The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 3

Due: 11:59pm, Oct. 18 (Friday), 2024

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- Please submit your assignment on Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student must not copy homework solutions from another student or from any other source.

Problem 1 (15pts).
Consider an LP in its standard form $\min \boldsymbol{c}^\top \boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$. Suppose that the matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with $m < n$, and that its rows are linearly independent. Assume $\hat{\boldsymbol{x}}$ is a basic feasible solution associated with the basis $B = \{B(1), B(2), \ldots, B(m)\}$. For each of the following statements, state whether it is true or false. Please explain your answers (if not true, you do **not** need to show a concrete counterexample, but you need to argue when/how it can be false).

(a) $\boldsymbol{A}_B\hat{\boldsymbol{x}}_B = \boldsymbol{b}$, and $\boldsymbol{A}_{B'}\hat{\boldsymbol{x}}_{B'} \neq \boldsymbol{b}$ for every basis $B' \neq B$;

(b) Every optimal solution of the LP is a basic feasible solution;

(c) If the LP is unbounded and we apply the simplex method, starting from $\hat{\boldsymbol{x}}$, using the Bland's rule, the algorithm will ultimately find a basic direction $\boldsymbol{d}$ such that $\boldsymbol{c}^\top\boldsymbol{d} < 0$ and $\boldsymbol{d} \geq \boldsymbol{0}$ (after finite number of iterations).

Problem 2 (15pts).
Assume we want to apply the two-phase simplex method to solve the following linear program:

$$\begin{array}{rll} \text{minimize} & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 & \\ \text{subject to} & x_1 - x_2 + 2x_3 & \geq 2 \\ & x_2 - x_3 + 2x_4 & \leq 4 \\ & 2x_1 + 3x_3 - x_4 & = 2 \\ & x_1, x_2, x_3, x_4 & \geq 0. \end{array}$$

(a) Write down the auxiliary LP of phase I (use variables $\boldsymbol{s}$ as slack variables in the standard form, variables $\boldsymbol{y}$ as auxiliary variables in phase I, and order the variables as $(\boldsymbol{x}; \boldsymbol{s}; \boldsymbol{y})$);

(b) For the auxiliary LP of phase I, consider the basis associated with variables x_1, x_4 and the auxiliary variable corresponding to the second constraint (i.e., y_2). Compute the associated basic feasible solution by solving the resultant linear system. Is this solution optimal for the auxiliary LP of phase I? Why?

(c) With the above basis, compute the inverse of the basis matrix (i.e., $\boldsymbol{A}_B^{-1}$), all basic directions and their reduced costs (you don't need to write down the steps, and you can use Python/MATLAB to help you calculate). Are all reduced costs nonnegative? If not, derive the next simplex iteration by following the smallest index rule.

Problem 3 (20pts).
Consider the following linear program:

$$\begin{array}{rll} \text{maximize} & x_1 + 2x_2 + 3x_3 + 8x_4 & \\ \text{subject to} & x_1 - x_2 + x_3 & \leq 2 \\ & x_3 - x_4 & \leq 1 \\ & 2x_2 + 3x_3 + 4x_4 & \leq 8 \\ & x_1, x_2, x_3, x_4 & \geq 0. \end{array}$$

Use simplex tableau to completely solve it. For each step, draw the simplex tableau. Clearly mark what is the current basis, the current basic solution, and the corresponding objective function value.

Problem 4 (25pts).
Apply the two-phase simplex method (implemented by simplex tableau) to solve the following linear program. For each step, draw the simplex tableau. Clearly mark what is the current basis, the current basic solution, and the corresponding objective function value.

$$\begin{array}{rll} \text{minimize} & x_1 - x_2 + 2x_3 & \\ \text{subject to} & 2x_1 - x_2 + 2x_3 & \leq -1 \\ & x_1 - x_2 - x_3 & \leq 4 \\ & x_2 - x_4 & = 0 \\ & x_1, x_2, x_3, x_4 & \geq 0. \end{array}$$

Problem 5 (25pts).
Use the two-phase simplex method (implemented by simplex tableau) to completely solve the linear optimization problem. For each step, draw the simplex tableau. Clearly mark what is the current basis, the current basic solution, and the corresponding objective function value.

$$\begin{array}{rll} \text{minimize} & x_1 + 3x_2 + x_4 - 2x_5 & \\ \text{subject to} & x_1 + 2x_2 + 4x_4 + x_5 & = 2 \\ & x_1 + 2x_2 - 2x_4 + x_5 & = 2 \\ & -x_1 - 4x_2 + 3x_3 & = 1 \\ & x_1, x_2, x_3, x_4, x_5 & \geq 0. \end{array}$$

Exercise (0pts. No need to submit your answer. Solution is attached.)
Consider a linear optimization problem in the standard form, described in terms of the following initial tableau (Table 2):

B								
	0	0	0	δ	3	γ	ξ	0
2	0	1	0	α	1	0	3	β
3	0	0	1	-2	2	η	-1	2
1	1	0	0	0	-1	2	1	3

Table 1

The entries α, β, γ, δ, η and ξ in the tableau are unknown parameters, and $B = \{2, 3, 1\}$. For each of the following statements, find (sufficient) conditions of the parameter values that will make the statement true.

1. This is an acceptable initial tableau (i.e., the basic variables are feasible for the problem).
2. The first row (in the constraint) indicates that the problem is infeasible.
3. The basic solution is feasible but we have not reached an optimal basic set B.
4. The basic solution is feasible and the first simplex iteration indicates that the problem is unbounded.
5. The basic solution is feasible, x_6 is a candidate for entering B, and when we choose x_6 as the entering basis, x_3 leaves B.

Solution.

1. $\beta \geq 0$.
2. $\alpha \geq 0, \beta < 0$. The sum of all the positive variables has a negative value, which indicates infeasibility.
3. $\beta > 0$, at least one of $\delta, \gamma, \xi < 0$. The reduced cost of one of the non-basic variables is negative. Note that $\beta > 0$ is required, as in the *degenerate* case, we might have reached optimal but with a negative reduced cost. In the next iterate, the simplex method updates the basis with $y = x$, and y might be an optimal solution. In the case, the original x is already optimal.
4. $\beta \geq 0$, $\alpha \leq 0$, $\delta < 0$. The fourth column has all entries negative or zero.
5. $\beta \geq 0$, $\gamma < 0$, $\frac{2}{\eta} < \frac{3}{2}$ and $\eta > 0$ gives $\eta > \frac{4}{3}$. By the minimum ratio test,we want η to be the pivot element.

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Problem 1:

(a) True, different basis $B' \neq B$ will lead to a different solution $A_{B'} \hat{x}_{B'} = b'$

$(b' \neq b)$

(b) True, by Fundamental LP theorem

(c) True, Bland's rule prevents cycling and ensures a basic direction d s.t. $c^T d < 0$, $d \geq 0$ will eventually be found.

Problem 2:

$$\min \quad c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$$

$$\text{s.t.} \quad \begin{aligned} x_1 - x_2 + 2x_3 \quad\quad\quad &\geq 2 \\ x_2 - x_3 + 2x_4 &\leq 4 \\ 2x_1 \quad\quad + 3x_3 - x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

a) Auxiliary LP:

$$\min \quad y_1 + y_2$$

$$\text{s.t.} \quad \begin{aligned} x_1 - x_2 + 2x_3 \quad\quad - s_1 + y_1 &= 2 \\ x_2 - x_3 + 2x_4 + s_2 &= 4 \\ 2x_1 + 3x_3 - x_4 \quad\quad + y_2 &= 2 \\ x_1, x_2, x_3, x_4, s_1, s_2, y_1, y_2 &\geq 0 \end{aligned}$$

b) consider x_1, x_4, y_2, setting all non-basic variables $=0$ in the constraints:

we get: $x_1 - x_2 + 2x_3 - s_1 + y_1 = 2 \Rightarrow x_1 = 2$

$x_2 - x_3 + 2x_4 + s_2 = 4 \Rightarrow 2x_4 = 4$
$\Rightarrow x_4 = 2$

$2x_1 + 3x_3 - x_4 + y_2 = 2 \Rightarrow y_2 = 0$

obtained BFS: $x_1 = 2, \ x_4 = 2, \ y_2 = 0$

$\therefore$ Solution is optimal for the aux LP.

Reason: $y_1 + y_2 = 0$ from the above aux LP solution, and there fore minimized.

All variables $x_1 = 2, x_4 = 2, y_2 = 0$ of the BFS are feasible, solution is optimal.

c) $A_B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Using Python / Graphing Calculator: $A_B^{-1} = \begin{bmatrix} -5/3 & 2/3 & 4/3 \\ 2/3 & 1/3 & -1/3 \\ 4/3 & -1/3 & -2/3 \end{bmatrix}$

and Reduced cost $= 6$ (nonnegative)

Problem 3 :

$$\max \quad x_1 + 2x_2 + 3x_3 + 8x_4$$

$$\begin{aligned} \text{s.t.} \quad x_1 - x_2 + x_3 &\le 2 \\ x_3 - x_4 &\le 1 \\ 2x_2 + 3x_3 + 4x_4 &\le 8 \end{aligned}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Rewrite in Standard Form

$$\min \quad -x_1 - 2x_2 - 3x_3 - 8x_4$$

$$\begin{aligned} \text{s.t.} \quad x_1 - x_2 + x_3 + s_1 &= 2 \\ x_3 - x_4 + s_2 &= 1 \\ 2x_2 + 3x_3 + 4x_4 + s_3 &= 8 \end{aligned}$$

$$x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0$$

Simplex Tableau :

B	-1	-2	-3	-8	0	0	0	0
5	1	-1	1	0	1	0	0	2
6	0	0	1	-1	0	1	0	1
7	0	2	3	4	0	0	1	8

B	0	-3	-2	-8	1	0	0	2
1	1	-1	1	0	1	0	0	2
6	0	0	1	-1	0	1	0	1
7	0	2	3	4	0	0	1	8

B	0	0	5/2	-2	1	0	3/2	14
1	1	0	5/2	2	1	0	1/2	6
6	0	0	1	-1	0	1	0	1
2	0	1	3/2	2	0	0	1/2	4

B	0	1	4	0	1	0	2	18
1	1	-1	1	0	1	0	0	2
6	0	1/2	7/4	0	0	1	1/4	3
4	0	1/2	3/4	1	0	0	1/4	2

All reduced costs are nonnegative, optimal solution found

Optimal value: 18 , optimal solution : $[2, 0, 0, 2, 0, 3, 0]^T$

Problem 4: min $x_1 - x_2 + 2x_3$

s.t. $2x_1 - x_2 + 2x_3 \leq -1$ ← rewrite as $-2x_1 + x_2 - 2x_3 \geq 1$

$x_1 - x_2 - x_3 \leq 4$

$x_2 - x_4 = 0$

$x_1, x_2, x_3, x_4 = 0$

Add slack and Aux variables

min $y_1 + y_2$

s.t. $-2x_1 + x_2 - 2x_3 - s_1 + y_1 = 1$

$x_1 - x_2 - x_3 + s_2 = 4$

$x_2 - x_4 + y_2 = 0$

$x_1, x_2, x_3, x_4, s_1, s_2, y_1, y_2 = 0$

B	0	0	0	0	0	1	1	1	0
7	-2	1	-2	0	-1	0	1	0	1
6	1	-1	-1	0	0	1	0	0	4
8	0	1	0	-1	0	0	0	1	0

initial Tableau:

B	1	-1	3	1	1	0	0	0	-5
7	-2	1	-2	0	-1	0	1	0	1
6	1	-1	-1	0	0	1	0	0	4
8	0	1	0	-1	0	0	0	1	0

B	1	0	3	0	1	0	0	1	-5
7	-2	0	-2	1	-1	0	1	-1	1
6	1	0	-1	-1	0	1	0	1	4
2	0	1	0	-1	0	0	0	1	0

B	0	0	2	1/2	1/2	0	1/2	1/2	-9/2
1	1	0	1	-1/2	1/2	0	-1/2	1/2	-1/2
6	0	0	-2	-3/2	-1/2	1	1/2	1/2	9/2
2	0	1	0	-1	0	0	0	1	0

Drop rows w/ Aux variables

Calculate $\bar{c}^T = c^T - c_B^T A_B^{-1} A$

$$= \begin{pmatrix} 1 & -1 & 2 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{bmatrix} 1 & 0 & 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & -2 & -3/2 & -1/2 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix}$$

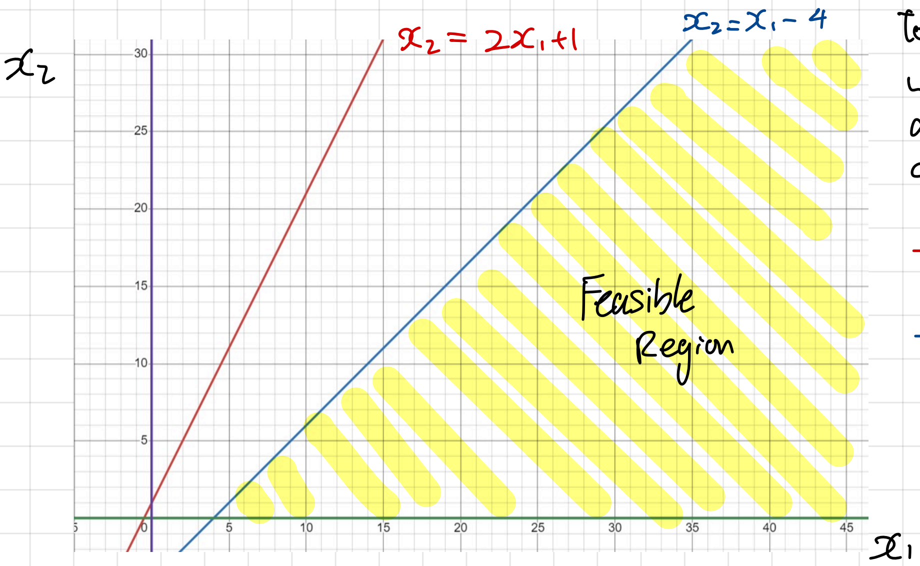
$$= \begin{pmatrix} 0 & -1 & 1 & -1/2 & -1/2 & 0 \end{pmatrix}$$

B	0	-1	1	-1/2	-1/2	0	4
1	1	0	1	-1/2	1/2	0	-1/2
6	0	0	-2	-3/2	-1/2	1	9/2
2	0	1	0	-1	0	0	0

B	0	0	1	$\boxed{-3/2}$	$-1/2$	0	4
1	1	0	1	$-1/2$	$1/2$	0	$-1/2$
6	0	0	-2	$-3/2$	$-1/2$	1	$9/2$
2	0	1	0	-1	0	0	0

At this point, we are unable to find the next index to leave/enter the basis. (unable to derive a valid pivot row/column)

$\Rightarrow$ There is no basic feasible solution that satisfies all constraints
$\Rightarrow$ The problem is infeasible.



To represent graphically (2D) we let $x_3 = x_4 = 0$ and express the remaining constraints with:

: $2x_1 - x_2 = -1$

: $x_1 - x_2 = 4$

$\Rightarrow$ Feasible region is unbounded.

Problem 5 :

$$\min \quad x_1 + 3x_2 + x_4 - 2x_5$$

$$\text{s.t.} \quad \begin{aligned} x_1 + 2x_2 + 4x_4 + x_5 &= 2 \\ x_1 + 2x_2 - 2x_4 + x_5 &= 2 \\ -x_1 - 4x_2 + 3x_3 &= 1 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Add Aux variables : $\min \; y_1 + y_2 + y_3$

$$\begin{aligned} x_1 + 2x_2 + 4x_4 + x_5 + y_1 &= 2 \\ x_1 + 2x_2 - 2x_4 + x_5 + y_2 &= 2 \\ -x_1 - 4x_2 + 3x_3 + y_3 &= 1 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3 \geq 0$$

B	0	0	0	0	0	1	1	1	0
6	1	2	0	4	1	1	0	0	2
7	1	2	0	-2	1	0	1	0	2
8	-1	-4	3	0	0	0	0	1	1

B	-1	0	-3	-2	0	0	0	0	-5
6	1	2	0	4	1	1	0	0	2
7	1	2	0	-2	1	0	1	0	2
8	-1	-4	3	0	0	0	0	1	1

B	0	2	-3	2	1	1	0	0	-3
1	1	2	0	4	1	1	0	0	2
7	0	0	0	-6	0	-1	1	0	0
3	0	-2	3	4	1	1	0	1	3

B	0	0	0	6	2	2	0	1	0
1	1	2	0	4	1	1	0	0	2
7	0	0	0	-6	0	-1	1	0	0
3	0	-2/3	1	4/3	1/3	1/3	0	1/3	1

- remove x_7 from basis. '-6' pivot point

B	0	0	0	0	2	1	1	1	0
1	1	2	0	0	1	1/3	-2/3	0	2
4	0	0	0	1	0	1/6	-1/6	0	0
3	0	-2/3	1	0	1/3	1/9	2/9	1/3	1

Calculate : $\bar{c}^T = c^T - c_B^T A_B^{-1} A = (1 \;\; 3 \;\; 0 \;\; 1 \;\; -2) - (1 \;\; 1 \;\; 0) \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -2/3 & 1 & 0 & 1/3 \end{bmatrix}$

$$= (0 \;\; 1 \;\; 0 \;\; 0 \;\; -3)$$

B	0	1	0	0	$\boxed{-3}$	-2
1	1	2	0	0	$\underline{1}$	2
4	0	0	0	1	0	0
3	0	-2/3	1	0	1/3	1

B	3	7	0	0	0	4
5	1	2	0	0	1	2
4	0	0	0	1	0	0
3	-1/3	-2	1	0	0	1/3

- All reduced costs are nonnegative, optimal solution reached
- Optimal value = -4, optimal solution = $[0, 0, 1/3, 0, 2]^T$