STA4020 Problem Set 4

Fall 2024

General Instructions:

This assignment contains 3 questions. Please ensure that all work is clearly shown and that final answers are boxed or highlighted. This assignment is due by **23:59:59 PM on November 15, 2024**. Late submissions will incur a penalty unless prior arrangements have been made.

Ethical and Responsible Usage of AI

While you are encouraged to explore modern tools such as AI and machine learning platforms to enhance your learning, it is essential to use these resources responsibly. All solutions must be your original work. You may use AI to assist with understanding concepts or to guide you, but copying AI-generated answers directly without comprehension is not permitted. Always strive to understand the underlying methods and derivations behind your solutions.

Additional Reminders:

1. Independent Work: Each student must submit their own work. Direct copying from others is considered academic misconduct.
2. Formatting: Please submit your answers as a typed document (preferably in PDF format) or as clearly scanned handwritten work. Both electronic submissions and hard copies are acceptable.
3. References: If you use any external sources, including textbooks, articles, or online resources, make sure to cite them appropriately.

If you have any questions or need clarification on any of the questions, feel free to reach out before the due date.

Good luck!

Question 1 (20 Points)

In 2022, the return on short-term government securities was about 3%. Suppose the expected return on a portfolio with a beta of 1.0 is 12%. According to CAPM: $\beta = 1$

- **(5 Points)** What is the expected return on the market portfolio?
- **(5 Points)** What are the expected returns on stocks with $\beta = 0.6$ and $\beta = 1.5$?
- **(10 Points)** Suppose you consider buying a stock for \$90. The stock is expected to pay \$5 dividend next year, and you expect to sell it then for \$97. You calculate the β of the stock as 0.9. Is the stock overpriced or underpriced?

Hint 1: The return on short-term government securities can be viewed as the risk-free rate R_f. This is because government securities are generally considered free from default risk.

Hint 2: In the last question, *overpriced* means the stock's current price is too high relative to its expected return, meaning investors are paying more than what the stock is worth based on the risk they are taking. *Underpriced* means the stock's current price is too low relative to its expected return, offering a higher return than expected for the level of risk.

Question 2 (20 Points)

In the lecture, we have derived the first principal component $\boldsymbol{w}_1$ by solving the following optimization problem:

$$\max_{\boldsymbol{w}_1} \boldsymbol{w}_1^\top \boldsymbol{\Sigma} \boldsymbol{w}_1 \quad \text{subject to} \quad \boldsymbol{w}_1^\top \boldsymbol{w}_1 = 1,$$

where $\boldsymbol{\Sigma}$ is the covariance matrix of asset returns, and $\boldsymbol{w}_1$ is the weight vector corresponding to the first principal component, associated with the largest eigenvalue of $\boldsymbol{\Sigma}$.

Now, derive the second principal component by solving the following optimization problem:

$$\max_{\boldsymbol{w}_2} \boldsymbol{w}_2^\top \boldsymbol{\Sigma} \boldsymbol{w}_2 \quad \text{subject to} \quad \boldsymbol{w}_2^\top \boldsymbol{w}_2 = 1 \quad \text{and} \quad \boldsymbol{w}_2^\top \boldsymbol{w}_1 = 0,$$

where $\boldsymbol{w}_1$ is the first principal component. In other words, you are required to show that $\boldsymbol{w}_2$ is the eigenvector of $\boldsymbol{\Sigma}$ corresponding to the second largest eigenvalue λ_2.

Hint 1: Recall that if $\boldsymbol{w}$ is an eigenvector of $\boldsymbol{\Sigma}$ associated with an eigenvalue λ, then $\boldsymbol{\Sigma}\boldsymbol{w} = \lambda \boldsymbol{w}$.

Hint 2: The constraint $\boldsymbol{w}_2^\top \boldsymbol{w}_1 = 0$ ensures that the second principal component is orthogonal to the first. Use the method of Lagrange multipliers to solve this optimization problem.

Question 3 (60 Points)

Background

Fama and French famously used Book-to-Market ratio (BM) and Market Capitalization (Size) to construct their well-known value and size factors. By performing an *independent* double sort on these two variables, they were able to capture the relationship between a stock's size, value characteristics, and its future returns. In this assignment, you will replicate this type of factor construction by performing an **independent double sort** on BM and Size.

You are provided with three datasets:

1. Book-to-Market Ratio (BM) (a DataFrame): Each row corresponds to the end of a month, and each column represents a stock. The data file name is `bm.csv`.

2. Market Capitalization (Size) (a DataFrame): Each row corresponds to the end of a month, and each column represents a stock. The data file name is `market_cap.csv`.

3. Monthly Stock Returns (a DataFrame): Each row corresponds to a month, and each column represents a stock. The data file name is `stock_returns.csv`.

Task

Your objective is to perform an independent double sort based on BM and Size, using the following guidelines:

- At the end of each month, divide stocks into **3 groups** based on their BM and Size.

- For BM: Sort the stocks into **low**, **medium**, and **high** groups using the **30th** and **70th** percentiles of BM values **for that month**.

- For Size: Sort the stocks into **small**, **medium**, and **large** groups using the **30th** and **70th** percentiles of market capitalization **for that month**.

This will result in **9 portfolios** (3 BM groups × 3 Size groups) for each month.

Steps

- **Portfolio construction**:

  - ⋆ For each month-end, assign stocks into one of the 9 portfolios based on their BM and Size values, using the 30th and 70th percentiles as cutoffs.

  - ⋆ Calculate the **value-weighted** return for each portfolio for the **following month**. In a value-weighted portfolio, each stock's weight is proportional to its market capitalization relative to the total market capitalization of all stocks in that portfolio.

  - ⋆ **If a stock's return is missing in any month, ignore that stock for the return calculation of that month.**

    - ⋆ Rebalance the portfolios at the end of each month.

- **Return Analysis**: Test whether the average return for each portfolio is statistically significant (i.e., deriving the t-statistic and the p-values).

- **Discussion**: Within each **Size** group (small, medium, large), examine whether there is a pattern in the returns across the three BM groups (low, medium, high). Provide a brief discussion on any observable patterns.

- **Submission and deliverables**:

    - ⋆ **(30 Points)** Please submit your Python code with clear explanation of key steps/functions. **We should be able to REPLICATE your results with your code!**
    - ⋆ **(15 Points)** The t-statistics and p-values for the average returns of the 9 portfolios.
    - ⋆ **(15 Points)** A brief discussion on whether you observe any pattern in returns across the BM groups within each Size group.

Problem 1:

a) By CAPM: $\mathbb{E}(R_i) - R_f = \beta_i (\mathbb{E}[R_m] - R_f)$

$$12 - 3 = 1 \, (\mathbb{E}(R_m) - 3)$$

$\mathbb{E}(R_m) = 12\%$, expected return on market portfolio is 12%

b) At $\beta = 0.6$: using above formula:

$$\mathbb{E}(R_m) = \frac{1}{0.6}(12-3) + 3$$
$$= 9 \times \frac{5}{3} + 3$$
$$= 15 + 3$$
$$= 18$$

At $\beta = 0.6$, expected return of market portfolio is 18%

At $\beta = 1.5$, $\mathbb{E}(R_m) = \frac{1}{1.5}(9) + 3 = 9$

expected returns of market portfolio at $\beta = 1.5$ is 9%

c) net asset return of stock: $R = \frac{97 + 5 - 90}{90} = \frac{12}{90} \approx 0.1333$

expected return on asset is 13.3%

By CAPM, Expected return: $R = R_f + \beta(\mathbb{E}(R_m) - R_f)$
$$= 3 + 0.9(12 - 3)$$
$$= 11.1\%$$

Comparing the 2 expected returns

1. Exp returns for holding the stock = 13.3%
2. Exp returns according to CAPM = 11.1%

Since there is an offer of higher returns than the required by CAPM

the stock is underpriced

Problem 2: 2nd Principal Component.

$$\max_{w_2} \; w_2^T \Sigma w_2 \quad \text{st} \quad w_2^T w_2 = 1 \quad \text{and} \quad w_2^T w_1 = 0$$

Set up the Lagrangian:

$$\mathcal{L} = w_2^T \Sigma w_2 - \lambda_1 (w_2^T w_2 - 1) - \lambda_2 (w_2^T w_1)$$

$$\nabla_{w_2} \mathcal{L} = 2 \Sigma w_2 - 2\lambda_1 w_2 - \lambda_2 w_1 = 0 \iff \Sigma w_2 = \lambda_1 w_2 + \frac{\lambda_2 w_1}{2} \quad \text{—— } (*)$$

Using constraints to solve $\lambda_{1,2}$: $2\Sigma w_2 - 2\lambda_1 w_2 - \lambda_2 w_1 = 0$

multiply w_2^T on both sides

$$2 w_2^T \Sigma w_2 - 2\lambda_1 \underbrace{w_2^T w_2}_{=1} - \lambda_2 \underbrace{w_2^T w_1}_{=0} = 0$$

$$w_2^T \Sigma w_2 = \lambda_1 \quad \text{—— } (1)$$

if we multiply w_1^T on both sides: $2 w_1^T \Sigma w_2 - 2\lambda_1 \underbrace{w_1^T w_2}_{=0} - \lambda_2 \underbrace{w_1^T w_1}_{=1} = 0$

$$2 w_1^T \Sigma w_2 - \lambda_2 = 0$$

$$\lambda_2 = 2 w_1^T \Sigma w_2 \quad \text{—— } (2)$$

sub (1), (2) into $(*)$: $\Sigma w_2 = (w_2^T \Sigma w_2) w_2 + (w_1^T \Sigma w_2) w_1$

$$\Rightarrow \Sigma (I - w_2^T w_2 - w_1^T w_1) w_2 = 0 \quad \text{—— } (**)$$

let $M = I - w_2^T w_2 - w_1^T w_1$, we see that M is a projection matrix that projects any given vector onto a subspace orthogonal to w_1 and w_2

Based on the 2 given constraints $\Rightarrow$ w_2 is orthogonal to w_1 and itself.

Thus, $(**)$ implies that w_2 is the eigenvector of Σ that corresponds to λ_2 (2nd largest eigenvalue)