Assignment 2 - Linear Models

Q2.6

```r
# set your path
data<-read.table(file = "hooker.txt",header = TRUE)
head(data)
```

```
##      BT     AP
## 1 210.8 29.211
## 2 210.2 28.559
## 3 208.4 27.972
## 4 202.5 24.697
## 5 200.6 23.726
## 6 200.1 23.369
```

```r
# Initialize variables from HW1 data
TEMP<-data$BT
AP<-data$AP

x<-100*log(AP)

x_mean<-mean(x)
y_mean<-mean(TEMP)

Sxx<-sum((x-x_mean)^2)
Sxy<-sum((x-x_mean)*TEMP)

beta_1<-Sxy/Sxx
beta_0<- y_mean-beta_1*x_mean
```

(d)(i)

```r
# 95% confidence interval for beta_1
n<-length(x)
alpha<-0.05
t<-qt(1-alpha/2,df=n-2)
SE<-sqrt(sum((TEMP-beta_0-beta_1*x)^2)/(n-2))/sqrt(Sxx)
CI<-c(beta_1-t*SE,beta_1+t*SE)
cat("95% confidence interval for beta_1 is: ",CI)
```

```
## 95% confidence interval for beta_1 is:  0.4699716 0.4863969
```

(d)(ii)

```r
# Calculate 95% confidence interval for the average temperature when AP = 25
x_25<-100*log(25)
y_25<-beta_0+beta_1*x_25

SE<-sqrt(sum((TEMP-beta_0-beta_1*x)^2)/(n-2))*sqrt(1/n+(x_25-x_mean)^2/Sxx)
```

```r
CI<-c(y_25-t*SE,y_25+t*SE)
cat("95% confidence interval for the average temperature when AP = 25 is:
",CI)
```

```
## 95% confidence interval for the average temperature when AP = 25 is:
202.9448 203.4351
```

```r
# Check the 95% confidence interval for the average temperature when AP = 25
using predict()
fit<-lm(TEMP~x)
predict(fit,newdata=data.frame(x=100*log(25)),interval="confidence",level=0.9
5)
```

```
##      fit      lwr      upr
## 1 203.19 202.9448 203.4351
```

Q2.8

```r
# Initialize data from question
company <- c("General Motors", "Ford/Volvo", "Renault/Nissan", "Volkswagen",
"DaimlerChrysler", "Toyota", "Fiat", "Honda", "PSA", "BMW")
cars_sold <- c(8149, 7316, 4778, 4580, 4506, 4454, 2535, 2291, 2278, 1187)
revenue <- c(1996, 2118, 1174, 943, 1813, 1175, 628, 605, 465, 447)

# Create data frame
df <- data.frame(company, cars_sold, revenue)

# Fit linear model y = revenue, x = cars sold (sales)
fit <- lm(revenue ~ cars_sold, data = df)

# Print summary
summary(fit)
```

```
##
## Call:
## lm(formula = revenue ~ cars_sold, data = df)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -291.21 -151.73  -48.85   71.08  598.21
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  31.9113   185.2190   0.172 0.867488
## cars_sold     0.2625     0.0393   6.680 0.000156 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 264 on 8 degrees of freedom
## Multiple R-squared:  0.848,  Adjusted R-squared:  0.829
## F-statistic: 44.62 on 1 and 8 DF,  p-value: 0.0001559
```

(a)

Hypothesis testing for the importance of cars sold in predicting revenue. - Null hypothesis: The slope of the linear model is 0. - Alternative hypothesis: The slope of the linear model is not 0.

Since the p-value = `0.000156` < `0.05`, we reject the null hypothesis that the slope is 0. There is a significant linear relationship between revenue and cars sold.

(b)

```
# 95% confidence interval for the regression coefficient of the number of
cars sold
n <- length(cars_sold)
alpha <- 0.05
t <- qt(1-alpha/2, df = n-2)
beta_0 <- coef(fit)[1]
beta_1 <- coef(fit)[2]
x_mean <- mean(cars_sold)
y_mean <- mean(revenue)
Sxx <- sum((cars_sold - x_mean)^2)
SE <- sqrt(sum((revenue - beta_0 - beta_1 * cars_sold)^2) / (n-2)) /
sqrt(Sxx)
CI <- c(beta_1 - t * SE, beta_1 + t * SE)
cat("95% confidence interval for beta_1 is: ", CI)

## 95% confidence interval for beta_1 is:  0.1718915 0.3531304

# Checking the 95% confidence interval using confint()
confint(fit, level = 0.95)

##                    2.5 %      97.5 %
## (Intercept) -395.2044054 459.0271079
## cars_sold      0.1718915   0.3531304
```

(c)

```
# 90% confidence interval for the regression coefficient of the numbers of
cars sold
alpha <- 0.1
t <- qt(1-alpha/2, df = n-2)
SE <- sqrt(sum((revenue - beta_0 - beta_1 * cars_sold)^2) / (n-2)) /
sqrt(Sxx)
CI <- c(beta_1 - t * SE, beta_1 + t * SE)
cat("90% confidence interval for beta_1 is: ", CI)

## 90% confidence interval for beta_1 is:  0.189436 0.335586

# Check again using confint()
confint(fit, level = 0.90)
```

```
##                       5 %      95 %
## (Intercept) -312.512258 376.334960
## cars_sold      0.189436   0.335586
```

(d)

```r
# Calculate the coefficient of determination by taking model sum of squares
divided by the total sum of squares
SST <- sum((revenue - y_mean)^2)
SSReg <- beta_1^2 * Sxx

# Coefficient of Determination
R2 <- SSReg / SST
cat("The coefficient of determination is: ", R2)

## The coefficient of determination is:  0.8479792

# get the coefficent of determination using summary()
R2 <- summary(fit)$r.squared
cat("The coefficient of determination is: ", R2)

## The coefficient of determination is:  0.8479792
```

(e)

```r
# Calculate standard deviation after factoring sales of cars
y_hat <- beta_0 + beta_1 * cars_sold
sigma_no_x <- sqrt(sum((revenue - y_hat)^2) / (n-2))
cat("The standard deviation is when factoring sales of cars: ", sigma_no_x,
"\n")

## The standard deviation is when factoring sales of cars:  263.9908

# Calculate standard deviation without factoring sales of cars
y_hat <- mean(revenue)
sigma_x <- sqrt(sum((revenue - y_hat)^2) / (n-1))
cat("The standard deviation is without factoring sales of cars: ", sigma_x,
"\n")

## The standard deviation is without factoring sales of cars:  638.3531
```

(f)

```r
# Calculate estimates for BMW
cars_sold_BMW <- 1187
revenue_BMW <- beta_0 + beta_1 * cars_sold_BMW
cat("The estimated revenue for BMW is: ", revenue_BMW)

## The estimated revenue for BMW is:  343.5119
```

2.12. Occasionally, a model is considered in which the intercept is known to be zero a priori. Such a model is given by

$$y_i = \beta_1 x_i + \epsilon_i,\ i = 1, 2, \ldots, n$$

where the errors ϵ_i follow the usual assumptions.

a. Obtain the LSEs $(\hat{\beta}_1, s^2)$ of (β_1, σ^2).

b. Define $e_i = y_i - \hat{\beta}_1 x_i$. Is it still true that $\sum_{i=1}^n e_i = 0$? Why or why not?

c. Show that $V(\hat{\beta}_1) = \sigma^2 / \sum_{i=1}^n x_i^2$.

$$y_i = \beta_1 x_i + \varepsilon_i \quad , \quad i = 1, 2, \ldots, n$$

a) $\varepsilon_i = y_i - \beta_1 x_i$

$$LSE = \min \sum_{i=1}^n \varepsilon_i^2$$

$$= \min \sum_{i=1}^n (y_i - \beta_1 x_i)^2$$

$$\frac{d}{d\beta_1} LSE = 2 \cdot \sum_{i=1}^n (-x_i)(y_i - \beta_1 x_i)$$

To minimize :

$$2 \sum_{i=1}^n (y_i - \beta_1 x_i)(-x_i) = 0$$

$$\sum_{i=1}^n y_i x_i - \beta_1 x_i^2 = 0$$

$$\sum_{i=1}^n y_i x_i = \beta_1 \sum_{i=1}^n x_i^2$$

hence, $$\hat{\beta}_1 = \frac{\sum_{i=1}^n y_i x_i}{\sum_{i=1}^n x_i^2}$$

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n \varepsilon_i^2$$

$$= \frac{1}{n-1} \sum_{i=1}^n (y_i - \beta_1 x_i)^2$$

b) let $\sum_{i=1}^{n} \varepsilon_i = 0$,

$$\Rightarrow \sum_{i=1}^{n} (y_i - \hat{\beta}_1 x_i) = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} y_i = \hat{\beta}_1 \sum_{i=1}^{n} x_i \quad \text{(needs to hold to be true)}$$

RHS: $\hat{\beta}_1 \sum_{i=1}^{n} x_i = \frac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2} \cdot \sum_{i=1}^{n} x_i$

$$\neq \sum_{i=1}^{n} y_i = \text{LHS}$$

$\therefore$ not true

c) $$Var(\hat{\beta}_1) = Var\left(\frac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2}\right)$$

$$= Var\left(\frac{\sum_{i=1}^{n} (\beta_1 x_i + \varepsilon_i)(x_i)}{\sum_{i=1}^{n} x_i^2}\right)$$

$$= Var\left(\frac{\beta_1 \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i \varepsilon_i}{\sum_{i=1}^{n} x_i^2}\right)$$

$$= Var\left(\beta_1 + \frac{\sum_{i=1}^{n} x_i \varepsilon_i}{\sum_{i=1}^{n} x_i^2}\right)$$

$$= Var\left(\frac{\sum_{i=1}^{n} x_i \varepsilon_i}{\sum_{i=1}^{n} x_i^2}\right)$$

$$= \frac{1}{\left(\sum_{i=1}^{n} x_i^2\right)^2} Var\left(\sum_{i=1}^{n} x_i \varepsilon_i\right)$$

$$= \frac{1}{\left(\sum_{i=1}^{n} x_i^2\right)^2} \sum_{i=1}^{n} Var(x_i \varepsilon_i)$$

$$= \frac{1}{\left(\sum_{i=1}^{n} x_i^2\right)^{\cancel{2}}} \cdot \cancel{\sum_{i=1}^{n}} \left(\cancel{x_i^2}\, \sigma^2\right) \quad , \text{ since } \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$$

$$= \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2} \qquad \text{(shown)}$$

Q2.14

(a)

Let ‘x’ denote s the quantities of calcium in carefully prepared solutions. Let ‘y’ denote the corresponding analytical results.

```r
# Initialize x and y variables
x <- c(4, 8, 12.5, 16, 20, 25, 31, 36, 40, 40)
y <- c(3.7, 7.8, 12.1, 15.6, 19.8, 24.5, 31.1, 35.5, 39.4, 39.5)

# Fit the linear model of y and x
x_mean <- mean(x)
y_mean <- mean(y)
Sxx <- sum((x - x_mean)^2)
Sxy <- sum((x - x_mean) * y)
beta_1 <- Sxy / Sxx
beta_0 <- y_mean - beta_1 * x_mean

# Calculate the number of observations and the t-value
n <- length(x)
alpha <- 0.05
t <- qt(1-alpha/2, df = n-2)

# Residuals and residual sum of squares
residuals <- y - (beta_0 + beta_1 * x)
RSS <- sum(residuals^2)

# Print the coefficients
cat("The estimated coefficients are: beta_0 = ", beta_0, ", beta_1 = ",
beta_1, "\n")

## The estimated coefficients are: beta_0 =  -0.2280899 , beta_1 =  0.9947566

# Fit a linear model using lm()
fit <- lm(y ~ x)

summary(fit)

##
## Call:
## lm(formula = y ~ x)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.16217 -0.10178 -0.07266  0.03979  0.49064
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.228090   0.137840  -1.655    0.137
```

```
## x            0.994757   0.005219 190.585 6.43e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.2067 on 8 degrees of freedom
## Multiple R-squared:  0.9998, Adjusted R-squared:  0.9998
## F-statistic: 3.632e+04 on 1 and 8 DF,  p-value: 6.429e-16
```

The assumptions made: 1. The data is normally distributed. 2. Each instance of x_i is independent of other instances, and the same goes for y_i. — #### (b)

```r
# Calculate standard error for beta_0
se_b0 <- sqrt(RSS / (n-2)) * sqrt(1/n + x_mean^2 / Sxx)

# Calculate 95% confidence interval for beta_0 (intercept)
CI_b0 <- c(beta_0 - t * se_b0, beta_0 + t * se_b0)
cat("95% confidence interval for beta_0 is: ", CI_b0, "\n")
```

```
## 95% confidence interval for beta_0 is:  -0.5459503 0.08977054
```

```r
# Check CI_b0 using confint()
confint(fit, level = 0.95)[1,]
```

```
##       2.5 %      97.5 %
## -0.54595031  0.08977054
```

(c)

```r
# Calculate standard error for beta_1
se_b1 <- sqrt(RSS / (n-2)) / sqrt(Sxx)

# 95% confidence interval for beta_1 (slope)
CI_b1 <- c(beta_1 - t * se_b1, beta_1 + t * se_b1)
cat("95% confidence interval for beta_1 is: ", CI_b1)
```

```
## 95% confidence interval for beta_1 is:  0.9827204 1.006793
```

```r
# Check CI_b1 using confint()
confint(fit, level = 0.95)[2,]
```

```
##     2.5 %    97.5 %
## 0.9827204 1.0067927
```

(d)

In this context, there are two expectations: i. When `x = 0`, `y = 0`. I.e. if there is no calcium in the solution, the analytical result should be 0. ii. The slope of the linear model should be 1, based on the empirical techniques.

Now we test if there is enough evidence for each claim (i) and (ii). (i)

```r
# Hypothesis testing for beta_0 = 0
# Null hypothesis: beta_0 = 0
# Alternative hypothesis: beta_0 != 0 (two tail test)

t_stat <- beta_0 / (sqrt(sum((y - beta_0 - beta_1 * x)^2) / (n-2)) * sqrt(1/n
+ x_mean^2 / Sxx))
p_value <- 2 * pt(-abs(t_stat), df = n-2)
cat("The p-value for testing beta_0 = 0 is: ", p_value, "\n")

## The p-value for testing beta_0 = 0 is:  0.1365732
```

Since the p-value = `0.1368` > `0.05`, we do not reject the null hypothesis that `beta_0 = 0`. There is not enough evidence to suggest that the analytical result is non-0 when there is no calcium in the solution.

```r
# Hypothesis testing for beta_1 = 1
# Null hypothesis: beta_1 = 1
# Alternative hypothesis: beta_1 != 1 (two tail test)

t_stat <- (beta_1 - 1) / (sqrt(sum((y - beta_0 - beta_1 * x)^2) / (n-2)) /
sqrt(Sxx))
p_value <- 2 * pt(-abs(t_stat), df = n-2)
cat("The p-value for testing beta_1 = 1 is: ", p_value, "\n")

## The p-value for testing beta_1 = 1 is:  0.3445086
```

Since the p-value = `0.34451` > `0.05`, we do not reject the null hypothesis that `beta_1 = 1`. There is not enough evidence to suggest that the slope of the linear model is not 1.

(e)

Assume that the condition in (d)(i) is true, i.e. `beta_0 = 0`. Then the linear model simplifies to `y = beta_1 * x + e`, where ‘e’ denotes error. We can now recalculate the confidence interval for beta_1.

```r
# Initialize new regression model based on known b0
lm_new <- lm(y ~ 0 + x)

summary(lm_new)

##
## Call:
## lm(formula = y ~ 0 + x)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.24861 -0.19054 -0.09167  0.00104  0.49827
##
## Coefficients:
##   Estimate Std. Error t value Pr(>|t|)
```

```
## x 0.987153   0.002704   365.1   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.2258 on 9 degrees of freedom
## Multiple R-squared:  0.9999, Adjusted R-squared:  0.9999
## F-statistic: 1.333e+05 on 1 and 9 DF,  p-value: < 2.2e-16
```

```
# Check the 95% confidence interval for beta_1 when beta_0 = 0 using
confint()
confint(lm_new, level = 0.95)
```

```
##       2.5 %    97.5 %
## x 0.9810362 0.9932693
```

Now we retest the statement in d(ii) if the slope is 1

```
# Conduct hypothesis testing for beta_1 = 1 given the new linear model with
known b0
b1_new = coef(lm_new)
se_new = summary(lm_new)$coefficients["x", "Std. Error"]
t_stat <- (b1_new - 1) / se_new
p_value <- 2 * pt(-abs(t_stat), df = n-1)
cat("The p-value for testing beta_1 = 1 is: ", p_value)
```

```
## The p-value for testing beta_1 = 1 is:  0.001042038
```

Since the p-value = `0.00104` < `0.05`, we reject the null hypothesis that beta_1 = 1. There is enough evidence to suggest that the slope of the linear model is not 1.

(f)

The results in (d) and (e) are different due to the assumption made in (e) that `beta_0 = 0`. This assumption simplifies the linear model by forcing the intercept value to be 0, and changes the degrees of freedom from `n-2` to `n-1`, which affects the t-statistic and p-value. The results in (d) are based on the original linear model, while the results in (e) are based on the simplified linear model with `beta_0 = 0`.

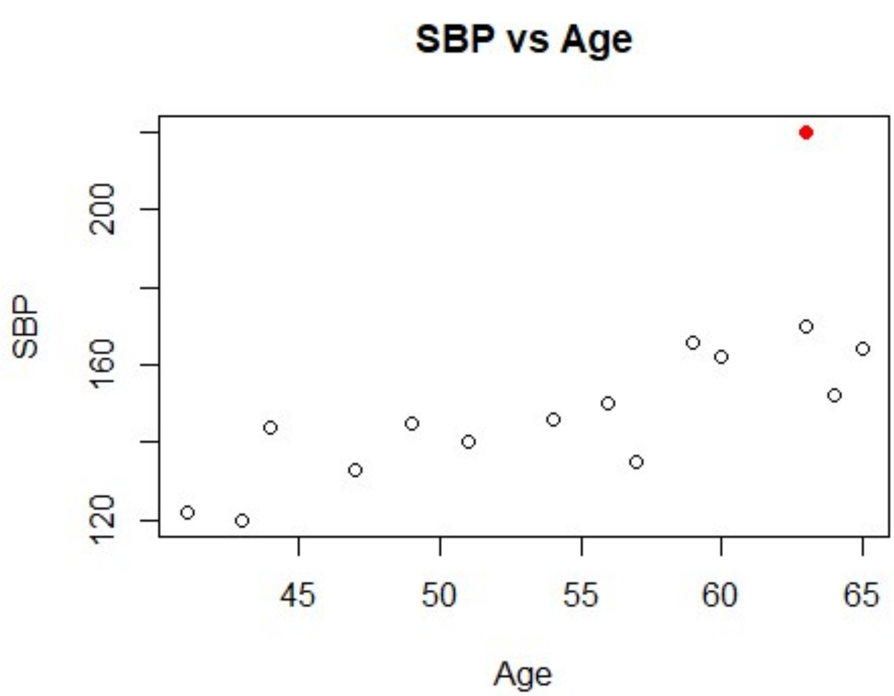
Q2.18

```
# Create vectors for SBP and Age
sbp <- c(164, 220, 133, 146, 162, 144, 166, 152, 140, 145, 135, 150, 170,
122, 120)
age <- c(65, 63, 47, 54, 60, 44, 59, 64, 51, 49, 57, 56, 63, 41, 43)

# Create a dataframe
data <- data.frame(SBP = sbp, Age = age)

# Display the dataframe
print(data)
```

```
##     SBP  Age
## 1  164  65
## 2  220  63
## 3  133  47
## 4  146  54
## 5  162  60
## 6  144  44
## 7  166  59
## 8  152  64
## 9  140  51
## 10 145  49
## 11 135  57
## 12 150  56
## 13 170  63
## 14 122  41
## 15 120  43
```

(a)

```r
# scatter plot sbp against age
plot(data$Age, data$SBP, xlab = "Age", ylab = "SBP", main = "SBP vs Age")

# Label the extreme point with a different colour
expoint <- data[data$Age == 63 & data$SBP == 220,]
points(expoint$Age, expoint$SBP, col = "red", pch = 19)
```



The plot shows an almost positive linear relationship between SBP and Age, indicating that as age increases, SBP also increases. There also seems to be a potential out-lier at age 63 with SBP 220 (marked in red).

(b)

Let ‘x’ denote the age and ‘y’ denote the SBP. Let the date assume the equation `y = beta_0 + beta_1 * x + e`, where ‘e’ denotes the error term.

```r
# Fit a linear model of SBP and Age
model <- lm(SBP ~ Age, data = data)

summary(model)

##
## Call:
## lm(formula = SBP ~ Age, data = data)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -21.904  -5.642  -2.221   2.422  50.085
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  33.3062    31.2162   1.067  0.30541
## Age           2.1684     0.5679   3.818  0.00213 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 17.3 on 13 degrees of freedom
## Multiple R-squared:  0.5286, Adjusted R-squared:  0.4923
## F-statistic: 14.58 on 1 and 13 DF,  p-value: 0.002133
```

Obtain the fitted equation:

```r
# Get the coefficients of the linear model
beta_0 <- coef(model)[1]
beta_1 <- coef(model)[2]

cat("The estimated coefficients are: beta_0 = ", beta_0, ", beta_1 = ",
beta_1, "\n")

## The estimated coefficients are: beta_0 =  33.30617 , beta_1 =  2.168392

cat("The fitted equation is: y_i = ", beta_0, " + ", beta_1, " * x_i")

## The fitted equation is: y_i =  33.30617  +  2.168392  * x_i
```

(c)

```r
# Construct an ANOVA table for the linear model in (b)
anova(model)
```

```
## Analysis of Variance Table
##
## Response: SBP
##           Df Sum Sq Mean Sq F value   Pr(>F)
## Age        1 4361.5  4361.5  14.578 0.002133 **
## Residuals 13 3889.4   299.2
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

(d)

```r
# Calculate F ratio for testing the significance of the linear relationship
f_value <- summary(model)$fstatistic[1]
cat("The F ratio for testing the significance of the linear relationship is:
", f_value, "\n")
```

```
## The F ratio for testing the significance of the linear relationship is:
14.57785
```

```r
# Calculate the p-value for the F ratio
p_value <- pf(f_value, df1 = 1, df2 = 13, lower.tail = FALSE)
cat("The p-value for the F ratio is: ", p_value)
```

```
## The p-value for the F ratio is:  0.00213278
```

Assuming alpha = `0.05`, since the p-value = `0.002133` < `0.05`, we reject the null hypothesis that there is no linear relationship between SBP and Age. There is a significant linear relationship between SBP and Age.

(e)

Test the hypothesis that b1 = 0 at alpha = 0.05.

```r
# Define null hypothesis
# Null hypothesis: beta_1 = 0

# Conduct t-test for beta_1 = 0
t_stat <- coef(model)[2] / summary(model)$coefficients["Age", "Std. Error"]
p_value <- 2 * pt(-abs(t_stat), df = 13)
cat("The p-value for t-testing beta_1 = 0 is: ", p_value)
```

```
## The p-value for t-testing beta_1 = 0 is:  0.00213278
```

Since the p-value = `0.002133` < `0.05`, we reject the null hypothesis that `beta_1 = 0`. There is a significant linear relationship between SBP and Age.

We notice that the observation in (e) matches that (d).