The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 4

Due: 11:59pm, Oct. 25 (Friday), 2024

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- Please submit your assignment on Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student must not copy homework solutions from another student or from any other source.

Problem 1 (20pts). Consider an LP in its standard form $\min \boldsymbol{c}^\top \boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ and its dual $\max \boldsymbol{b}^\top \boldsymbol{y}$ s.t. $\boldsymbol{A}^\top \boldsymbol{y} \leq \boldsymbol{c}$. Suppose that the matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with $m < n$, and that its rows are linearly independent. For each of the following statements, state whether it is true or false. Please explain your answers (if not true, please show a counterexample).

(a) If the primal LP has a unique optimal solution, then its dual has a unique optimal solution. F

(b) Suppose $\boldsymbol{y}^*$ is a dual basic feasible solution, i.e., $\boldsymbol{A}^\top \boldsymbol{y}^* \leq \boldsymbol{c}$ and there exists a basis B such that $(\boldsymbol{y}^*)^\top = \boldsymbol{c}_B^\top \boldsymbol{A}_B^{-1}$. If $\boldsymbol{A}_B^{-1}\boldsymbol{b} \geq \boldsymbol{0}$, then $\boldsymbol{y}^*$ is optimal to the dual problem. T

(c) Let $(\boldsymbol{x}, \boldsymbol{y})$ be a pair of solutions feasible to the primal and the dual, respectively. Let v^* denote the optimal objective value of the primal problem. Then $\boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{x} \leq v^*$. WDT T

(d) If the polyhedron $P = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ is nonempty and bounded, than the polyhedron $Q = \{\boldsymbol{y} : \boldsymbol{A}^\top \boldsymbol{y} \leq \boldsymbol{c}\}$ is nonempty and bounded. T

Problem 2 (20pts). Consider the following linear program:

$$
\begin{array}{rl}
\text{maximize} & x_1 + 6x_2 + 6x_3 \\
\text{subject to} & x_1 + 2x_2 + 3x_3 \leq 1 \\
& x_1 + 3x_2 + 2x_3 \leq 1 \\
& x_1, x_2, x_3 \geq 0
\end{array}
$$

1. What is the corresponding dual problem?
2. Solve the dual problem graphically.
3. Use complementarity slackness to solve the primal problem.

Problem 3 (20pts). Suppose $\boldsymbol{M}$ is a square matrix such that $\boldsymbol{M} = -\boldsymbol{M}^\top$, for example,

$$\boldsymbol{M} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -4 \\ -2 & 4 & 0 \end{pmatrix}.$$

Consider the following optimization problem:

$$\begin{array}{rc} \text{minimize}_{\boldsymbol{x}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{M}\boldsymbol{x} \geq -\boldsymbol{c} \\ & \boldsymbol{x} \geq 0. \end{array}$$

(a) Show that the dual problem of it is equivalent to the primal problem.

(b) Argue that the problem has an optimal solution if and only if there is a feasible solution.

Problem 4 (20pts). We consider the general linear optimization problem:

$$\min_{\boldsymbol{x}} \ \boldsymbol{c}^\top \boldsymbol{x} \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \quad \boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}, \tag{1}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{C} \in \mathbb{R}^{p \times n}$, $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\boldsymbol{d} \in \mathbb{R}^p$ are given. Its dual problem is a maximization problem. Write it as an equivalent minimization problem, and show that the dual of the equivalent minimization problem is equivalent to problem (1) (i.e., the dual of the dual of (1) is equivalent to (1)).

Problem 5 (20pts). Let $\boldsymbol{x}, \boldsymbol{y}$ be two vectors. Then $\boldsymbol{x} \gneqq \boldsymbol{y}$ means $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$, and $\boldsymbol{x} > \boldsymbol{y}$ means $x_i > y_i$, $i = 1, \ldots, n$. Use the strong duality theorem to prove Gordan's theorem: Given matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, either $\boldsymbol{A}\boldsymbol{x} > \boldsymbol{0}$ has a solution, or $\boldsymbol{A}^\top \boldsymbol{y} = \boldsymbol{0}, \boldsymbol{y} \gneqq \boldsymbol{0}$ has a solution. (Hint: $\boldsymbol{A}\boldsymbol{x} > \boldsymbol{0}$ has a solution if and only if $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{1}$ has a solution. $\boldsymbol{A}^\top \boldsymbol{y} = \boldsymbol{0}, \boldsymbol{y} \gneqq \boldsymbol{0}$ has a solution if and only if $\boldsymbol{1}^\top \boldsymbol{y} > 0, \boldsymbol{A}^\top \boldsymbol{y} = \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}$ has a solution. Here $\boldsymbol{1}$ denotes the vector whose entries are all 1.)

Problem 1:

a) False

counter example: (P) $\min\ 2x_1 + 2x_2$

st $x_1 + x_2 \geq 2$

$x_1, x_2 \geq 0$

(D) $\max\ 2y_1$

st $y_1 \leq 2$

$y_1 \leq 2$

$y_1 \geq 0$

→ geometrically, the intersection of the constraint lines denote the feasible region for (P) with an objective value of 4.

→ $x_1 = 1,\ x_2 = 1$

⇒ Any value between 0 and 2 is feasible and optimal since they satisfy the constraint.

b) True

y^* is dual basic feasible and $A_B^{-1}b \geq 0$, by Complementary Slackness:

(1) $x_i \cdot (c_i - A_i^T y) = 0 \quad , \forall i$

(2) $y_j \cdot (A_j^T y - c_j) = 0 \quad , \forall j$

And since $A^T y^* = c$, and from (1), we obtain

$x_i \cdot (c_i - c_i) = 0 \quad , \forall i$

⇒ Confirms y^* is optimal to dual.

c) True.

By Weak Duality, since (x, y) is a feasible solution pair to (P) and (D).

⇒ $b^T y \leq c^T x$

Since the optimal obj val of primal is V^*

⇒ $b^T y \leq V^*$

And since $b^T y$ in dual can be expressed as $y^T A x$ in primal:

⇒ $y^T A x \leq V^*$

d) True.

- By Fundamental LP theorem, P nonempty & bounded ⇒ there exist a BFS And hence there is an optimal solution.

- By Strong Duality, the dual Polyhedron Q must also be nonempty & unbounded.

Problem 2:

primal:

$$\max \quad x_1 + 6x_2 + 6x_3$$

$$\text{st} \quad x_1 + 2x_2 + 3x_3 \le 1$$
$$x_1 + 3x_2 + 2x_3 \le 1$$
$$x_1, x_2, x_3 \ge 0$$

① dual:

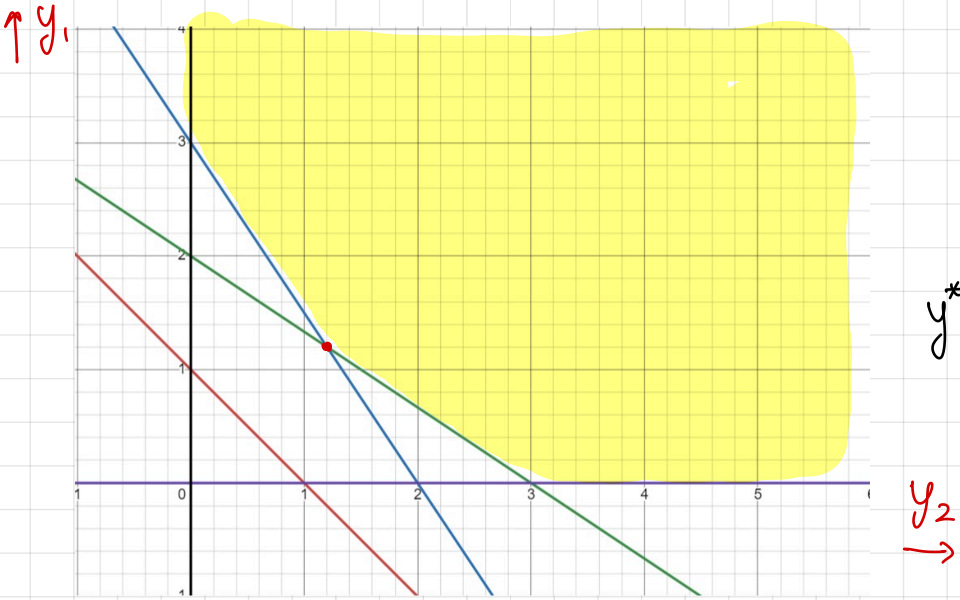
$$\min \quad y_1 + y_2$$

$$\text{st} \quad y_1 + y_2 \ge 1$$
$$2y_1 + 3y_2 \ge 6$$
$$3y_1 + 2y_2 \ge 6$$
$$y_1, y_2 \ge 0$$

$C = (1\ 6\ 6)^T$, $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $b = (1\ 1)^T$

$A^T = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 2 \end{pmatrix}$

② Rewrite dual constraints: $y_1 \to y$, $y_2 \to x$ $\Rightarrow$ ① $y = -x + 1$ ② $2y = -3x + 6$ ③ $3y = -2x + 6$



$y^* = (1.2 \quad 1.2)^T$

③

By Complementary slackness: $y_i(A_i^T x - b_i) = 0$,

$$\begin{cases} \text{if } y_i > 0 \Rightarrow A_i^T x = b_i \\ \text{if } A_i^T x < b_i \Rightarrow y_i = 0 \end{cases}$$

Since $y^* = (1.2 \quad 1.2)^T$, verify the complementary Conditions:

$x_1 \cdot (1 - y_1 - y_2) = 0 \rightarrow$ (not useful)

$$\left.\begin{aligned} x_2 \cdot (6 - 2y_1 - 3y_2) = 0 \\ x_3 \cdot (6 - 3y_1 - 2y_2) = 0 \end{aligned}\right\} \text{solving}$$

Since x_2, x_3 are active, the 2nd & 3rd dual constraints have to be active

since $y^* = (1.2, 1.2)$ is feasible for dual, by complementarity => optimal solution for dual.

Thus: $x^* = (0, 0.2, 0.2)$

and the optimal value is 2.4

Problem 3:

$$M = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -4 \\ -2 & 4 & 0 \end{bmatrix}, \quad M^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$$

a)

$$\min_x \quad c^T x$$
$$\text{st} \quad Mx \geq -c$$
$$x \geq 0$$

The dual:

$$\therefore \max_x \ (-c)^T y \quad \Rightarrow \quad -\max \ c^T y \quad \Rightarrow \quad \min \ c^T y$$
$$\text{s.t.} \ M^T y \leq c \qquad \text{s.t.} \ -M^T y \geq -c \qquad \text{s.t.} \ My \geq -c$$
$$y \geq 0 \qquad y \geq 0 \qquad y \geq 0$$

primal

(shown)

b)

Case ①:

if (P) is infeasible:

By Strong Duality $\Rightarrow$ (D) is unbounded

But since (D) = (P), (P) is unbounded also

$\therefore$ if (P) infeasible $\Rightarrow$ (P) is unbounded $\Rightarrow$ no optimal solution for (P)

Case ②:

if (P) is feasible:

(P) holds $\Rightarrow$ optimal value of (D) exists (strong duality)

Since (P) = (D) $\Rightarrow$ optimal value of (D) = optimal value of (P)

$\Rightarrow$ An optimal value exists.

Considering Both cases ① and ②, We see that an optimal solution only exist if and only if (P) is feasible. Since (P) = (D), We can generalise our statement:

Optimal solution to the problem exists iff there exist a feasible solution.

Problem 4:

$$(P) \quad \min_x \; c^T x \quad \text{st} \quad Ax \le b \,, \quad Cx = d \; \text{———————} \; (1)$$

$A \in \mathbb{R}^{m \times n}$ $\quad c \in \mathbb{R}^n$

$C \in \mathbb{R}^{p \times n}$ $\quad b \in \mathbb{R}^m$

$\quad d \in \mathbb{R}^p$

In standard form: $(P) \; \min_x \; c^T x$

$$\text{st} \quad Ax + s = b, \quad Cx = d, \quad x, s \geq 0$$

$\Rightarrow$

$$(D) \quad \max_y \; b^T y + d^T z \quad \text{st} \quad A^T y + C^T z \leq c, \quad y \geq 0, \quad z \text{ free}$$

Rewrite (D) as min:

$$\min_x \; -(b^T y + d^T z) \quad \text{st} \quad A^T y + C^T z = c, \quad y \geq 0,\; z \text{ free}$$

$\Rightarrow$

(DmD) - Dual of min Dual

We consider c as $c = c_1 + c_2$
(to manage $b^T y$ and $d^T z$ separately)

$$\max_{x_1} \; -c_1^T x_{c_1} \quad \text{st} \quad A x_{c_1} \le b$$

and

$$\max_{x_2} \; -c_2^T x_{c_2} \quad \text{st} \quad C x_{c_2} = d$$

Combining the constituent max problems:

$$\max_{x_1, x_2} \; -[c_1^T x_{c_1} + c_2^T x_{c_2}] \quad \text{st} \quad Ax \le b, \quad Cx = d$$

$\Rightarrow$

$$\min_x \; c^T x \quad \text{st} \quad Ax \le b, \quad Cx = d$$

Problem 5

We rename the Qn and Gordan's Theorem:

a) $Ax > 0$
b) $A^T y = 0$, $y \geq\neq 0$

Show that either one of the 2 systems (a), (b) has a solution.

→ We observe that we can deal with strict inequalities:
i.e. $Ax \geq 1$ has no solution $\rightleftharpoons$ $Ax > 0$ has no solution

Constructing the following primal dual:

(P) $\min\ 0^T x$
$\text{st}\ Ax \geq 1$

(D) $\max\ 1^T y$
$\text{st}\ A^T y = 0$
$y \geq 0$

if (P) is feasible:
(a) holds true, and by strong duality, optimal value of (D) $= 0$,
$\Rightarrow y = 0$
∴ (a) True, (b) not True

if (P) is not feasible: (D) unbounded $\Rightarrow$ (b) holds True and (a) not True