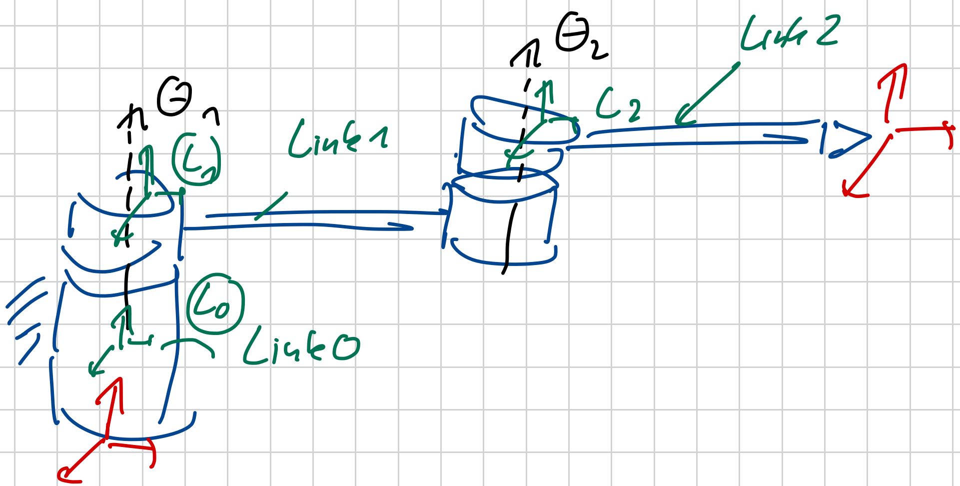
Lecture #5: 27 Jan

# Ch 2 Manipulator Kinematics

## 2.1 Forward Kinematics of Segmented Chains

FK solves $g_{ST}(\bar{\theta})$ where $\bar{\theta}$ is vector of all joint positions and $g_{ST}$ maps from base of manipulator to its end-effector

### (A) Basic Conventions




Ⓢ Spatial (Base) Frame

Ⓣ Tool frame (end-effector frame)

- joints are numbered from 1 to n, starting at base
- Links are numbered from 0 to n, with base being Link 0.

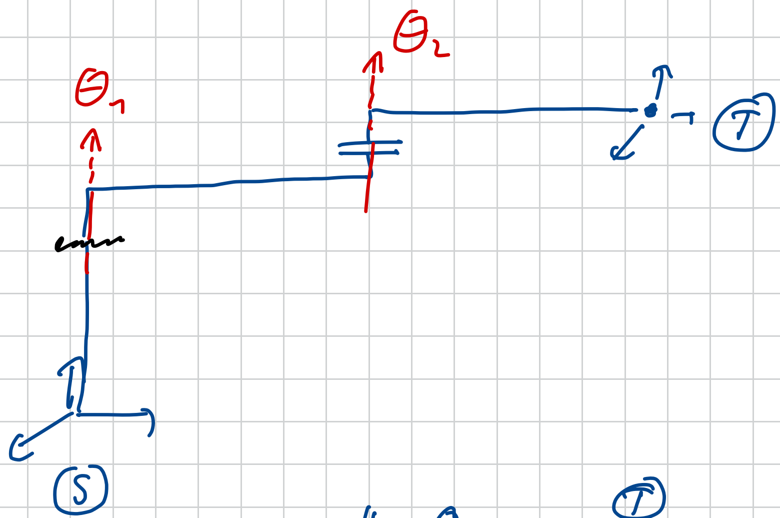
## Joint Space $Q$:

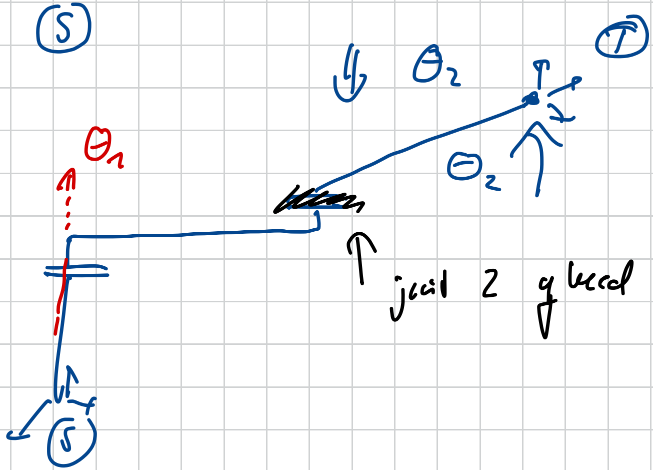
- revolute: $\theta_i \in [0, 2\pi) \Rightarrow \theta_i \in S^1$ : unit circle
- prismatic: $\theta_i \in \mathbb{R}$ (also called "angle")
- $Q = S^1 \times S^1 \times \mathbb{R} \times \ldots \times$
- $\bar{\theta} = (\theta_1, \theta_2, \ldots, \theta_n) \in Q$ : Manipulator Configuration.

$$\boxed{\text{FK problem: Given a set of joint angles } \bar{\theta} \in Q \text{, what is pose of end-effector (tool frame) } \bar{\theta} \Rightarrow g_{ST}(\bar{\theta})}$$

$$g_{ST}(\bar{\theta}) = g_{se_1}(\theta_1) \cdot g_{e_1e_2}(\theta_2) \cdot \ldots \cdot g_{e_{n-1}e_n}(\theta_n) \cdot g_{e_nt}$$

## (B) Solving FK in Screw Kinematics: What is the product of exponentials formula?




**Step 1:** let's assume that joint 1 is welded or glued together

- recall from chapter 1:

$$g_{ST}(\theta_2) = e^{\hat{\xi}_2 \theta_2} \cdot g_{ST}(0)$$

$$\left( g_{as}(\theta) = e^{\hat{\xi} \theta} g_{as}(0) \right)$$

**Step 2:**

$$g_{ST}(\theta_2)$$

$$\Longrightarrow g_{ST}(\theta_1) = e^{\hat{\xi}_1 \theta_1} \cdot g_{ST}(\theta_2)$$

$$\Longrightarrow \boxed{g_{ST}(\theta_1, \theta_2) = e^{\hat{\xi}_1 \theta_1} \cdot e^{\hat{\xi}_2 \theta_2} \cdot g_{ST}(0)}$$

In general:

$$\boxed{g_{ST}(\bar{\theta}) = e^{\hat{\xi}_1 \theta_1} \cdot e^{\hat{\xi}_2 \theta_2} \cdot \ldots \cdot e^{\hat{\xi}_n \theta_n} \, g_{ST}(\bar{0})}$$

"product of exponentials" formula

Remarks:

- $\bar{\xi}_i$ must be numbered sequentially starting from base,
- $g_{ST}(\bar{\theta})$ can be computed w/o regarding specific order in which individual joint rotations are performed.
- product of exponentials formula solves FK using only one frame, the base frame S!
- Specifically $\bar{\xi}_i$'s are defined in base frame!

- <u>Procedure:</u>

1) Define <u>any</u> reference config corresponding to $\bar{\theta} = \bar{0}$

2) Compute $g_{ST}(\bar{0})$

3) Construct for each joint the twist $\bar{\xi}_i$

if $\xi_i$ is revolute: $\bar{\xi}_i = \begin{bmatrix} -\bar{\omega}_i \times \bar{q}_i \\ \bar{\omega}_i \end{bmatrix}$

where $\bar{\omega}_i$ is unit vector along axis of rotation

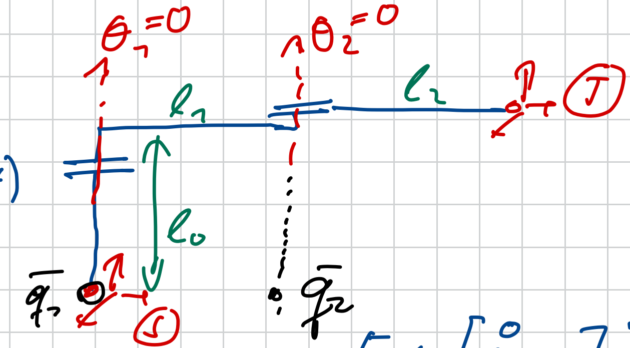
$\bar{q}_i$ is <u>any</u> point on this axis

$\xi_i$ is prismatic: $\bar{\xi}_i = \begin{bmatrix} \bar{v}_i \\ \bar{0} \end{bmatrix}$

where $\bar{v}_i$ unit vector along translational axis.

4) Enjoy computing FK using product of exponentials formula!

1)




2) $g_{ST}(\bar{0}) = \begin{bmatrix} \mathbb{1} & \begin{bmatrix} 0 \\ l_1+l_2 \\ l_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$

3) $\bar{\xi}_1$: $\left.\begin{array}{l} \bar{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \bar{q}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{array}\right\} \quad \bar{\xi}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\bar{\xi}_2$: $\left.\begin{array}{l} \bar{\omega}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \bar{q}_2 = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} \end{array}\right\} \quad -\bar{\omega}_2 \times \bar{q}_2 = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}$

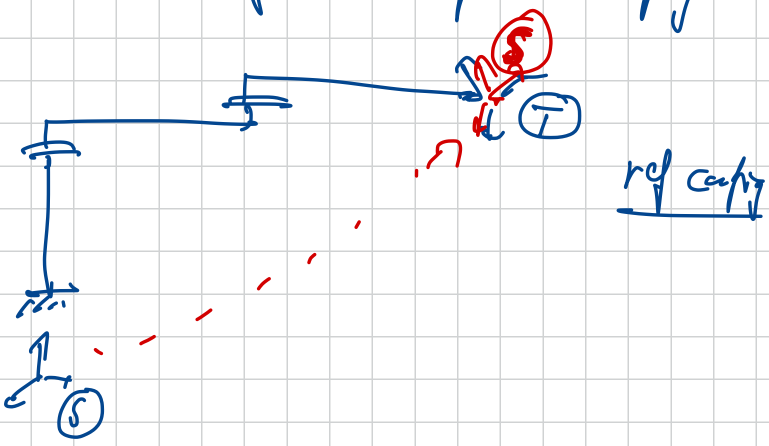
$\Rightarrow \quad \bar{\xi}_2 = \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$g_{ST}(\bar{\theta}) = e^{\hat{\xi}_1 \theta_1} \cdot e^{\hat{\xi}_2 \theta_2} \cdot \begin{bmatrix} \mathbb{1} & \begin{bmatrix} 0 \\ l_1+l_2 \\ l_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$

## (c) Choices that simplify FK calculation

$$g_{ST}(\bar{\theta}) = e^{\hat{S}_1 \theta_1} \cdot \ldots \cdot e^{\hat{S}_n \theta_n} \cdot g_{ST}(\bar{0})$$

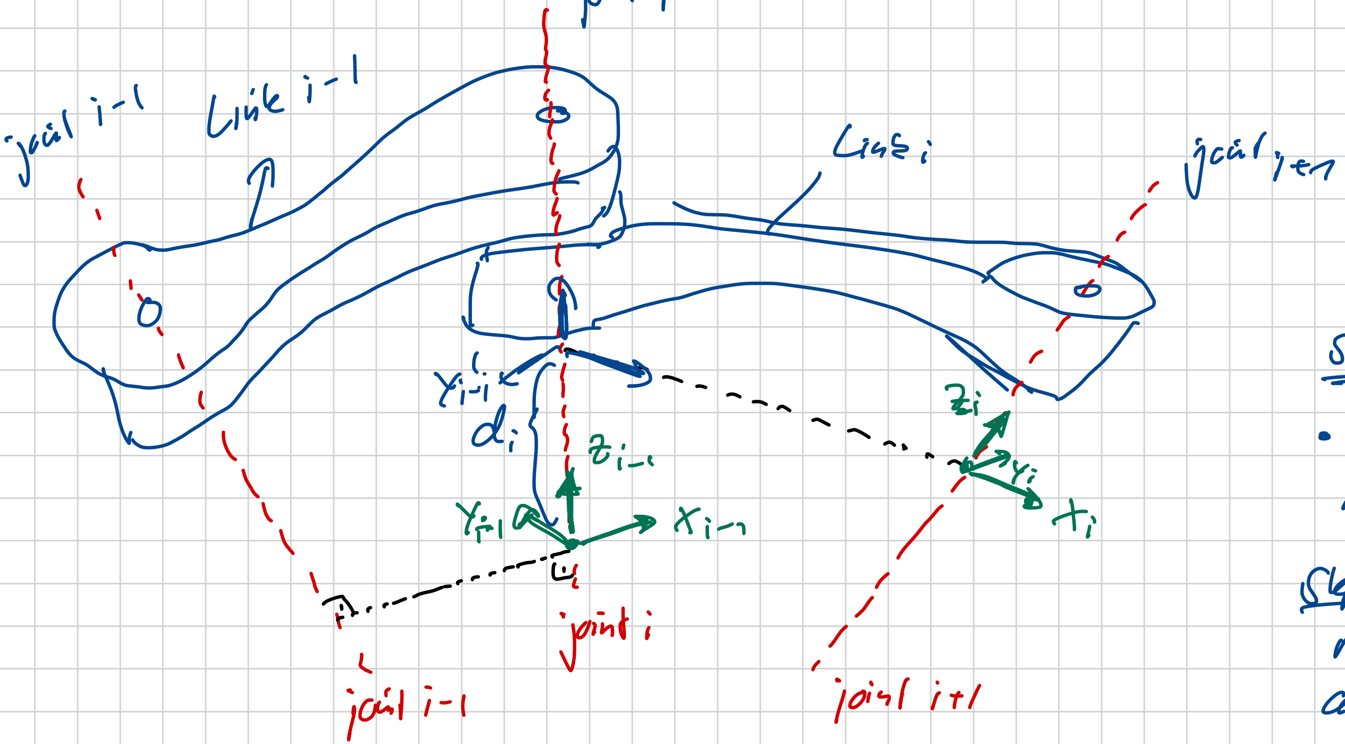
- if at reference config (S) is collocated w/ (T) $\Rightarrow$ $g_{ST}(\bar{0}) = \mathbb{1}$




- Choose reference config such that kinematic analysis simplifies:
  - outstretched positions
  - choose points $\bar{q}_i$ on axes $i$ such that they have many zeros

## (D) Relationship w/ Denavit - Hartenberg Parameterization (DH)

- DH Parameterization uses least number of parameters to describe FK.
  $\Rightarrow$ DH requires only 4 parameters for each degree of freedom.
- ( we have 6 parameters in Screw Kinematics )
- Price we pay for reducing # of parameters : Link frames have to be placed in specific locations.




**Step 1:**

- translation ~~along~~ along $z_{i-1}$ by distance $d_i$

**Step 2:**

rotation about $z_{i-1}$ to align w/ new x axis

$$\Rightarrow g_{12} = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Steps 3 and 4:

$$g_{34} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_{i-1\,i} = g(d_i, \theta_i, a_i, \alpha_i)$$

$$= g_{12} \cdot g_{34}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i \cdot c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i\, s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## (E) Manipulator Workspace : Where can a ~~[illegible]~~ manipulator reach?

**Def:** The workspace of a manipulator is set of all reachable end-effector poses (configurations): $W = \{ g_{ST}(\bar{\theta}) : \bar{\theta} \in Q \}$

more easily understood:

a) reachable workspace: All positions $\bar{p}(\bar{\theta})$ that end-effector can reach w/o considering orientation.

b) dexterous workspace: All positions $p(\bar{\theta})$ that end-effector can reach w/ <u>any</u> orientation.