The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 9

Due: 23:59, Dec 12, 2024

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must submit your assignment in Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student **must not copy** homework solutions from another student or from any other source.

Problem 1: True of False (20 pts).
For each of the following statements, state whether it is true or false. Please explain your answers (if not true, please show a counterexample).

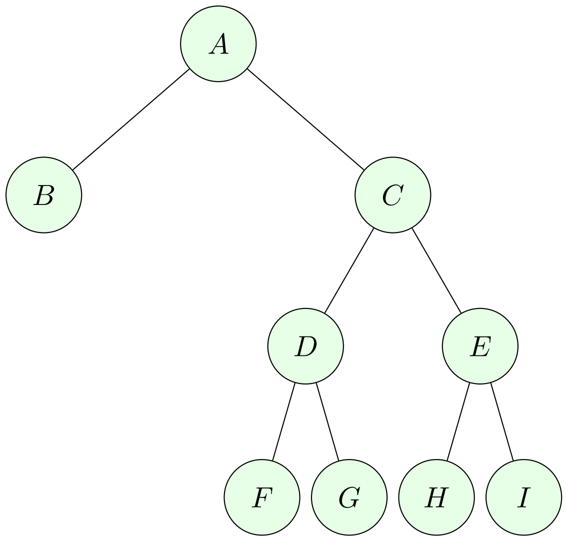
(a) If the linear programming relaxation of an integer programming problem is infeasible, then the integer program itself is also infeasible.

(b) If the linear programming relaxation of an integer programming problem is unbounded, then the integer program itself is also unbounded.

(c) If x^* is an optimal solution of the linear programming relaxation of an integer programming problem, and x^* satisfies the integer restrictions, then x^* is an optimal solution to the integer programming problem.

(d) If the optimal solution to the linear programming relaxation of integer programming problem has exactly one variable taking a non-integer value, then the branch and bound algorithm will terminate after branching once and exploring the resulting two nodes.

Problem 2: Branch-and-Bound Tree (20 pts).
Suppose, the branch-and-bound algorithm is being executed for an integer programming problem with **maximization** objective, and the current state of the search corresponds to the following tree:



Node	LP opt. obj.	x^{LP} integer?
A	70.5	no
B	64.5	no
C	67	no
D	65.5	no
E	66	no
F	57	yes
G	60.5	no
H	Infeasible	n/a
I	61.5	yes

The letters in the nodes correspond to the order in which the LP relaxations have been solved (in alphabetical order). The table above right lists for each node the optimal objective function value for the corresponding node LP relaxation, and indicates whether the resulting solution satisfies all integer restrictions of the original integer program. Assume the original integer program has five decision variables, $x_1, \ldots, x_5$, and requires all the decision variables to take integer values. Let z^* denote the optimal value of the integer program being solved.

(a) Let z^* be the optimal objective function value for the integer program. What is the **largest lower bound** $\underline{z}$ that you can determine given the current state of the search (i.e., the largest number $\underline{z}$ so that $\underline{z} \leq z^*$)?

(b) Let z^* be the optimal objective function value for the integer program. What is the **smallest upper bound** $\overline{z}$ that you can determine given the current state of the search (i.e., the smallest number $\overline{z}$ so that $\overline{z} \geq z^*$)?

(c) Identify the leaf nodes (among B, F, G, H, I) that can be fathomed (i.e., leaf nodes that do not require further exploration), and for each of these, give the reason why it can be fathomed (use the node labels to identify them).

(d) Suppose the optimal solution to the LP relaxation at node C is $(x_1, x_2, x_3, x_4, x_5) = (1, 0, 0, 0.75, 3)$. What additional constraints would have been added to create nodes D and E?

Problem 3: Branch-and-Bound Method (30 pts).
Use the branch-and-bound method to solve the following integer program:

$$
\begin{array}{lll}
\text{maximize} & 17x + 12y & \\
\text{subject to} & 10x + 7y & \leq 40 \\
 & x + y & \leq 5 \\
 & x, y & \geq 0 \\
 & x, y \in \mathbb{Z}. &
\end{array}
$$

Form the branch-and-bound tree and indicate the solution associated with each node (similar to the procedures introduced in the lecture). Please include your calculations in your answer.

Problem 4: Manufacturing Company (30 pts).
A manufacturing company plans to build new factories (variables x_1 and x_2) and warehouses (variables x_3 and x_4) in Shenzhen and/or Beijing. The company wants to solve the following binary integer program to determine the location and number of the potential factories and warehouses:

$$
\begin{array}{lll}
\text{maximize} & 9x_1 + 5x_2 + 6x_3 + 4x_4 & \\
\text{subject to} & 6x_1 + 3x_2 + 5x_3 + 2x_4 & \leq 10 \\
& x_3 + x_4 & \leq 1 \\
& x_3 - x_1 & \leq 0 \\
& x_4 - x_2 & \leq 0 \\
& x_1, x_2, x_3, x_4 & \in \{0, 1\}.
\end{array}
$$

(a) Discuss and interpret the meaning of the constraints "$x_3 + x_4 \leq 1$", "$x_3 - x_1 \leq 0$", and "$x_4 - x_2 \leq 0$".

(b) Use the branch-and-bound method to solve the integer problem. You are allowed to use an LP solver to solve each of the relaxed linear programs. Please specify the branch-and-bound tree and what you did at each node.

Problem 1: True of False (20 pts).

For each of the following statements, state whether it is true or false. Please explain your answers (if not true, please show a counterexample).

(a) If the linear programming relaxation of an integer programming problem is infeasible, then the integer program itself is also infeasible. T

(b) If the linear programming relaxation of an integer programming problem is unbounded, then the integer program itself is also unbounded. T

(c) If x^* is an optimal solution of the linear programming relaxation of an integer programming problem, and x^* satisfies the integer restrictions, then x^* is an optimal solution to the integer programming problem. T

(d) If the optimal solution to the linear programming relaxation of integer programming problem has exactly one variable taking a non-integer value, then the branch and bound algorithm will terminate after branching once and exploring the resulting two nodes. F

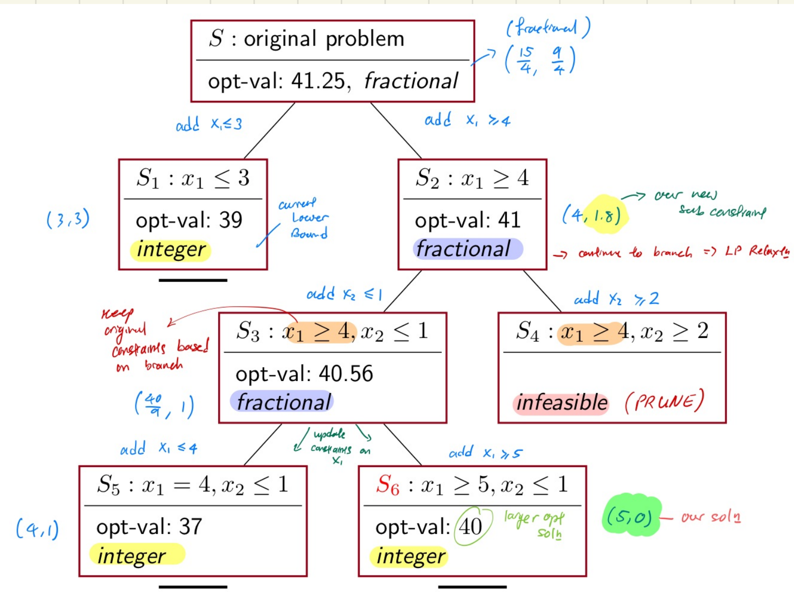
a) IP is a more strict version of IP. If LP relaxation infeasible => Original IP is also infeasible (T)

b) Similar to (a). But since IP keeps all existing constraints of LP, (T)
If LP unbounded => IP unbounded.

c) If x^* is optimal for LP relaxation and satisfies IP restrictions (integer value)
then it is also optimal for IP since it achieves the best solution w/o violating any constraints
(T)

d) Branch and bound might take more than 1 branch since the resulting 2 nodes (F)
might also produce optimal solutions each taking a non-integer value.

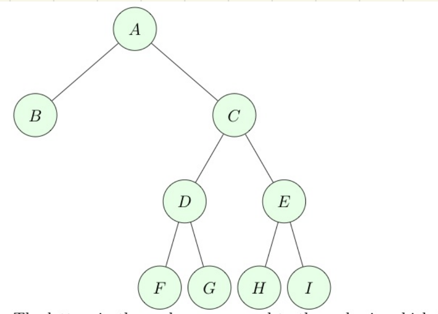
Example from Lecture:



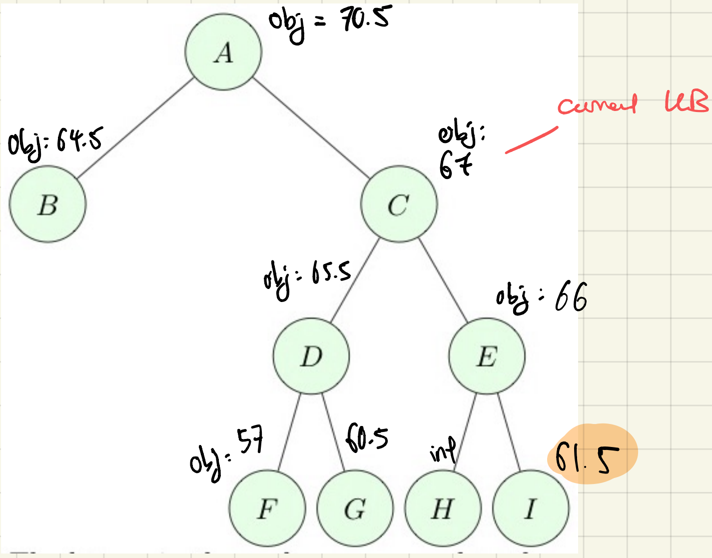
From the 2nd branch (S_2)
we have a cases where the optimal solution has exactly 1 non-integer value.
i.e. (4, 1.8)

=> The 1st branch after (S_3, S_4) still results in cases with nodes with non-integer values (S_3) which requires another branch.

Problem 2: Branch-and-Bound Tree (20 pts).
Suppose, the branch-and-bound algorithm is being executed for an integer programming problem with **maximization** objective, and the current state of the search corresponds to the following tree:



Node	LP opt. obj.	x^{LP} integer?
A	70.5	no
B	64.5	no
C	67	no
D	65.5	no
E	66	no
F	57	yes
G	60.5	no
H	Infeasible	n/a
I	61.5	yes



The letters in the nodes correspond to the order in which the LP relaxations have been solved (in alphabetical order). The table above right lists for each node the optimal objective function value for the corresponding node LP relaxation, and indicates whether the resulting solution satisfies all integer restrictions of the original integer program. Assume the original integer program has five decision variables, $x_1, \ldots, x_5$, and requires all the decision variables to take integer values. Let z^* denote the optimal value of the integer program being solved.

(a) Let z^* be the optimal objective function value for the integer program. What is the **largest lower bound** $\underline{z}$ that you can determine given the current state of the search (i.e., the largest number $\underline{z}$ so that $\underline{z} \leq z^*$)?

(b) Let z^* be the optimal objective function value for the integer program. What is the **smallest upper bound** $\overline{z}$ that you can determine given the current state of the search (i.e., the smallest number $\overline{z}$ so that $\overline{z} \geq z^*$)?

(c) Identify the leaf nodes (among B, F, G, H, I) that can be fathomed (i.e., leaf nodes that do not require further exploration), and for each of these, give the reason why it can be fathomed (use the node labels to identify them).

(d) Suppose the optimal solution to the LP relaxation at node C is $(x_1, x_2, x_3, x_4, x_5) = (1, 0, 0, 0.75, 3)$. What additional constraints would have been added to create nodes D and E?

a) Largest Lower bound = 61.5

b) Smallest upper bound = 64.5

c)

Leaf node:	Can be fathomed:	Reason:
B	X no	B cannot be fathomed since 61.5 < 64.5 -> have not explored B
F	✓ Yes	The objective value of B = 64.5, this is the current Lower bound. Since obj value (F) = 57 which is less than 64.5, no better solution can come from this branch => F is fathomed.
G	✓ Yes	Since no solution from G can produce an obj value > 60.5, no better solution will come from node G that produces an obj value > 61.5 (node I)
H	✓ Yes	Since H is infeasible => no need to consider this sub problem further.
I	✓ Yes	This is the optimal solution

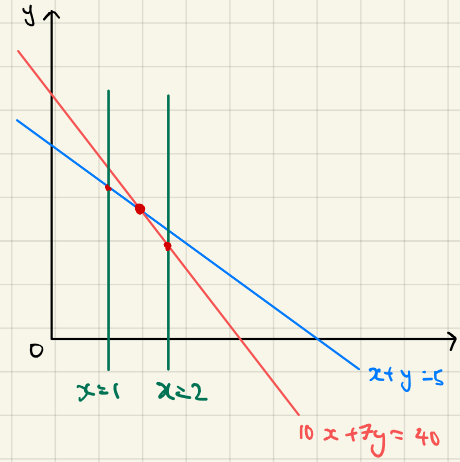
d) $X_4 \leq 0$ for node D

$X_4 \geq 1$ for node E

Problem 3: Branch-and-Bound Method (30 pts).
Use the branch-and-bound method to solve the following integer program:

$$\begin{array}{lll} \text{maximize} & 17x + 12y & \\ \text{subject to} & 10x + 7y & \le 40 \\ & x + y & \le 5 \\ & x, y & \ge 0 \\ & x, y \in \mathbb{Z}. & \end{array}$$

Form the branch-and-bound tree and indicate the solution associated with each node (similar to the procedures introduced in the lecture). Please include your calculations in your answer.



S-original, Relaxed LP:

$y = 5 - x \} \to 10x + 7y = 40$

$\Rightarrow 10x + 7(5 - x) = 40$

$10x + 35 - 7x = 40$

$3x = 5$

$x = 5/3$

$\therefore y = 10/3$, branch for x

$\therefore$ Opt value $= 17(5/3) + 10(10/3) = 185/3$

S_1: $x \le 1$, opt solⁿ = $(1, 4)$, opt val: 65

Current L-bound = 65, stop here, expand S_2

S_2: $x \ge 2$, $x = 2 \} \to 10x + 7y = 40$

$\Rightarrow 7y = 20 \iff y = 20/7$

opt solⁿ: $(2, 20/7)$, opt val: $478/7$

branch for y

S_3: $y \le 2$, graphically, opt solⁿ: $(2.6, 2)$

and opt val: 68.2 $\Rightarrow$ branch for x

S_4: $y \ge 3$, infeasible $\Rightarrow$ prune

S_5: $x \le 2$ with $(S_3) \Rightarrow x = 2$, graphically:

opt solⁿ: $(2, 2)$, opt val: $58 <$ L-bound

$\Rightarrow$ no branching, expand S_6

S_6: $x \ge 3$, $x = 3 \} \to 10x + 7y = 40$

$\Rightarrow 7y = 10 \Rightarrow y = 10/7$

opt solⁿ: $(3, 10/7)$, opt val: 68.143

$\Rightarrow$ branch for y

S_7: $y \le 1$, $y = 1 \} \to 10x + 7y = 40$

$\Rightarrow 10x = 33 \Rightarrow x = 3.3$, opt solⁿ: $(3.3, 1)$

opt val: 68.1

$\Rightarrow$ branch for x

S_8: $y \ge 2$, infeasible $\Rightarrow$ prune

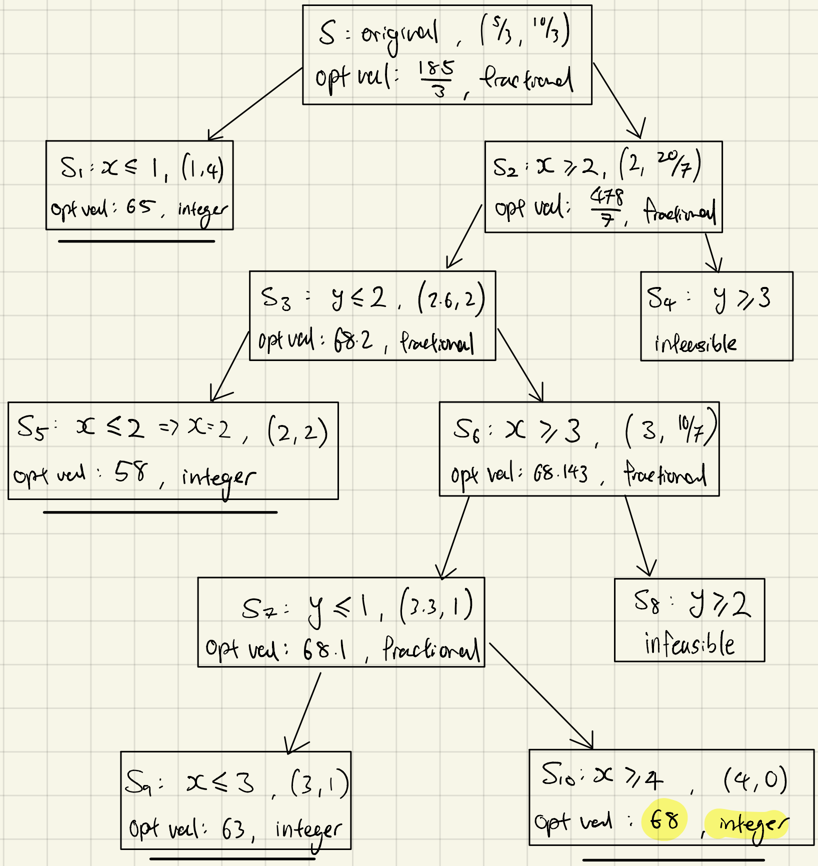
S_9: $x \le 3$ with $(S_6) \Rightarrow x = 3$

Graphically: opt solⁿ: $(3, 1)$

opt val: $63 <$ L-bound (stop, no branching)

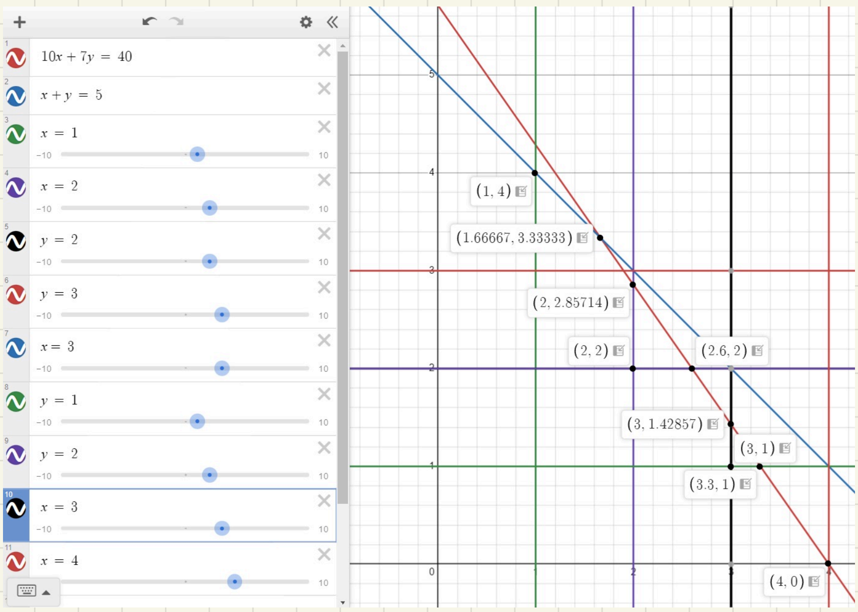
S_{10}: $x \ge 4$, graphically, opt solⁿ: $(4, 0)$

opt val: 68 $\Rightarrow$ Final optimal solⁿ ✓✓



Optimal solution: $x = 4$, $y = 0$

optimal value: 68



Problem 4: Manufacturing Company (30 pts).
A manufacturing company plans to build new factories (variables x_1 and x_2) and warehouses (variables x_3 and x_4) in Shenzhen and/or Beijing. The company wants to solve the following binary integer program to determine the location and number of the potential factories and warehouses:

$$\begin{array}{lll} \text{maximize} & 9x_1 + 5x_2 + 6x_3 + 4x_4 & \\ \text{subject to} & 6x_1 + 3x_2 + 5x_3 + 2x_4 & \leq 10 \\ & x_3 + x_4 & \leq 1 \\ & x_3 - x_1 & \leq 0 \\ & x_4 - x_2 & \leq 0 \\ & x_1, x_2, x_3, x_4 & \in \{0, 1\}. \end{array}$$

(a) Discuss and interpret the meaning of the constraints "$x_3 + x_4 \leq 1$", "$x_3 - x_1 \leq 0$", and "$x_4 - x_2 \leq 0$".

(b) Use the branch-and-bound method to solve the integer problem. You are allowed to use an LP solver to solve each of the relaxed linear programs. Please specify the branch-and-bound tree and what you did at each node.

a) "$x_3 + x_4 \leq 1$"
=> implies that the company can only have one warehouse in shenzhen or beijing, but not both.

"$x_3 - x_1 \leq 0$" i.e. $x_3 = 1 \Rightarrow x_1 = 1$
=> if the warehouse is in Shenzhen, the factory must also be in Shenzhen

"$x_4 - x_2 \leq 0$" i.e. $x_4 = 1 \Rightarrow x_2 = 1$
=> if the warehouse is in Beijing, the factory must also be in Beijing.

b) Using an LP solver: relax $0 \leq x_1, x_2, x_3, x_4 \leq 1$

S - Original

```
# Code for Solving the Original Problem
```

```
**** S - Original Problem ****
Status: Optimal
x1 = 0.83333333
x2 = 1.0
x3 = 0.0
x4 = 1.0
Objective value: 16.49999997
```

S - Original:

- Fractional
- proceed to branch on x1
- add `x1 = 0` (S1)
- add `x1 = 1` (S2)

S - 1 : Add `x1 = 0`

```
# Code for Solving the Problem with Constraint S1 ...
```

```
**** S - 1 : x1 == 0 ****
Status: Optimal
x1 = 0.0
x2 = 1.0
x3 = 0.0
x4 = 1.0
Objective value: 9.0
```

S - 1 `x1 = 0`:

- Integer
- Current Lbound: 9
- no need for further branching

S2 - Add `x1 = 1`

```
# Code for Solving the Problem with Constraint S2 ...
```

```
**** S - 2 : x1 == 1 ****
Status: Optimal
x1 = 1.0
x2 = 0.8
x3 = 0.0
x4 = 0.8
Objective value: 16.2
```

S - 2 `x1 = 1`:

- Fractional
- Current Ubound (LP - relaxed): 16.2
- Branch on x2
- Add `x2 = 0` (S3)
- Add `x2 = 1` (S4)

S3 - Add `x2 = 0`

```
# Code for Solving the Problem with Constraint S3 ...
```

```
**** S - 3 : x2 == 0 ****
Status: Optimal
x1 = 1.0
x2 = 0.0
x3 = 0.8
x4 = 0.0
Objective value: 13.8
```

S - 3 `x2 = 0`:

- Fractional
- Branch on x3
- Add `x3 = 0` (S5)
- Add `x3 = 1` (S6)

⋮

S4 - Add `x2 = 1`

```
# Code for Solving the Problem with Constraint S4 ...
```

```
**** S - 4 : x2 == 1 ****
Status: Optimal
x1 = 1.0
x2 = 1.0
x3 = 0.0
x4 = 0.5
Objective value: 16.0
```

S - 4 `x2 = 1`:

- Fractional
- Branch on x4
- Add `x4 = 0` (S7)
- Add `x4 = 1` (S8)

⋮

Continuing from node S3

S3 - Add `x2 = 0`

```
# Code for Solving the Problem with Constraint S3 ...
```

```
**** S - 3 : x2 == 0 ****
Status: Optimal
x1 = 1.0
x2 = 0.0
x3 = 0.8
x4 = 0.0
Objective value: 13.8
```

S - 3 `x2 = 0`:

- Fractional
- Branch on x3
- Add `x3 = 0` (S5)
- Add `x3 = 1` (S6)

S5 - Add `x3 = 0`

```
# Code for Solving the Problem with Constraint S5 ...
```

```
**** S - 5 : x3 == 0 ****
Status: Optimal
x1 = 1.0
x2 = 0.0
x3 = 0.0
x4 = 0.0
Objective value: 9.0
```

S - 5 `x3 = 0`:

- Integer
- No need for further branching

S6 - Add `x3 = 1`

```
# Code for Solving the Problem with Constraint S6 ...
```

```
**** S - 6 : x3 == 1 ****
Status: Infeasible
x1 = 1.0
x2 = 0.0
x3 = 1.0
x4 = -0.5
Objective value: 13.0
```

S - 6 `x3 = 1`:

- Infeasible
- Prune this branch

Continuing from node S4

S4 - Add `x2 = 1`

`# Code for Solving the Problem with Constraint S4 ···`

```
**** S - 4 : x2 == 1 ****
Status: Optimal
x1 = 1.0
x2 = 1.0
x3 = 0.0
x4 = 0.5
Objective value: 16.0
```

S - 4 `x2 = 1`:

- Fractional
- Branch on x4
- Add `x4 = 0` (S7)
- Add `x4 = 1` (S8)

S7 - Add `x4 = 0`

`# Code for Solving the Problem with Constraint S7 ···`

```
**** S - 7 : x4 == 0 ****
Status: Optimal
x1 = 1.0
x2 = 1.0
x3 = 0.2
x4 = 0.0
Objective value: 15.2
```

S - 7 `x4 = 0`:

- Fractional
- Branch on x3
- Add x3 = 0 (S9)
- Add x3 = 1 (S10)

S8 - Add `x4 = 1`

`# Code for Solving the Problem with Constraint S8 ···`

```
**** S - 8 : x4 == 1 ****
Status: Infeasible
x1 = 1.0
x2 = 0.66666667
x3 = 0.0
x4 = 1.0
Objective value: 16.33333335
```

S - 8 `x4 = 1`:

- Infeasible
- Prune this branch

S9 - Add `x3 = 0`

`# Code for Solving the Problem with Constraint S9 ···`

```
**** S - 9 : x3 == 0 ****
Status: Optimal
x1 = 1.0
x2 = 1.0
x3 = 0.0
x4 = 0.0
Objective value: 14.0
```

S - 9 `x3 = 0`:

- Integer
- No further branching required

S10 - Add `x3 = 1`

`# Code for Solving the Problem with Constraint S10 ···`

```
**** S - 10 : x3 == 1 ****
Status: Infeasible
x1 = 1.0
x2 = -0.33333333
x3 = 1.0
x4 = 0.0
Objective value: 13.33333335
```

S - 10 `x3 = 1`:

- Infeasible
- Prune this branch

Optimal solution: (1, 1, 0, 0)   Optimal value: 14.0