Homework 1 – Rigid Body Kinematics

Homework 1 concentrates on rigid body kinematics. Problem 1 will familiarize you with quaternions and how they can be used to track rigid bodies. Problem 2 concentrates on screw theory in the plane ($\mathbb{R}^2$). It provides you with the theoretical background to solve kinematics tasks for robot mechanisms that operate in a plane while helping you to revisit important concepts of screw theory.

Q1. Unit Quaternions (6pts + 2pts)

Let $Q = (q_0, \vec{q})$ and $P = (p_0, \vec{p})$ be quaternions, where $q_0, p_0 \in \mathbb{R}$ are the scalar parts of Q and P and $\vec{q}, \vec{p}$ are the vector parts.

1. (2pts) Let x be a point and let X be a quaternion whose scalar part is zero and whose vector part is equal to x (such a quaternion is called a *pure* quaternion). Show that if Q is a unit quaternion, the product QXQ^* is a pure quaternion and the vector part of QXQ^* satisfies
$$(q_0^2 - \vec{q} \cdot \vec{q})\vec{x} + 2\bigg(q_0(\vec{q} \times \vec{x}) + (x \cdot \vec{q})\vec{q} \bigg).$$
Verify that the vector part describes the point to which x is rotated under the rotation associated with Q.

2. (2pts) Show that the set of unit quaternions is a two-to-one covering of $SO(3)$. That is, for each $R \in SO(3)$, there exist two distinct unit quaternions which can be used to represent this rotation.

3. (2pts) Compare the number of additions and multiplications needed to perform the following operations:
   i. Compose two rotation matrices.
   ii. Compose two quaternions.
   iii. Apply a rotation matrix to a vector.
   iv. Apply a quaternion to a vector [as in part (a)].

   Count a subtraction as an addition, and a division as a multiplication.

4. (extra 2pts) Show that a rigid body rotating at unit velocity about a unit vector in $\omega \in \mathbb{R}^3$ can be represented by the quaternion differential equations
$$\dot{Q} \cdot Q^* = (0, \omega/2),$$
where $\cdot$ represents quaternion multiplication.

Q2. Planar Rigid Body Transformations (8pts)

A transformation $g = (p, R) \in SE(2)$ consists of a translation $p \in \mathbb{R}^2$ and a 2×2 rotation matrix R. We represent this in homogeneous coordinates as a 3×3 matrix:

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

A twist $\hat{\xi} \in se(2)$ can be represented by a 3×3 matrix of the form:

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \qquad \hat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \qquad \omega \in \mathbb{R}, v \in \mathbb{R}^2.$$

The twist coordination for $\hat{\xi} \in se(2)$ have the form $\xi = (v, w) \in \mathbb{R}^3$. Note that v is a vector in the plane and ω is a scalar.

1. (2pt) Show that the exponential of a twist in $se(2)$ gives a rigid body transformation in $SE(2)$. Consider both the pure translation case, $\xi = (v, 0)$, and the general case, $\xi = (v, \omega), \omega \neq 0$. (2pts)
2. (1pt) Show that the planar twists which correspond to pure rotation about a point q with coordinates $(q_x, q_y) \in \mathbb{R}^2$ with unit angular velocity and pure translation in a direction v with velocity $(v_x, v_y) \in \mathbb{R}^2$ are given by

$$\xi = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix} \text{(pure rotation)} \qquad \xi = \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} \text{(pure translation)}.$$

3. (2pt) Show that every planar rigid body motion can be described as either pure rotation about a point (called the *pole* of the motion) or pure translation.
4. (2pt) Show that the matrices $\hat{V}^s = \dot{g}g^{-1}$ and $\hat{V}^b = g^{-1}\dot{g}$ are both twists. Define and interpret the spatial twist $V^s \in \mathbb{R}^3$ and the body twist $V^b \in \mathbb{R}^3$ both in words and graphically with schematics.
5. (1pt) The adjoint transformation is used to map body velocities $V^b \in \mathbb{R}^3$ into spatial velocities $V^s \in \mathbb{R}^3$. Show that the adjoint transformation for planar rigid motions is given by

$$\mathrm{Ad}_g = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$