1 Review of Classical Mechanics

1.1. Consider the following system, called a *harmonic oscillator*. The block has a mass m and lies on a frictionless surface. The spring has a force constant k. Write the Lagrangian and get the equation of motion.

From the diagram, we can immediately write

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

This gives us a conjugate momentum and generalized force of

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial \mathcal{L}}{\partial x} = -kx$$

which can be combined to give the equation of motion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x} = -kx = \frac{\partial \mathcal{L}}{\partial x}$$

1.2. Do the same for the coupled-mass problem discussed at the end of Section 1.8. Compare the equations of motion with Eqs. (1.8.24) and (1.8.25).

Returning to Figure 1.5, we can write

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 \\ &= \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) \\ V &= \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k(x_2 - x_1)^2 \\ &= k(x_1^2 - x_1x_2 + x_2^2) \end{aligned}$$

and hence

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - k(x_1^2 - x_1x_2 + x_2^2)$$

Feeding the above into the Euler-Lagrange equations gives us

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(m\dot{x}_1\Big) = m\ddot{x}_1 = -2kx_1 + kx_2 = \frac{\partial \mathcal{L}}{\partial x_1} \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(m\dot{x}_2\Big) = m\ddot{x}_2 = x_1 - 2kx_2 = \frac{\partial \mathcal{L}}{\partial x_2} \end{aligned}$$

which is exactly Eqs. (1.8.24) and (1.8.25).

1.3. A particle of mass m moves in three dimensions under a potential $V(r, \theta, \phi) = V(r)$. Write its $\mathcal{L}$ and find the equations of motion.

Using a similar geometric argument to Shankar, we see that the distance covered by a particle in time Δt is

$$\mathrm{d}S = [(\mathrm{d}r)^2 + (r\sin(\theta)\mathrm{d}\phi)^2 + (r\mathrm{d}\theta)^2]$$

where ϕ is the azimuthal angle and θ is the inclination. This gives us a squared velocity of

$$v^2 = \dot{r}^2 + r^2\sin^2(\theta)\dot{\phi}^2 + r^2\dot{\theta}^2$$

and thus a Lagrangian of

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2(\theta)\dot{\phi}^2 + r^2\dot{\theta}^2) - V(r)$$

The equations of motion for this particle are given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{r}} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(m\dot{r}\Big) = m\ddot{r} = mr\sin^2(\theta)\dot{\phi}^2 + mr\dot{\theta}^2 - \frac{\partial V(r)}{\partial r} = \frac{\partial\mathcal{L}}{\partial r}\\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(mr^2\sin^2(\theta)\dot{\phi}\Big) = 2mr\dot{r}\sin^2(\theta)\dot{\phi} + 2mr^2\sin(\theta)\cos(\theta)\dot{\theta}\dot{\phi} + mr^2\sin^2(\theta)\ddot{\phi} = 0 = \frac{\partial\mathcal{L}}{\partial\phi}\\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(mr^2\dot{\theta}\Big) = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = mr^2\sin(\theta)\cos(\theta)\dot{\phi}^2 = \frac{\partial\mathcal{L}}{\partial\theta}
\end{aligned}$$

Simplifying, these become

$$\begin{aligned}
m\ddot{r} &= mr(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) - \frac{\partial V(r)}{\partial r}\\
m\ddot{\phi} &= -2m\dot{\phi}\Big(\frac{\dot{r}}{r} - \dot{\theta}\cot\theta\Big)\\
m\ddot{\theta} &= m\Big(\dot{\phi}^2\sin\theta\cos\theta - 2\frac{\dot{r}}{r}\dot{\theta}\Big)
\end{aligned}$$

1.4. Derive Eq. (2.3.6) from (2.3.5) by changing variables.

This is a straightforward exercise of algebra,

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}|\dot{\mathbf{r}}_2|^2 - V(\mathbf{r}_1 - \mathbf{r}_2)\\
&= \frac{1}{2}m_1\Big|\dot{\mathbf{r}}_{\mathrm{CM}} + \frac{m_2\dot{\mathbf{r}}}{m_1+m_2}\Big|^2 + \frac{1}{2}m_2\Big|\dot{\mathbf{r}}_{\mathrm{CM}} - \frac{m_1\dot{\mathbf{r}}}{m_1+m_2}\Big|^2 - V(\mathbf{r})\\
&= \frac{1}{2}m_1\Big(|\dot{\mathbf{r}}_{\mathrm{CM}}|^2 + \frac{2m_2\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}_{\mathrm{CM}}}{m_1+m_2} + \frac{m_2^2|\dot{\mathbf{r}}|^2}{(m_1+m_2)^2}\Big) + \frac{1}{2}m_2\Big(|\dot{\mathbf{r}}_{\mathrm{CM}}|^2 - \frac{2m_1\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}_{\mathrm{CM}}}{m_1+m_2} + \frac{m_1^2|\dot{\mathbf{r}}|^2}{(m_1+m_2)^2}\Big) - V(\mathbf{r})\\
&= \frac{1}{2}(m_1+m_2)|\dot{\mathbf{r}}_{\mathrm{CM}}|^2 + \frac{m_1m_2\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}_{\mathrm{CM}}}{m_1+m_2} - \frac{m_1m_2\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}_{\mathrm{CM}}}{m_1+m_2} + \frac{1}{2}\frac{m_1m_2^2 + m_1^2m_2}{(m_1+m_2)^2}|\dot{\mathbf{r}}|^2 - V(r)\\
&= \frac{1}{2}(m_1+m_2)|\dot{\mathbf{r}}_{\mathrm{CM}}|^2 + \frac{1}{2}\frac{m_1m_2(m_1+m_2)}{(m_1+m_2)^2}|\dot{\mathbf{r}}|^2 - V(r)\\
&= \frac{1}{2}(m_1+m_2)|\dot{\mathbf{r}}_{\mathrm{CM}}|^2 + \frac{1}{2}\frac{m_1m_2}{m_1+m_2}|\dot{\mathbf{r}}|^2 - V(r)
\end{aligned}$$

1.5. Show that if $T = \sum_i\sum_j T_{ij}(q)\dot{q}^i\dot{q}^j$, where $\dot{q}$'s are generalized velocities, $\sum_i p_i\dot{q}^i = 2T$.

Assuming the Lagrangian built from T contains a potential term independent of velocity, the conjugate momentum to q is

$$\begin{aligned}
p_i &= \frac{\partial L}{\partial \dot{q}^i} \\
&= \frac{\partial}{\partial \dot{q}^i}\Big(\sum_j \sum_k T_{kj}(q)\dot{q}^j \dot{q}^k\Big) \\
&= \sum_j \sum_k T_{kj}(q)\frac{\partial \dot{q}^j}{\partial \dot{q}^i}\dot{q}^k + \sum_j \sum_k T_{kj}(q)\dot{q}^j \frac{\partial \dot{q}^k}{\partial \dot{q}^i} \\
&= \sum_j \sum_k T_{kj}(q)\delta^j_i \dot{q}^k + \sum_j \sum_k T_{kj}(q)\dot{q}^j \delta^k_i \\
&= \sum_k T_{ki}(q)\dot{q}^k + \sum_j T_{ij}(q)\dot{q}^j \\
&= 2\sum_j T_{ij}(q)\dot{q}^j
\end{aligned}$$

where we have assumed that T_{ij} is symmetric in the last equality. From the above, we see

$$\sum_i p_i \dot{q}^i = 2\sum_i \sum_j T_{ij}(q)\dot{q}^i \dot{q}^j = 2T.$$

1.6. Using the conservation of energy, show that the trajectories in phase space for the oscillator are ellipses of the form $(x/a)^2 + (p/b)^2 = 1$, where $a^2 = 2E/k$ and $b^2 = 2mE$.

The Hamiltonian (and thus the energy) for the classical harmonic oscillator is given by

$$\mathcal{H} = \frac{p^2}{2m} + \frac{k}{2}x^2 \equiv E.$$

Since energy is conserved, $\partial\mathcal{H}/\partial t = 0$ and we can divide by E (a constant) to get

$$\Big(\frac{p}{\sqrt{2mE}}\Big)^2 + \Big(\frac{x}{\sqrt{2E/k}}\Big)^2 = 1,$$

or, defining $a^2 = 2E/k$ and $b^2 = 2mE$,

$$\Big(\frac{x}{a}\Big)^2 + \Big(\frac{p}{b}\Big)^2 = 1.$$

1.7. Solve Exercise 2.1.2 using the Hamiltonian formalism.

In this simple case, we can make the replacement $\dot{x}_i^2 \to p_i^2/m^2$ and flip the sign of V in the found $\mathcal{L}$ to arrive at

$$\mathcal{H} = T + V = \frac{p_1^2 + p_2^2}{2m} + k(x_1^2 - x_1 x_2 + x_2^2)$$

To obtain the dynamical equations for the system, we compute

$$\frac{\partial \mathcal{H}}{\partial p_i} = \frac{p_i}{m} = \dot{x}_i \quad \text{and} \quad -\frac{\partial \mathcal{H}}{\partial x_i} = kx_j - 2kx_i = \dot{p}_i$$

where $j \neq i$. Taking the time derivative of $\partial\mathcal{H}/\partial p_i$ allows us to substitute the resulting equations into $-\partial\mathcal{H}/\partial x_i$ to obtain

$$\ddot{x}_i = \frac{k}{m}(x_j - 2x_i)$$

which is exactly what we found in Exercise 2.1.2.

1.8. Show that $\mathcal{H}$ corresponding to $\mathcal{L}$ in Eq. (2.3.6) is $\mathcal{H} = |\mathbf{p}_{\text{CM}}|^2/2M + |\mathbf{p}|^2/2\mu + V(\mathbf{r})$, where M is the total mass, μ is the reduced mass, $\mathbf{p}_{\text{CM}}$ and $\mathbf{p}$ are the momenta conjugate to $\mathbf{r}_{\text{CM}}$ and $\mathbf{r}$, respectively.

Starting from the Lagrangian,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(m_1 + m_2)|\dot{\mathbf{r}}_{\text{CM}}|^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}|\dot{\mathbf{r}}|^2 - V(r),\\ &= \frac{M}{2}|\dot{\mathbf{r}}_{\text{CM}}|^2 + \frac{\mu}{2}|\dot{\mathbf{r}}|^2 - V(r),\end{aligned}$$

we can find the conjugate momenta to $\mathbf{r}$ and $\mathbf{r}_{\text{CM}}$ via

$$\begin{aligned}\mathbf{p}_{\text{CM}} &= \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_{\text{CM}}} = M\dot{\mathbf{r}}_{\text{CM}},\\ \mathbf{p} &= \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \mu\dot{\mathbf{r}}.\end{aligned}$$

Performing the necessary Legendre transform reveals

$$\begin{aligned}\mathcal{H} &= \mathbf{p}\cdot\dot{\mathbf{r}} + \mathbf{p}_{\text{CM}}\cdot\dot{\mathbf{r}}_{\text{CM}} - \mathcal{L}\\ &= \frac{|\mathbf{p}|^2}{\mu} + \frac{|\mathbf{p}_{\text{CM}}|^2}{M} - \Big(\frac{M}{2}\frac{|\mathbf{p}_{\text{CM}}|^2}{M^2} + \frac{\mu}{2}\frac{|\mathbf{p}|^2}{\mu^2} - V(r)\Big)\\ &= \frac{|\mathbf{p}|^2}{2\mu} + \frac{|\mathbf{p}_{\text{CM}}|^2}{2M} + V(r)\end{aligned}$$

1.9. Show that

$$\begin{aligned}\{\omega, \lambda\} &= -\{\lambda, \omega\}\\ \{\omega, \lambda + \sigma\} &= \{\omega, \lambda\} + \{\omega, \sigma\}\\ \{\omega, \lambda\sigma\} &= \{\omega, \lambda\}\sigma + \lambda\{\omega, \sigma\}\end{aligned}$$

Starting from the definition, we see

$$
\begin{aligned}
\{\omega, \lambda\} &= \sum_i \Big(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\Big) \\
&= -\sum_i \Big(\frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i} - \frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i}\Big) \\
&= -\{\lambda, \omega\}
\end{aligned}
$$

while the linearity of the partial derivative produces

$$
\begin{aligned}
\{\omega, \lambda + \sigma\} &= \sum_i \Big(\frac{\partial \omega}{\partial q_i}\frac{\partial (\lambda+\sigma)}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial (\lambda+\sigma)}{\partial q_i}\Big) \\
&= \sum_i \Big(\frac{\partial \omega}{\partial q_i}\Big[\frac{\partial \lambda}{\partial p_i} + \frac{\partial \sigma}{\partial p_i}\Big] - \frac{\partial \omega}{\partial p_i}\Big[\frac{\partial \lambda}{\partial q_i} + \frac{\partial \sigma}{\partial q_i}\Big]\Big) \\
&= \sum_i \Big(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} + \frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_i}\Big) \\
&= \sum_i \Big(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\Big) + \sum_i \Big(\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_i}\Big) \\
&= \{\omega, \lambda\} + \{\omega, \sigma\}
\end{aligned}
$$

and the product rule gives

$$
\begin{aligned}
\{\omega, \lambda\sigma\} &= \sum_i \Big(\frac{\partial \omega}{\partial q_i}\frac{\partial (\lambda\sigma)}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial (\lambda\sigma)}{\partial q_i}\Big) \\
&= \sum_i \Big(\frac{\partial \omega}{\partial q_i}\Big[\frac{\partial \lambda}{\partial p_i}\sigma + \lambda\frac{\partial \sigma}{\partial p_i}\Big] - \frac{\partial \omega}{\partial p_i}\Big[\frac{\partial \lambda}{\partial q_i}\sigma + \lambda\frac{\partial \sigma}{\partial q_i}\Big]\Big) \\
&= \sum_i \Big(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i}\sigma + \lambda\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\sigma - \lambda\frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_i}\Big) \\
&= \sum_i \Big(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\Big)\sigma + \lambda\sum_i \Big(\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_i}\Big) \\
&= \{\omega, \lambda\}\sigma + \lambda\{\omega, \sigma\}
\end{aligned}
$$

1.10. (i) Verify Eqs. (2.7.4) and (2.7.5). (ii) Consider a problem in two dimensions given by $\mathcal{H} = p_x^2 + p_y^2 + ax^2 + by^2$. Argue that if $a = b$, $\{l_z, \mathcal{H}\}$ must vanish. Verify by explicit computation.

Eq. (2.7.4) is immediately obvious from the fact that $\partial q_i/\partial q_j = \partial p_i/\partial p_j = \delta_{ij}$, and so $\{q_i, q_j\} = \{p_i, p_j\} = 0$. Furthermore,

$$
\begin{aligned}
\{q_i, p_j\} &= \sum_k \Big(\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k}\Big) \\
&= \sum_k \delta_{ik}\delta_{jk} \\
&= \delta_{ij}
\end{aligned}
$$

For eq (2.7.5), we see

$$\begin{aligned}
\{q_i, \mathcal{H}\} &= \sum_j \Big(\frac{\partial q_i}{\partial q_j}\frac{\partial \mathcal{H}}{\partial p_j} - \frac{\partial q_i}{\partial p_j}\frac{\partial \mathcal{H}}{\partial q_j}\Big) \\
&= \sum_j \delta_{ij}\frac{\partial \mathcal{H}}{\partial p_j} \\
&= \frac{\partial \mathcal{H}}{\partial p_i} \\
&= \dot{q}_i
\end{aligned}$$

and

$$\begin{aligned}
\{p_i, \mathcal{H}\} &= \sum_j \Big(\frac{\partial p_i}{\partial q_j}\frac{\partial \mathcal{H}}{\partial p_j} - \frac{\partial p_i}{\partial p_j}\frac{\partial \mathcal{H}}{\partial q_j}\Big) \\
&= -\sum_j \delta_{ij}\frac{\partial \mathcal{H}}{\partial q_j} \\
&= -\frac{\partial \mathcal{H}}{\partial q_i} \\
&= \dot{p}_i
\end{aligned}$$

If $a = b$ in the given Hamiltonian, the potential energy is dependent only on the radial distance from the origin, i.e. the Hamiltonian is circularly symmetric. With no preferred direction in space, we expect l_z to be conserved, or $\{l_z, \mathcal{H}\} = 0$. We can verify this explicitly via by noting that

$$\begin{aligned}
\frac{\partial l_z}{\partial x} &= \frac{\partial}{\partial x}(xp_y - yp_x) = p_y \\
\frac{\partial l_z}{\partial y} &= \frac{\partial}{\partial y}(xp_y - yp_x) = -p_x \\
\frac{\partial l_z}{\partial p_x} &= \frac{\partial}{\partial p_x}(xp_y - yp_x) = -y \\
\frac{\partial l_z}{\partial p_y} &= \frac{\partial}{\partial p_y}(xp_y - yp_x) = x
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial x} &= \frac{\partial}{\partial x}(p_x^2 + p_y^2 + ax^2 + by^2) = 2ax \\
\frac{\partial \mathcal{H}}{\partial y} &= \frac{\partial}{\partial y}(p_x^2 + p_y^2 + ax^2 + by^2) = 2by \\
\frac{\partial \mathcal{H}}{\partial p_x} &= \frac{\partial}{\partial p_x}(p_x^2 + p_y^2 + ax^2 + by^2) = 2p_x \\
\frac{\partial \mathcal{H}}{\partial p_y} &= \frac{\partial}{\partial p_y}(p_x^2 + p_y^2 + ax^2 + by^2) = 2p_y
\end{aligned}$$

and so, with $a = b$,

$$\begin{aligned}
\{l_z, \mathcal{H}\} &= \frac{\partial l_z}{\partial x}\frac{\partial \mathcal{H}}{\partial p_x} - \frac{\partial l_z}{\partial p_x}\frac{\partial \mathcal{H}}{\partial x} + \frac{\partial l_z}{\partial y}\frac{\partial \mathcal{H}}{\partial p_y} - \frac{\partial l_z}{\partial p_y}\frac{\partial \mathcal{H}}{\partial y} \\
&= (p_y)(2p_x) - (-y)(2ax) + (-p_x)(2p_y) - (x)(2ay) \\
&= 2p_xp_y - 2p_xp_y + 2axy - 2axy \\
&= 0
\end{aligned}$$

1.11. Fill in the missing steps leading to Eq. (2.7.18) starting from Eq. (2.7.14).

If we view $\mathcal{H}$ as a function of $\bar{q}$ and $\bar{p}$, we find

$$\begin{aligned}
\dot{\bar{q}}_j &= \{\bar{q}, \mathcal{H}\} \\
&= \sum_i \Big(\frac{\partial \bar{q}_j}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial \bar{q}_j}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \Big) \\
&= \sum_i \Big(\frac{\partial \bar{q}_j}{\partial q_i} \Big[\sum_k \frac{\partial \mathcal{H}}{\partial \bar{q}_k} \frac{\partial \bar{q}_k}{\partial p_i} + \frac{\partial \mathcal{H}}{\partial \bar{p}_k} \frac{\partial \bar{p}_k}{\partial p_i} \Big] - \frac{\partial \bar{q}_j}{\partial p_i} \Big[\sum_l \frac{\partial \mathcal{H}}{\partial \bar{q}_l} \frac{\partial \bar{q}_l}{\partial q_i} + \frac{\partial \mathcal{H}}{\partial \bar{p}_l} \frac{\partial \bar{p}_l}{\partial q_i} \Big] \Big) \\
&= \sum_i \sum_k \Big(\frac{\partial \bar{q}_j}{\partial q_i} \frac{\partial \mathcal{H}}{\partial \bar{q}_k} \frac{\partial \bar{q}_k}{\partial p_i} + \frac{\partial \bar{q}_j}{\partial q_i} \frac{\partial \mathcal{H}}{\partial \bar{p}_k} \frac{\partial \bar{p}_k}{\partial p_i} - \frac{\partial \bar{q}_j}{\partial p_i} \frac{\partial \mathcal{H}}{\partial \bar{q}_k} \frac{\partial \bar{q}_k}{\partial q_i} - \frac{\partial \bar{q}_j}{\partial p_i} \frac{\partial \mathcal{H}}{\partial \bar{p}_k} \frac{\partial \bar{p}_k}{\partial q_i} \Big) \\
&= \sum_k \Big(\frac{\partial \mathcal{H}}{\partial \bar{q}_k} \Big[\sum_i \frac{\partial \bar{q}_j}{\partial q_i} \frac{\partial \bar{q}_k}{\partial p_i} - \frac{\partial \bar{q}_j}{\partial p_i} \frac{\partial \bar{q}_k}{\partial q_i} \Big] + \frac{\partial \mathcal{H}}{\partial \bar{p}_k} \Big[\sum_i \frac{\partial \bar{q}_j}{\partial q_i} \frac{\partial \bar{p}_k}{\partial p_i} - \frac{\partial \bar{q}_j}{\partial p_i} \frac{\partial \bar{p}_k}{\partial q_i} \Big] \Big) \\
&= \sum_k \Big(\frac{\partial \mathcal{H}}{\partial \bar{q}_k} \{\bar{q}_j, \bar{q}_k\} + \frac{\partial \mathcal{H}}{\partial \bar{p}_k} \{\bar{q}_j, \bar{p}_k\} \Big)
\end{aligned}$$

We can find $\dot{\bar{p}}_j$ by exchanging $\bar{q}_j$ for $\bar{p}_j$ in the result above,

$$\dot{\bar{p}}_j = \Big(\frac{\partial \mathcal{H}}{\partial \bar{q}_k} \{\bar{p}_j, \bar{q}_k\} + \frac{\partial \mathcal{H}}{\partial \bar{p}_k} \{\bar{p}_j, \bar{p}_k\} \Big)$$

In order for these to reduce to the canonical equations,

$$\dot{\bar{q}}_k = \frac{\partial \mathcal{H}}{\partial \bar{p}_k} \qquad \dot{\bar{p}}_k = -\frac{\partial \mathcal{H}}{\partial \bar{q}_k}$$

we must have $\{\bar{q}_j, \bar{q}_k\} = \{\bar{p}_j, \bar{p}_k\} = 0$ and $\{\bar{q}_j, \bar{p}_k\} = -\{\bar{p}_k, \bar{q}_j\} = \delta_{jk}$.

1.12. Verify that the change to a rotated frame

$$\begin{aligned}
\bar{x} &= x \cos\theta - y \sin\theta \\
\bar{y} &= x \sin\theta + y \cos\theta \\
\bar{p}_x &= p_x \cos\theta - p_y \sin\theta \\
\bar{p}_y &= p_x \sin\theta + p_y \cos\theta
\end{aligned}$$

is a canonical transformation.

From the above, we immediately see $\{\bar{x}, \bar{y}\} = \{\bar{p}_x, \bar{p}_y\} = 0$ and

$$\begin{aligned}
\{\bar{x}, \bar{p}_x\} &= \frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{p}_x}{\partial p_x} - \frac{\partial \bar{x}}{\partial p_x}\frac{\partial \bar{p}_x}{\partial x} + \frac{\partial \bar{x}}{\partial y}\frac{\partial \bar{p}_x}{\partial p_y} - \frac{\partial \bar{x}}{\partial p_y}\frac{\partial \bar{p}_x}{\partial y} \\
&= \cos^2\theta + \sin^2\theta \\
&= 1 \\
\{\bar{x}, \bar{p}_y\} &= \frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{p}_y}{\partial p_x} - \frac{\partial \bar{x}}{\partial p_x}\frac{\partial \bar{p}_y}{\partial x} + \frac{\partial \bar{x}}{\partial y}\frac{\partial \bar{p}_y}{\partial p_y} - \frac{\partial \bar{x}}{\partial p_y}\frac{\partial \bar{p}_y}{\partial y} \\
&= \cos\theta\sin\theta - \sin\theta\cos\theta \\
&= 0 \\
\{\bar{y}, \bar{p}_y\} &= \frac{\partial \bar{y}}{\partial x}\frac{\partial \bar{p}_y}{\partial p_x} - \frac{\partial \bar{y}}{\partial p_x}\frac{\partial \bar{p}_y}{\partial x} + \frac{\partial \bar{y}}{\partial y}\frac{\partial \bar{p}_y}{\partial p_y} - \frac{\partial \bar{y}}{\partial p_y}\frac{\partial \bar{p}_y}{\partial y} \\
&= \sin^2\theta + \cos^2\theta \\
&= 1 \\
\{\bar{y}, \bar{p}_x\} &= \frac{\partial \bar{y}}{\partial x}\frac{\partial \bar{p}_x}{\partial p_x} - \frac{\partial \bar{y}}{\partial p_x}\frac{\partial \bar{p}_x}{\partial x} + \frac{\partial \bar{y}}{\partial y}\frac{\partial \bar{p}_x}{\partial p_y} - \frac{\partial \bar{y}}{\partial p_y}\frac{\partial \bar{p}_x}{\partial y} \\
&= \sin\theta\cos\theta - \sin\theta\cos\theta \\
&= 0
\end{aligned}$$

i.e. $\{\bar{q}_j, \bar{q}_k\} = \{\bar{p}_j, \bar{p}_k\} = 0$ and $\{\bar{q}_j, \bar{p}_k\} = -\{\bar{p}_k, \bar{q}_j\} = \delta_{jk}$—the transformation is canonical.

1.13. Show that the polar variables $\rho = (x^2+y^2)^{1/2}$, $\phi = \tan^{-1}(y/x)$,

$$p_\rho = \hat{e}_\rho \cdot \mathbf{p} = \frac{xp_x + yp_y}{(x^2+y^2)^{1/2}}, \quad p_\phi = xp_y - yp_x (= l_z)$$

are canonical. ($\hat{e}_\rho$ is the unit vector in the radial direction.)

First we collect the necessary derivatives,

$$\begin{gathered}
\frac{\partial \rho}{\partial x} = \frac{x}{(x^2+y^2)^{1/2}} \qquad \frac{\partial \rho}{\partial y} = \frac{y}{(x^2+y^2)^{1/2}} \qquad \frac{\partial \rho}{\partial p_x} = 0 \qquad \frac{\partial \rho}{\partial p_y} = 0 \\
\frac{\partial \phi}{\partial x} = -\frac{y}{x^2+y^2} \qquad \frac{\partial \phi}{\partial y} = \frac{x}{x^2+y^2} \qquad \frac{\partial \phi}{\partial p_x} = 0 \qquad \frac{\partial \phi}{\partial p_y} = 0 \\
\frac{\partial p_\rho}{\partial x} = -\frac{y(p_y x - p_x y)}{(x^2+y^2)^{3/2}} \qquad \frac{\partial p_\rho}{\partial y} = -\frac{x(p_x y - p_y x)}{(x^2+y^2)^{3/2}} \qquad \frac{\partial p_\rho}{\partial p_x} = \frac{x}{(x^2+y^2)^{1/2}} \qquad \frac{\partial p_\rho}{\partial p_y} = \frac{y}{(x^2+y^2)^{1/2}} \\
\frac{\partial p_\phi}{\partial x} = p_y \qquad \frac{\partial p_\phi}{\partial y} = -p_x \qquad \frac{\partial p_\phi}{\partial p_x} = -y \qquad \frac{\partial p_\phi}{\partial p_y} = x
\end{gathered}$$

From these, we can compute the Poisson bracket of all variable combinations. Clearly the Poisson bracket of any coordinate (or conjugate momenta) with itself is 0, so we need only check those that differ:

$$
\begin{aligned}
\{\rho, \phi\} &= \frac{\partial \rho}{\partial x}\frac{\partial \phi}{\partial p_x} - \frac{\partial \rho}{\partial p_x}\frac{\partial \phi}{\partial x} + \frac{\partial \rho}{\partial y}\frac{\partial \phi}{\partial p_y} - \frac{\partial \rho}{\partial p_y}\frac{\partial \phi}{\partial y} \\
&= 0 \\
\{p_\rho, p_\phi\} &= \frac{\partial p_\rho}{\partial x}\frac{\partial p_\phi}{\partial p_x} - \frac{\partial p_\rho}{\partial p_x}\frac{\partial p_\phi}{\partial x} + \frac{\partial p_\rho}{\partial y}\frac{\partial p_\phi}{\partial p_y} - \frac{\partial p_\rho}{\partial p_y}\frac{\partial p_\phi}{\partial y} \\
&= \frac{y^2(p_y x - p_x y)}{(x^2+y^2)^{3/2}} - \frac{p_y x}{(x^2+y^2)^{1/2}} - \frac{x^2(p_x y - p_y x)}{(x^2+y^2)^{3/2}} + \frac{p_x y}{(x^2+y^2)^{1/2}} \\
&= \frac{p_y x - p_x y}{(x^2+y^2)^{1/2}} - \frac{p_y x - p_x y}{(x^2+y^2)^{1/2}} \\
&= 0 \\
\{\rho, p_\rho\} &= \frac{\partial \rho}{\partial x}\frac{\partial p_\rho}{\partial p_x} - \frac{\partial \rho}{\partial p_x}\frac{\partial p_\rho}{\partial x} + \frac{\partial \rho}{\partial y}\frac{\partial p_\rho}{\partial p_y} - \frac{\partial \rho}{\partial p_y}\frac{\partial p_\rho}{\partial y} \\
&= \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} \\
&= 1 \\
\{\rho, p_\phi\} &= \frac{\partial \rho}{\partial x}\frac{\partial p_\phi}{\partial p_x} - \frac{\partial \rho}{\partial p_x}\frac{\partial p_\phi}{\partial x} + \frac{\partial \rho}{\partial y}\frac{\partial p_\phi}{\partial p_y} - \frac{\partial \rho}{\partial p_y}\frac{\partial p_\phi}{\partial y} \\
&= -xy + xy \\
&= 0 \\
\{\phi, p_\rho\} &= \frac{\partial \phi}{\partial x}\frac{\partial p_\rho}{\partial p_x} - \frac{\partial \phi}{\partial p_x}\frac{\partial p_\rho}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial p_\rho}{\partial p_y} - \frac{\partial \phi}{\partial p_y}\frac{\partial p_\rho}{\partial y} \\
&= -\frac{xy}{(x^2+y^2)^{3/2}} + \frac{xy}{(x^2+y^2)^{3/2}} \\
&= 0 \\
\{\phi, p_\phi\} &= \frac{\partial \phi}{\partial x}\frac{\partial p_\phi}{\partial p_x} - \frac{\partial \phi}{\partial p_x}\frac{\partial p_\phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial p_\phi}{\partial p_y} - \frac{\partial \phi}{\partial p_y}\frac{\partial p_\phi}{\partial y} \\
&= \frac{y^2}{x^2+y^2} + \frac{x^2}{x^2+y^2} \\
&= 1
\end{aligned}
$$

1.14. Verify that the change from the variables $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{p}_1$, $\mathbf{p}_2$ to $\mathbf{r}_{\mathrm{CM}}$, $\mathbf{p}_{\mathrm{CM}}$, $\mathbf{r}$, and $\mathbf{p}$ is a canonical transformation. (See Exercise 2.5.4).

The new variables are defined by

$$
\begin{aligned}
\mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 \\
\mathbf{p} &= \frac{m_1 m_2}{m_1+m_2}\left(\frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2}\right) \\
\mathbf{r}_{\mathrm{CM}} &= \frac{1}{m_1+m_2}\left(m_1\mathbf{r}_1 + m_2\mathbf{r}_2\right) \\
\mathbf{p}_{\mathrm{CM}} &= \mathbf{p}_1 + \mathbf{p}_2
\end{aligned}
$$

Because these are linear relationships, we can work entirely in the Poisson algebra,

$$\begin{aligned}
\{\mathbf{r}, \mathbf{p}\} &= \frac{m_1 m_2}{m_1 + m_2}\{\mathbf{r}_1 - \mathbf{r}_2, \frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2}\} \\
&= \frac{m_2}{m_1 + m_2}\{\mathbf{r}_1, \mathbf{p}_1\} + \frac{m_1}{m_1 + m_2}\{\mathbf{r}_2, \mathbf{p}_2\} \\
&= \frac{m_2 + m_1}{m_1 + m_2} \\
&= 1 \\
\{\mathbf{r}, \mathbf{p}_{\mathrm{CM}}\} &= \{\mathbf{r}_1 - \mathbf{r}_2, \mathbf{p}_1 + \mathbf{p}_2\} \\
&= \{\mathbf{r}_1, \mathbf{p}_1\} - \{\mathbf{r}_2, \mathbf{p}_2\} \\
&= 1 - 1 \\
&= 0 \\
\{\mathbf{r}_{\mathrm{CM}}, \mathbf{p}\} &= \frac{m_1 m_2}{(m_1 + m_2)^2}\{m_1\mathbf{r}_1 + m_2\mathbf{r}_2, \frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2}\} \\
&= \frac{m_1 m_2}{(m_1 + m_2)^2}(\{\mathbf{r}_1, \mathbf{p}_1\} - \{\mathbf{r}_2, \mathbf{p}_2\}) \\
&= \frac{m_1 m_2}{(m_1 + m_2)^2}(1 - 1) \\
&= 0 \\
\{\mathbf{r}_{\mathrm{CM}}, \mathbf{p}_{\mathrm{CM}}\} &= \frac{1}{m_1 + m_2}\{m_1\mathbf{r}_1 + m_2\mathbf{r}_2, \mathbf{p}_1 + \mathbf{p}_2\} \\
&= \frac{m_1}{m_1 + m_2}\{\mathbf{r}_1, \mathbf{p}_1\} + \frac{m_2}{m_1 + m_2}\{\mathbf{r}_2, \mathbf{p}_2\} \\
&= \frac{m_1 + m_2}{m_1 + m_2} \\
&= 1
\end{aligned}$$

All other combinations are trivially 0, implying that the transformation is canonical.

1.15. Verify that

$$\begin{aligned}
\bar{q} &= \ln(q^{-1}\sin p) \\
\bar{p} &= q\cot p
\end{aligned}$$

is a canonical transformation.

Once again collecting derviatives,

$$\begin{aligned}
\frac{\partial \bar{q}}{\partial q} &= -\frac{1}{q} \qquad & \frac{\partial \bar{q}}{\partial p} &= \cot p \\
\frac{\partial \bar{p}}{\partial q} &= \cot p \qquad & \frac{\partial \bar{p}}{\partial p} &= -q\csc^2 p
\end{aligned}$$

we find

$$\begin{aligned}
\{\bar{q}, \bar{p}\} &= \csc^2 p - \cot^2 p \\
&= 1
\end{aligned}$$

where this identity can be seen from

$$\cos^2 p + \sin^2 p = 1$$

and so

$$\cot^2 p + 1 = \csc^2 p.$$

As this is the only nontrivial combination we are done: the transformation is canonical.

1.16. We would like to derive here Eq. (2.7.9), which gives the transformation of the momenta under a coordinate transformation in configuration space:

$$q_i \to \bar{q}_i(q_1, \ldots, q_n)$$

(1) Argue that if we invert the above equation to get $q = q(\bar{q})$, we can derive the following counterpart of Eq. (2.7.7):

$$\dot{q}_i = \sum_j \frac{\partial q_i}{\partial \bar{q}_j} \dot{\bar{q}}_j$$

(2) Show from the above that

$$\left(\frac{\partial \dot{q}_i}{\partial \dot{\bar{q}}_j}\right)_{\bar{q}} = \frac{\partial q_i}{\partial \bar{q}_j}$$

(3) Now calculate

$$\bar{p}_i = \left[\frac{\partial \mathcal{L}(\bar{q}, \dot{\bar{q}})}{\partial \dot{\bar{q}}_i}\right]_{\bar{q}} = \left[\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{\bar{q}}_i}\right]_{\bar{q}}$$

Use the chain rule and the fact that $q = q(\bar{q})$ and note $q(\bar{q}, \dot{\bar{q}})$ to derive Eq. (2.7.9).

(4) Verify, by calculating the PB in Eq. (2.7.18), that the point transformation is canonical.

The first step is simply applying the chain rule when taking the total time derivative of $q_i = q_i(\bar{q}_1, \ldots \bar{q}_n)$, yielding

$$\dot{q}_i = \sum_k \frac{\partial q_i}{\partial \bar{q}_k} \dot{\bar{q}}_k.$$

Taking the partial derivative of this with respect to a particular $\dot{\bar{q}}_j$ gives

$$\frac{\partial \dot{q}_i}{\partial \dot{\bar{q}}_j} = \sum_k \frac{\partial q_i}{\partial \bar{q}_k} \frac{\partial \dot{\bar{q}}_k}{\partial \dot{\bar{q}}_j} = \sum_k \frac{\partial q_i}{\partial \bar{q}_k} \delta_{kj} = \frac{\partial q_i}{\partial \bar{q}_j},$$

where we have used the fact that $\partial q_i / \partial \bar{q}_k$ is independent of $\dot{\bar{q}}_j$.

Differentiating the Lagrangian with respect to $\dot{\bar{q}}_i$ gives

$$\bar{p}_i = \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{\bar{q}}_i} = \sum_j \frac{\partial \mathcal{L}}{\partial q_j} \frac{\partial q_j}{\partial \dot{\bar{q}}_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \dot{\bar{q}}_i} = \sum_j \frac{\partial q_j}{\partial \bar{q}_i} p_j$$

where we have used $q = q(\bar{q})$ and so $\partial q_j / \partial \dot{\bar{q}}_i = 0$.

We skip verification of the invariance of the Poisson bracket, noting the fact that $\bar{p}_i$ defined this way guarantees canonical phase space coordiantes.

1.17. Verify Eq. (2.7.19) by direct computation. Use the chain rule to go from q, p derivatives to $\bar{q}, \bar{p}$ derivatives. Collect terms that represent PB of the latter.

Canonical transformations must obey $\dot{\bar{q}}_i = \frac{\partial \mathcal{H}}{\partial \bar{p}}$. Expanded out, this requirement becomes

$$\begin{aligned}
\frac{\mathrm{d}\bar{q}_i}{\mathrm{d}t} &= \sum_j \frac{\partial \bar{q}_i}{\partial q_j}\frac{\partial q_j}{\partial t} + \frac{\partial \bar{q}_i}{\partial p_j}\frac{\partial p_j}{\partial t} \\
&= \sum_j \frac{\partial \bar{q}_i}{\partial q_j}\frac{\partial \mathcal{H}}{\partial p_j} - \frac{\partial \bar{q}_i}{\partial p_j}\frac{\partial \mathcal{H}}{\partial q_j} \\
&= \frac{\partial \mathcal{H}}{\partial \bar{p}_i} \\
&= \sum_j \frac{\partial \mathcal{H}}{\partial q_j}\frac{\partial q_j}{\partial \bar{p}_i} + \frac{\partial \mathcal{H}}{\partial p_j}\frac{\partial p_j}{\partial \bar{p}_i}
\end{aligned}$$

i.e. $\partial q_j/\partial \bar{p}_i = -\partial \bar{q}_i/\partial p_j$ and $\partial p_j/\partial \bar{p}_i = \partial \bar{q}_i/\partial q_j$. Using this last relationship makes this verification especially easy, as

$$\begin{aligned}
\{\omega, \sigma\}_{q,p} &= \sum_i \frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_i} \\
&= \sum_{i,j,k} \frac{\partial \omega}{\partial \bar{q}_j}\frac{\partial \bar{q}_j}{\partial q_i}\frac{\partial \sigma}{\partial \bar{p}_k}\frac{\partial \bar{p}_k}{\partial p_i} - \frac{\partial \omega}{\partial \bar{p}_k}\frac{\partial \bar{p}_k}{\partial p_i}\frac{\partial \sigma}{\partial \bar{q}_j}\frac{\partial \bar{q}_j}{\partial q_i} \\
&= \sum_{j,k} \left(\frac{\partial \omega}{\partial \bar{q}_j}\frac{\partial \sigma}{\partial \bar{p}_k} - \frac{\partial \omega}{\partial \bar{p}_k}\frac{\partial \sigma}{\partial \bar{q}_j}\right) \sum_i \frac{\partial \bar{q}_j}{\partial q_i}\frac{\partial \bar{p}_k}{\partial p_i} \\
&= \sum_{j,k} \left(\frac{\partial \omega}{\partial \bar{q}_j}\frac{\partial \sigma}{\partial \bar{p}_k} - \frac{\partial \omega}{\partial \bar{p}_k}\frac{\partial \sigma}{\partial \bar{q}_j}\right) \sum_i \frac{\partial p_i}{\partial \bar{p}_j}\frac{\partial \bar{p}_k}{\partial p_i} \\
&= \sum_{j,k} \left(\frac{\partial \omega}{\partial \bar{q}_j}\frac{\partial \sigma}{\partial \bar{p}_k} - \frac{\partial \omega}{\partial \bar{p}_k}\frac{\partial \sigma}{\partial \bar{q}_j}\right) \frac{\partial \bar{p}_k}{\partial p_j} \\
&= \sum_{j,k} \left(\frac{\partial \omega}{\partial \bar{q}_j}\frac{\partial \sigma}{\partial \bar{p}_k} - \frac{\partial \omega}{\partial \bar{p}_k}\frac{\partial \sigma}{\partial \bar{q}_j}\right) \delta_{kj} \\
&= \sum_j \frac{\partial \omega}{\partial \bar{q}_j}\frac{\partial \sigma}{\partial \bar{p}_j} - \frac{\partial \omega}{\partial \bar{p}_j}\frac{\partial \sigma}{\partial \bar{q}_j} \\
&= \{\omega, \sigma\}_{\bar{q},\bar{p}}
\end{aligned}$$

1.18. Show that $p = p_1 + p_2$, the total momentum, is the generator of infinitesimal translations for a two-particle system.

A generic two-particle system has the Hamiltonian

$$\mathcal{H} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(q_1 - q_2)$$

Consider the generator

$$g(q, p) = p_1 + p_2$$

which has the partial derivatives

$$\frac{\partial g}{\partial q_i} = 0 \quad \text{and} \quad \frac{\partial g}{\partial p_i} = 1 \quad \text{for } i = 1, 2.$$

Under the canonical transformation

$$\begin{aligned} q_i &\to q_i + \varepsilon \frac{\partial g}{\partial p_i} = q_i + \varepsilon \\ p_i &\to p_i - \varepsilon \frac{\partial g}{\partial q_i} = p_i \end{aligned}$$

the Hamiltonian is clearly left unchanged. Physically, this transformation can be seen as a spatial translation, offsetting the positions of the particles by an amount ε.

1.19. Verify that the infinitesimal transformation generated by any dynamical variables g is a canonical transformation. (Hint: Work, as usual, to first order in ε.)

This process is made easier if we first examine the Poisson bracket of $\{q_i, f\}$ and $\{p_i, f\}$ for an arbitrary well-behaved function $f(q, p)$,

$$\begin{aligned} \{q_i, f\} &= \sum_k \frac{\partial q_i}{\partial q_k}\frac{\partial f}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial f}{\partial q_k} \\ &= \sum_k \delta_{ik}\frac{\partial f}{\partial p_k} \\ &= \frac{\partial f}{\partial p_i} \\ \{p_i, f\} &= \sum_k \frac{\partial p_i}{\partial q_k}\frac{\partial f}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial f}{\partial q_k} \\ &= -\sum_k \delta_{ik}\frac{\partial f}{\partial q_k} \\ &= -\frac{\partial f}{\partial q_i} \end{aligned}$$

where we have used the fact that phase space coordinates are independent of one another. Remembering the Jacobi identity, we proceed to check that the given transformation preserves the Poisson bracket. For the position coordinates we have

$$\begin{aligned} \{\bar{q}_i, \bar{q}_j\} &= \{q_i + \varepsilon\frac{\partial g}{\partial p_i}, q_j + \varepsilon\frac{\partial g}{\partial p_j}\} \\ &= \{q_i, q_j\} + \varepsilon\Big(\{q_i, \frac{\partial g}{\partial p_j}\} + \{\frac{\partial g}{\partial p_i}, q_j\}\Big) + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon\Big(\{q_i, \{q_j, g\}\} + \{\{q_i, g\}, q_j\}\Big) \\ &= \varepsilon\Big(\{q_i, \{q_j, g\}\} + \{q_j, \{g, q_i\}\}\Big) \\ &= -\varepsilon\{g, \{q_i, q_j\}\} \\ &= -\varepsilon\{g, 0\} \\ &= 0 \end{aligned}$$

while the momenta coordinates are

$$\begin{aligned}
\{\bar{p}_i, \bar{p}_j\} &= \{p_i - \varepsilon\frac{\partial g}{\partial q_i}, p_j - \varepsilon\frac{\partial g}{\partial q_j}\} \\
&= \{p_i, p_j\} - \varepsilon\Big(\{p_i, \frac{\partial g}{\partial q_j}\} + \{\frac{\partial g}{\partial q_i}, p_j\}\Big) + \mathcal{O}(\varepsilon^2) \\
&= -\varepsilon\Big(\{p_i, \{g, p_j\}\} + \{\{g, p_i\}, p_j\}\Big) \\
&= -\varepsilon\Big(\{p_i, \{g, p_j\}\} + \{p_j, \{p_i, g\}\}\Big) \\
&= \varepsilon\{g, \{p_i, p_j\}\} \\
&= \varepsilon\{g, 0\} \\
&= 0
\end{aligned}$$

Finally, the Poisson bracket of an arbitrary pair of position and momentum coordinates is given by

$$\begin{aligned}
\{\bar{q}_i, \bar{p}_j\} &= \{q_i + \varepsilon\frac{\partial g}{\partial p_i}, p_j - \varepsilon\frac{\partial g}{\partial q_j}\} \\
&= \{q_i, p_j\} - \varepsilon\Big(\{q_i, \frac{\partial g}{\partial q_j}\} - \{\frac{\partial g}{\partial p_i}, p_j\}\Big) + \mathcal{O}(\varepsilon^2) \\
&= \delta_{ij} - \varepsilon\Big(\{q_i, \{g, p_j\}\} - \{\{q_i, g\}, p_j\}\Big) \\
&= \delta_{ij} - \varepsilon\Big(\{q_i, \{g, p_j\}\} + \{p_j, \{q_i, g\}\}\Big) \\
&= \delta_{ij} + \varepsilon\{g, \{p_j, q_i\}\} \\
&= \delta_{ij} - \varepsilon\{g, \delta_{ji}\} \\
&= \delta_{ij}
\end{aligned}$$

1.20. Consider

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2)$$

whose invariance under the rotation of the coordinates *and* momenta leads to the conservation of l_z. But $\mathcal{H}$ is also invariant under the rotation of *just the coordinates*. Verify that this is a *noncanonical* transformation. Convince yourself that in this case it is not possible to write $\delta\mathcal{H}$ as $\varepsilon\{\mathcal{H}, g\}$ for any g, i.e. that no conservation law follows.

If we transform solely the coordinates as

$$\begin{aligned}
\bar{x} &= x\cos\theta + y\sin\theta \\
\bar{y} &= -x\sin\theta + y\cos\theta
\end{aligned}$$

the momenta become

$$\begin{aligned}
p_x(\bar{x}, \bar{y}) &= m\dot{x} = m(\dot{\bar{x}}\cos\theta - \dot{\bar{y}}\sin\theta) = p_{\bar{x}}\cos\theta - p_{\bar{y}}\sin\theta \\
p_y(\bar{x}, \bar{y}) &= m\dot{y} = m(\dot{\bar{x}}\sin\theta + \dot{\bar{y}}\cos\theta) = p_{\bar{x}}\sin\theta + p_{\bar{y}}\cos\theta
\end{aligned}$$

where $p_{\bar{x}}$ and $p_{\bar{y}}$ are the canonical momenta conjugate to x and y.

The only possible nonzero Poisson brackets are

$$
\begin{aligned}
\{\bar{x}, p_x\} &= \{\bar{x}, p_{\bar{x}} \cos\theta - p_{\bar{y}} \sin\theta\} \\
&= \{\bar{x}, p_{\bar{x}}\} \cos\theta - \{\bar{x}, p_{\bar{y}}\} \sin\theta \\
&= \cos\theta \\
\{\bar{x}, p_y\} &= \{\bar{x}, p_{\bar{x}} \sin\theta + p_{\bar{y}} \cos\theta\} \\
&= \{\bar{x}, p_{\bar{x}}\} \sin\theta + \{\bar{x}, p_{\bar{y}}\} \cos\theta \\
&= \sin\theta \\
\{\bar{y}, p_x\} &= \{\bar{y}, p_{\bar{x}} \cos\theta - p_{\bar{y}} \sin\theta\} \\
&= \{\bar{y}, p_{\bar{x}}\} \cos\theta - \{\bar{y}, p_{\bar{y}}\} \sin\theta \\
&= -\sin\theta \\
\{\bar{y}, p_y\} &= \{\bar{y}, p_{\bar{x}} \sin\theta + p_{\bar{y}} \cos\theta\} \\
&= \{\bar{y}, p_{\bar{x}}\} \sin\theta + \{\bar{y}, p_{\bar{y}}\} \cos\theta \\
&= \cos\theta
\end{aligned}
$$

Because the Poisson bracket no longer reproduces $\{q_i, p_j\} = \delta_{ij}$, this is a noncanonical coordinate transformation.

We know that $\{\mathcal{H}, g\}$ should capture the time derivative of g, which depends on the system's coordinates. Because the new coordinates are noncanonical, their time derivatives are *not* captured by their Poisson bracket with $\mathcal{H}$, and so those functions depending on them lose this property, too.

1.21. Consider $\mathcal{H} = \frac{1}{2}p^2 + \frac{1}{2}x^2$, which is invariant under infinitesimal rotations in *phase space* (the x-p plane). Find the generator of this transformation (after verifying that it is canonical). (You could have guessed the answer based on Exercise 2.5.2).

A rotation in the x-p plane by an angle θ can be written

$$
\begin{aligned}
\bar{x} &= x\cos\theta + p\sin\theta \\
\bar{p} &= -x\sin\theta + p\cos\theta
\end{aligned}
$$

which, as θ becomes infinitesimal, transforms to

$$
\begin{aligned}
\bar{x} &= x + \varepsilon p \stackrel{?}{=} x + \varepsilon \frac{\partial g}{\partial p} \\
\bar{p} &= p - \varepsilon x \stackrel{?}{=} p - \varepsilon \frac{\partial g}{\partial x}
\end{aligned}
$$

We can simply integrate to find g,

$$
\begin{aligned}
\frac{\partial g}{\partial p} = p &\implies g(x,p) = \frac{1}{2}p^2 + \mathcal{O}(x) \\
\frac{\partial g}{\partial x} = x &\implies g(x,p) = \frac{1}{2}x^2 + \mathcal{O}(p)
\end{aligned}
$$

i.e. $g(x,p) = \mathcal{H}$ up to the addition of a constant factor.

To verify the canonical nature of the transformation, observe

$$\begin{aligned}
\{\bar{x}, \bar{x}\} &= \{x + \varepsilon p, x + \varepsilon p\} \\
&= \{x, x\} + \varepsilon\Big(\{x, p\} + \{p, x\}\Big) + \mathcal{O}(\varepsilon^2) \\
&= 0 \\
\{\bar{x}, \bar{p}\} &= \{x + \varepsilon p, p - \varepsilon x\} \\
&= \{x, p\} - \varepsilon\Big(\{x, x\} - \{p, p\}\Big) + \mathcal{O}(\varepsilon^2) \\
&= 1 \\
\{\bar{p}, \bar{p}\} &= \{p - \varepsilon x, p - \varepsilon x\} \\
&= \{p, p\} - \varepsilon\Big(\{p, x\} + \{x, p\}\Big) + \mathcal{O}(\varepsilon^2) \\
&= 0
\end{aligned}$$

1.22. Why is it that a *non*canonical transformation that leaves $\mathcal{H}$ invariant does not map a solution into another? Or, in view of the discussion on consequence II, why is it that an experiment and its transformed version do not give the same result when the transformation that leaves $\mathcal{H}$ invariant is not canonical? It is best to consider an example. Consider the potential given in Exercise 2.8.3. Suppose I release a particle at $(x = a, y = 0)$ with $(p_x = b, p_y = 0)$ and you release one in the transformed state in which $(x = 0, y = a)$ and $(p_x = b, p_y = 0)$, i.e., you rotate the coordinates but not the momenta. This is a noncanonical transformation that leaves $\mathcal{H}$ invariant. Convince yourself that at later times the states of the two particles are no related by the same transformation. Try to understand what goes wrong in the general case.

1.23. Show that $\partial S_{\text{cl}}/\partial x_f = p(t_f)$.

Consider the classical action with the path parameterized by both t and x_f. Its partial derivative with respect to x_f is then

$$\begin{aligned}
\frac{\partial S_{\text{cl}}}{\partial x_f} &= \frac{\partial}{\partial x_f}\int_0^{t_f} \mathcal{L}(x_{\text{cl}}(x_f, t), \dot{x}_{\text{cl}}(x_f, t))\,\mathrm{d}t \\
&= \int_0^{t_f} \frac{\partial \mathcal{L}}{\partial x_{\text{cl}}}\frac{\partial x_{\text{cl}}}{\partial x_f} + \frac{\partial \mathcal{L}}{\partial \dot{x}_{\text{cl}}}\frac{\partial \dot{x}_{\text{cl}}}{\partial x_f}\,\mathrm{d}t \\
&= \int_0^{t_f} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}_{\text{cl}}}\frac{\partial x_{\text{cl}}}{\partial x_f} + \frac{\partial \mathcal{L}}{\partial \dot{x}_{\text{cl}}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial x_{\text{cl}}}{\partial x_f}\,\mathrm{d}t \\
&= \int_0^{t_f} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}_{\text{cl}}}\frac{\partial x_{\text{cl}}}{\partial x_f}\Big)\mathrm{d}t \\
&= p(t)\frac{\partial x_{\text{cl}}}{\partial x_f}\Big|_0^{t_f} \\
&= p(t_f)
\end{aligned}$$

where $x_{\text{cl}}(0) = x_1$ and $x_{\text{cl}}(t_f) = x_f$, and so $\partial x_{\text{cl}}/\partial x_f|_0^{t_f} = 1 - 0 = 1$.

1.24. Consider the harmonic oscillator, for which the general solution is

$$x(t) = A\cos\omega t + B\sin\omega t.$$

Express the energy in terms of A and B and note that it does not depend on time. Now choose A and B such that $x(0) = x_1$ and $x(T) = x_2$. Write down the energy in terms of x_1, x_2, and T. Show that the action for the trajectory connecting x_1 and x_2 is

$$S_{\mathrm{cl}}(x_1, x_2, T) = \frac{m\omega}{2\sin\omega T}[(x_1^2 + x_2^2)\cos\omega T - 2x_1x_2].$$

Verify that $\partial S_{\mathrm{cl}}/\partial T = -E$.

The kinetic energy of the harmonic oscillator is given by

$$\begin{aligned}
\frac{1}{2}m\dot{x}^2 &= \frac{1}{2}m(-\omega A\sin\omega t + \omega B\cos\omega t)^2 \\
&= \frac{1}{2}m\omega^2(B\cos\omega t - A\sin\omega t)^2 \\
&= \frac{1}{2}m\omega^2\Big(B\frac{e^{i\omega t} + e^{-i\omega t}}{2} - A\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\Big)^2 \\
&= \frac{1}{2}m\omega^2\Big((B + iA)\frac{e^{i\omega t}}{2} + (B - iA)\frac{e^{-i\omega t}}{2}\Big)^2 \\
&= \frac{1}{2}m\omega^2(A^2 + B^2)\Big(\frac{e^{i(\omega t + \tan^{-1}(A/B))} + e^{-i(\omega t + \tan^{-1}(A/B))}}{2}\Big)^2 \\
&= \frac{1}{2}m\omega^2(A^2 + B^2)\cos^2(\omega t + \tan^{-1}(A/B)) \\
&\leq \frac{1}{2}m\omega^2(A^2 + B^2)
\end{aligned}$$

Since the total energy of the harmonic oscillator is equal to the maximum of the kinetic energy, $E = \frac{1}{2}m\omega^2(A^2 + B^2)$.

To satisfy the first initial condition, $A = x_1$. The second is equivalent to

$$x_2 = x_1\cos\omega T + B\sin\omega T,$$

which, upon solving for B, yields

$$B = \frac{x_2 - x_1\cos\omega T}{\sin\omega T}.$$

Using the above, we can rewrite the energy in terms of x_1 and x_2,

$$\begin{aligned}
E &= \frac{1}{2}m\omega^2\Big(x_1^2 + \frac{x_2^2 - 2x_1x_2\cos\omega T + x_1^2\cos^2\omega T}{\sin^2\omega T}\Big) \\
&= \frac{1}{2}m\omega^2\Big(\frac{x_1^2 - 2x_1x_2\cos\omega T + x_2^2}{\sin^2\omega T}\Big)
\end{aligned}$$

The Lagrangian for the simple harmonic oscillator is given by

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2$$

or, substituting in the classical path,

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m(\dot{x}^2 - \omega^2x^2) \\
&= \frac{1}{2}m\omega^2(B^2\cos^2\omega t + A^2\sin^2\omega t - A^2\cos^2\omega t - B^2\sin^2\omega t - 4AB\cos\omega t\sin\omega t) \\
&= \frac{1}{2}m\omega^2\Big((B^2 - A^2)\cos 2\omega t - 2AB\sin 2\omega t\Big)
\end{aligned}$$

The action is thus

$$\begin{aligned} S_{\text{cl}} &= \frac{1}{2}m\omega^2(B^2 - A^2)\int_0^T \cos 2\omega t \, \mathrm{d}t - m\omega^2 AB \int_0^T \sin 2\omega t \, \mathrm{d}t \\ &= \frac{1}{4}m\omega(B^2 - A^2)\sin 2\omega T + \frac{1}{2}m\omega AB(\cos 2\omega T - 1) \end{aligned}$$

where in our calculations of both $\mathcal{L}$ and S we have relied on the following identities

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \end{aligned}$$

Using a CAS, S simplifies to the solution,

$$S_{\text{cl}}(x_1, x_2, T) = \frac{m\omega}{2\sin\omega T}[(x_1^2 + x_2^2)\cos\omega T - 2x_1x_2].$$

Finally,

$$\begin{aligned} \frac{\partial S_{\text{cl}}}{\partial T} &= \frac{-m\omega^2\cos\omega T}{2\sin^2\omega T}[(x_1^2 + x_2^2)\cos\omega T - 2x_1x_2] - \frac{m\omega^2}{2}(x_1^2 + x_2^2) \\ &= -\frac{1}{2}m\omega^2\Big(\frac{x_1^2 - 2x_1x_2\cos\omega T + x_2^2}{\sin^2\omega T}\Big) \\ &= -E \end{aligned}$$