Funciones de probabilidad

Definición: Una función f(x) es llamada densidad de probabilidad si

1. $\int_{Dom_x} f(x)dx = 1$, Dom_x es el dominio de la variable aleatoria (VA)
2. $f(x) \geq 0$

Además, existe una función F(x) llamada función de distribución, que se define como:

$$F(x) = \int_a^x f(t)dt = P(X \leq x)$$

Estas funciones cumplen

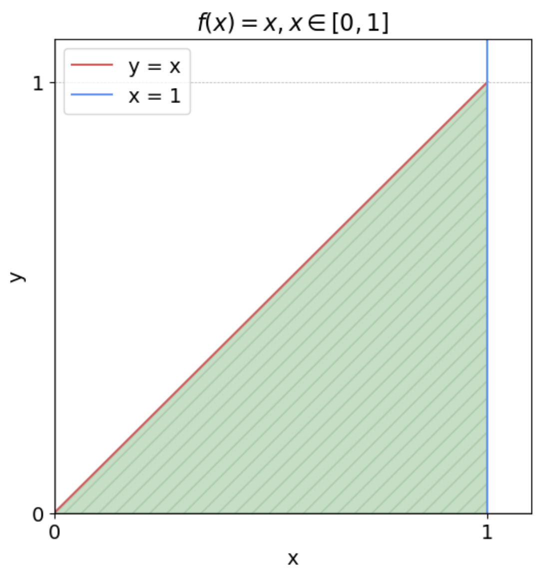
$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

$$\frac{dF(x)}{dx} = f(x)$$

Ejemplo:

Verifica si $f(x) = x, x \in [0,1]$ es una fdp (función de densidad de probabilidad)

a) Como $f(x) = x \geq 0$ en $[0,1]$ se cumple 2 de la definición



b)

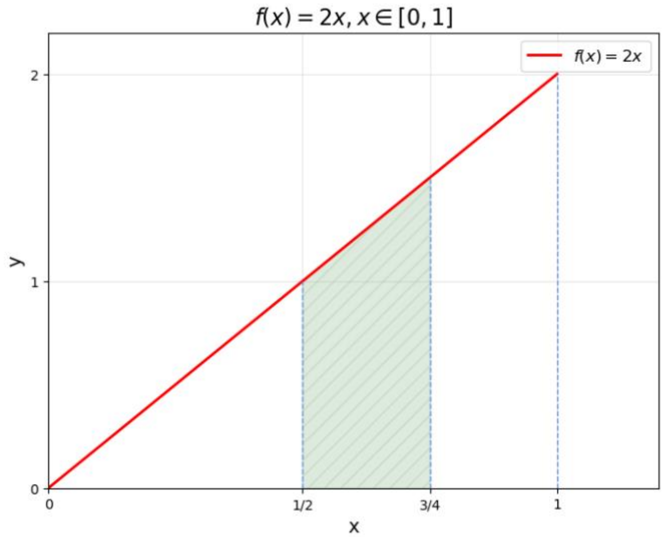
$$\int_0^1 xdx = \frac{x^2}{2}|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$$

Por lo tanto, $f(x) = x$ no es una fdp

Sugerimos $f(x) = Kx$ en $[0,1]$

$$\int_0^1 Kxdx = K\int_0^1 xdx = K\left(\frac{1}{2}\right) = 1 \therefore K = 2$$

$$f(x) = 2x, x \in [0,1]$$



La FDP es (función acumulativa de probabilidad):

$$F(x) = \int_0^x 2t dt = t^2|_0^x = x^2 - 0^2 = x^2$$

Por ejemplo:

$$P\left(\frac{1}{2} < X < \frac{3}{4}\right) = \int_{\frac{1}{2}}^{\frac{3}{4}} 2x dx = F\left(\frac{3}{4}\right) - F\left(\frac{1}{2}\right)$$

$$= x^2|_{\frac{1}{2}}^{\frac{3}{4}} = \left(\frac{3}{4}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{9}{16} - \frac{1}{4} = \frac{5}{16}$$

$$= F\left(\frac{3}{4}\right) - F\left(\frac{1}{2}\right)$$

Primer momento con respecto al origen

$$\mu_1{}' = E[X] = \int_{Dom_x} x f(x)dx = \mu$$

Segundo momento

$$\mu_2' = E[x^2] = \int_{Dom_x} x^2 f(x)dx$$

Segundo momento con respecto a la media

$$\mu_2 = E[(x-\mu)^2] = \mu_2' - (\mu_1)^2 = E[x^2] - E[x]^2 = \sigma^2$$

Para nuestro ejemplo

$$\mu = \int_0^1 x f(x)dx = \int_0^1 x(2x)dx = 2\int_0^1 x^2 dx = \frac{2}{3}x^3|_0^1 = \frac{2}{3} = E[x]$$

$$E[x^2] = \int_0^1 x^2 f(x)dx = \int_0^1 x^2(2x)dx = 2\int_0^1 x^3 dx = \frac{2}{4}x^4|_0^1 = \frac{1}{2}$$

$$\sigma^2 = E[x^2] - E[x]^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$$

Función generadora de momentos

$$M(t) = \int_{Dom_x} e^{xt} f(x)dx = \int_0^1 e^{xt}(2x)dx$$

U	dv
$2x$	e^{xt}
2	$\frac{1}{t}e^{xt}$
0	$\frac{1}{t^2}e^{xt}$

$$= \frac{2x}{t}e^{xt} - \frac{2}{t^2}e^{xt}|_0^1 = \frac{2}{t}e^t - \frac{2}{t^2}e^t - \left(0 - \frac{2}{t^2}\right) = \frac{2e^t}{t} - \frac{2e^t}{t^2} + \frac{2}{t^2}$$

Nota cultural

$$M'(0) = \mu$$
$$M''(0) = E[x^2]$$
$$\sigma^2 = M''(0) - \left(M'(0)\right)^2$$

Definición:

La función de densidad de probabilidad conjunta de las VA X, Y es una función $f(x,y)$ que cumple:

a) $f(x,y) \geq 0$

b) $\iint_R f(x,y)dA = 1$

Y donde

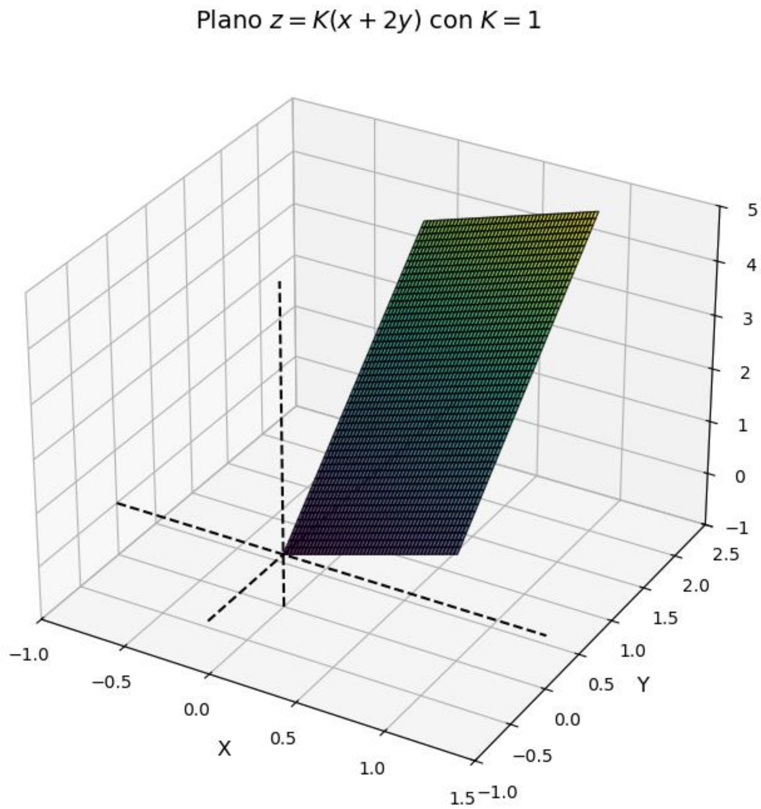
$$P(a \leq X \leq b), c \leq Y \leq d) = \int_a^b \int_c^d f(x,y)dydx$$

Donde el rectángulo $(a,b) \times (c,d) \in R$, y la función de distribución es

$$F(x,y) = \int_a^x \int_c^y f(t,s)dsdt$$

Ejemplo:

$$f(x,y) = K(x+2y), 0 \leq x \leq 1, 0 \leq y \leq 2$$



$$K \int_0^1 \int_0^2 (x+2y)dydx = 1$$

$$K \int_0^1 (xy + y^2)|_0^2 dx = K \int_0^1 (2x+4)dx = K(x^2+4x)|_0^1 = 5K = 1 \therefore K = \frac{1}{5}$$

$$f(x,y) = \frac{1}{5}(x+2y), 0 \leq x \leq 1, 0 \leq y \leq 2$$

Función de densidad marginal

$$g(x) = \int_{Dom_y} f(x,y)dy$$

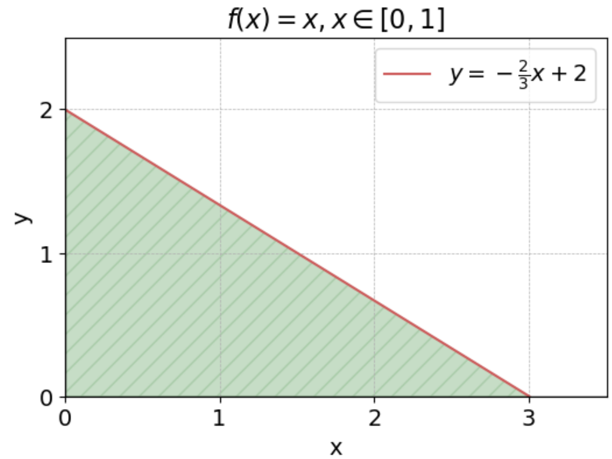
$$h(y) = \int_{Dom_x} f(x,y)dx$$

Para nuestro ejemplo

$$g(x) = \int_0^2 \frac{1}{5}(x+2y)dy = \frac{1}{5}(xy+y^2)_0^2 = \frac{1}{5}(2x+4), 0 \leq x \leq 1$$

$$h(y) = \int_0^1 \frac{1}{5}(x+2y)dx = \frac{1}{5}\left(\frac{x^2}{2} + 2xy\right)_0^1 = \frac{1}{5}\left(\frac{1}{2} + 2y\right), 0 \leq x \leq 2$$

Ejemplo maldito para la función $f(x,y) = K$



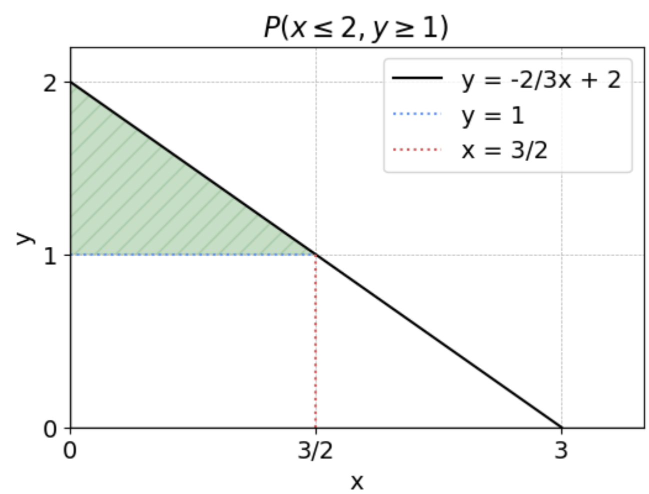
$$P_1(0{,}2)\ P_2(3{,}0)$$
$$m = \frac{0-2}{3-0} = -\frac{2}{3}$$
$$y = m(x - x_0) + y_o$$
$$y = -\frac{2}{3}(x-0) + 2 = -\frac{2}{3}x + 2$$

Para calcular K

$$\iint_R K dA = \int_0^3 \int_0^{-\frac{2}{3}x+2} K dy dx = \int_0^2 \int_0^{-\frac{3}{2}(y-2)} K dx dy$$
$$= K \int_0^3 y\Big|_0^{-\frac{2}{3}x+2} dx = K \int_0^3 \left(2 - \frac{2}{3}x\right) dx = K\left(2x - \frac{x^2}{3}\right)_0^3 = K(6-3) = 3K = 1 \therefore K = \frac{1}{3}$$

Determina

a) $P(x \leq 2, y \geq 1)$



$$P(x \le 2, y \ge 1) = \int_0^{\frac{3}{2}} \int_1^{-\frac{2}{3}x+2} \frac{1}{5}(x+2y)dydx = \frac{1}{5}\int_0^{\frac{3}{2}} xy + y^2|_1^{-\frac{2}{3}x+2} dx$$
$$= \frac{1}{5}\int_0^{\frac{3}{2}} x\left(-\frac{2}{3}x+2\right) + \left(-\frac{2}{3}x+2\right)^2 - x - 1dx$$
$$= \frac{1}{5}\int_0^{\frac{3}{2}} -\frac{2}{3}x^2 + 2x + \frac{4}{9}x^2 - \frac{8}{3}x + 4 - x - 1dx$$
$$= \frac{1}{5}\int_0^{\frac{3}{2}} -\frac{2}{9}x^2 - \frac{5}{3}x + 3dx = \frac{1}{5}\left(-\frac{2}{27}x^3 - \frac{5}{6}x^2 + 3x\right)|_0^{\frac{3}{2}}$$
$$= \frac{1}{5}\left(-\frac{2}{27}\left(\frac{3}{2}\right)^3 - \frac{5}{6}\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right)\right) = \frac{19}{40}$$

b) $g(x)$

$$g(x) = \int_0^{-\frac{2}{3}x+2} \frac{1}{5}(x+2y)dy = \frac{1}{5}(xy+y^2)|_0^{-\frac{2}{3}x+2}$$
$$= \frac{1}{5}\left(x\left(-\frac{2}{3}x+2\right) + \left(-\frac{2}{3}x+2\right)^2\right)$$
$$= \frac{1}{5}\left(-\frac{2}{3}x^2 + 2x + \frac{4}{9}x^2 - \frac{8}{3}x + 4\right) = \frac{1}{5}\left(-\frac{2}{9}x^2 - \frac{2}{3}x + 4\right)$$
$$= -\frac{2}{45}x^2 - \frac{2}{15}x + \frac{4}{5}$$

c) $h(y)$

$$h(y) = \int_0^{\frac{3}{2}y-3} \frac{1}{5}(x+2y)dx = \frac{1}{5}\left(\frac{x^2}{2} + 2xy\right)_0^{\frac{3}{2}y-3} = \frac{1}{5}\left(\frac{\left(\frac{3}{2}y-3\right)^2}{2} + 2\left(\frac{3}{2}y-3\right)y\right)$$
$$= \frac{1}{5}\left(\frac{\frac{9}{4}y^2 - 9y + 9}{2} + 3y^2 - 6y\right) = \frac{1}{5}\left(\frac{33}{8}y^2 - 21y + \frac{9}{2}\right)$$
$$= \frac{33}{40}y^2 - \frac{21}{5}y + \frac{9}{10}$$

d) $E[XY] = \iint_R \ xyf(x,y)dA$

$$E[XY] = \int_0^3 \int_0^{-\frac{2}{3}x+2} \frac{1}{5} xy(x+2y)dydx = \frac{1}{5}\int_0^3 \int_0^{-\frac{2}{3}x+2} x^2y + 2xy^2 dydx$$
$$= \frac{1}{5}\int_0^3 \frac{x^2y^2}{2} + \frac{2xy^3}{3} |_0^{-\frac{2}{3}x+2} dx$$
$$= \frac{1}{5}\left(\frac{1}{6}\right)\int_0^3 3x^2\left(-\frac{2}{3}x+2\right)^2 + 4x\left(-\frac{2}{3}x+2\right)^3 dx$$
$$= \frac{1}{30}\int_0^3 3x^2\left(\frac{4}{9}x^2 - \frac{8}{3}x + 4\right) + 4x\left(-\frac{8}{27}x^3 + \frac{8}{3}x^2 - 8x + 8\right) dx$$
$$= \frac{1}{30}\int_0^3 \frac{4}{3}x^4 - 8x^3 + 12x^2 - \frac{32}{27}x^4 + \frac{32}{3}x^3 - 32x^2 + 32xdx$$
$$= \frac{1}{30}\int_0^3 \frac{4}{27}x^4 + \frac{8}{3}x^3 - 20x^2 + 32xdx$$
$$= \frac{1}{30}\left(\frac{4}{135}x^5 + \frac{2}{3}x^4 - \frac{20}{3}x^3 + 16x^2\right)_0^3$$
$$= \frac{1}{30}\left(\frac{4}{135}(3)^5 + \frac{2}{3}(3)^4 - \frac{20}{3}(3)^3 + 16(3)^2\right) = \frac{21}{25}$$