1 The entangled quantum world

1.1. Can you explain this vanishing? Recall the 4-dimensional notion of 'divergence' described in §19.3; here we need the 3-space version. *Hint*: See Exercise [19.2].

> Since we are concerned with the case where no sources are present, Gauss's law for the electric field is 0. The divergence of the magnetic field is always taken to be 0.

1.2. If $|\uparrow\rangle$ and $|\downarrow\rangle$ are normalized, what factor does $|\Omega\rangle$ need to make it normalized? (You may assume that $\||\alpha\rangle|\beta\rangle\| = \|\alpha\|\|\beta\|$.)

> The norm of $|\Omega\rangle$, as currently defined, is
>
> $$\begin{aligned}\langle\Omega|\Omega\rangle &= \big(\langle\downarrow|\langle\uparrow| - \langle\uparrow|\langle\downarrow|\big)\big(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\big)\\ &= \langle\downarrow|\langle\uparrow|\uparrow\rangle|\downarrow\rangle - \langle\downarrow|\langle\uparrow|\downarrow\rangle|\uparrow\rangle - \langle\uparrow|\langle\downarrow|\uparrow\rangle|\downarrow\rangle + \langle\uparrow|\langle\downarrow|\downarrow\rangle|\uparrow\rangle\\ &= \langle\downarrow|\downarrow\rangle + \langle\uparrow|\uparrow\rangle\\ &= 2\end{aligned}$$
>
> and so $|\Omega\rangle$ must be multiplied by $1/\sqrt{2}$ make it normalized.

1.3. Can you see quickly why this has spin 0? *Hint*: One way is to use the index notation to show that any such anti-symmetrical combination must essentially be a scalar, bearing in mind that the spin space is 2-dimensional.

1.4. Why not? Find a way of doing this, however, if $|\alpha\rangle$ and $|\beta\rangle$ are not so localized.

1.5. Confirm this parenthetic comment, and give a direct calculational verification of the above expression for $|\Omega\rangle$. *Hint*: See Exercise [22.26].

> A general normalized 2-spinor can be written in component form as
>
> $$|\swarrow\rangle = \begin{pmatrix} e^{i\phi/2}\cos\frac{\theta}{2} \\ e^{-i\phi/2}\sin\frac{\theta}{2}\end{pmatrix}$$
>
> where θ and ϕ correspond to the usual coordinates parameterizing S^2. In the standard up-down spin basis,
>
> $$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$
>
> we see
>
> $$|\swarrow\rangle = a|\uparrow\rangle + b|\downarrow\rangle$$
>
> where $a = e^{i\phi/2}\cos\frac{\theta}{2}$ and $b = e^{-i\phi/2}\sin\frac{\theta}{2}$. To change this to a spinor representing a state pointing in the opposite direction, we must make the substitutions
>
> $$\begin{aligned}\theta &\to \pi - \theta\\ \phi &\to \pi + \phi\end{aligned}$$

which enacts the changes

$$a \to ie^{i\phi/2}\sin\tfrac{\theta}{2} = i\bar{b}$$
$$b \to -ie^{-i\phi/2}\cos\tfrac{\theta}{2} = -i\bar{a}$$

We can ignore the common factor of i here, including it in the arbitrary phase factor of the state. Therefore, we have found

$$|{\swarrow}\rangle = a|{\uparrow}\rangle + b|{\downarrow}\rangle$$
$$|{\nearrow}\rangle = \bar{b}|{\uparrow}\rangle - \bar{a}|{\downarrow}\rangle$$

To show that the state $|{\swarrow}\rangle|{\nearrow}\rangle - |{\nearrow}\rangle|{\swarrow}\rangle$ is proportional to $|\Omega\rangle = |{\uparrow}\rangle|{\downarrow}\rangle - |{\downarrow}\rangle|{\uparrow}\rangle$, we expand out the former using what we have found above

$$\begin{aligned}
|{\swarrow}\rangle|{\nearrow}\rangle - |{\nearrow}\rangle|{\swarrow}\rangle &= \big(a|{\uparrow}\rangle + b|{\downarrow}\rangle\big)\big(\bar{b}|{\uparrow}\rangle - \bar{a}|{\downarrow}\rangle\big) - \big(\bar{b}|{\uparrow}\rangle - \bar{a}|{\downarrow}\rangle\big)\big(a|{\uparrow}\rangle + b|{\downarrow}\rangle\big) \\
&= (a\bar{b} - \bar{b}a)|{\uparrow}\rangle|{\uparrow}\rangle - (a\bar{a} + \bar{b}b)|{\uparrow}\rangle|{\downarrow}\rangle + (b\bar{b} + \bar{a}a)|{\downarrow}\rangle|{\uparrow}\rangle - (b\bar{a} - \bar{a}b)|{\downarrow}\rangle|{\downarrow}\rangle \\
&= -(a\bar{a} + b\bar{b})\big(|{\uparrow}\rangle|{\downarrow}\rangle - |{\downarrow}\rangle|{\uparrow}\rangle\big) \\
&\propto |{\uparrow}\rangle|{\downarrow}\rangle - |{\downarrow}\rangle|{\uparrow}\rangle
\end{aligned}$$

1.6. Why?

The Majorana description calls for a state of n spins to be the symmetric product of them, and so we have

$$|{\leftarrow}{\nearrow}\rangle = |{\leftarrow}\rangle|{\nearrow}\rangle + |{\nearrow}\rangle|{\leftarrow}\rangle$$

1.7. See if you can prove these. *Hint*: Use the coordinate and/or geometric desicrptions of §22.9.

It's easiest to rewrite our states in terms of the up and down basis states. We can do this using our results from 23.5,

$$|{\leftarrow}\rangle = \frac{\sqrt{2}}{2}|{\uparrow}\rangle + \frac{\sqrt{2}}{2}|{\downarrow}\rangle$$
$$|{\nearrow}\rangle = \cos\alpha|{\uparrow}\rangle + \sin\alpha|{\downarrow}\rangle$$

where

$$\alpha = -\frac{1}{2}\Big(\frac{\pi}{2} - \cos^{-1}\frac{3}{5}\Big).$$

It is useful to note that $\sin^2\alpha = 1/10$ and $\cos^2\alpha = 9/10$, and so $\sin\alpha = -1/\sqrt{10}$ and $\cos\alpha = 3/\sqrt{10}$. Using this, we can rewrite the state as

$$\begin{aligned}
|{\leftarrow}{\nearrow}\rangle &= \Big(\frac{\sqrt{2}}{2}|{\uparrow}\rangle + \frac{\sqrt{2}}{2}|{\downarrow}\rangle\Big)\Big(\cos\alpha|{\uparrow}\rangle + \sin\alpha|{\downarrow}\rangle\Big) + \Big(\cos\alpha|{\uparrow}\rangle + \sin\alpha|{\downarrow}\rangle\Big)\Big(\frac{\sqrt{2}}{2}|{\uparrow}\rangle + \frac{\sqrt{2}}{2}|{\downarrow}\rangle\Big) \\
&= \frac{3\sqrt{5}}{5}|{\uparrow}\rangle|{\uparrow}\rangle + \frac{\sqrt{5}}{5}|{\uparrow}\rangle|{\downarrow}\rangle + \frac{\sqrt{5}}{5}|{\downarrow}\rangle|{\uparrow}\rangle - \frac{\sqrt{5}}{5}|{\downarrow}\rangle|{\downarrow}\rangle
\end{aligned}$$

This is clearly not orthogonal to $|{\downarrow}\rangle|{\downarrow}\rangle$. As for the other states, let's examine them one at a time.

$$\begin{aligned}
\Big(\langle{\downarrow}|\langle{\leftarrow}|\Big)|{\leftarrow}{\nearrow}\rangle &= \Big(\frac{\sqrt{2}}{2}\langle{\downarrow}|\langle{\uparrow}| + \frac{\sqrt{2}}{2}\langle{\downarrow}|\langle{\downarrow}|\Big)|{\leftarrow}{\nearrow}\rangle \\
&= \frac{3}{2}\sin\alpha + \frac{1}{2}\cos\alpha \\
&= 0 \\
\Big(\langle{\leftarrow}|\langle{\downarrow}|\Big)|{\leftarrow}{\nearrow}\rangle &= \Big(\frac{\sqrt{2}}{2}\langle{\uparrow}|\langle{\downarrow}| + \frac{\sqrt{2}}{2}\langle{\downarrow}|\langle{\downarrow}|\Big)|{\leftarrow}{\nearrow}\rangle \\
&= \frac{3}{2}\sin\alpha + \frac{1}{2}\cos\alpha \\
&= 0 \\
\Big(\langle{\rightarrow}|\langle{\rightarrow}|\Big)|{\leftarrow}{\nearrow}\rangle &= \Big(\frac{\sqrt{2}}{2}\langle{\uparrow}| - \frac{\sqrt{2}}{2}\langle{\downarrow}|\Big)\Big(\frac{\sqrt{2}}{2}\langle{\uparrow}| - \frac{\sqrt{2}}{2}\langle{\downarrow}|\Big)|{\leftarrow}{\nearrow}\rangle \\
&= \frac{1}{2}\Big(\langle{\uparrow}|\langle{\uparrow}| - \langle{\uparrow}|\langle{\downarrow}| - \langle{\downarrow}|\langle{\uparrow}| + \langle{\downarrow}|\langle{\downarrow}|\Big)|{\leftarrow}{\nearrow}\rangle \\
&= \frac{\sqrt{2}}{2}\cos\alpha + \frac{\sqrt{2}}{2}\sin\alpha - \frac{\sqrt{2}}{4}\Big(2\cos\alpha + 2\sin\alpha\Big) \\
&= 0
\end{aligned}$$

1.8. Show this.

We must first normalize $|{\leftarrow}{\nearrow}\rangle$. Its norm is

$$\begin{aligned}
\langle{\leftarrow}{\nearrow}|{\leftarrow}{\nearrow}\rangle &= 2\cos^2\alpha + 2\sin^2\alpha + (\cos\alpha + \sin\alpha)^2 \\
&= 3\cos^2\alpha + 2\cos\alpha\sin\alpha + 3\sin^2\alpha \\
&= \frac{12}{5}
\end{aligned}$$

and so we multiply it by $\sqrt{5/12}$. Both parties measuring $|{\downarrow}\rangle|{\downarrow}\rangle$ results in a probability of

$$\Big[\Big(\langle{\downarrow}|\langle{\downarrow}|\Big)|{\leftarrow}{\nearrow}\rangle\Big]^2 = \frac{5}{6}\sin^2\alpha = \frac{1}{12}.$$

1.9. Explain all these numbers in both the boson and fermion cases.

The fermion case is quite simple. The antisymmety condition rules out permutations, so we are left with only combinations of states,

$$\binom{10}{n} = \frac{10!}{n!(10-n)!}$$

The boson case is one of assigning n indistinguishable quantities to 10 bins, i.e. it is a variant of the famous 'stars and bars' combinatorics problem, and so may be parameterized by

$$\binom{10+n-1}{n} = \frac{(9+n)!}{9!n!}.$$