1 Symmetry and Groups

1.1. The center of a group G (denoted by Z) is defined to be the set of elements $\{z_1, z_2, \cdots\}$ that commute with all elements of G, that is $z_i g = g z_i$ for all g. Show that Z is an abelian subgroup of G.

We must show that Z contains the identity, is closed under multiplication, and contains the inverses of all its elements.

Because the left inverse and right inverse are the same, we have $Ig = gI$, and so, by the definition of Z, $I \in Z$.

Assume that a product of two elements in Z produces an element not in Z. That is,

$$z_i z_j = g_l \notin Z.$$

This would imply $g_l g_k \neq g_k g_l$, or $z_i z_j g_k \neq g_k z_i z_j$. But, by definition of the elements of Z, $z_i z_j g_k = z_i g_k z_j = g_k z_i z_j$, which contradicts the previous result. We see Z is closed.

Now, assume there exists an out-of-set inverse z_i^{-1} of an element z_i. Then $z_i^{-1} g_j \neq g_j z_i^{-1}$. Right multiplying by z_i gives us

$$z_i^{-1} g_j z_i \neq g_j z_i^{-1} z_i = g_j I = g_j,$$

or $z_i^{-1} g_j z_i \neq g_j$. But, by virtue of the nature of Z,

$$z_i^{-1} g_j z_i = z_i^{-1} z_i g_j = I g_j = g_j,$$

and so our above result becomes $g_j \neq g_j$. By contradiction, Z must be abelian subgroup of G.

1.2. Let $f(g)$ be a function of the elements in a finite group G, and consider the sum $\sum_{g \in G} f(g)$. Prove the identity $\sum_{g \in G} f(g) = \sum_{g \in G} f(gg') = \sum_{g \in G} f(g'g)$ for g' an arbitrary element of G. We will need this identity again and again in chapters II.1 and II.2.

This identity holds if both gg' and $g'g$ cycle through all elements of G as g goes through all available values. Imagine creating a multiplication table for our group. Cycling through g, we see the products gg' and $g'g$ move one-by-one through a row or column. As each row and column contains every element of G, the products gg' and $g'g$ touch upon every element in the group. That

$$\sum_{g \in G} f(g) = \sum_{g \in G} f(gg') = \sum_{g \in G} f(g'g)$$

follows.

1.3. Show that $Z_2 \otimes Z_4 \neq Z_8$.

From the discussion preceding this problem set, we know that the possibility of $Z_2 \otimes Z_4$ being isomorphic to Z_8 would require 2 and 4 to be coprime (which they clearly aren't). Geometrically, we can think of Z_8 of lying on a circle, while $Z_2 \otimes Z_4$ lies on a torus. These shapes are not homeomorphic, and so it is not a surprise that our groups aren't isomorphic.

To give an explicit example of their difference, consider the elements that square to the identity. In Z_8, we have $(-1)^2 = 1$ and $(1)^2 = 1$, whereas $Z_2 \otimes Z_4$ gives us

$$(1,1)^2 = (1,-1)^2 = (-1,1)^2 = (-1,-1)^2.$$

1.4. Find all groups of order 6.

By Lagrange's theorem, all subgroups of a group of order 6 must be Z_2, Z_3, or Z_6. Clearly, if we take Z_6 to be a subgroup, then that subgroup *is* the group. What if we take Z_2 to be a subgroup? Forming the direct product of this with Z_3 is the only possible construction that gives us 6 elements: $Z_3 \otimes Z_3$ is of order 9 and $Z_2 \otimes Z_2 \otimes Z_2$ is of order 8.

To restate our findings more concisely: there are two groups of order 6, Z_6 and $Z_2 \otimes Z_3$.