12 Rotational Invariance and Angular Momentum

12.1 Translations in Two Dimensions

12.1.1. Verify that $\hat{a} \cdot \mathbf{P}$ is the generator of infinitesimal translations along $\mathbf{a}$ by considering the relation

$$\langle x, y| I - \frac{i}{\hbar} \boldsymbol{\delta} a \cdot \mathbf{P} |\psi\rangle = \psi(x - \delta a_x, y - \delta a_y)$$

Noting that $\boldsymbol{\delta} a$ is of order ε, we have

$$\begin{aligned}
\langle x, y| I - \frac{i}{\hbar} \boldsymbol{\delta} a \cdot \mathbf{P} |\psi\rangle &= \langle x, y| I - \frac{i}{\hbar} \delta a_x P_x + I - \frac{i}{\hbar} \delta a_y P_y + I - I |\psi\rangle \\
&= \langle x, y| I - \frac{i}{\hbar} \delta a_x P_x |\psi\rangle + \langle x, y| I - \frac{i}{\hbar} \delta a_y P_y |\psi\rangle - \langle x, y|\psi\rangle \\
&= \psi(x - \delta a_x, y) + \psi(x, y - \delta a_y) - \psi(x, y) \\
&\approx \psi(x, y) - \frac{\partial \psi}{\partial x}\bigg|_{x,y} \delta a_x + \psi(x, y) - \frac{\partial \psi}{\partial y}\bigg|_{x,y} \delta a_y - \psi(x, y) \\
&= \psi(x, y) - \frac{\partial \psi}{\partial x}\bigg|_{x,y} \delta a_x - \frac{\partial \psi}{\partial y}\bigg|_{x,y} \delta a_y \\
&\approx \psi(x - \delta a_x, y - \delta a_y)
\end{aligned}$$

where each approximation becomes an equality as $\delta a \to 0$.

12.2 Rotations in Two Dimensions

12.2.1. Provide the steps linking Eq. (12.2.8) to Eq. (12.2.9). [Hint: Recall the derivation of Eq. (11.2.8) from Eq. (11.2.6).]

Working backwards, we see

$$\begin{aligned}
\langle x, y| I - \frac{i\varepsilon_z L_z}{\hbar} |\psi\rangle &= \iint \langle x, y| U[R] |x', y'\rangle \langle x', y'|\psi\rangle \, \mathrm{d}x' \mathrm{d}y' \\
&= \iint \langle x, y| x' - y'\varepsilon_z, x'\varepsilon_z + y'\rangle \psi(x', y') \, \mathrm{d}x' \mathrm{d}y' \\
&= \iint \langle x, y| x', y'\rangle \psi(x' + y'\varepsilon_z, y' - x'\varepsilon_z) \, \mathrm{d}x' \mathrm{d}y' \\
&= \iint \delta(x - x')\delta(y - y') \psi(x' + y'\varepsilon_z, y' - x'\varepsilon_z) \, \mathrm{d}x' \mathrm{d}y' \\
&= \psi(x + y\varepsilon_z, y - x\varepsilon_z)
\end{aligned}$$

In the above, we were able to make the changes

$$\begin{aligned}
x' &\to x' + y'\varepsilon_z \\
y' &\to y' - x'\varepsilon_z
\end{aligned}$$

because both $\mathrm{d}y'\varepsilon_z$ and $\mathrm{d}x'\varepsilon_z$ vanish to first order.

12.2.2. Using these commutation relations (and your keen hindsight) derive $L_z = XP_y - YP_x$. At least show that Eqs. (12.2.16) and (12.2.17) are consistent with $L_z = XP_y - YP_x$.

It is unclear to me how one can fix L_z using only these commutation relations, as it could also depend on Z and P_z while satisfying (12.2.16) and (12.2.17). Still, we can complete the second part of the problem:

$$\begin{aligned}
[X, L_z] &= [X, XP_y - YP_x] \\
&= [X, XP_y] - [X, YP_x] \\
&= [X, X]P_y + X[X, P_y] - [X, Y]P_x - Y[X, P_x] \\
&= -i\hbar Y \\
[Y, L_z] &= [Y, XP_y - YP_x] \\
&= [Y, XP_y] - [Y, YP_x] \\
&= [Y, X]P_y + X[Y, P_y] - [Y, Y]P_x - Y[Y, P_x] \\
&= i\hbar X \\
[P_x, L_z] &= [P_x, XP_y - YP_x] \\
&= [P_x, XP_y] - [P_x, YP_x] \\
&= [P_x, X]P_y + X[P_x, P_y] - [P_x, Y]P_x - Y[P_x, P_x] \\
&= -i\hbar P_y \\
[P_y, L_z] &= [P_y, XP_y - YP_x] \\
&= [P_y, XP_y] - [P_y, YP_x] \\
&= [P_y, X]P_y + X[P_y, P_y] - [P_y, Y]P_x - Y[P_y, P_x] \\
&= i\hbar P_x
\end{aligned}$$

12.2.3. Derive Eq. (12.2.19) by doing a coordinate transformation on Eq (12.2.10), and also by the direct method mentioned above.

We must transform both ∂/∂_x and ∂/∂_y, for which we will need the relations

$$\begin{aligned}
\rho &= (x^2 + y^2)^{1/2} \\
\phi &= \tan^{-1}(\tfrac{y}{x}) \\
x &= \rho\cos\phi \\
y &= \rho\sin\phi
\end{aligned}$$

Using these and the chain rule applied to $f(\rho(x, y), \phi(x, y))$, we find

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} \\
&= \frac{1}{2}\frac{2x}{(x^2 + y^2)^{1/2}}\frac{\partial f}{\partial \rho} + \frac{1}{1 + (\frac{y}{x})^2}\Big(-\frac{y}{x^2}\Big)\frac{\partial f}{\partial \phi} \\
&= \frac{x}{(x^2 + y^2)^{1/2}}\frac{\partial f}{\partial \rho} - \frac{y}{x^2 + y^2}\frac{\partial f}{\partial \phi} \\
&= \frac{\rho\cos\phi}{\rho}\frac{\partial f}{\partial \rho} - \frac{\rho\sin\phi}{\rho^2}\frac{\partial f}{\partial \phi}
\end{aligned}$$

$$
\begin{aligned}
&= \cos\phi\frac{\partial f}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial f}{\partial \phi} \\
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y} \\
&= \frac{1}{2}\frac{2y}{(x^2+y^2)^{1/2}}\frac{\partial f}{\partial \rho} + \frac{1}{1+(\frac{y}{x})^2}\frac{1}{x}\frac{\partial f}{\partial \phi} \\
&= \frac{y}{(x^2+y^2)^{1/2}}\frac{\partial f}{\partial \rho} + \frac{x}{x^2+y^2}\frac{\partial f}{\partial \phi} \\
&= \frac{\rho\sin\phi}{\rho}\frac{\partial f}{\partial \rho} + \frac{\rho\cos\phi}{\rho^2}\frac{\partial f}{\partial \phi} \\
&= \sin\phi\frac{\partial f}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial f}{\partial \phi}
\end{aligned}
$$

which implies

$$
\begin{aligned}
\frac{\partial}{\partial x} &\to \cos\phi\frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial y} &\to \sin\phi\frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}
\end{aligned}
$$

Substituting this into $L_z = XP_y - YP_x$ reveals

$$
\begin{aligned}
L_z &= -i\hbar x\frac{\partial}{\partial y} + i\hbar y\frac{\partial}{\partial x} \\
&= -i\hbar\rho\cos\phi\Big(\sin\phi\frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}\Big) + i\hbar\rho\sin\phi\Big(\cos\phi\frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi}\Big) \\
&= -i\hbar\rho\Big(\cos\phi\sin\phi - \cos\phi\sin\phi\Big)\frac{\partial}{\partial \rho} - i\hbar(\cos^2\phi + \sin^2\phi)\frac{\partial}{\partial \phi} \\
&= -i\hbar\frac{\partial}{\partial \phi}
\end{aligned}
$$

Alternatively, if we require that L_z generate infinitesimal translations, i.e.

$$
\langle \rho, \phi | I - \frac{i}{\hbar}\varepsilon_z L_z | \psi \rangle = \psi(\rho,\phi) - \frac{i}{\hbar}\varepsilon_z \langle \rho,\phi | L_z | \psi \rangle = \psi(\rho, \phi - \varepsilon_z) = \psi(\rho,\phi) - \frac{\partial \psi}{\partial \phi}\varepsilon_z
$$

then we immediately have

$$
L_z = -i\hbar\frac{\partial}{\partial \phi}
$$

12.2.4. Rederive the equivalent of Eq. (12.2.23) keeping terms of order $\varepsilon_x\varepsilon_z^2$. (You may assume $\varepsilon_y = 0$.) Use this information to rewrite Eq. (12.2.24) to order $\varepsilon_x\varepsilon_z^2$. By equating coefficients of this term deduce the constraint

$$
-2L_zP_xL_z + P_xL_z^2 + L_z^2P_x = \hbar^2 P_x
$$

This seems to conflict with statement (1) made above, but not really, in view of the identity

$$
-2\Lambda\Omega\Lambda + \Omega\Lambda^2 + \Lambda^2\Omega \equiv [\Lambda,[\Lambda,\Omega]]
$$

Using the identity, verify that the new constraint coming from the $\varepsilon_x \varepsilon_z^2$ term is satisfied given the commutation relations between P_x, P_y, and L_z.

Tracing the steps outlined in the text, we consider the following sequence of operators

$$U[R(-\varepsilon_z \mathbf{k})]T(-\varepsilon_x \mathbf{i})U[R(\varepsilon_z \mathbf{k})]T(\varepsilon_x \mathbf{i})$$

Applied to a point (x, y), this has the effect of

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &\to \begin{bmatrix} x+\varepsilon_x \\ y \end{bmatrix} \\
&\to \begin{bmatrix} (x+\varepsilon_x) - y\varepsilon_z \\ (x+\varepsilon_x)\varepsilon_z + y \end{bmatrix} \\
&\to \begin{bmatrix} x - y\varepsilon_z \\ (x+\varepsilon_x)\varepsilon_z + y \end{bmatrix} \\
&\to \begin{bmatrix} (x-y\varepsilon_z) + (x\varepsilon_z + \varepsilon_x\varepsilon_z + y)\varepsilon_z \\ -(x-y\varepsilon_z)\varepsilon_z + x\varepsilon_z + \varepsilon_x\varepsilon_z + y \end{bmatrix} \\
&= \begin{bmatrix} x(1+\varepsilon_z^2) + \varepsilon_x\varepsilon_z^2 \\ y(1+\varepsilon_z^2) + \varepsilon_x\varepsilon_z \end{bmatrix}
\end{aligned}$$

Writing this out in terms of operators, we see that we must have

$$\left(I + \frac{i}{\hbar}\varepsilon_z L_z\right)\left(I + \frac{i}{\hbar}\varepsilon_x P_x\right)\left(I - \frac{i}{\hbar}\varepsilon_z L_z\right)\left(I - \frac{i}{\hbar}\varepsilon_x P_x\right) = I + I\varepsilon_z^2 - \frac{i}{\hbar}\varepsilon_x\varepsilon_z^2 P_x - \frac{i}{\hbar}\varepsilon_x\varepsilon_z P_y$$

We can expand the lefthand side (keeping terms up to $\mathcal{O}(\varepsilon^3)$) to find

$$\frac{i}{\hbar}\frac{\varepsilon_x^2\varepsilon_z}{\hbar^2}[L_z, P_x]P_x + \frac{i}{\hbar}\frac{\varepsilon_x\varepsilon_z^2}{\hbar^2}L_z[P_x, L_z] + \frac{\varepsilon_x^2}{\hbar^2}P_x^2 + \frac{\varepsilon_x\varepsilon_z}{\hbar^2}[P_x, L_z] + \frac{\varepsilon_z^2}{\hbar^2}L_z^2 + I$$

As this problem is formulated, I believe it is unsolvable. At the very least, there is some necessary step that cannot be found by looking at the mathematics, as using a CAS yields multiple untrue requirements.

12.3 The Eigenvalue Problem of L_z

12.3.1. Provide the steps linking Eq. (12.3.5) to Eq. (12.3.6).

Imposing the Hermiticity of L_z gives

$$\begin{aligned}
-i\hbar\int_0^\infty\int_0^{2\pi} \psi_1^*\frac{\partial\psi_2}{\partial\phi}\rho\,\mathrm{d}\phi\,\mathrm{d}\rho &= \left[-i\hbar\int_0^\infty\int_0^{2\pi} \psi_2^*\frac{\partial\psi_1}{\partial\phi}\rho\,\mathrm{d}\phi\,\mathrm{d}\rho\right]^* \\
&= i\hbar\int_0^\infty\int_0^{2\pi} \psi_2\frac{\partial\psi_1^*}{\partial\phi}\rho\,\mathrm{d}\phi\,\mathrm{d}\rho \\
&= i\hbar\int_0^\infty \left(\psi_1^*\psi_2\right)\Big|_{\phi=0}^{\phi=2\pi}\rho\,\mathrm{d}\rho - i\hbar\int_0^\infty\int_0^{2\pi} \psi_1^*\frac{\partial\psi_2}{\partial\phi}\rho\,\mathrm{d}\phi\,\mathrm{d}\rho
\end{aligned}$$

Clearly, this equality holds only if

$$\psi_1^*(\rho, 0)\psi_2(\rho, 0) = \psi_1^*(\rho, 2\pi)\psi_2(\rho, 2\pi)$$

which, given each ψ_i is arbitrary, implies that

$$\psi(\rho, 0) = \psi(\rho, 2\pi).$$

12.3.2. Let us try to deduce the restriction on l_z from another angle. Consider a superposition of two allowed l_z eigenstates:

$$\psi(\rho, \phi) = A(\rho)e^{i\phi l_z/\hbar} + B(\rho)e^{i\phi l'_z/\hbar}$$

By demanding that upon a 2π rotation we get the same physical state (not necessarily the same state vector), show that $l_z - l'_z = m\hbar$, where m is an integer. By arguing on the grounds of symmetry that the allowed values of l_z must be symmetric about zero, show that these values are *either* $\ldots, 3\hbar/2, \hbar/2, -\hbar/2, -3\hbar/2, \ldots$ or $\ldots, 2\hbar, \hbar, 0 - \hbar, -2\hbar, \ldots$. It is not possible to restrict l_z any further this way.

The physical state of the system is described by its probability distribution,

$$|\psi(\rho, \phi)|^2 = A^2(\rho) + A(\rho)B^*(\rho)e^{i\phi(l_z - l'_z)/\hbar} + A^*(\rho)B(\rho)e^{-i\phi(l_z - l'_z)/\hbar} + B(\rho)^2$$

In order for this to remain undisturbed under a rotation by 2π, we must have

$$e^{i2\pi(l_z - l'_z)/\hbar} = 1$$

and

$$e^{-i2\pi(l_z - l'_z)/\hbar} = 1$$

or, equivalently,

$$l_z - l'_z = m\hbar, \quad m \in \mathbb{Z}.$$

Given that there is no preference to positive or negative l_z values in nature, the eigenvalues of L_z should be spaced evenly about 0. The only possibilities for l_z that satisfy these two constraints are

$$l_z = m\hbar$$

or

$$l_z = \frac{m\hbar}{2}$$

12.3.3. A particle is described by a wave function

$$\psi(\rho, \phi) = Ae^{-\rho^2/2\Delta^2} \cos^2 \phi$$

Show (by expressing $\cos^2 \phi$ in terms of Φ_m) that

$$P(l_z = 0) = 2/3$$
$$P(l_z = 2\hbar) = 1/6$$
$$P(l_z = -2\hbar) = 1/6$$

(Hint: Argue that the radial part $e^{-\rho^2/2\Delta^2}$ is irrelevant here.)

First, note that we can use

$$\Phi_m(\phi) = (2\pi)^{-1/2} e^{im\phi},$$

to write

$$\begin{aligned}
\cos^2\phi &= \Big(\frac{e^{i\phi} + e^{-i\phi}}{2}\Big)^2 \\
&= \frac{1}{2} + \frac{e^{i2\phi}}{4} + \frac{e^{-i2\phi}}{4} \\
&= \frac{(2\pi)^{1/2}}{2}\Phi_0(\phi) + \frac{(2\pi)^{1/2}}{4}\Phi_2(\phi) + \frac{(2\pi)^{1/2}}{4}\Phi_{-2}(\phi)
\end{aligned}$$

Now, given the fact that the wave function is separable, we may write $\psi(\rho, \phi) = \Omega(\rho)\theta(\phi)$ and focus only on the angular component, $\theta(\phi)$. We can normalize this as

$$\begin{aligned}
\int_0^{2\pi} |\theta(\phi)|^2 \,\mathrm{d}\phi &= B^2 \int_0^{2\pi} \cos^4(\phi)\,\mathrm{d}\phi \\
&= B^2 \int_0^{2\pi} \Big(\frac{1}{2} + \frac{1}{2}\cos(2\phi)\Big)^2 \mathrm{d}\phi \\
&= B^2 \int_0^{2\pi} \frac{1}{4} + \frac{1}{2}\cos(2\phi) + \frac{1}{4}\cos^2(2\phi)\,\mathrm{d}\phi \\
&= B^2 \int_0^{2\pi} \frac{1}{4} + \frac{1}{2}\cos(2\phi) + \frac{1}{8} + \frac{1}{8}\cos(4\phi)\,\mathrm{d}\phi \\
&= B^2 \cdot \frac{6\pi}{8}
\end{aligned}$$

or $B = (4/3\pi)^{1/2}$. Putting everything together, the angular wave function is

$$\theta(\phi) = \Big(\frac{2}{3}\Big)^{1/2}\Big(\Phi_0(\phi) + \frac{1}{2}\Phi_2(\phi) + \frac{1}{2}\Phi_{-2}(\phi)\Big)$$

The probabilities associated with finding the particle in various states of definite angular momentum are

$$\begin{aligned}
P(l_z = 0) &= \frac{2}{3}\Big|\int_0^{2\pi} \Phi_0^*(\phi)\Big(\Phi_0(\phi) + \frac{1}{2}\Phi_2(\phi) + \frac{1}{2}\Phi_{-2}(\phi)\Big)\,\mathrm{d}\phi\Big|^2 \\
&= \frac{2}{3}\cdot 1 \\
&= \frac{2}{3} \\
P(l_z = 2\hbar) &= \frac{2}{3}\Big|\int_0^{2\pi} \Phi_2^*(\phi)\Big(\Phi_0(\phi) + \frac{1}{2}\Phi_2(\phi) + \frac{1}{2}\Phi_{-2}(\phi)\Big)\,\mathrm{d}\phi\Big|^2 \\
&= \frac{2}{3}\cdot\frac{1}{4} \\
&= \frac{1}{6} \\
P(l_z = -2\hbar) &= \frac{2}{3}\Big|\int_0^{2\pi} \Phi_{-2}^*(\phi)\Big(\Phi_0(\phi) + \frac{1}{2}\Phi_2(\phi) + \frac{1}{2}\Phi_{-2}(\phi)\Big)\,\mathrm{d}\phi\Big|^2 \\
&= \frac{2}{3}\cdot\frac{1}{4} \\
&= \frac{1}{6}
\end{aligned}$$

12.3.4. A particle is described by a wave function

$$\psi(\rho, \phi) = Ae^{-\rho^2/2\Delta^2}\Big(\frac{\rho}{\Delta}\cos\phi + \sin\phi\Big)$$

Show that

$$P(l_z = \hbar) = P(l_z = -\hbar) = \tfrac{1}{2}$$

Though it is not required to solve the problem, we first find the normalization factor A,

$$\begin{aligned}
&A^2\int_0^\infty\!\!\int_0^{2\pi} e^{-\rho^2/\Delta^2}\Big(\frac{\rho}{\Delta}\cos\phi + \sin\phi\Big)^2 \rho\,\mathrm{d}\phi\,\mathrm{d}\rho\\
=&A^2\int_0^\infty\!\!\int_0^{2\pi} e^{-\rho^2/\Delta^2}\Big(\frac{\rho^2}{\Delta^2}\cos^2\phi + \frac{2\rho}{\Delta}\cos\phi\sin\phi + \sin^2\phi\Big)\rho\,\mathrm{d}\phi\,\mathrm{d}\rho\\
=&A^2\int_0^\infty\!\!\int_0^{2\pi} e^{-\rho^2/\Delta^2}\Big[\frac{\rho^2}{\Delta^2}\Big(\frac{1}{2}+\frac{1}{2}\cos(2\phi)\Big) + \frac{\rho}{\Delta}\sin(2\phi) + \Big(\frac{1}{2}-\frac{1}{2}\cos(2\phi)\Big)\Big]\rho\,\mathrm{d}\phi\,\mathrm{d}\rho\\
=&\pi A^2\int_0^\infty e^{-\rho^2/\Delta^2}\Big[\frac{\rho^2}{\Delta^2}+1\Big]\rho\,\mathrm{d}\rho\\
&=\pi A^2\Big(\frac{1}{\Delta^2}\int_0^\infty \rho^3 e^{-\rho^2/\Delta^2}\mathrm{d}\rho + \int_0^\infty \rho e^{-\rho^2/\Delta^2}\mathrm{d}\rho\Big)\\
&=\pi A^2\Big(\frac{1}{\Delta^2}\frac{\Delta^4}{2} + \frac{\Delta^2}{2}\Big)\\
&=\pi A^2\Delta^2
\end{aligned}$$

i.e. $A = (\pi\Delta^2)^{-1/2}$. Now we rewrite $\psi(\rho,\phi)$ in terms of $\Phi_m(\phi)$ as

$$\begin{aligned}
\psi(\rho,\phi) &= \frac{e^{-\rho^2/2\Delta^2}}{(\pi\Delta^2)^{1/2}}\Big(\frac{\rho}{2\Delta}(e^{i\phi}+e^{-i\phi}) - \frac{i}{2}(e^{i\phi}-e^{-i\phi})\Big)\\
&= \frac{e^{-\rho^2/2\Delta^2}}{(\pi\Delta^2)^{1/2}}\frac{(2\pi)^{1/2}}{2}\Big(\frac{\rho}{\Delta}\Phi_1(\phi) + \frac{\rho}{\Delta}\Phi_{-1}(\phi) - i\Phi_1(\phi) + i\Phi_{-1}(\phi)\Big)\\
&= \frac{e^{-\rho^2/2\Delta^2}}{(2\Delta^2)^{1/2}}\Big(\Phi_1(\phi)\Big[\frac{\rho}{\Delta} - i\Big] + \Phi_{-1}(\phi)\Big[\frac{\rho}{\Delta}+i\Big]\Big)
\end{aligned}$$

This is a superposition of two angular momentum eigenstates: one of $l_z = \hbar$ and the other of $l_z = -\hbar$. Since these states are orthogonal to one another, we know

$$\begin{aligned}
\int_0^{2\pi}\Phi_1^*(\phi)\psi(\rho,\phi)\mathrm{d}\phi &= \frac{e^{-\rho^2/2\Delta^2}}{(2\Delta^2)^{1/2}}\Big(\frac{\rho}{\Delta} - i\Big)\\
\int_0^{2\pi}\Phi_2^*(\phi)\psi(\rho,\phi)\mathrm{d}\phi &= \frac{e^{-\rho^2/2\Delta^2}}{(2\Delta^2)^{1/2}}\Big(\frac{\rho}{\Delta} + i\Big)
\end{aligned}$$

Since the magnitude of the inner product of the wave function with the $l_z = \hbar$ and $l_z = -\hbar$ angular momentum eigenfunctions is the same no matter which one we choose, they are equally likely to occur. That is, $P(l_z = \hbar) = P(l_z = -\hbar) = \frac{1}{2}$.

12.3.5. Note that the angular momentum seems to generate a repulsive potential in Eq. (12.3.13). Calculate its gradient and identify it as the centrifugal force.

Isolating $V(\rho)$ gives

$$\begin{aligned} V(\rho) &= \frac{1}{R(\rho)}\Big[ER(\rho) + \frac{\hbar^2}{2\mu}\Big(\frac{\mathrm{d}^2R(\rho)}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}R(\rho)}{\mathrm{d}\rho} - \frac{m^2}{\rho^2}R(\rho)\Big)\Big] \\ &= E + \frac{\hbar^2}{2\mu}\frac{1}{R(\rho)}\frac{\mathrm{d}^2R(\rho)}{\mathrm{d}\rho^2} + \frac{\hbar^2}{2\mu}\frac{1}{\rho R(\rho)}\frac{\mathrm{d}R(\rho)}{\mathrm{d}\rho} - \frac{\hbar^2}{2\mu}\frac{m^2}{\rho^2} \end{aligned}$$

The gradient of this is

$$\begin{aligned} \frac{\mathrm{d}V(\rho)}{\mathrm{d}\rho}\mathbf{e}_\rho &= \frac{\hbar^2}{2\mu}\Big(-\frac{1}{R^2}\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}\frac{\mathrm{d}R}{\mathrm{d}\rho} + \frac{1}{R}\frac{\mathrm{d}^3R}{\mathrm{d}\rho^3} - \frac{1}{\rho^2R}\frac{\mathrm{d}R}{\mathrm{d}\rho} - \frac{1}{\rho R^2}\frac{\mathrm{d}R}{\mathrm{d}\rho} + \frac{1}{\rho R}\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2} + 2\frac{m^2}{\rho^3}\Big)\mathbf{e}_\rho \\ &= \frac{\hbar^2}{2\mu}\Big(\frac{\mathrm{d}^3R}{\mathrm{d}\rho^3}\frac{1}{R} + \frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}\Big[\frac{1}{\rho R} - \frac{1}{R^2}\frac{\mathrm{d}R}{\mathrm{d}\rho}\Big] - \frac{\mathrm{d}R}{\mathrm{d}\rho}\Big[\frac{1}{\rho^2R} + \frac{1}{\rho R^2}\Big] + 2\frac{m^2}{\rho^3}\Big)\mathbf{e}_\rho \end{aligned}$$

12.3.6. Consider a particle of mass μ constrained to move on a circle of radius a. Show that $H = L_z^2/2\mu a^2$. Solve the eigenvalue problem of H and interpret the degeneracy.

For a (quasi) free particle constrained to move on a circle of radius a, the kinetic energy (and thus classical Hamiltonian) is given by

$$\begin{aligned} T &= \frac{1}{2}\mu v^2 \\ &= \frac{1}{2}\mu a^2\dot{\phi}^2 \\ &= \frac{1}{2}\mu a^2\Big(\frac{\mathrm{d}}{\mathrm{d}t}\tan^{-1}(\tfrac{y}{x})\Big)^2 \\ &= \frac{1}{2}\mu a^2\Big[\frac{1}{1+(\frac{y}{x})^2}\Big(\frac{\dot{y}}{x} - \frac{y}{x^2}\dot{x}\Big)\Big]^2 \\ &= \frac{1}{2}\mu a^2\Big(\frac{x\dot{y} - y\dot{x}}{x^2+y^2}\Big)^2 \\ &= \frac{1}{2}\frac{a^2}{\mu}\Big(\frac{x\mu\dot{y} - y\mu\dot{x}}{a^2}\Big)^2 \\ &= \frac{(xp_y - yp_x)^2}{2\mu a^2} \end{aligned}$$

Identifying the classical angular momentum operator with its quantum variant gives

$$H = \frac{L_z^2}{2\mu a^2}$$

We know that rotationally invariant Hamiltonians permit solutions of the form

$$\psi_m(\rho,\phi) = R_{Em}(\rho)\Phi_m(\phi)$$

From the problem statement, $R_{Em}(\rho) \propto \delta(\rho - a)$. Using the angular coordinate form of L_z produces the time-independent equation

$$-\frac{\hbar^2}{2\mu a^2}\frac{\partial^2}{\partial\phi^2}\psi_m(\rho,\phi) = \frac{m^2\hbar^2}{2\mu a^2}\psi_m(\rho,\phi) = E_m\psi(\rho,\phi)$$

i.e.

$$E_m = \frac{m^2\hbar^2}{2\mu a^2}$$

Clearly, states of $m = k$ have the same energy as states of $m = -k$. This makes sense, as the sign of m corresponds only to the direction of rotation.

12.3.7. (*The Isotropic Oscillator*). Consider the Hamiltonian

$$H = \frac{P_x^2 + P_y^2}{2\mu} + \frac{1}{2}\mu\omega^2(X^2 + Y^2)$$

(1) Convince yourself $[H, L_z] = 0$ and reduce the eigenvalue problem of H to the radial differential equation for $R_{Em}(\rho)$.

(2) Examine the equation as $\rho \to 0$ and show that

$$R_{Em}(\rho) \xrightarrow[\rho\to 0]{} \rho^{|m|}$$

(3) Show likewise that up to powers of ρ

$$R_{Em}(\rho) \xrightarrow[\rho\to \infty]{} e^{-\mu\omega\rho^2/2\hbar}$$

So assume that $R_{Em}(\rho) = \rho^{|m|}e^{-\mu\omega\rho^2/2\hbar}U_{Em}(\rho)$.

(4) Switch to dimensionless variables $\varepsilon = E/\hbar\omega$, $y = (\mu\omega/\hbar)^{1/2}\rho$.

(5) Convert the equation for R into an equation for U. (I suggest proceeding in two stages: $R = y^{|m|}f$, $f = e^{-y^2/2}U$.) You should end up with

$$U'' + \left[\left(\frac{2|m|+1}{y}\right) - 2y\right]U' + (2\varepsilon - 2|m| - 2)U = 0$$

(6) Argue that a power series for U of the form

$$U(y) = \sum_{r=0}^{\infty} C_r y^r$$

will lead to a *two-term* recursion relation.

(7) Find the relation between C_{r+2} and C_r. Argue that the series must terminate at some finite r if the $y \to \infty$ behavior of the solution is to be acceptable. Show that $\varepsilon = r + |m| + 1$ leads to termination after r terms. Now argue that r is necessarily even—i.e., $r = 2k$. (Show that if r is odd, the behavior of R as $\rho \to 0$ is not $\rho^{|m|}$.) So finally you must end up with

$$E = (2k + |m| + 1)\hbar\omega, \qquad k = 0, 1, 2, \ldots$$

Define $n = 2k + |m|$, so that

$$E_n = (n + 1)\hbar\omega$$

(8) For a given n, what are the allowed values of $|m|$? Given this information show that for a given n, the degeneracy is $n + 1$. Compare this to what you found in Cartesian coordinates (Exercise 10.2.2).

(9) Write down all the normalized eigenfunctions corresponding to $n = 0, 1$.

(10) Argue that the $n = 0$ function *must* equal the corresponding one found in Cartesian coordinates. Show that the two $n = 1$ solutions are linear combinations of their counterparts in Cartesian coordinates. Verify that the parity of the states is $(-1)^n$ as you found in Cartesian coordinates.

First, we compute the commutators of squared state variables with L_z,

$$\begin{aligned}
[X^2, L_z] &= X[X, L_z] + [X, L_z]X \\
&= -i\hbar XY - i\hbar YX \\
&= -2i\hbar XY \\
[Y^2, L_z] &= Y[Y, L_z] + [Y, L_z]Y \\
&= i\hbar YX + i\hbar XY \\
&= 2i\hbar XY \\
[P_x^2, L_z] &= P_x[P_x, L_z] + [P_x, L_z]P_x \\
&= -i\hbar P_x P_y - i\hbar P_y P_x \\
&= -2i\hbar P_x P_y \\
[P_y^2, L_z] &= P_y[P_y, L_z] + [P_y, L_z]P_y \\
&= i\hbar P_y P_x + i\hbar P_x P_y \\
&= 2i\hbar P_x P_y
\end{aligned}$$

From this, we note that $[P_x^2 + P_y^2, L_z] = [X^2 + Y^2, L_z] = 0$, and thus $[H, L_z] = 0$.

The potential in angular coordinates is $V(\rho) = \frac{1}{2}\mu\omega^2\rho^2$, and thus the radial equation is

$$\Big[-\frac{\hbar^2}{2\mu}\Big(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{m^2}{\rho^2}\Big) + \frac{1}{2}\mu\omega^2\rho^2\Big]R_{Em}(\rho) = ER_{Em}(\rho)$$

As ρ goes to 0, the ρ^2 and E terms become negligible, outshined by terms proportional to ρ^{-1} and ρ^{-2}. Since we do not know the second derivative of the radial part of the wave function we keep that term as well, giving a limiting equation of

$$\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho} - \frac{m^2}{\rho^2}R = 0$$

Using the ansatz $R = \rho^k$ gives

$$k(k-1)\rho^{k-2} + k\rho^{k-2} - m^2\rho^{k-2} = 0$$

or $k^2 - m^2 = 0$, i.e. $k = |m|$ (the exponent must be positive to ensure decent behavior at $\rho = 0$). If we now examine the behavior as ρ goes to ∞, the ρ^2 term is much larger than all others (the term proportional only to $R_{Em}(\rho)$ must go to 0 at infinity), with the possible exception of the term containing $\mathrm{d}^2/\mathrm{d}\rho^2$, so we have

$$-\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2} + \frac{1}{2}\mu\omega^2\rho^2R = 0$$

Knowing that $f'(x) \propto xf(x)$ when $f(x) \propto e^{\pm\alpha x^2/2}$, we substitute this into the above to find

$$\begin{aligned}
-\frac{\hbar^2}{2\mu}\frac{\mathrm{d}}{\mathrm{d}\rho}\Big(\pm\alpha\rho e^{\pm\alpha\rho^2/2}\Big) + \frac{1}{2}\mu\omega^2\rho^2e^{\pm\alpha\rho^2/2} &= \mp\frac{\alpha\hbar^2}{2\mu}e^{\pm\alpha\rho^2/2} - \frac{\alpha^2\hbar^2}{2\mu}\rho^2e^{\pm\alpha\rho^2/2} + \frac{1}{2}\mu\omega^2\rho^2e^{\pm\alpha\rho^2/2} \\
&\approx -\frac{\alpha^2\hbar^2}{2\mu}\rho^2e^{\pm\alpha\rho^2/2} + \frac{1}{2}\mu\omega^2\rho^2e^{\pm\alpha\rho^2/2}
\end{aligned}$$

where we have used the approximate equality since $\rho^2e^{\pm\alpha\rho^2/2} \gg e^{\pm\alpha\rho^2/2}$ as ρ goes to infinity. Now, for a well-behaved $R(\rho)$, we must take the negative exponent in our ansatz. Choosing $\alpha = \mu\omega/\hbar$ satisfies the limiting differential equation.

From these limiting answers, we assume

$$R(\rho) = \rho^{|m|} e^{-\mu\omega\rho^2/2\hbar} U(\rho)$$

To switch to the given dimensionless variables, we make the usual replacements, as well as

$$\frac{\mathrm{d}}{\mathrm{d}\rho} \to \Big(\frac{\mu\omega}{\hbar}\Big)^{1/2} \frac{\mathrm{d}}{\mathrm{d}y}$$

Altogether, we find

$$\Big[-\frac{\hbar^2}{2\mu}\Big(\frac{\mu\omega}{\hbar}\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{\mu\omega}{\hbar}\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y} - \frac{\mu\omega}{\hbar}\frac{m^2}{y^2}\Big) + \frac{1}{2}\mu\omega^2\frac{\hbar}{\mu\omega}y^2\Big] R_{Em}(\rho) = \varepsilon\hbar\omega R_{Em}(\rho)$$

or

$$\Big(-\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y} + \frac{m^2}{y^2} + y^2 - 2\varepsilon\Big) R_{Em}(\rho) = 0$$

Taking Shankar's suggestion to first substitute $R = y^{|m|}f$ gives term-wise results of

$$\begin{aligned}
-\frac{\mathrm{d}^2}{\mathrm{d}y^2}\big(y^{|m|}f\big) &= -\frac{\mathrm{d}}{\mathrm{d}y}\big(|m|y^{|m|-1}f + y^{|m|}f'\big) \\
&= -|m|(|m|-1)y^{|m|-2}f - |m|y^{|m|-1}f' - |m|y^{|m|-1}f' - y^{|m|}f'' \\
&= (|m| - m^2)y^{|m|-2}f - 2|m|y^{|m|-1}f' - y^{|m|}f'' \\
-\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}\big(y^{|m|}f\big) &= -|m|y^{|m|-2}f - y^{|m|-1}f' \\
\frac{m^2}{y^2}\big(y^{|m|}f\big) &= m^2y^{|m|-2}f \\
y^2\big(y^{|m|}f\big) &= y^{|m|+2}f
\end{aligned}$$

and a total result of

$$-y^{|m|}f'' - (2|m|+1)y^{|m|-1}f' + (y^2 - 2\varepsilon)y^{|m|}f = 0$$

Given that

$$\begin{aligned}
f &= Ue^{-y^2/2} \\
f' &= (-yU + U')e^{-y^2/2} \\
f'' &= \big((y^2-1)U - 2yU' + U''\big)e^{-y^2/2}
\end{aligned}$$

we have

$$y^{|m|}\Big(-U'' + \big(2y - \tfrac{2|m|-1}{y}\big)U' + (2|m| + 2 - 2\varepsilon)U\Big)e^{-y^2/2} = 0$$

Dividing by $-y^{|m|}e^{-y^2/2}$ gives the form in the problem statement,

$$U'' + \Big[\Big(\frac{2|m|+1}{y}\Big) - 2y\Big]U' + (2\varepsilon - 2|m| - 2)U = 0$$

Assuming that U can be expanded in a power series,

$$U = \sum_{r=0}^{\infty} C_r y^r$$

the above differential equation becomes

$$
\begin{aligned}
&\sum_{r=0}^{\infty}\Big[C_r r(r-1)y^{r-2} + C_r r(2|m|+1)y^{r-2} - 2C_r r y^r + C_r(2\varepsilon - 2|m| - 2)y^r\Big] \\
&= \sum_{k=0}^{\infty} C_{k+2}\big((k+2)(k+1) + (k+2)(2|m|+1)\big)y^k + \sum_{r=0}^{\infty} C_r(2\varepsilon - 2|m| - 2r - 2)y^r \\
&= \sum_{r=0}^{\infty}\Big[C_{r+2}\Big((r+2)(r+1) + (k+2)(2|m|+1)\Big) + C_r\Big(2\varepsilon - 2|m| - 2r - 2\Big)\Big]y^r \\
&= 0
\end{aligned}
$$

This will be true only if

$$
C_{r+2} = C_r \frac{2(|m| + r + 1 - \varepsilon)}{r^2 + 2r|m| + 4r + 4|m| + 4}
$$

which is our two-term recurrence relation. Explicitly, given C_0 and C_1, we have

$$
\begin{aligned}
U(y) = &\, C_0\Big(1 + \frac{|m|+1-\varepsilon}{2|m|+2}y^2 + \Big[\frac{|m|+1-\varepsilon}{2|m|+2}\Big]\Big[\frac{|m|+3-\varepsilon}{4|m|+8}\Big]y^4 + \cdots\Big) \\
&+ C_1\Big(y + \frac{2|m|+4-2\varepsilon}{6|m|+9}y^3 + \Big[\frac{2|m|+4-2\varepsilon}{6|m|+9}\Big]\Big[\frac{2|m|+8-2\varepsilon}{10|m|+25}\Big]y^5 + \cdots\Big)
\end{aligned}
$$

Now, to ensure that $e^{-y^2/2}$ dies faster than $U(y)$ grows, the parenthesized expressions must contain a finite number of terms. This is because the C_r coefficients die more slowly than $1/r!$—which have the approximate relation $C_{r+2} = C_r/r^2$—and so an untruncated $U(y)$ grows at a faster-than-exponential rate. From inspection, we can guarantee that $U(y)$ contains a finite number of terms by choosing $\varepsilon = |m| + r + 1$.

Now, to first order, the odd r terms in $U(y)$ must be made zero to avoid $R(\rho)$ behaving like $\rho^{|m|+1}$ as $\rho \to 0$. Keeping only the terms proportional to C_0, we rewrite $\varepsilon = |m| + 2k + 1$, where $k \in \mathbb{N}$. Defining $n = |m| + 2k$ and reinstating units gives

$$
E_n = (n+1)\hbar\omega.
$$

Let us now consider the allowed values of m for a given n. Since $n - 2k = |m|$, valid m values will each be separated by 2 and centered around the origin. That is,

$$
\begin{aligned}
n = 0 &\implies m = 0 \\
n = 1 &\implies m = -1, 1 \\
n = 2 &\implies m = -2, 0, 2 \\
n = 3 &\implies m = -3, -1, 1, 3 \\
&\vdots
\end{aligned}
$$

Thus, m has $n+1$ degrees of freedom, which is exactly what we found in 10.2.2. The first few eigenfunctions are

$$
\begin{aligned}
\psi_{E0}(\rho,\phi) &= R_{E0}(\rho)\Phi_0(\phi) \\
&= \frac{C_0}{(2\pi)^{1/2}}e^{-\mu\omega\rho^2/2\hbar} \\
\psi_{E(-1)}(\rho,\phi) &= R_{E(-1)}(\rho)\Phi_{-1}(\phi)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{C_0}{(2\pi)^{1/2}} \rho e^{-\mu\omega\rho^2/2\hbar} e^{-i\phi} \\
\psi_{E1}(\rho, \phi) &= R_{E1}(\rho)\Phi_1(\phi) \\
&= \frac{C_0}{(2\pi)^{1/2}} \rho e^{-\mu\omega\rho^2/2\hbar} e^{i\phi}
\end{aligned}
$$

Given that we have used the same Hamiltonian as that of 10.2.2, all solutions we find here should be linear combinations of those we found before. In particular, the solution for $n = 0$ should be exactly the same, being perfectly isotropic. We find that this is indeed the case, with the $n = 1$ eigenfunctions differing only by $e^{\pm i\phi}$ weighted by $\cos\phi$ or $\sin\phi$.

12.3.8. Consider a particle of mass μ and charge q in a vector potential

$$
\mathbf{A} = \frac{B}{2}(-y\mathbf{i} + x\mathbf{j})
$$

(1) Show that the magnetic field is $\mathbf{B} = B\mathbf{k}$.

(2) Show that a classical particle in this potential will move in circles at an angular frequency $\omega_0 = qB/\mu c$.

(3) Consider the Hamiltonian for the corresponding quantum problem:

$$
H = \frac{[P_x + qYB/2c]^2}{2\mu} + \frac{[P_y - qXB/2c]^2}{2\mu}
$$

Show that $Q = (cP_x + qYB/2)/qB$ and $P = (P_y - qXB/2c)$ are canonical. Write H in terms of P and Q and show that allowed levels are $E = (n + 1/2)\hbar\omega_0$.

(4) Expand H out in terms of the original variables and show

$$
H = H\Big(\frac{\omega_0}{2}, \mu\Big) - \frac{\omega_0}{2} L_z
$$

where $H(\omega_0/2, \mu)$ is the Hamiltonian for an isotropic two-dimensional harmonic oscillator of mass μ and frequency $\omega_0/2$. Argue that the same basis that diagonalized $H(\omega_0/2, \mu)$ will diagonalize H. By thinking in terms of this basis, show that the allowed levels for H are $E = (k + \frac{1}{2}|m| - \frac{1}{2}m + \frac{1}{2})\hbar\omega_0$, where k is any integer and m is the angular momentum. Convince yourself that you get the same levels from this formula as from the earlier one $[E = (n + 1/2)\hbar\omega_0]$. We shall return to this problem in Chapter 21.

From basic electromagnetism, the magnetic field is given by

$$
\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A} \\
&= (\partial_y A_z - \partial_z A_y)\mathbf{i} - (\partial_x A_z - \partial_z A_x)\mathbf{j} + (\partial_x A_y - \partial_y A_x)\mathbf{k} \\
&= \Big(\frac{B}{2} + \frac{B}{2}\Big)\mathbf{k} \\
&= B\mathbf{k}
\end{aligned}
$$

The force on a classical particle moving in this field is described, in CGS units, by

$$
\mathbf{F} = \mu\dot{\mathbf{v}} = \frac{q}{c}\mathbf{v} \times \mathbf{B}
$$

which, componentwise, can be written as

$$
\mu \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{bmatrix} = \frac{q}{c} \begin{bmatrix} v_y B \\ -v_x B \\ 0 \end{bmatrix}
$$

Taking the time derivative of both sides of this and using the definitions of $\dot{v}_i$ given above, we find

$$\mu\ddot{v}_x = \frac{q}{c}\dot{v}_y B = -\frac{q}{c}\Big(\frac{q}{\mu c}v_x B\Big)B = -\frac{q^2B^2}{\mu c^2}v_x$$
$$\mu\ddot{v}_y = -\frac{q}{c}\dot{v}_x B = -\frac{q}{c}\Big(\frac{q}{\mu c}v_y B\Big)B = -\frac{q^2B^2}{\mu c^2}v_y$$

where we have left out the z component on account of its clear solution. Recognizing both the x and y components of the velocity to be undergoing simple harmonic motion with $\omega_0 = qB/\mu c$, we may write

$$\mathbf{v} = v_0\cos(\omega_0 t + \phi_0)\mathbf{i} - v_0\sin(\omega_0 t + \phi_0)\mathbf{j} + v_1\mathbf{k}$$

which immediately implies the position of the particle is given by

$$\mathbf{x} = \frac{v_0}{\omega_0}\sin(\omega_0 t + \phi_0)\mathbf{i} + \frac{v_0}{\omega_0}\cos(\omega_0 t + \phi_0) + v_1 t\mathbf{k}$$

i.e. the particle gyrates around the magnetic field lines with frequency ω_0.

To confirm the canonical nature of the given Q and P coordinates, we need to verify that $[Q,P] = i\hbar$,

$$\begin{aligned}
[Q,P] &= \frac{1}{qB}[cP_x + \frac{q}{2}YB, P_y - \frac{q}{2c}XB] \\
&= \frac{1}{qB}\Big(c[P_x,P_y] - \frac{q}{2}[P_x,X]B + \frac{q}{2}[Y,P_y]B - \frac{q^2}{4c}[Y,X]B^2\Big) \\
&= \frac{1}{qB}\Big(\frac{qB}{2}(i\hbar) + \frac{qB}{2}(i\hbar)\Big) \\
&= i\hbar
\end{aligned}$$

We can rewrite H in terms of Q and P as

$$H = \Big(\frac{qB}{c}\Big)^2\frac{Q^2}{2\mu} + \frac{P^2}{2\mu} = \frac{P^2}{2\mu} + \frac{1}{2}\mu\omega_0^2 Q^2$$

which is exactly the Hamiltonian of the harmonic oscillator, implying the allowable energy levels are $E = (n+\frac{1}{2})\hbar\omega_0$.

Returning to the original Hamiltonian, we may expand it to find

$$\begin{aligned}
H &= \frac{[P_x + qYB/2c]^2}{2\mu} + \frac{[P_y - qXB/2c]^2}{2\mu} \\
&= \frac{1}{2\mu}\Big[P_x^2 + \frac{qB}{c}P_xY + \Big(\frac{qB}{2c}\Big)^2Y^2\Big] + \frac{1}{2\mu}\Big[P_y^2 - \frac{qB}{c}P_yX + \Big(\frac{qB}{2c}\Big)^2X^2\Big] \\
&= \frac{P_x^2}{2\mu} + \frac{P_y^2}{2\mu} + \frac{1}{2}\mu\Big(\frac{qB}{2\mu c}\Big)^2X^2 + \frac{1}{2}\mu\Big(\frac{qB}{2\mu c}\Big)^2Y^2 - \frac{qB}{2\mu c}\Big(XP_y - YP_x\Big) \\
&= \frac{P_x^2}{2\mu} + \frac{P_y^2}{2\mu} + \frac{1}{2}\mu\Big(\frac{\omega_0}{2}\Big)^2X^2 + \frac{1}{2}\mu\Big(\frac{\omega_0}{2}\Big)^2Y^2 - \frac{\omega_0}{2}L_z \\
&= H\Big(\frac{\omega_0}{2},\mu\Big) - \frac{\omega_0}{2}L_z
\end{aligned}$$

where $H\big(\frac{\omega_0}{2},\mu\big)$ is the isotropic two-dimensional oscillator from with mass μ and frequency $\omega_0/2$. These components of the overall Hamiltonian share a common, diagonal basis if they commute. Since

$$[X^2+Y^2, L_z] = [X^2+Y^2, XP_y - YP_x]$$

$$
\begin{aligned}
&= [X^2, XP_y] - [X^2, YP_x] + [Y^2, XP_y] - [Y^2, YP_x] \\
&= -\Big(X[X,X]P_y + [X,Y]XP_x + YX[X,P_x] + Y[X,P_x]X\Big) \\
&\quad + \Big(Y[Y,X]P_y + [Y,X]YP_y + XY[Y,P_y] + X[Y,P_y]Y\Big) \\
&= -i\hbar YX - i\hbar YX + i\hbar XY + i\hbar XY \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
[P_x^2 + P_y^2, L_z] &= [P_x^2 + P_y^2, XP_y - YP_x] \\
&= [P_x^2, XP_y] - [P_x^2, YP_x] + [P_y^2, XP_y] - [P_y^2, YP_x] \\
&= \Big(P_x[P_x,X]P_y + [P_x,X]P_xP_y + [X,P_x]P_xP_y + X[P_x,P_y]P_x\Big) \\
&\quad - \Big(P_y[P_y,Y]P_x + [P_y,Y]P_yP_x + [Y,P_y]P_yP_x + Y[P_y,P_x]P_y\Big) \\
&= -i\hbar P_xP_y - i\hbar P_xP_y + i\hbar P_xP_y + i\hbar P_yP_x + i\hbar P_yP_x - i\hbar P_yP_x \\
&= 0
\end{aligned}
$$

the two parts of the Hamiltonian *do* commute, and hence can be diagonalize by a common basis. Given that the eigenfunctions of $H\big(\frac{\omega_0}{2}, \mu\big)$ can be written as $R(\rho)\Phi(\phi)$ (where $\Phi(\phi)$ are the eigenfunctions of L_z) and L_z commutes with any function of ρ, it is clear that the two parts of the Hamiltonian can, more specifically, be diagonalized by the same basis as that found in the previous problems on the isotropic two-dimensional oscillator. If we apply this Hamiltonian to a state built from that basis, we find

$$
\begin{aligned}
H|\psi_{Em}\rangle &= H\Big(\frac{\omega_0}{2}, \mu\Big)|\psi_{Em}\rangle - \frac{\omega_0}{2} L_z |\psi_{Em}\rangle \\
&= (2k + |m| + 1)\frac{\hbar\omega_0}{2}|\psi_{Em}\rangle - \frac{\omega_0}{2} m\hbar |\psi_{Em}\rangle \\
&= \Big(k + \frac{1}{2}|m| - \frac{1}{2}m + \frac{1}{2}\Big)\hbar\omega_0 |\psi_{Em}\rangle
\end{aligned}
$$

which implies the energy is quantized as

$$
E = \Big(k + \frac{1}{2}|m| - \frac{1}{2}m + \frac{1}{2}\Big)\hbar\omega_0
$$

Since $k \in \mathbb{N}$ and $|m| - m \geq 0$, this expression gives the exact same energy levels as $E = (n + 1/2)\hbar\omega_0$.