KDC S2012

Lecture #1 : 17 Jan 2012

Slides 1-5

KDC : RI6-711 Kinematics, Dynamic Systems, and Control

- Introductory Slides: KDC is introductory course to robotic mobility and manipulation at graduate level
- Goal of Course: Develop fundamental concepts and implementation skills ~~for~~ to analyze, model and control robotic systems that interact with their physical environment.
- Syllabus, Recommended books, Assignment + Projects, Blackboard.
- Refined Syllabus for Kinematics part.

Notes:

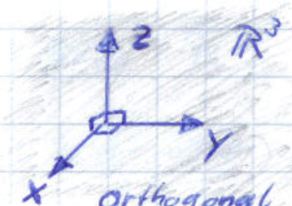
BOARD 1

Chapter 1: Rigid Body Motions

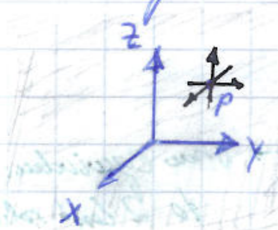
1.1 Fundamentals

- The chapter introduces kinematics of rigid body. Kinematics describes **motion of a rigid body in space**. What exactly are we meaning by these three terms? Answering this question will lead to fundamental assumptions of Kinematics Theory.

(A) Concept of Space: What is space in which rigid bodies move?



- Not trivial after giving it some thought
- Kinematics Theory: Cartesian coordinate space $\mathbb{R}^3$
- Q Real world? Objects possessing mass deform space-time (Einstein)
- Curved space no problem on earth for robotics; however: space robotics!
- Important to realize: $\mathbb{R}^3$ (and $\mathbb{R}^2$) are mathematical approximations
- How many directions of motion do exist in $\mathbb{R}^3$ for point ~~, point cloud~~?

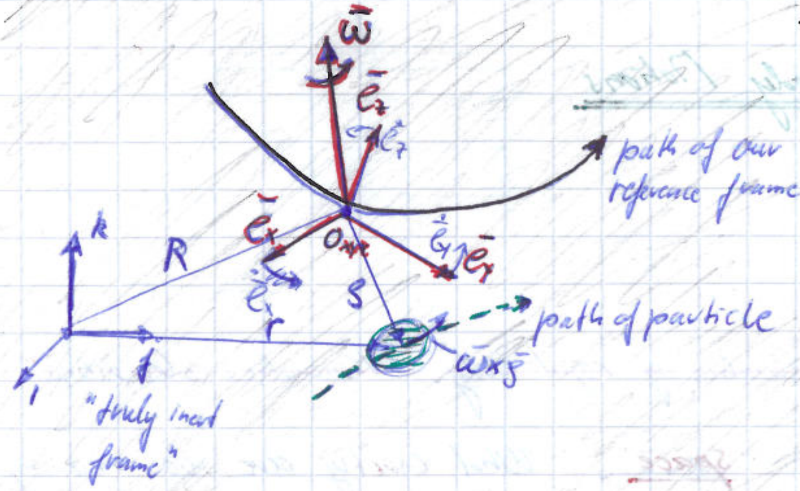


3-DoFs: point is free to move in three directions: degrees of freedom

BOARD 2

(B) Concept of World or Inertial Frame: Where is the origin of $\mathbb{R}^3$?

- Not a trivial problem either
- Fundamental issue: additional 'constraint' forces in frames that do not rest



• Rates observed in rotating frame

• $\bar r = \bar R + \bar\rho$

• $\bar v = d\bar r/dt = \frac{d\bar R}{dt} + \frac{d\bar\rho}{dt}$

$= \dot R + V_r + \underbrace{\bar\omega\times\rho}$

$\frac{d\rho}{dt} = (\dot\rho_1 e_x + \dot\rho_2 e_y + \dot\rho_3 e_z) + \underbrace{\bar\omega\times\bar\rho}_{\text{uses}}$

$\dot{\bar e}_i = \bar\omega\times e_i$

• with acceleration we get: $\bar a = \ddot R + \bar a_r + \bar\omega\times(\bar\omega\times\rho) + \dot{\bar\omega}\times\rho + 2\bar\omega\times\bar V_r$

• only $\bar V_r$ and $\bar a_r$ are observed in moving frame, yet total acceleration has also

$\ddot R$: acceleration of moving frame

$\omega\times(\omega\times\rho)$: Centripetal acceleration correction

$\dot\omega\times\rho$: angular acceleration correction

$2\bar\omega\times\bar V_r$: Coriolis acceleration correction

} true acceleration $\bar a$ at cost of "constraint" forces

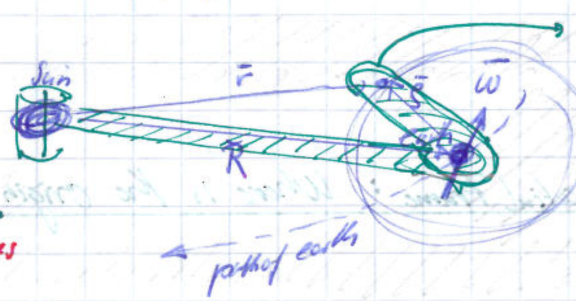
$\Rightarrow$ Complicates dynamics

• Basic assumption in robotics ~~Phy~~: we can define inertial frame.

↯ we do not know an inertial frame

• Natural choice: geographic location; however:

→ Q
Coriolis forces?
• Rails N-S
• Rivers N-S
• Weather patterns
• not a problem for terrestrial robotics



→ show equivalence to 2 link robot
→ mention $\bar V = \dot R + \bar V_r + \bar\omega\times\rho$: kinematics
→ $\bar a = \dots$: dynamics.

• Shake fundamentals of Newtonian Physics: requires inertial frame. Modern interpretation fixed stars.

- How is world frame being measured in robotics? (GPS, Vision, IMUs)

⇒ Although clear math concept, measuring motions to world frame not trivial.

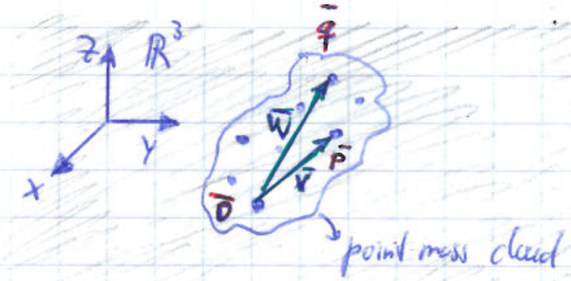
BOARD 3

(C) Concept of Rigid Body: What exactly is a rigid body?

→ Interactive: Book vs. Cloth

- Between soft and hard object which of the two is rigid body?
- In the end neither. There are large enough forces which can bend every object.
- Question of scale.

Def.: A rigid body is completely undistortable object that consists of point masses with distances ∧ relative orientations preserved throughout time.



- distance: $\underline{\|\bar{p}-\bar{o}\| = \text{const}} = \sqrt{\sum_i (p_i - o_i)^2}$
- not sufficient due to reflection (all z components to −z)
- additional requirement: $\underline{\bar{v}\times\bar{w} = \text{const.}}$

→ Thought Experiment:

- point cloud: travel into body:
  atoms: electrons 10^{10} ~~nucleus~~: 10^{-15}
  100000 ≈ order of distance earth - moon. (384.403 km)
- nothing in between.

- Notes:
  - Cross product defines orientation due to RHR

→ Q

- How many DOFS does a rigid body have? ⇒ 6: 3 trans + 3 orientation
- Screw theory will take advantage of $\mathbb{R}^6$ to represent Rigid body motions in compact way.

BOARD 4

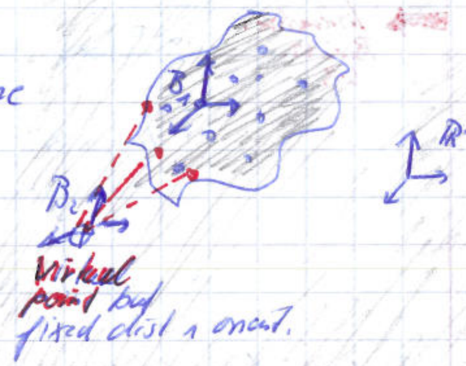
(D) Body frame: We don't need to track all points to describ RB motion.

- distances and orientations constant $\forall$ points of RB $\forall t$

$\Rightarrow$ position $\wedge$ orientation known for any point of RB wrt World, it is known $\forall$ points.

- Concept of Body Frame: Frame attached to any point of RB at $t=0$.

$\Rightarrow$ Tracking RB $\Leftrightarrow$ Tracking Body Frame



LEARNING OBJECTIVES Section 1:

- Every theory is only as good as its underlying assumptions! We have covered these fundamental assumptions for Kinematics
- You understand that $\mathbb{R}^3$ ($\mathbb{R}^2$) ~~etc~~, Inertial Frame, Rigid Body and Body Frame are math concepts.
- You can explain the abstractions they represent.

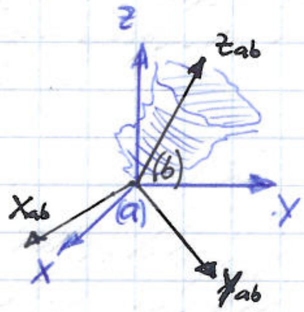
BOARDS

1.2 Tracking Rigid Bodies in Space

- Tracking RBs requires (1) to define **Body Frame** at $t=0$ and
  (2) to track its position $\wedge$ orientation $\forall t>0$

- start w/ orientation

(A) Rotation Matrix: How can we track orientations in $\mathbb{R}^3$?



(a): world frame, or frame A
(b): body frame defined at $t=0$

- Orientation is equivalent to knowing $\bar{x}_{ab}$, $\bar{y}_{ab}$ and $\bar{z}_{ab}$ $\forall t$ $\Leftrightarrow$ principal axis vectors of body frame in components of world frame.

- Note: $\bar{x}_{ab}$, $\bar{y}_{ab}$ $\wedge$ $\bar{z}_{ab}$ are unit vectors!

$\Rightarrow$ Definiton: $R_b^a = [\bar{x}_{ab}\ \bar{y}_{ab}\ \bar{z}_{ab}]$ is "Rotation Matrix" $\in \mathbb{R}^{3\times3}$

- R_b^a not only describes orientation of body, it also transforms points between coordinate frames: $\bar{q}^a = R_b^a \bar{q}^b$

- Helpful analogy: mathematical object "vector" describes not only position of points but also difference between two points $\bar{o}$ $\bar{r}_2$ $d\bar{r}$ $\bar{r}_1$

- Tracking across frames: $R_c^a = R_b^a \cdot R_c^b$ with $\cdot$ replacing normal matrix multiplication (composition rule)

$\Rightarrow$ - Composition does not commute! $R_b^a R_c^b \neq R_c^b R_a^b$

Interactive Spatial Frame $\overset{z}{\underset{x}{\perp}}{}^{y}$ and then compose rotations by turning your bodies!

Lecture #2: 19 Jan 2012

BOARD 1

(B) Rotational Transformation: Algebraic Properties of R

- Like a vector, R is a mathematical object
- R is member of special orthogonal group $SO(3)$

$$SO(n) = \{ R \in \mathbb{R}^{n \times n} : RR^T = I \wedge \det R = +1 \}$$

$R = [\bar{r}_1 \bar{r}_2 \bar{r}_3]$

- RR^T: $\bar{r}_i^T \bar{r}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$ because $\bar{r}_i$ orthogonal unit vectors

$$\Rightarrow RR^T = I$$

- $\det R = \bar{r}_1 (\bar{r}_2 \times \bar{r}_3) = +1$ due to definition of cross product (RHR)

- Group ~~member~~: $R_1 R_2 \in SO(3)$

- Rotations are rigid body transformations: They preserve distances and relative orientations of points belonging to RB

- $\|Rp - Rq\| = \|p - q\|$: math trick $\|Rp - Rq\|^2 = (Rp - Rq)^T (Rp - Rq)$
$$= [R(p-q)]^T [R(p-q)]$$
$$= (p-q)^T R^T R (p-q)$$
$$= (p-q)^T (p-q) = \|p-q\|^2$$

- $R(\bar{v} \times \bar{w}) = R\bar{v} \times R\bar{w}$: proof requires algebraic property of cross product

- $$\underbrace{\bar{a} \times \bar{b}}_{\text{vector notation}} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{\hat{a}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \underbrace{\hat{a} b}_{\text{matrix notation}}$$

$\hat{a}$ is skew symmetric $\hat{a}^T = -\hat{a}$

- Now proof of preservation of orientation follows from ~~brute~~ direct calculation. (Try it!)

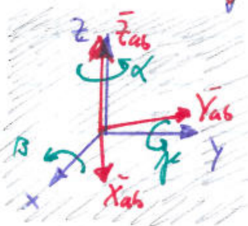
BOARD 2

(C) Euler Angles: What are least parameter presentations of R?

- $R \in \mathbb{R}^{3\times 3}$ has 9 element, but only 3 independent parameters (describes 3 dofs)

Interactive Question Why?

- Constraint $R^T R = I_3$ creates the constraint equations
- Derive **Elementary Rotations** from dofs:

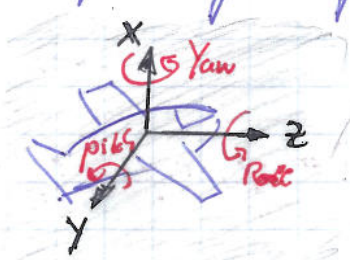


- about z-axis: $\bar{X}_{ab} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix}$ $\bar{Y}_{ab} = \begin{bmatrix} -\sin\alpha \\ +\cos\alpha \\ 0 \end{bmatrix}$ $\bar{Z}_{ab} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\Rightarrow R_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underbrace{R_x(\beta) \wedge R_y(\gamma)}_{\text{equivalent}}$$

- Now composition rule: $R \overset{!}{=} R_z(\alpha) \cdot R_x(\beta) \cdot R_y(\gamma) \Rightarrow$ arbitrary rotation expressed with only 3 parameters α, β, γ <u>and</u> rule of composition!
- issue: Compositions do not commute $R_b^a R_c^b \neq R_c^b R_b^a \rightarrow$ DEMO
- <u>Remarks:</u>

  - Only 12 independent rotation compositions
  - Each combination has set of so called $\underset{\text{Euler, 1776}}{\underline{\text{Euler}}}$ angles
  - Example: current frame method ZYZ angles
  - Example: fixed frame method ZYX angles $\hat{=}$ Roll Pitch Yaw



- **Fundamental Problem with Least parameter Representations:**

  Singularities can not be avoided in <u>any</u> 3d representation of $SO(3)$

- What is meant by that? $(\alpha,\beta,\gamma) \underset{\text{not always!}}{\overset{\text{always}}{\rightleftarrows}} SO(3)$

- Example: $R_{ab} = R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)$

$$= \begin{bmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & \cdots \\ \vdots & \ddots \end{bmatrix}$$

Inverse calculation leads to
$$\beta = \operatorname{atan2}\left(\sqrt{r_{31}^2 + r_{32}^2},\ r_{33}\right)$$
$$\alpha = \operatorname{atan2}\left(r_{23}/s_\beta\ ;\ r_{13}/s_\beta\right)$$
$$\gamma = \operatorname{atan2}\left(r_{32}/s_\beta\ ;\ -r_{31}/s_\beta\right)$$

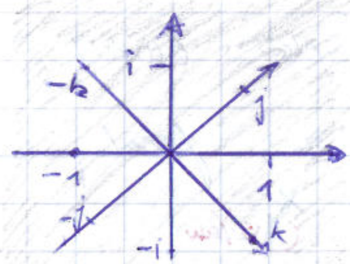
$\Rightarrow$ not invertable if $\sin\beta = 0$

$\Longrightarrow$ Slides on Apollo 11 Mission and Gimbal Lock
BE SURE TO HIGHLIGHT 3 Param Representation of R by Gimbal 'Manipulator'

BOARD 3

(D) Quaternions: A safe parametrization of R without singularities

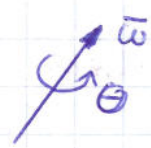
- Quaternions generalize complex numbers (Hamilton 1843)
- Used in mechanics, computer vision, computer graphics to represent Rotations in 3d
- ~~go to~~
- In robotics used alongside matrix representation to keep track of orientation with few parameters. ~~when tracking~~



$$\boxed{Q = q_0 + q_1 \bar{i} + q_2 \bar{j} + q_3 \bar{k}}$$

- 4 parameters, globally

- $Q = (q_0, \bar{q})$
$$= \left(\cos\frac{\theta}{2},\ \bar{\omega}\sin\left(\frac{\theta}{2}\right)\right)$$



with $\bar{\omega}$: axis of rotation in space
Θ: amount of rotation.

- Composition rule: $Q_1 Q_2 = (q_0^1 q_0^2 - \bar{q}_1 \bar{q}_2,\ q_0^1 \bar{q}_2 + q_0^2 \bar{q}_1 + \bar{q}_1 \times \bar{q}_2)$
  and $Q_c^a = Q_b^a Q_c^b$

- inversion:
$$Q = (q_0, \bar{q}) \rightarrow \theta = 2\arccos q_0,\ \bar{\omega} = \begin{cases} \frac{\bar{q}}{\sin(\theta/2)}, & \theta \neq 0 \\ 0, & \theta = 0 \end{cases}$$

Lecture #3

BOARD 1

(E) Exponential Coordinates: Intuitive and Canonical representation of R

- In previous section on quaternions, we have assumed that R can be represented by $R(\bar{\omega}, \theta)$ w/ θ: amount of rotation, $\bar{\omega}$: axis of rotation.

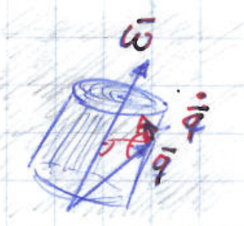
- **Euler Theorem:** (equivalent axis representation)

  Any orientation $R \in SO(3)$ is equivalent to a rotation about a fixed axis $\bar{\omega} \in \mathbb{R}^3$ through an angle $\theta \in [0, 2\pi)$

- Relationship between R and $(\bar{\omega}, \theta)$ is given by:

$$R(\bar{\omega}; \theta) = e^{\hat{\omega}\theta}$$
where components of vector $\bar{\omega}\theta$ are "exponential ~~canonical~~ coordinates" and $\|\bar{\omega}\| = 1$.

- Intuitive motivation:



• rotation about fixed axis

$\underline{\dot{\bar{q}}} = \bar{\omega} \times \bar{q} = \underline{\hat{\omega}\bar{q}}$ Looks like 1st order diff eqn if $\bar{q}$ were scalar and $\hat{\omega}$ as well.

$\Rightarrow$ Intuitive solution: $q(t) = e^{\bar{\omega}t} q(0)$

Analog: With "Matrix Exponential"
$$e^{\hat{\omega}t} = I_3 + \hat{\omega}t + \frac{(\hat{\omega}t)^2}{2!} + \frac{(\hat{\omega}t)^3}{3!} + \dots$$

We have $\bar{q} = e^{\hat{\omega}t}\bar{q}(0)$

- Instead of time, we rotate through angle $\leadsto \bar{q}(\theta) = e^{\hat{\omega}\theta}\, \bar{q}(0)$

- Problem: $e^{\hat{\omega}\theta}$ defined as infinite series: not very useful

Hence: $e^{\hat{\omega}\theta} = I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)\hat{\omega} + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \dots\right)\hat{\omega}^2$

[after usage of relations $\hat{a}^2 = aa^T - \|a\|^2 I$, $\hat{a}^3 = -\|a\|^2 \hat{a}$]

$$\Rightarrow \underline{e^{\hat{\omega}\theta} = I_3 + \hat{\omega}\sin\theta + \hat{\omega}^2 (1-\cos\theta)}$$ Rodrigues' formula

- Inversion: Given $R \in SO(3)$, we find

$$\theta = \operatorname{acos}\left(\frac{\operatorname{trace}(R) - 1}{2}\right), \quad \omega = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

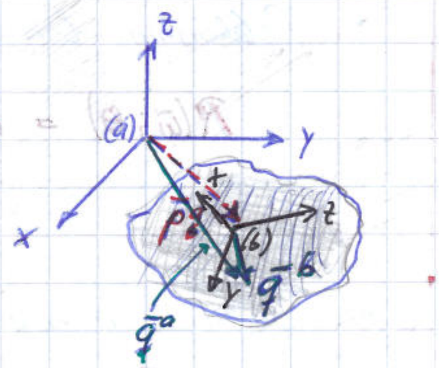
Note: ~~not unique~~ singular for $R = I \Rightarrow \operatorname{trace}(R) = 3 \Rightarrow \theta = 0 \wedge \hat{\omega}$ arbitrary and not unique for $\omega' = -\omega$ and $\theta' = 2\pi - \theta$ give same result.

Remark: Screw theory uses exponential coordinates, also called canonical coordinates of rotation group.

BOARD 2

(F) Homogeneous Transformation: How does rotation combine with translation?

- Tracking translation in addition to rotation: $\underline{\bar{q}^a_a = \bar{p}^a_b + R^a_b\, \bar{q}^b}$



→ try using $\bar{s}$ instead of $\bar{q}$ to avoid confusion in (6)

- Goal for matrix representation

$$\bar{q}^a = M \bar{q}^b$$

Challenge Question:

- Initial step clear: $\bar{q}^{a'} = R^a_b\, \bar{q}^b$ is multiplication. But how do we get $\bar{q}^a = T \bar{q}^{a'}$? $\left(= \begin{bmatrix} \bar{q}^{a'}_x + p_x \\ q'_y + p_y \\ q'_z + p_z \end{bmatrix} \right)$

- Solution through additional matrix element:

$$\begin{bmatrix} q_x \\ q_y \\ q_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & & & p_x \\ & 1 & & p_y \\ & & 1 & p_z \\ & & & 1 \end{bmatrix} \begin{bmatrix} q'_x \\ q'_y \\ q'_z \\ 1 \end{bmatrix}$$ trick!

- Concept of homogeneous representation

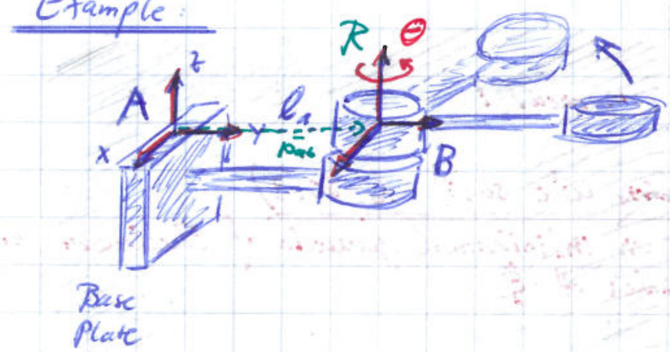
- Definition: Homogeneous Transformation $g_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}$ is homogeneous representation of $SE(3)$ with $SE(3) = \{(\bar{p}, R) : p \in \mathbb{R}^3, R \in SO(3)\}$ Special Euclidean group.

- Advantage:

Composition rule applies just as for rotations:

$$g_{ac} = g_{ab}\, g_{bc} = \begin{bmatrix} R_{ab} R_{bc} & R_{ab} p_c^b + p_{ab} \\ 0 & 1 \end{bmatrix}$$

- Example:



$$\Rightarrow \bar{p}_{ab} = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}$$

$$R_{ab} = \begin{pmatrix} C_\theta & -S_\theta & 0 \\ S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow g_{ab} = \begin{bmatrix} C_\theta & -S_\theta & 0 & 0 \\ S_\theta & C_\theta & 0 & l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

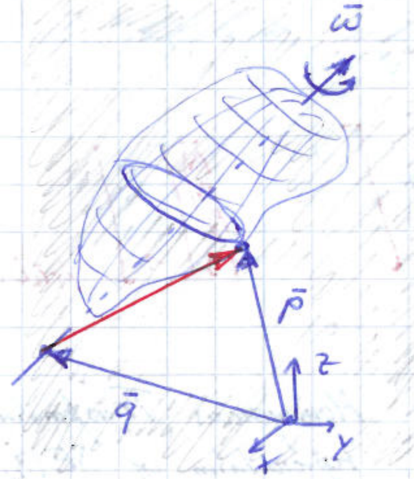
BOARD 3

(G) Twists: How exponential coordinates generalize to full motions

- Exponential mapping that we introduced for rotations $SO(3)$ can be generalized to $SE(3)$.

- The generalization will allow an elegant and rigorous, geometric treatment of tracking rigid bodies in space.

• Consider rotation about $\bar\omega$ with $\|\bar\omega\| = 1$, away from origin:



• Like in previous derivation, look at displacements that are small: $\dot{\bar p} = \bar\omega \times (\bar p - \bar q)$

• Goal: homogeneous coordinates for matrix multiplication: $\frac{d}{dt}\begin{bmatrix}\bar p\\ 1\end{bmatrix} = \begin{bmatrix} & \end{bmatrix}\begin{bmatrix}\bar p\\ 1\end{bmatrix}$

$$\Leftrightarrow \begin{bmatrix}\dot{\bar p}\\ 0\end{bmatrix} = \begin{bmatrix} & \end{bmatrix}\begin{bmatrix}\bar p\\ 1\end{bmatrix} \Rightarrow \begin{bmatrix}\dot{\bar p}\\ 0\end{bmatrix} = \begin{array}{cc} 1 & 4 \\ \multicolumn{2}{c}{\begin{bmatrix}\hat\omega & -\bar\omega\times\bar q\\ 0 & 0\end{bmatrix}} \\ 2 & 3 \end{array}\begin{bmatrix}\bar p\\ 1\end{bmatrix} \Rightarrow \boxed{\dot{\tilde p} = \hat\xi\,\tilde p}$$

• $-\bar\omega \times \bar q =: \bar v$ is translational component! (explain

)

• ① $\hat\xi$ is generalized skew symmetric matrix $\hat\omega \in so(3)$ to $se(3)$

② $\hat\xi$ is called a twist or infinitesimal generator of Euclidean group.
$\begin{bmatrix}\bar v\\ \bar\omega\end{bmatrix}$ called twist coordinates of $\hat\xi$

Show linear first before definition

• Same format also works for pure translation:

$$\dot{\tilde p} = \begin{bmatrix}\dot{\bar p}\\ 0\end{bmatrix} \overset{!}{=} \begin{bmatrix}\bar v\\ 0\end{bmatrix} = \begin{bmatrix}0 & \bar v\\ 0 & 0\end{bmatrix}\begin{bmatrix}\bar p\\ 1\end{bmatrix} = \hat\xi\,\tilde p$$

$\Rightarrow$ Def.: $\hat\xi = \begin{bmatrix}\hat\omega & \bar v\\ 0 & 0\end{bmatrix}$ is generalization of skew symmetric matrix $\hat\omega$

• Same procedure as before yields: $\tilde p(t) = \underbrace{e^{\hat\xi t}}_{\text{matrix exponential}}\tilde p(0)$

• Calculating out $e^{\hat\xi\theta}$ for ~~unit displacement~~ displacement of unit velocity in the amount of θ yield

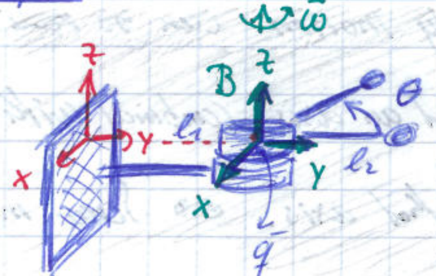
$$= \begin{bmatrix}I & v\theta\\ 0 & 1\end{bmatrix},\ \bar\omega = 0$$

$$e^{\hat\xi\theta} = I + \hat\xi\theta + \frac{(\hat\xi\theta)^2}{2!} + \frac{(\hat\xi\theta)^3}{3!} + \dots \overset{\text{Math}}{=} \begin{bmatrix} e^{\hat\omega\theta} & (I - e^{\hat\omega\theta})(\bar\omega\times\bar v) + \bar\omega\bar\omega^T\bar v\theta \\ 0 & 1\end{bmatrix},\ \bar\omega \neq 0$$

- Exponential map $g_{ab}(\theta) = e^{\hat{\xi}\theta} g_{ab}(0)$ gives motion of rigid body within <u>one</u> coordinate frame.

- <u>Example</u>

BOARD4



- Coordinates of B relative to A given by $g_{ab}(\theta) = e^{\hat{\xi}\theta} g_{ab}(0)$

- twist given by $\xi = \begin{bmatrix} \bar{v} \\ \bar{\omega} \end{bmatrix} = \begin{bmatrix} -\bar{\omega} \times \bar{q} \\ \bar{\omega} \end{bmatrix}$

$= \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$g_{ab}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}$

- Per Definition:

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(\bar{\omega} \times \bar{v}) + \bar{\omega}\bar{\omega}^T \bar{v}\,\theta \\ 0 & 1 \end{bmatrix}$$

$$\bar{\omega}^T \bar{v} = [0\,0\,1]\begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$= \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(\bar{\omega} \times \bar{v}) \\ 0 & 1 \end{bmatrix}$$

Def. $e^{\hat{\omega}\theta}$ (Rodrigues)

$$\underset{\text{(Rodrigues)}}{\overset{\text{Def. } e^{\hat{\omega}\theta}}{=}} \begin{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} + \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\hat{\omega}} \sin\theta + \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\hat{\omega}^2} \cancel{\cos\theta}(1-\cos\theta) & (I - e^{\hat{\omega}\theta})(\bar{\omega} \times \bar{v}) \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{matrix} & \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)(\bar{\omega} \times \bar{v}) \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_z(\theta) & \begin{bmatrix} 1-\cos\theta & \sin\theta & 0 \\ -\sin\theta & 1-\cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}(\bar{\omega} \times (-\bar{\omega} \times \bar{q})) \\ 0 & 1 \end{bmatrix}$$

$$-\bar{\omega} \times \bar{q} = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} +l_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{\omega} \times \begin{bmatrix} +l_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ +l_1 \\ 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} R_z(\theta) & \begin{bmatrix} 1-\cos\theta & \sin\theta & 0 \\ -\sin\theta & 1-\cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ +l_1 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}}_{e^{\hat{\xi}\theta}} \; g_{ab}(0) = g_{ab}(\theta)$$

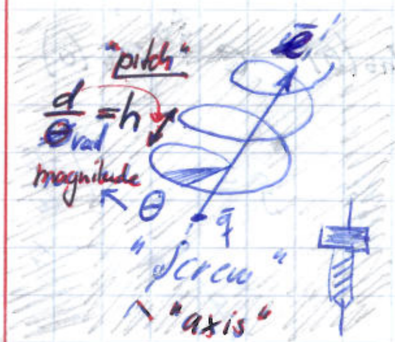
$$\Rightarrow \begin{bmatrix} R_z(\theta) & \begin{bmatrix} +l_1\sin\theta \\ +l_1(1-\cos\theta) \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} g_{ab}(0) = \begin{bmatrix} R_z(\theta) & p_{ab} \\ 0 & 1 \end{bmatrix}$$ by inspection ok!

BOARD 5 (H) Screws: What is the geometric meaning of twists?

recall

- Euler theorem (recall): every rotation can be realized ... $R(\bar\omega, \theta)$
- Generalization:

Chasles Theorem: Every rigid body motion can be realized by a rotation about an axis combined with a translation ~~along~~ parallel to that axis. $\Leftrightarrow$ Screw motion



1) Axis: $\bar l = \begin{cases} \left\{ \frac{\bar\omega \times \bar v}{\|\bar\omega\|^2} + \lambda \bar\omega : \lambda \in \mathbb{R} \right\}, & \bar\omega \neq 0 \\ \{ 0 + \lambda \bar v : \lambda \in \mathbb{R} \}, & \bar\omega = 0 \end{cases}$

$\bar q$: free on axis $\bar\omega$

2) Pitch: $h = \frac{\bar\omega^T \bar v}{\|\bar\omega\|^2}$ (ratio of trans to rot)

3) Magnitude: $M = \begin{cases} \|\bar\omega\|, & \bar\omega \neq 0 \quad \text{(net rotation)} \\ \|\bar v\|, & \bar\omega = 0 \quad \text{(net translation)} \end{cases}$

$= \theta$ if $\|\bar\omega\| = 1$ (or $\|\bar v\| = 1$)

$\Rightarrow$ Screw coordinates of a twist with coordinates

$$\begin{pmatrix} \bar v \\ \bar\omega \end{pmatrix} = \begin{pmatrix} -\bar\omega \times \bar q + h\bar\omega \\ \bar\omega \end{pmatrix}$$

and

$$g = \begin{bmatrix} e^{\hat\omega\theta} & (I - e^{\hat\omega\theta})\bar q + h\theta\bar\omega \\ 0 & 1 \end{bmatrix}$$

Remark:

important

pure rotation $\Leftrightarrow$ $h = 0$

pure translation $\Leftrightarrow$ $h = \infty$ $\left(g = \begin{bmatrix} I & \theta \bar v \\ 0 & 1 \end{bmatrix} \right)$

BOARD 6

Learning Objectives for Section 2:

1) You can explain concept of rotation matrix R, and can track rigid body orientation
2) You know that the cross product $a \times b$ in matrix form is $\hat{a}$
3) You can compute rotations based on Euler angles for pitch, roll + yaw and based on quaternions.
4) You know how to compute the matrix exponential $e^{\hat{\omega}\theta}$ using Rodriguez formula
5) You can track full rigid body motions using homogeneous transformations
6) You understand the concept of twists as generators for homogeneous trafos.
7) You understand the geometric interpretation of twists by screw motions.

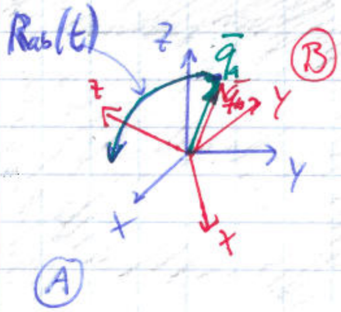
LECTURE #4

BAMO 1

1.3 Differential Kinematics of Rigid Body Motions

- $g(t)$ describes motion in $\mathbb{R}^3$ of rigid body, but
- $\dot{g}(t)$ is not RB velocity!
- Find proper computation of RB velocity by focusing on rotation first, and then generalize to full motions.

(A) Rotational Velocity of RB: Generalization of $\hat{\omega}$ to $\dot{R}R^{-1}$



- Pure rotation with constant velocity $\bar{\omega}$ $\Rightarrow$ $\dot{\bar{q}}_a = \bar{\omega} \times \bar{q}_a = \hat{\omega}\, \bar{q}_a$
- Formal: $\dot{\bar{q}}_a = \frac{d}{dt}(\bar{q}_a) = \frac{d}{dt}(R_{ab}\bar{q}_b) \overset{\dot{\bar{q}}_b = 0}{=} \dot{R}_{ab}\bar{q}_b$
- Shift to A-frame:

$$\dot{\bar{q}}_a = \dot{R}_{ab} \underbrace{R_{ab}^{-1} R_{ab}}_{I} \bar{q}_b = \dot{R}_{ab} R_{ab}^{-1} \bar{q}_a$$

Generalization

- Analogy: $\dot{R}_{ab} R_{ab}^{-1} = \hat{\omega}$

Def.: Instantaneous spatial angular velocity

$$\hat{\omega}_{ab}^{s} := \dot{R}_{ab} R_{ab}^{-1}$$

Example: $R_{ab} = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \hat{\omega}_{ab}^{s} = \dot{R}_{ab} R_{ab}^{-1} = \dot{\theta} \begin{bmatrix} -s & -c & 0 \\ c & -s & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \checkmark$$

Remarks:

- proof that $\dot{R}_{ab} R_{ab}^{-1}$ is skew symmetric ($S^T = -S$) uses $\frac{d}{dt} I = \frac{d}{dt}(RR^T) \overset{!}{=} 0$
- body angular velocity: $\bar{\omega}_{ab}^{b} = R_{ab}^{-1}\, \bar{\omega}_{ab}^{s}$ (velocity viewed from body frame)

$$= R_{ab}^{-1} \dot{R}_{ab}$$

- $\bar{v}_{q_a}(t) = \bar{\omega}_{ab}^{s}(t) \times \bar{q}_a(t)$
- $\bar{v}_{q_b}(t) = \bar{\omega}_{ab}^{b}(t) \times \bar{q}_b(t)$

BOARD2 (B) Full Rigid Body Velocity: generalization of $\hat{\xi}$ to $\dot g_{as}\, g_{as}^{-1}$

- Approach generalization to full motions by analogy w/ rotation calculation
- Homogeneous transformation $g_{as}(t) = \begin{bmatrix} R_{as}(t) & p_{as}(t) \\ 0 & 1 \end{bmatrix}$ describes motion over time. What is instantaneous velocity?

- $\dot{\bar q}_a = \frac{d}{dt}\left(g_{as}\,\bar q_b\right) \overset{\dot{\bar q}_b = 0}{=} \dot g_{as}\,\bar q_b = \dot g_{as}\, g_{as}^{-1}\, g_{as}\,\bar q_b = \dot g_{ab}\, g_{as}^{-1}\,\bar q_a$

$\Longrightarrow$

Def.: Instantaneous Spatial Velocity

$$\boxed{\hat V_{as}^{s} := \dot g_{as}\, g_{as}^{-1}}$$

follows from comparing $g g^{-1} \overset{!}{=} I$
$\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- Formal Computation: $\hat V_{as}^{s} = \dot g_{as}\, g_{as}^{-1} = \begin{bmatrix} \dot R & \dot p \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R^T & -R^T \bar p \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \dot R R^T & -\dot R R^T \bar p + \dot{\bar p} \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $\hat V_{as}^{s}$ is **twist** with coordinates $\begin{pmatrix} \bar v_{as}^{s} \\ \bar \omega_{as}^{s} \end{pmatrix} = \begin{pmatrix} -\dot R R^T \bar p + \dot{\bar p} \\ (\dot R R^T)^{\vee} \end{pmatrix}$ coordinates out of skew symmetric matrix

$= \begin{pmatrix} \vec\omega^{s} \times \bar p + \dot{\bar p} \\ \bar\omega^{s} \end{pmatrix}$

- Interpretation:

- $\bar\omega^{s}$ is instantaneous ~~spatial~~ angular velocity of RB viewed from inertial frame.
- $\bar v^{s} = \bar\omega \times \bar p + \dot{\bar p}$ is velocity of RB origin corrected by "false" rotation about inertial origin

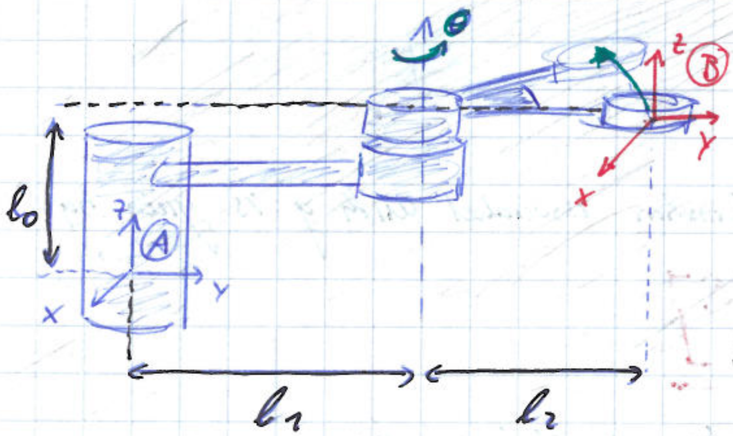
- Body Frame instantaneous spatial velocity:

$\hat V_{as}^{B} = g_{as}^{-1}\, \dot g_{as} = \begin{bmatrix} R^T \dot R & R^T \dot p \\ 0 & 0 \end{bmatrix}$ $\quad V_{as}^{B} = \begin{pmatrix} v_{as}^{B} \\ \omega_{as}^{B} \end{pmatrix} = \begin{pmatrix} R^T \dot p \\ (R^T \dot R)^{\vee} \end{pmatrix}$

velocity of origin

angular velocity in body frame.

- Example:



- after rotation by θ, homogeneous trafo is given by

$$g_{as} = \begin{bmatrix} R_z(\theta) & \begin{bmatrix} -l_2 \sin\theta \\ l_1 + l_2 \cos\theta \\ l_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

- Per Definition:

$$\vec{V}_{as}^{s} = \left(\dot{g}_{as} g_{as}^{-1}\right)^{\vee} = \begin{pmatrix} -\dot{R}R^T\vec{p} + \dot{\vec{p}} \\ (\dot{R}R^T)^{\vee} \end{pmatrix}$$

- $\dot{R}R^T = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ from previous example

$$\Rightarrow (\dot{R}R^T)^{\vee} = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix} = \vec{\omega}^s$$

- $\vec{V}^s = \hat{\omega}_s \vec{p} + \dot{\vec{p}} = \begin{bmatrix} 0 & +\dot{\theta} & 0 \\ -\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -l_2 \sin\theta \\ l_1 + l_2 \cos\theta \\ l_0 \end{bmatrix} + \begin{bmatrix} -l_2 c_\theta \\ -l_2 s_\theta \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} \dot{\theta} l_1 \\ 0 \\ 0 \end{bmatrix}$$

- Same procedure in Body frame yields:

$$\vec{V}^b = \begin{pmatrix} \vec{v}_1^b \\ \vec{\omega}^b \end{pmatrix} = \begin{pmatrix} R^T \dot{\vec{p}} \\ (R^T \dot{R})^{\vee} \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} -l_2 \dot{\theta} \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \end{pmatrix}$$

BOARD 3

(C) The Adjunct Transformation: How do velocities transform between frames?

- Homogeneous Trafo g_{as} transforms motions between frames.
- A powerful tool that is adjunct to g_{as} is the Adjunct transformation, which transforms the corresponding velocities!
- We have:

$$\hat{V}_{as}^{s} = \dot{g}_{as} g_{as}^{-1} = (g_{as} g_{as}^{-1}) \dot{g}_{as} g_{as}^{-1} = g_{as} g_{as}^{-1} \dot{g}_{as} g_{as}^{-1} = g_{as} \hat{V}_{as}^{B} g_{as}^{-1}$$

• Formal Computation leads to

(cross product)

$$V_{ab}^{s} = \begin{bmatrix} v_{ab}^{s} \\ \omega_{ab}^{s} \end{bmatrix} = \begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} v_{ab}^{b} \\ \omega_{ab}^{b} \end{bmatrix} = V_{ab}^{B}$$

<u>Def.</u>: The adjoint transformation associated with g is given by:

$$Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

Ad_g transforms velocities associated w/ homogeneous trafo.

Remarks:

- Ad_g is very powerful tool which we will use extensively
- For <u>any</u> twist $\hat{\xi}$, we have $g\hat{\xi}g^{-1}$ is twist with coords: $Ad_g \bar{\xi}$
- $Ad_g^{-1} = Ad_{g^{-1}} = \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}$
- Mapping across multiple frames: $\bar{V}_{ac}^{s} = \bar{V}_{ab}^{s} + Ad_{g_{ab}} \bar{V}_{bc}^{s}$ (composition rule)

not matrix but coordinates!

BOARD 1

LECTURE #5

2/5/2012

(D) Forces and Moment applied to Rigid Body: What is a wrench?

- Merriam Webster: "A wrench is a violent twist"
- Concept of wrench in Screw theory as a generalized force

$$\bar{F} = \begin{bmatrix} \bar{f} \\ \bar{\tau} \end{bmatrix}$$

where $\bar{f}$ is linear force acting on body (pure force)

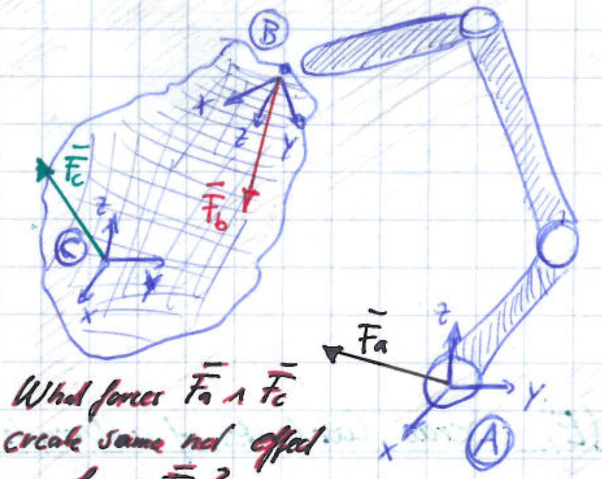
$\bar{\tau}$ is rotational component. (pure moment)

Remarks:

a) <u>Value</u> of $\bar{F}$ depends on coord system in which it is represented!

b) $\bar{F}_b = (\bar{f}_b, \bar{\tau}_b)$ is wrench <u>applied at origin of</u> B!

• Differential kinematics provides powerful tools to understand force transmission:



What forces $\bar F_a$ ∧ $\bar F_c$ create same net effect as force $\bar F_b$?

Def.: Two wrenches are equivalent if they generate same work for every possible rigid body motion.

• $W = \int \bar V^T \bar F \, dt = \int (\bar v^T \bar f + \bar\omega^T \bar\tau) \, dt$

⇒ infinitessimal work: $\delta W = \bar V_{ac}^{b\,T} \bar F_c \overset{!}{=} \bar V_{ab}^{b\,T} \bar F_b$ (*)

• $\hat V_{ac}^{b} \overset{Def}{=} g_{ac}^{-1} \dot g_{ac} \overset{Composition}{=} (g_{ab} g_{bc})^{-1} (\dot g_{ab} g_{bc} + g_{ab} \dot g_{bc})$

$(uv)^{-1} = v^{-1} u^{-1}$

$= g_{bc}^{-1} g_{ab}^{-1} \dot g_{ab} g_{bc} + g_{bc}^{-1} g_{ab}^{-1} g_{ab} \dot g_{bc}$

$= g_{bc}^{-1} \hat V_{ab}^{b} g_{bc} + \cancel{\hat V_{bc}^{b}}$ 0 (no motion b/w B ∧ C)

⇒ in coords $\bar V_{ac}^{b} = Ad_{g_{bc}^{-1}} \bar V_{ab}^{b}$

⇒ $\bar V_{ab}^{b} = Ad_{g_{bc}} \bar V_{ac}^{b}$

⇒ in (*): $\bar V_{ac}^{b\,T} \bar F_c = (Ad_{g_{bc}} \bar V_{ac}^{b})^T \bar F_b$

$= \bar V_{ac}^{b\,T} Ad_{g_{bc}}^T \bar F_b$

⇒ $\boxed{\bar F_c = Ad_{g_{bc}}^T \bar F_b}$

Remarks:

a) $\begin{bmatrix} \bar f_c \\ \bar\tau_c \end{bmatrix} = \begin{bmatrix} R_{bc}^T & 0 \\ -R_{bc}^T \hat p_{bc} & R_{bc}^T \end{bmatrix} \begin{bmatrix} \bar f_b \\ \bar\tau_b \end{bmatrix}$

• rotates f ∧ τ into C and adds torque generated by $-\bar p_{bc} \times \bar f_b$ (confirms intuition)

b) In Picture: $\bar F_a = Ad_{g_{ba}}^T \bar F_b$: equivalent force as if applied at origin of A. Not simply coord-trafo!

c) If several wrenches act on RB, they can be added if represented in the same frame!

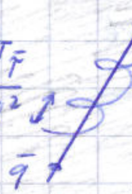
Procedure: 1) write all wrenches wrt single frame
2) add within that frame

d) Body vs Spatial representation: $F^b = Ad_g^T F^s$
$(V^b = Ad_{g^{-1}} V^s)$

BOARD 2

(E) Screw Coordinates of a Wrench

Poinsot Theorem: Every collection of wrenches applied to a rigid body is equivalent to a force applied ~~along~~ a fixed axis plus a torque about the same axis.



pitch: $h = \frac{\bar{f}^T \bar{\tau}}{\|\bar{f}\|^2}$

axis: $\bar{l} = \begin{cases} \underbrace{\frac{\bar{f} \times \bar{\tau}}{\|\bar{f}\|^2}}_{\bar{q}} + \lambda \bar{f}, & \bar{f} \neq 0 \\ \lambda \bar{\tau}, & \bar{f} = 0 \end{cases}$ ($\bar{q}$ one of many possible)

magnitude: $M = \begin{cases} \|\bar{f}\|, & \bar{f} \neq 0 \\ \|\bar{\tau}\|, & \bar{f} = 0 \end{cases}$

Can be skipped if time needed

- Poinsot and Chasles Theorem form theoretical basis of Screw theory
- Corresponding wrenches are: $\bar{F} = \begin{cases} M \cdot \begin{bmatrix} \bar{\omega} \\ -\bar{\omega} \times \bar{q} + h\bar{\omega} \end{bmatrix}, & h \text{ finite} \\ M \cdot \begin{bmatrix} 0 \\ \bar{\omega} \end{bmatrix}, & h = \infty \end{cases}$

  $\bar{\omega} = \frac{\bar{f}}{M}$ (h finite), $\bar{\omega} = \frac{\bar{\tau}}{M}$ ($h = \infty$)
- Dual nature of twists and wrenches:

Zero pitch $(h=0)$ ~~wrench~~ twist is pure rotation, but zero twist wrench is pure lin. force
$h=\infty$ twist is pure translation while $h=\infty$ wrench is pure rotational force

BOARD 3

1.4 Summary of Key Concepts of Rigid Body Motions

- Fundamentals:

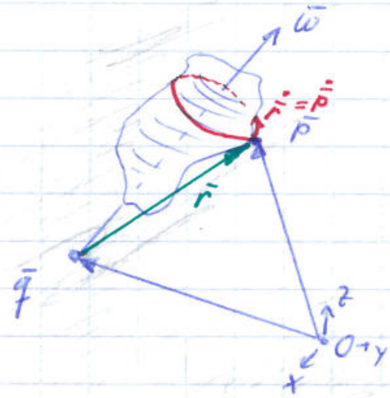
Concepts of inertial frame and rigid body and Euclidean Space

- Tracking Rigid Bodies:

Concepts of: - homogeneous transformation $g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$: $\bar{q}_a = g_{ab}\,\bar{q}_b$

- representation by exponentials of twists $g = e^{\hat{\xi}\theta}$

with $\hat{\xi} = \begin{bmatrix} \hat{\omega} & \bar{v} \\ 0 & 0 \end{bmatrix}$



→ most important examples:

a) we know $\dot{\bar{r}} = \bar{\omega} \times \bar{r}$

$$\frac{d}{dt}(\bar{p} - \bar{q}) = \bar{\omega} \times (\bar{p} - \bar{q})$$

true velocity

$$\dot{\bar{p}} = \bar{\omega} \times \bar{p} - \bar{\omega} \times \bar{q}$$
$$= \hat{\omega}\bar{p} - \bar{\omega} \times \bar{q}$$

⇓ SE(3) representation → $\bar{v}$ in $\hat{\xi}$ is velocity correction here!

$$\begin{bmatrix} \dot{\bar{p}} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & -\bar{\omega} \times \bar{q} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{p} \\ 1 \end{bmatrix}$$

b) Solution of $\dot{\bar{p}} = \hat{\xi}\,\bar{p}$ is given by $\bar{p}(t) = e^{\hat{\xi}t}\,\bar{p}(0)$

⇝ transition to θ : $\bar{p}(\theta) = \underbrace{e^{\hat{\xi}\theta}}_{}\,p(0)$

requires "initial condition", cannot cross frames on its own

$$\Rightarrow g_{as}(\theta) = e^{\hat{\xi}\theta}\,g_{as}(0)$$

- Differential Kinematics:

  - Velocity of rigid body motion $g(t) \in SE(3)$ is specified by $\hat{V}^s = \dot{g} g^{-1}$ (spatial velocity)

    $\hat{V}^b = g^{-1} \dot{g}$ (body velocity)

    with $\hat{V}^s$, $\hat{V}^b$ are twists and $\dot{\bar{q}}_a = \hat{V}^s_{ab} \bar{q}_a$

    $\dot{\bar{q}}_b = \hat{V}^b_{ab} \bar{q}_b$

  - Transformation of velocities through Adjoint Trafo:

    $V^s = Ad_g V^b$ with $Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$

  - Forces generalize to Wrenches $\bar{F} = \begin{bmatrix} \bar{f} \\ \bar{\tau} \end{bmatrix} \in \mathbb{R}^6$

  - Equivalent forces/wrenches are given by $\bar{F}_c = Ad^T_{g_{bc}} \bar{F}_b$

  - For rigid body w/ configuration g_{ab} : $\underbrace{\bar{F}^s = \bar{F}_a}_{\text{spatial wrench}}$, $\underbrace{\bar{F}^b = \bar{F}_b}_{\text{body wrench.}}$

- Auxiliary:

Concepts of Rotation matrix, elementary rotations, Euler Angles and Quaternions.