1 The Postulates—a General Discussion

1.1. Consider the following operators on a Hilbert space $\mathbb{V}^3(C)$:

$$L_x = \frac{1}{2^{1/2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_y = \frac{1}{2^{1/2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(a) What are the possible values one can obtain if L_z is measured?

(b) Take the state in which $L_z = 1$. In this state what are $\langle L_x \rangle$, $\langle L_x^2 \rangle$, and ΔL_x?

(c) Find the normalized eigenstates and the eigenvalues of L_x in the L_z basis.

(d) If the particle is in the state with $L_z = -1$, and L_x is measured, what are the possible outcomes and their probabilities?

(e) Consider the state

$$|\psi\rangle = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2^{1/2} \end{bmatrix}$$

in the L_z basis. If L_z^2 is measured in this state and a result $+1$ is obtained, what is the state after measurement? How probable was this result? If L_z is measured immediately afterwards, what are the outcomes and respective probabilities?

(f) A particle is in a state for which the probabilites are $P(L_z = 1) = 1/4$, $P(L_z = 0) = 1/2$, and $P(L_z = -1) = 1/4$. Convince yourself that the most general, normalized state with this property is

$$|\psi\rangle = \frac{e^{i\delta_1}}{2}|L_z = 1\rangle + \frac{e^{i\delta_2}}{2^{1/2}}|L_z = 0\rangle + \frac{e^{i\delta_3}}{2}|L_z = -1\rangle$$

It was stated earlier on that if $|\psi\rangle$ is a normalized state then the state $e^{i\theta}|\psi\rangle$ is a physically equivalent normalized state. Does this mean that the factors $e^{i\delta_i}$ multiplying the L_z eigenstates are irrelevant? [Calculate for example $P(L_x = 0)$.]

(a) These are the eigenvalues of L_z,

$$L_z = 1, 0, \text{ or } -1$$

(b) $L_z = 1$ when the state has the form

$$|\psi\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Using this, we calculate

$$
\begin{aligned}
\langle L_x \rangle &= \langle \psi | L_x | \psi \rangle \\
&= \frac{1}{2^{1/2}} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \frac{1}{2^{1/2}} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
&= 0 \\
\langle L_x^2 \rangle &= \langle \psi | L_x^2 | \psi \rangle \\
&= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{2} \\
\Delta L_x &= \langle \psi | (L_x - \langle L_x \rangle)^2 | \psi \rangle \\
&= \langle \psi | L_x^2 - L_x \langle L_x \rangle - \langle L_x \rangle L_x + \langle L_x \rangle^2 | \psi \rangle \\
&= \langle \psi | L_x^2 | \psi \rangle - 2 \langle L_x \rangle \langle \psi | L_x | \psi \rangle + \langle L_x \rangle^2 \langle \psi | \psi \rangle \\
&= \langle L_x^2 \rangle - \langle L_x \rangle^2 \\
&= \langle L_x^2 \rangle \\
&= \frac{1}{2}
\end{aligned}
$$

(c) Noting that we are already in the L_z basis (L_z is diagonal), this amounts to simply finding the eigenvalues and eigenvectors of L_x. We do so by examining the characteristic equation,

$$
\begin{aligned}
\det(\lambda I - L_x) &= \begin{vmatrix} \lambda & -\frac{1}{2^{1/2}} & 0 \\ -\frac{1}{2^{1/2}} & \lambda & -\frac{1}{2^{1/2}} \\ 0 & -\frac{1}{2^{1/2}} & \lambda \end{vmatrix} \\
&= \lambda \Big(\lambda^2 - \frac{1}{2} \Big) + \frac{1}{2^{1/2}} \Big(- \frac{\lambda}{2^{1/2}} \Big) \\
&= \lambda^3 - \frac{\lambda}{2} - \frac{\lambda}{2} \\
&= \lambda(\lambda^2 - 1)
\end{aligned}
$$

i.e. the eigenvalues of L_x are

$$L_x = 1, 0, \text{ and } -1$$

The eigenstate $|\lambda_i\rangle$ is found by solving $\ker(\lambda_i I - L_x)$,

$$(I - L_x)|L_x = 1\rangle = \begin{bmatrix} 1 & -\frac{1}{2^{1/2}} & 0 \\ -\frac{1}{2^{1/2}} & 1 & -\frac{1}{2^{1/2}} \\ 0 & -\frac{1}{2^{1/2}} & 1 \end{bmatrix} |L_x = 1\rangle = 0$$

$$-L_x|L_x = 0\rangle = \begin{bmatrix} 0 & -\frac{1}{2^{1/2}} & 0 \\ -\frac{1}{2^{1/2}} & 0 & -\frac{1}{2^{1/2}} \\ 0 & -\frac{1}{2^{1/2}} & 0 \end{bmatrix} |L_x = 0\rangle = 0$$

$$(-I - L_x)|L_x = -1\rangle = \begin{bmatrix} -1 & -\frac{1}{2^{1/2}} & 0 \\ -\frac{1}{2^{1/2}} & -1 & -\frac{1}{2^{1/2}} \\ 0 & -\frac{1}{2^{1/2}} & -1 \end{bmatrix} |L_x = -1\rangle = 0$$

By inspection, we can write

$$|L_x = 1\rangle \propto \begin{bmatrix} \frac{1}{2^{1/2}} \\ 1 \\ \frac{1}{2^{1/2}} \end{bmatrix}$$

$$|L_x = 0\rangle \propto \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$|L_x = -1\rangle \propto \begin{bmatrix} -\frac{1}{2^{1/2}} \\ 1 \\ -\frac{1}{2^{1/2}} \end{bmatrix}$$

Normalizing these yields

$$|L_x = 1\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

$$|L_x = 0\rangle = \frac{1}{2^{1/2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$|L_x = -1\rangle = \frac{1}{2}\begin{bmatrix} -1 \\ \sqrt{2} \\ -1 \end{bmatrix}$$

(d) $L_z = -1$ when the state has the form

$$|\psi\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The only possible measurements for L_x are its eigenvalues, 1, 0, and -1. We project the state onto the L_x eigenbasis via the projection operator $P_{L_x} = \sum_k |\lambda_k\rangle\langle\lambda_k|$

$$
\begin{aligned}
|\psi\rangle &= \sum_k |\lambda_k\rangle\langle\lambda_k|\psi\rangle \\
&= \langle L_x = 1|\psi\rangle|L_x = 1\rangle + \langle L_x = 0|\psi\rangle|L_x = 0\rangle + \langle L_x = -1|\psi\rangle|L_x = -1\rangle \\
&= \frac{1}{2}\begin{bmatrix}1 & \sqrt{2} & 1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}|L_x = 1\rangle + \frac{1}{2^{1/2}}\begin{bmatrix}1 & 0 & -1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}|L_x = 0\rangle + \frac{1}{2}\begin{bmatrix}-1 & \sqrt{2} & -1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}|L_x = -1\rangle \\
&= \frac{1}{2}|L_x = 1\rangle - \frac{1}{2^{1/2}}|L_x = 0\rangle - \frac{1}{2}|L_x = -1\rangle
\end{aligned}
$$

From this, we can read off the probabilities of the various measurements,

$$
\begin{aligned}
P(L_x = 1) &= \left(\frac{1}{2}\right)^2 = \frac{1}{4} \\
P(L_x = 0) &= \left(-\frac{1}{2^{1/2}}\right)^2 = \frac{1}{2} \\
P(L_x = -1) &= \left(-\frac{1}{2}\right)^2 = \frac{1}{4}
\end{aligned}
$$

(e) L_z^2 will have eigenvalues that are the square of the eigenvalues of L_z, i.e.

$$L_z^2 = 1 \text{ or } 0.$$

Since

$$L_z^2 = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix},$$

a measurement of 1 projects the state $|\psi\rangle$ onto the eigenspace associated with this eigenvalue,

$$|1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad |1'\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix},$$

and so the new (normalized) state is

$$|\psi'\rangle = \frac{2}{3^{1/2}}\begin{bmatrix}1/2\\0\\1/2^{1/2}\end{bmatrix}$$

The probability of this result is

$$|\langle 1|\psi\rangle|^2 + |\langle 1'|\psi\rangle|^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

If we measure L_z in this new state, we find two possibilities: $L_z = -1, 1$ (the $L_z = 0$ state is excluded). The first has the associated state and probability

$$|\psi_{+1}\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad P = |\langle +1|\psi'\rangle|^2 = \left(\frac{2}{3^{1/2}} \cdot \frac{1}{2}\right)^2 = \frac{1}{3}$$

while the second has the associated state and probability

$$|\psi_{-1}\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}, \quad P = |\langle -1|\psi'\rangle|^2 = \left(\frac{2}{3^{1/2}} \cdot \frac{1}{2^{1/2}}\right)^2 = \frac{2}{3}$$

(f) This is clear from how we measure the probability of a state. If the probability of finding $|\psi\rangle$ in the eigenstate $|\alpha\rangle$ is p and $\langle\alpha|\psi\rangle = \psi_\alpha \in \mathbb{C}$, then

$$\begin{aligned} p &= |\langle\alpha|\psi\rangle|^2 \\ &= \langle\alpha|\psi\rangle\langle\psi|\alpha\rangle \\ &= \psi_\alpha \cdot \bar{\psi}_\alpha \\ &= re^{i\delta} \cdot re^{-i\delta} \\ &= r^2 \end{aligned}$$

where we have written ψ_α in polar form as $re^{i\delta}$. From this, it is clear that the most general form for ψ_α that guarantees $p = |\langle\alpha|\psi\rangle|^2$ is

$$\psi_\alpha = \sqrt{p}\, e^{i\delta}$$

Despite the seeming arbitrariness of δ, it plays a role when performing a measurement with an operator whose eigenstates are different than the currently used eigenstates. To take the given example,

$$\begin{aligned} P(L_x = 0) &= |\langle L_x = 0|\psi\rangle|^2 \\ &= \left|\frac{1}{2^{1/2}}\left(\frac{e^{i\delta_1}}{2} - \frac{e^{i\delta_3}}{2}\right)\right|^2 \\ &= \frac{1}{8}(e^{i\delta_1} - e^{i\delta_3})(e^{-i\delta_1} - e^{-i\delta_3}) \\ &= \frac{1}{8}(2 - e^{i(\delta_1-\delta_3)} - e^{-i(\delta_1-\delta_3)}) \\ &= \frac{1}{4}(1 - \cos(\delta_1 - \delta_3)) \end{aligned}$$

From this, we see that it is the *relative* phase values that significantly alter our measurements. This is similar to what we find in electromagnetism: the relative phase of two waves creates measurable effects, while the phase of a single, monochromatic plane wave is undetectable.

1.2. Show that for a real wave function $\psi(x)$, the expectation value of momentum $\langle P\rangle = 0$. (Hint: Show that the probabilities for the momenta $\pm p$ are equal.) Generalize this result to the case $\psi = c\psi_r$, where ψ_r is real and c an arbitrary (real or complex) constant. (Recall that $|\psi\rangle$ and $\alpha|\psi\rangle$ are physically equivalent.)

Following the hint, we compute

$$\begin{aligned} P(P = +p) &= |\langle p|\psi\rangle|^2 \\ &= \langle\psi|p\rangle\langle p|\psi\rangle \\ &= \left(\int_{-\infty}^{\infty} \langle\psi|x\rangle\langle x|p\rangle \,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} \langle p|x'\rangle\langle x'|\psi\rangle \,\mathrm{d}x'\right) \\ &= \left(\int_{-\infty}^{\infty} \bar{c}\psi_r(x)\frac{e^{ipx/\hbar}}{(2\pi\hbar)^{1/2}} \,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} \frac{e^{-ipx'/\hbar}}{(2\pi\hbar)^{1/2}} c\psi_r(x') \,\mathrm{d}x'\right) \\ &= \frac{|c|^2}{2\pi\hbar}\left(\int_{-\infty}^{\infty} \psi_r(x)e^{ipx/\hbar} \,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} \psi_r(x')e^{-ipx'/\hbar} \,\mathrm{d}x'\right) \end{aligned}$$

Since this is invariant under the exchange $p \to -p$, the probability of finding a particle at momentum

p is exactly the same as finding that same particle at momentum $-p$. But any probability distribution $f_P(p)$ with this property is even about $p = 0$, and so $pf_P(p)$ is odd and $\langle P \rangle = 0$.

1.3. Show that if $\psi(x)$ has mean momentum $\langle P \rangle$, $e^{ip_0 x/\hbar}\psi(x)$ has mean momentum $\langle P \rangle + p_0$.

Assuming the state $|\psi'\rangle$ is normalized, we have

$$\begin{aligned}
\langle \psi' | P | \psi' \rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \psi' | x \rangle \langle x | P | x' \rangle \langle x' | \psi' \rangle \, \mathrm{d}x \, \mathrm{d}x' \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ip_0 x/\hbar} \bar{\psi}(x) \Big(-i\hbar \delta(x - x') \frac{\mathrm{d}}{\mathrm{d}x'} \Big) e^{ip_0 x'/\hbar} \psi(x') \, \mathrm{d}x \, \mathrm{d}x' \\
&= \int_{-\infty}^{\infty} e^{-ip_0 x/\hbar} \bar{\psi}(x) \Big(-i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \Big) e^{ip_0 x/\hbar} \psi(x) \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{-ip_0 x/\hbar} \bar{\psi}(x) \Big(-i\hbar \cdot \frac{ip_0}{\hbar} \cdot e^{ip_0 x/\hbar} \psi(x) - i\hbar \cdot e^{ip_0 x/\hbar} \frac{\mathrm{d}}{\mathrm{d}x} \psi(x) \Big) \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \bar{\psi}(x) \Big(p_0 - i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \Big) \psi(x) \, \mathrm{d}x \\
&= p_0 \int_{-\infty}^{\infty} |\psi(x)|^2 \, \mathrm{d}x + \int_{-\infty}^{\infty} \bar{\psi}(x) \Big(-i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \Big) \psi(x) \, \mathrm{d}x \\
&= p_0 + \langle P \rangle
\end{aligned}$$

where in the last step we used the fact that $\langle P \rangle$ in the position basis is

$$\langle P \rangle = \int_{-\infty}^{\infty} \bar{\psi}(x) \Big(-i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \Big) \psi(x) \, \mathrm{d}x.$$

Another way to show this relation is to recall that position space and momentum space are dual in the Fourier sense, and so a multiplication by $e^{ip_0 x}$ (or e^{ipx_0}) in one space becomes addition by p_0 (or x_0) in the other.