## Assignment 2 – Manipulator Kinematics

Assignment 2 will let you deepen your knowledge about manipulator kinematics and implement forward and inverse kinematics algorithms. Problem 1 lets you revisit key concepts of manipulator kinematics. In problem 2 you develop a forward kinematics algorithm for the Barrett WAM™ arm robot. In problem 3 you extend your algorithm to solving inverse kinematics numerically. In problem 4, you will solve the inverse kinematics problem using the Paden-Kahan method.

**Deliverables:**

1. A .pdf file with your solutions and results
2. Starter code has been provided with this assignment. You can use that to implement your solutions. Submit the folder in Gradescope.

### Problem 1: Main Concepts of Manipulator Kinematics (4pts)

For the three degree of freedom manipulator shown in Figure 1:

(a) (2pts) Find the forward kinematics map.

(b) (2pts) Derive the spatial and body Jacobians.

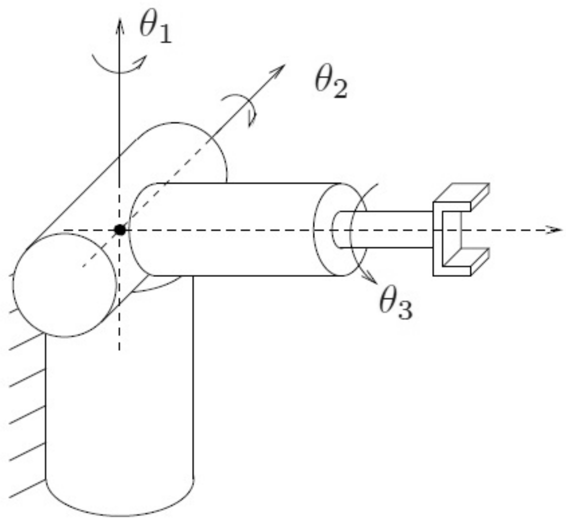


**Figure 1:** A simple three degree of freedom manipulator.

### Problem 2: Forward Kinematics (10+2pts)

You will create a forward kinematics implementation for a 7-dof robotic arm: the Barrett WAM™. A description of the arm's kinematics is provided with in `barret_WAM.pdf`. We will use the joint conventions from this document. The arm is shown in its home configuration, i.e., when all joint positions are zeros. The red curved arrows show the positive directions for each joint. The straight arrows represent frames. Their colors follow the convention: red-green-blue $\rightarrow$ $x$-$y$-$z$.

The WAM is mounted to a mobile platform, which is modelled as a cuboid. A world frame is fixed to the floor. The mobile platform is stationary, and the configuration of its centroid, i.e.,

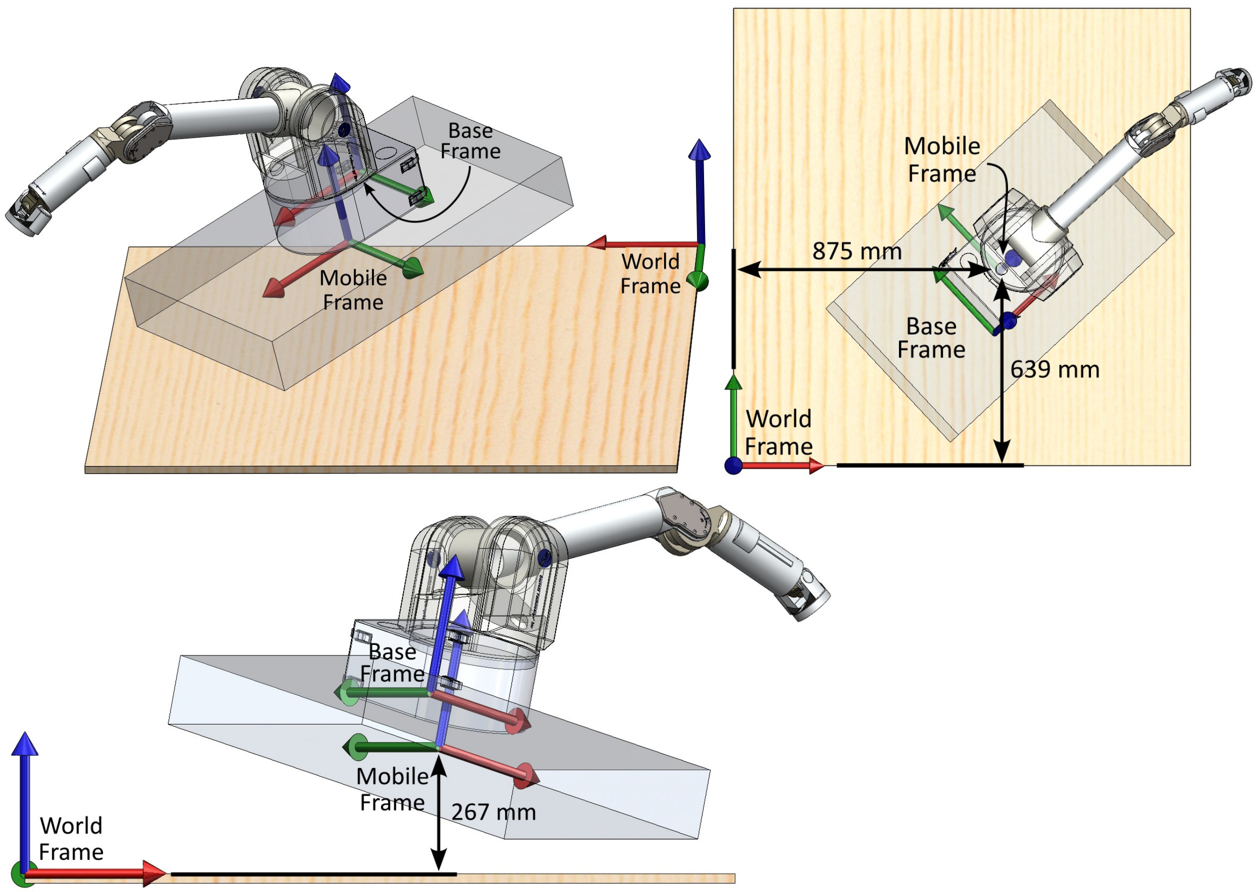


**Figure 2:** The WAM on the mobile platform as seen in the world frame.

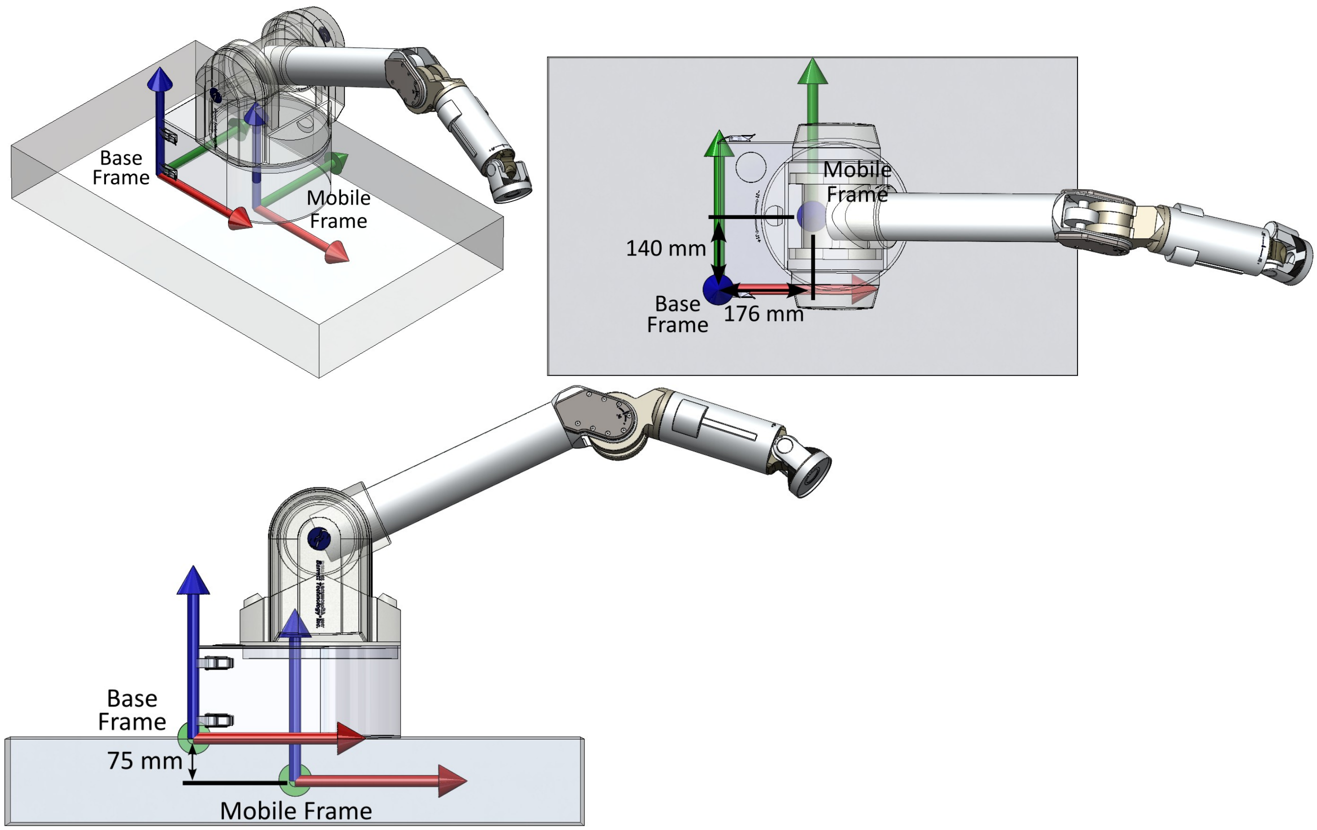


**Figure 3:** The WAM on the mobile platform as seen in mobile platform's body frame.

the mobile frame in the schematics, relative to the world frame is given by

$$T = \begin{bmatrix} R_z\left(43°\right) & \begin{bmatrix} 875 \\ 639 \\ 267 \end{bmatrix} \text{mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_y\left(14°\right) & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix},$$

where $R_z\left(\cdot\right), R_y\left(\cdot\right)$ are the standard rotation matrix about $z$-, $y$-axes, respectively.

With respect to the mobile platform's centroid, the back-right corner of the WAM (the location of the "Base" frame in the Barrett WAM PDF) is located at $-176$mm, $y = -140$mm, $z = 75$mm. The orientation of the WAM's base frame is aligned with the orientation of the mobile platform's body frame. Above are schematics of this scenario with the WAM and the mobile platform.

A whiteboard marker has been attached rigidly to the WAM's end plate, such that the marker is vertical and centered on the end plate when all joints are at zero (as shown in the Barrett WAM PDF). The marker is $12$cm long, so the drawing marker tip is $12$cm from the end plate.

A whiteboard is mounted nearby (in some unknown position and orientation).

Use millimeter (mm) as the length unit, and express all results in the world frame in this problem, unless required otherwise.

1. (2pts) In the home configuration, what is the position of the marker tip, given with respect to the WAM's base frame? (You should be able to answer this without any code.)

The robot has decided to draw something! Its joint trajectory is given in the file `qdata.txt`. Each line is a point on the joint trajectory; the file should have $7835$ points. Each point consists of seven space-separated numbers, one for each joint J1-J7. Joint values are given in radians. The robot is drawing for the entire trajectory.

1. (6pts) Create a program which implements the forward kinematics of the 7-dof Barrett WAM™. Use this program to convert the above joint trajectory into the $x$-$y$-$z$ trajectory of the marker tip in the world frame. Save the marker tip trajectory in a similar format (one line per point, three space-separated values $x$, $y$, $z$). Submit your code as well.

2. (1pt) Where is the whiteboard? Find the whiteboard's position as the centroid of the robot's drawing in the world frame, and compute the upward-pointing unit vector normal to the whiteboard.

3. (1pt) What did the robot write? Include a figure of the drawing, the drawing's centroid, and the whiteboard.

4. (extra 2pt) Create an animation of the robot drawing on the whiteboard. The animation needs to include at least the robot, the drawing, and the whiteboard.

## Problem 3: Numerical Inverse Kinematics (6pts+2pts)

In problem 2, you developed the forward kinematics map $x = \text{FK}(\theta)$ for the Barrett WAM™ arm. Whereas the FK map is unique and easy to compute analytically, the inverse kinematics map

$$\theta = \text{IK}(x)$$

can be quite difficult to compute analytically, and is generally not unique. In this problem, we will explore numerical methods which use the manipulator's Jacobian matrix to iteratively solve the IK problem of the Barrett WAM™ arm.

The document `iksurvey.pdf` serves as a useful overview of these methods. Another good resource for inverse kinematics algorithms is SSVO pp. 132–147.

Each of these methods relies on the manipulator Jacobian. This matrix is the *derivative* of the forward kinematics map function FK. In other words, this matrix maps from joint velocities $\dot{\theta}$ to end-effector velocities $v$:

$$v = J(\theta)\,\dot{\theta}$$

In fact, there are different Jacobians, depending on the representation of the velocity $v$. $J$ depends on the reference point (e.g. the marker tip), whether the rotational and/or translational components of velocity are reported, and how the velocities themselves are represented (e.g. velocity vectors, spatial twists, euler angle/quaternion parameter derivatives, etc).

For this problem, the arm is mounted to a tabletop as in problem 2 and we will ignore joint limits.

1. (2pts) Implement a function which computes a Jacobian matrix for the marker tip of the Barrett WAM™arm. The Jacobian matrix must produce both the linear and angular components of the marker tip velocity. Indicate the representation of velocities that you used. Evaluate your function at the following joint configuration,

   $$\theta_s = [\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7\ ]^T,$$

   and include the result in your submission.

2. (2pts) The marker tip location $x_s$ for the joint configuration $\theta_s$ above is given by

   $$x_s = [\ 0.44543,\ 1.12320,\ 2.22653,\ -0.29883,\ 0.44566,\ 0.84122,\ -0.06664\ ]^T.$$

   Note that $x_s$ is given as a position plus a quaternion $[x, y, z, q_i, q_j, q_k, q_0]^T$

   Suppose the *desired* marker tip location is

   $$x_d = [\ 1.362,\ 1.275,\ 1.406,\ -0.069,\ 0.599,\ 0.687,\ 0.406\ ]^T$$

and compute a velocity $v$, in the same representation as your Jacobian, which would move the marker tip from $x_s$ in the direction of $x_d$.

3. (2pts) Implement the Jacobian pseudoinverse iterative method using your results from questions 1 and 2. Indicate any choices for parameters that you chose (step sizes, stopping conditions, etc). Use your implementation to move from the starting configuration $\theta_s$ above to each of these goal marker tip poses:

$$x_{d1} = [\ 1.362,\ 1.275,\ 1.406,\ -0.069,\ 0.599,\ 0.687,\ 0.406\ ]^T$$

$$x_{d2} = [\ 1.707,\ 1.201,\ 1.23,\ -0.054,\ 0.472,\ 0.388,\ 0.79\ ]^T$$

For each goal pose, submit a joint trajectory file giving the joint values at each iteration (each file should start with $\theta_s$). Indicate the final joint configurations $\theta_{di}$ for each problem. Briefly discuss the performance of your algorithm.

4. (+2pts) Implement the damped least squares method. Use your implementation to solve each of the goal poses from the previous question. Again, submit a joint trajectory file and indicate the final joint configurations for each. Briefly discuss its performance.

### Problem 4: Analytical Inverse Kinematics (8 pts)

In this problem, you will solve the Paden-Kahan subproblems to calculate the inverse kinematics of the elbow manipulator studied in class. Refer to Chapter 3, Section 3.2 of *A Mathematical Introduction to Robotic Manipulators*. You have been provided with a starter code to organize your implementations.

1. (3 pts) Implement functions that can solve each of the following subproblems. (You are not required to derive the solutions; these are known Paden-Kahen subproblems for which you can look up the solutions.)

   a) **Subproblem 1:** Let $\xi$ be a zero-pitch twist with unit magnitude and $p, q \in \mathbb{R}^3$ be two points. Find $\theta$ such that
   $$e^{\hat{\xi}\theta} p = q$$

   b) **Subproblem 2:** Let $\xi_1, \xi_2$ be two zero-pitch twists with unit magnitude such that they intersect at least one point, and $p, q \in \mathbb{R}^3$ be two points. Find $\theta_1, \theta_2$ such that
   $$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = q$$

   c) **Subproblem 3:** Let $\xi$ be a zero-pitch twist with unit magnitude and $p, q \in \mathbb{R}^3$ be two points. Let $\delta$ be some positive real number. Find $\theta$ such that
   $$||e^{\hat{\xi}\theta} p - q|| = \delta$$

2. (5 pts) Consider the elbow manipulator given below. The manipulator has 6 revolute joints. Any revolute joint can be expressed as a twist with zero pitch and unit magnitude for rotation. Consider 2 points, $p_b, q_w \in \mathbb{R}^3$ represented in the base frame. These are chosen strategically at the intersection of multiple axes, as seen from the figure. In this section, we will extend and implement what you have already studied in class.

   Let $g_d$ be some desired pose of the end-effector frame in the space frame and $g_{st}(0)$ be the pose of the end-effector frame when all the joints are set to 0's. We want to solve the equation
   $$\prod_{i=1}^{6} e^{\hat{\xi}_i\theta_i} \; g_{st}(0) = g_d$$
   for $\theta_i$, given $\xi_{1:6}$

   Finish the starter code to solve this inverse kinematics problem using the Paden-Kahan Method. Follow these steps to finish the implementation.

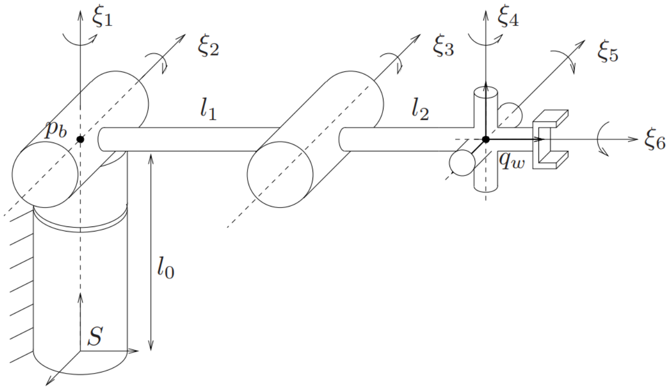


**Figure 4:** Elbow Manipulator

a) Define coordinate frame $S$ and calculate twists $\xi_{1:6}$ given $l_0, l_1, l_2$

b) Calculate points $p_b, q_w$ using the calculated twists. Also, calculate some point $p \in \mathbb{R}^3$ which is on the axis of $\xi_6$ and not on the axis of $\xi_4, \xi_5$.

c) Using the product of the exponential formula, apply forward kinematics to the point $q_w$ to derive this equation.

$$\prod_{i=1}^{3} e^{\hat{\xi}_i \theta_i} \; q_w = g_d g_{st}^{-1}(0) q_w$$

Subtract $p_b$ from both sides of this equation to derive the following result.

$$||e^{\hat{\xi}_3 \theta_3} q_w - p_b|| = ||g_d g_{st}^{-1}(0) q_w - p_b||$$

Note that his result is identical to subproblem 3. Use implementation of Subproblem 3 to calculate $\theta_3$

d) Substitute $\theta_3$ in your result from part (a) to get a new equation that is similar to subproblem 2. Use the implementation of subproblem 2 to calculate $\theta_1, \theta_2$.

e) Finally substitute $\theta_{1-3}$ in the PoE formula to get the following result

$$\prod_{i=4}^{6} e^{\hat{\xi}_i \theta_i} = \prod_{i=1}^{3} e^{-\hat{\xi}_i \theta_i} \; g_d g_{st}^{-1}(0)$$

Multiply this with $p$ to obtain a result solvable using subproblem 2 and finally calculate all the angles.