (5)\\
&= (\lambda y_1+\phi z_1)(x_1-x_2)+(\lambda y_2+\phi z_2)(x_2-x_1) && (6)\\
&= \lambda(x,\lambda y+\phi z) && (7)
\end{aligned}$$

It is satisfied with bilinear mapping.

For symmetric axiom:
$< x, y >= y_1(x_1-x_2)+y_2(x_2-x_1) = x_1y_1-x_2y_1+x_2y_2-x_1y_2$
$< y, x >= x_1(y_1-y_2)+x_2(y_2-y_1) = x_1y_1-x_1y_2+x_2y_2-x_2y_1$
Therefore $< x, y >=< y, x >$ is symmetric

It is satisfied with symmertic axiom.

For positive definite axiom:
$< x, x >= x_1(x_1-x_2)+x_2(x_2-x_1) = x_1^2-2x_1x_2+x_2^2 = (x_1-x_2)^2$
if $x_1 = x_2$ then the formula equal to 0 instead of larger than 0.
Therefore, it is not positive definite matrix.

(b) Prove that $<.,.>$ defined for all $x = [x_1, x_2]^T \in R^2$ and y $= [y_1, y_2]^T \in R^2$ as $<$x,y$>$ $= x_1y_1 - x_2y_2$ is not an inner product.

An inner product must satisfy positive definite axiom.
$\forall X \in R^2 \ / \ 0, < x, x >= x_1y_1 - x_2y_2 = x_1^2 - x_2^2$
if $x_1 < x_2$ then $x_1^2 - x_2^2 < 0$. Therefore, it is not satisfied with positive definite axiom. i.e. it is not an inner product.

Exercise 6
(a) Prove that if x and y are linearly dependant vectors, then $| < x, y > | = ||x||||y||$
As x and y are linearly dependent vectors, then $c_1x + c_2y = 0, (c_1, c_2$ not all equal to 0). And we can derive that y = kx.
$| < x, y > | = | < x, kx > | = |k|| < x, x > | = |k|||x||^2$
$||x||||y|| = ||x||||kx|| = |k|||x||^2$

Therefore $| < x, y > | = ||x||||y||$
(b) Show that we can retrieve the inner product from the norm via the following expression:
$< x, y >= \frac{1}{2}(||x + y||^2 - ||x||^2 - ||y||^2)$

$\frac{1}{2}(< x + y, x + y > - < x, x > - < y, y >)$
$= \frac{1}{2}(< x, x + y > + < y, x + y > - < x, x > - < y, y >)$
$= \frac{1}{2}(< x, x > + < x, y > + < y, x > + < y, y > - < x, x > - < y, y >)$
$= \frac{1}{2} * 2 < x, y > = < x, y >$

Therefore the expression is proved.

(c) Show that norm equivalence is an equivalence relation, that is, that norm equivalence is reflexive, symmetric and transitive

reflexive: $||v||_a$ is equivalent to $||v||_a$.
We can get $M1||v||_a \leq ||v||_a \leq M2||v||_a$ when M1 = M2 = 1.

symmetric: Assume "$||v||_a$ and $||v||_b$ are equivalent if $M_1||v||_a \leq ||v||_b \leq M_2||v||_a$ when $M_1 > 0, M_2 > 0$ such that for any $v \in V$" is true.

We can derive that $||v||_a \leq \frac{||v||_b}{M_1}$, $||v||_a \geq \frac{||v||_b}{M_2}$, $i.e. \frac{||v||_b}{M_2} \leq ||v_a|| \leq \frac{||v||_b}{M_1}$
Set $M_3 = \frac{1}{M_2}$, $M_4 = \frac{1}{M_1}$, we can conclude that **"$||v||_b$ and $||v||_a$ are equivalent if $M_3||v||_b \leq ||v||_a \leq M_4||v||_b$ when $M_3 > 0, M_4 > 0$ such that for any $v \in V$"** is also established

transitive: Assume "$||v||_a$ and $||v||_b$ are equivalent if $M_1||v||_a \leq ||v||_b \leq M_2||v||_a$ when $M_1 > 0, M_2 > 0$ such that for any $v \in V$" and "$||v||_b$ and $||v||_c$ are equivalent if $M_3||v||_b \leq ||v||_c \leq M_4||v||_b$ when $M_3 > 0, M_4 > 0$ such that for any $v \in V$" is true.

We can derive that $||v||_a \leq \frac{||v||_b}{M_1}$, $||v||_a \geq \frac{|v||_b}{M_2}$, $||v||_b \leq \frac{||v||_c}{M_3}$, $||v||_b \geq \frac{||v||_c}{M_4}$

$\implies ||v||_a \leq \frac{||v||_b}{M_1} \leq \frac{||v||_c}{M_1 M_3}$, $||v||_a \geq \frac{||v||_b}{M_2} \geq \frac{||v||_c}{M_2 M_4}$
Set $M_1' = \frac{1}{M_2 M_4}$ and $M_2' = \frac{1}{M_1 M_3}$, we can get that $M_1'||v||_c \leq ||v||_a \leq M_2'||v||_c$.
We can conclude that $||v||_a$ and $||v||_c$ are equivalent when $M_1' > 0, M_2' > 0$.

Therefore, it satisfied with reflective, symmetric and transitive properties.

(d) Assuming that V = $\mathbb{R}$, show that $|| \cdot ||_1$ and $|| \cdot ||_2$ are equivalent norms.
Assume that v = $[x_1, x_2]^T \in \mathbb{R}^2$

$$||v||_1 = |x_1| + |x_2|, ||v||_2 = \sqrt{x_1^2 + x_2^2} \tag{8}$$

Assume that $x_2 = kx_1$ (k $\in \mathbb{R}$)

$$||v||_1 = |x_1| + |k||x_1| = (1+|k|)|x_1| \tag{9}$$

$$||v||_2 = \sqrt{x_1^2 + (kx_1)^2} = \sqrt{k^2+1}|x_1| \tag{10}$$

Divide $||v||_2$ by $|v||_1$:

$$\frac{||v||_2}{||v||_1} = \frac{\sqrt{k^2+1}}{1+|k|} \tag{11}$$

Let f(k) $=$ $\frac{\sqrt{k^2+1}}{1+|k|}$, f(x) is an even function and f(x) $>$0 ($\forall x \in \mathbb{R}$)
When k $=$ 0,

$$f(0) = 1 \tag{12}$$

When k$>$0,

$$f(k)^2 = \frac{k^2+1}{k^2+2k+1} = \frac{k+\frac{1}{k}}{k+\frac{1}{k}+2} \tag{13}$$

As $k + \frac{1}{k} \geq 2$ (if k $>$ 0)
We can get that:

$$\frac{2}{2+2} = \frac{1}{2} \leq f(k)^2 = \frac{k+\frac{1}{k}}{k+\frac{1}{k}+2} < 1 \tag{14}$$

$$\frac{\sqrt{2}}{2} \leq f(k) < 1 \tag{15}$$

When k$<$0, it has the same condition with k$>$0 which is $\frac{\sqrt{2}}{2} \leq f(k) < 1$
Now, we can say that the range of the function f(x) is $[\frac{\sqrt{2}}{2}, 1]$

Which is to say, $\frac{\sqrt{2}}{2} \leq \frac{||v||_2}{||v||_1} \leq 1$

i.e., $\frac{\sqrt{2}}{2}||v||_1 \leq ||v||_2 \leq ||v||_1$ which $M_1 = \frac{\sqrt{2}}{2}, M_2 = 1$

Exercise 7 Consider the Euclidean vector space $\mathbb{R}^3$ with the dot product. A subspace U *subset* $\mathbb{R}^3$ and vector x $\in \mathbb{R}^3$ are given by

$$U = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}, \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

a) Show that x $\notin$ U

Solve $a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 3 \end{array}\right]$$

$$\xrightarrow[r_3 = r_3 - r_1]{r_2 = r_2 - r_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[r_3 = r_3 + 2r_2]{r_1 = r_1 + r_2} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

In the last row is $0 \cdot$ a + $0 \cdot$ b = 1, which has no solutions. Therefore x $\notin$ U.

b) Determine the orthogonal projection π_U(x) of x onto U. Show that π_U(x) can be written as a linear combination of $[1, 1, 1]^T$ and $[2, 2, 3]^T$

As U = span$[1,1,1]^T$,$[2,2,3]^T$, x = $[1,0,3]^T$
The basis of Vector is:

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

Compute the B^TB and the vector B^Tx:

$$B^TB = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 7 & 17 \end{pmatrix} \tag{16}$$

$$B^Tx = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix} \tag{17}$$

Solve the normal equation $B^TB\lambda = B^Tx$ to get the value of λ:

$$\lambda = \begin{pmatrix} -9/2 \\ 5/2 \end{pmatrix} \tag{18}$$

$$\pi_U(x) = B\lambda = \begin{pmatrix} 1/2 \\ 1/2 \\ 3 \end{pmatrix} \tag{19}$$

$$\begin{pmatrix} 1 & 2 & 1/2 \\ 1 & 2 & 1/2 \\ 1 & 3 & 3 \end{pmatrix} \tag{20}$$

Minus the second row with the first row; minus the second row with the first row.

$$\begin{pmatrix} 1 & 2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 1 & 5/2 \end{pmatrix} \tag{21}$$

Minus the first row with two times of third row.

$$\begin{pmatrix} 1 & 0 & -9/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{pmatrix} \tag{22}$$

$\pi_U(x) = -\frac{9}{2} \cdot [1,1,1]^T + \frac{5}{2} \cdot [2,2,3]^T$
Therefore, π_U(x) can be written as a linear combination of $[1,1,1]^T$ and $[2,2,3]^T$

(c) Determine the distance d(x, U)
d(x,U) is the distance between the original vector and its projection onto U.

$$||x - \pi_U(x)|| = ||[-1/2, 1/2, 0]^T|| = \frac{\sqrt{2}}{2} \tag{23}$$