1 Rotations and the Notion of Lie Algebra

1.1. Suppose you are given two vectors $\vec{p}$ and $\vec{q}$ in ordinary 3-dimensional space. Consider this array of three numbers:

$$\begin{pmatrix} p^2q^3 \\ p^3q^1 \\ p^1q^2 \end{pmatrix}$$

Prove that it is not a vector, even though it looks like a vector. (Check how it transforms under rotation!) In contrast,

$$\begin{pmatrix} p^2q^3 - p^3q^2 \\ p^3q^1 - p^1q^3 \\ p^1q^2 - p^2q^1 \end{pmatrix}$$

does transform like a vector. It is in fact the vector cross product $\vec{p} \otimes \vec{q}$.

To prove that this is not a vector, we merely need to find one rotation for which it does not transform like a vector. We will take the test rotation to be about the x-axis,

$$R_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix}.$$

Under this transformation, the vectors $\vec{p}$ and $\vec{q}$ transform as

$$\vec{p} = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix} \to \begin{pmatrix} p^1 \\ p^2\cos\theta_x + p^3\sin\theta_x \\ -p^2\sin\theta_x + p^3\cos\theta_x \end{pmatrix}$$

$$\vec{q} = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix} \to \begin{pmatrix} q^1 \\ q^2\cos\theta_x + q^3\sin\theta_x \\ -q^2\sin\theta_x + q^3\cos\theta_x \end{pmatrix}$$

and so the given array of numbers transform as

$$\begin{aligned} \begin{pmatrix} p^2q^3 \\ p^3q^1 \\ p^1q^2 \end{pmatrix} &\to \begin{pmatrix} (p^2\cos\theta_x + p^3\sin\theta_x)(-q^2\sin\theta_x + q^3\cos\theta_x) \\ (-p^2\sin\theta_x + p^3\cos\theta_x)q^1 \\ p^1(q^2\cos\theta_x + q^3\sin\theta_x) \end{pmatrix} \\ &= \begin{pmatrix} p^2q^3\cos^2\theta_x + (p^3q^3 - p^2q^2)\cos\theta_x\sin\theta_x - p^3q^2\sin^2\theta_x \\ -p^2q^1\sin\theta_x + p^3q^1\cos\theta_x \\ p^1q^2\cos\theta_x + p^1q^3\sin\theta_x \end{pmatrix} \end{aligned}$$

This is clearly not the transformation law characterizing vectors, and so the given array is not a vector.

By contrast, if we compute the transformation of a similar array,

$$\begin{aligned} \begin{pmatrix} p^3q^2 \\ p^1q^3 \\ p^2q^1 \end{pmatrix} &\to \begin{pmatrix} (-p^2\sin\theta_x + p^3\cos\theta_x)(q^2\cos\theta_x + q^3\sin\theta_x) \\ p^1(-q^2\sin\theta_x + q^3\cos\theta_x) \\ (p^2\cos\theta_x + p^3\sin\theta_x)q^1 \end{pmatrix} \\ &= \begin{pmatrix} p^3q^2\cos^2\theta_x + (p^3q^3 - p^2q^2)\cos\theta_x\sin\theta_x - p^2q^3\sin^2\theta_x \\ -p^1q^2\sin\theta_x + p^1q^3\cos\theta_x \\ p^2q^1\cos\theta_x + p^3q^1\sin\theta_x \end{pmatrix} \end{aligned}$$

we can subtract this from the first array to find that their combination does transform as a vector,

$$\begin{pmatrix} p^2q^3 - p^3q^2 \\ p^3q^1 - p^1q^3 \\ p^1q^2 - p^2q^1 \end{pmatrix} \to \begin{pmatrix} p^2q^3 - p^3q^2 \\ (p^3q^1 - p^1q^3)\cos\theta_x + (p^3q^1 - p^1q^3)\sin\theta_x \\ -(p^1q^2 - p^2q^1)\sin\theta_x + (p^1q^2 - p^2q^1)\cos\theta_x \end{pmatrix}$$

1.2. Verify that $R \simeq I + A$, with A given by $A = \theta_x \mathcal{J}_x + \theta_y \mathcal{J}_y + \theta_z \mathcal{J}_z$, satisfies the condition $\det R = 1$.

We assume that A is infinitesimal, as this is not true otherwise. Let us make this explicit by redefining $R = I + \epsilon A$ and denote the eigenvalues of A by λ_i. Then the eigenvalues of R are given by $1 + \epsilon\lambda_i$ and the determinant is

$$\begin{aligned} \det(R) = \det(I + A) &= (1 + \epsilon\lambda_1)(1 + \epsilon\lambda_2)(1 + \epsilon\lambda_3) \\ &= 1 + \epsilon(\lambda_1 + \lambda_2 + \lambda_3) + \mathcal{O}(\epsilon^2) \\ &= 1 + \epsilon \mathrm{Tr}(A) + \mathcal{O}(\epsilon^2) \end{aligned}$$

Since A is traceless (being the sum of the traceless generating matrices), this becomes

$$\det(R) = 1.$$

1.3. Using (14), show that a rotation around the x-axis through angle θ_x is given by

$$R_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix}$$

Write down $R_y(\theta_y)$. Show explicitly that $R_x(\theta_x)R_y(\theta_y) \neq R_y(\theta_y)R_x(\theta_x)$.

To go from the Lie algebra of $SO(3)$ to the Lie group element $R_x(\theta_x)$, group theory tells us to exponentiate the associated element of the former by θ_x. Before doing so, it will be useful to calculate

$$\mathcal{J}_x^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \equiv -I_x.$$

Note that $\mathcal{J}_x I_x = I_x \mathcal{J}_x = \mathcal{J}_x$. Exponentiating J_x through an angle θ_x yields

$$\begin{aligned} R_x(\theta_x) &= \exp(\theta_x \mathcal{J}_x) \\ &= 1 + \theta_x \mathcal{J}_x + \frac{1}{2!}(\theta_x \mathcal{J}_x)^2 + \frac{1}{3!}(\theta_x \mathcal{J}_x)^3 + \frac{1}{4!}(\theta_x \mathcal{J}_x)^4 + \frac{1}{5!}(\theta_x \mathcal{J}_x)^5 + \cdots \\ &= 1 + \theta_x \mathcal{J}_x - \frac{1}{2!}\theta_x^2 I_x - \frac{1}{3!}\theta_x^3 \mathcal{J}_x + \frac{1}{4!}\theta_x^4 I_x + \frac{1}{5!}\theta_x^5 \mathcal{J}_x + \cdots \\ &= (I - I_x) + \left(1 - \frac{1}{2!}\theta_x^2 + \frac{1}{4!}\theta_x^4 + \cdots\right) I_x + \left(\theta_x - \frac{1}{3!}\theta_x^3 + \frac{1}{5!}\theta_x^5 + \cdots\right) \mathcal{J}_x \\ &= (I - I_x) + \cos\theta_x I_x + \sin\theta_x \mathcal{J}_x \end{aligned}$$

or, in component form,

$$R_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix}$$

Following the same procedure for the y-axis gives

$$R_y(\theta_y) = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix}$$

To check their commutation, we compute

$$\begin{aligned} R_x(\theta_x)R_y(\theta_y) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ \sin\theta_x\sin\theta_y & \cos\theta_x & \sin\theta_x\cos\theta_y \\ \cos\theta_x\sin\theta_y & -\sin\theta_x & \cos\theta_x\cos\theta_y \end{pmatrix} \\ R_y(\theta_y)R_x(\theta_x) &= \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta_y & \sin\theta_y\sin\theta_x & -\sin\theta_y\cos\theta_x \\ 0 & \cos\theta_x & \sin\theta_x \\ \sin\theta_y & -\cos\theta_y\sin\theta_x & \cos\theta_y\cos\theta_x \end{pmatrix} \end{aligned}$$

Clearly, $R_x(\theta_x)R_y(\theta_y) \neq R_y(\theta_y)R_x(\theta_x)$.

1.4. Use the hermiticity of J to show that c_{ijk} in (18) are real numbers.

Since each J_i is hermitian, we have

$$\begin{aligned} [J_i, J_j]^\dagger &= (J_iJ_j - J_jJ_i)^\dagger \\ &= J_j^\dagger J_i^\dagger - J_i^\dagger J_j^\dagger \\ &= J_jJ_i - J_iJ_j \\ &= [J_j, J_i] \\ &= -[J_i, J_j] \\ &= -ic_{ijk}J_k \end{aligned}$$

Simultaneously, (18) implies

$$\begin{aligned} [J_i, J_j]^\dagger &= (ic_{ijk}J_k)^\dagger \\ &= -ic_{ijk}^\dagger J_k^\dagger \\ &= -ic_{ijk}^\dagger J_k \end{aligned}$$

Subtracting one from the other gives

$$0 = -i(c_{ijk} - c_{ijk}^\dagger)J_k.$$

Since J_k is not zero, this implies
$$c_{ijk} = c^{\dagger}_{ijk},$$
i.e. $c_{ijk} \in \mathbb{R}$.

1.5. Calculate $[J_{(mn)}, J_{(pq)}]$ by brute force using (24).

Jumping straight to calculation,

$$\begin{aligned}
[J_{(mn)}, J_{(pq)}] &= (J_{(mn)})^{ij}(J_{(pq)})^{jk} - (J_{(pq)})^{ij}(J_{(mn)})^{jk} \\
&= -(\delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni})(\delta^{pj}\delta^{qk} - \delta^{pk}\delta^{qj}) + (\delta^{pi}\delta^{qj} - \delta^{pj}\delta^{qi})(\delta^{mj}\delta^{nk} - \delta^{mk}\delta^{nj}) \\
&= -(\delta^{mi}\delta^{np}\delta^{qk} - \delta^{mi}\delta^{nq}\delta^{pk})
\end{aligned}$$

1.6. Of the six 4-by-4 matrices J_{12}, J_{23}, J_{31}, J_{14}, J_{24}, J_{34} that generate $SO(4)$, what is the maximum number that can be simultaneously diagonalized?

1.7. Verify (31).