1 The standard model of particle physics

1.1. By referring to Weyl's neutrino equation, given i §25.3, explain why it is reasonable to take the view that α_A and β'_A each describe massless particles, coupled by an interaction converting each into the other.

The Weyl equation describing a massless neutrino is given by

$$\nabla^A_{B'}\alpha_A = 0.$$

From our studies of differential equations, we know that adding a term to the righthand side will act as a 'source' or 'driver' for α_A. In this case, the only free index is B', so such a source term would take on the form $M\beta_{B'}$.

We can do the same for the other Weyl equation,

$$\nabla^{B'}_A\beta_{B'} = 0,$$

this time adding a source term of the form $M\alpha_A$. Together, we have the following equations

$$\nabla^A_{B'}\alpha_A = 2^{-1/2}M\beta_{B'}, \qquad \nabla^{B'}_A\beta_{B'} = 2^{-1/2}M\alpha_A$$

which represent two massless particles, each being the source of the other.

1.2. Show both of these things.

We know that the gamma matrices satisfy

$$\gamma_0^2 = 1, \quad \gamma_1^2 = -1, \quad \gamma_2^2 = -1, \quad \gamma_3^2 = -1$$

and

$$\gamma_i\gamma_j = -\gamma_j\gamma_i \quad (i \neq j).$$

From this, let us compute the necessary anticommutators,

$$\begin{aligned}
\{\gamma_5, \gamma_0\} &= -i\gamma_0\gamma_1\gamma_2\gamma_3\gamma_0 - i\gamma_0^2\gamma_1\gamma_2\gamma_3 \\
&= i\gamma_0^2\gamma_1\gamma_2\gamma_3 - i\gamma_1\gamma_2\gamma_3 \\
&= i(\gamma_1\gamma_2\gamma_3 - \gamma_1\gamma_2\gamma_3) \\
&= 0 \\
\{\gamma_5, \gamma_1\} &= -i\gamma_0\gamma_1\gamma_2\gamma_3\gamma_1 - i\gamma_1\gamma_0\gamma_1\gamma_2\gamma_3 \\
&= -i\gamma_0\gamma_2\gamma_3\gamma_1^2 + i\gamma_0\gamma_1^2\gamma_2\gamma_3 \\
&= i(\gamma_0\gamma_2\gamma_3 - \gamma_0\gamma_2\gamma_3) \\
&= 0 \\
\{\gamma_5, \gamma_2\} &= -i\gamma_0\gamma_1\gamma_2\gamma_3\gamma_2 - i\gamma_2\gamma_0\gamma_1\gamma_2\gamma_3 \\
&= i\gamma_0\gamma_1\gamma_2^2\gamma_3 - i\gamma_0\gamma_1\gamma_2^2\gamma_3 \\
&= -i(\gamma_0\gamma_1\gamma_3 - \gamma_0\gamma_1\gamma_3) \\
&= 0 \\
\{\gamma_5, \gamma_3\} &= -i\gamma_0\gamma_1\gamma_2\gamma_3^3 - i\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3 \\
&= i\gamma_0\gamma_1\gamma_2 + i\gamma_0\gamma_1\gamma_2\gamma_3^2 \\
&= i(\gamma_0\gamma_1\gamma_2 - \gamma_0\gamma_1\gamma_2) \\
&= 0
\end{aligned}$$

Finally, we check the square of γ_5,

$$\begin{aligned}
\gamma_5^2 &= (-i\gamma_0\gamma_1\gamma_2\gamma_3)^2 \\
&= -\gamma_0\gamma_1\gamma_2\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3 \\
&= \gamma_0^2\gamma_1\gamma_2\gamma_3\gamma_1\gamma_2\gamma_3 \\
&= \gamma_1^2\gamma_2\gamma_3\gamma_2\gamma_3 \\
&= \gamma_2^2\gamma_3^2 \\
&= 1
\end{aligned}$$

1.3. Find this normal subgroup. *Hint*: Think of the determinant of a 3×3 matrix.

$\mathbb{Z}_3$ is often represented by the roots of unity

$$\{1, e^{i\pi/3}, e^{i2\pi/3}\}.$$

This group can be embedded in SU(3) by multiplying the identity by each of the above elements, i.e.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} e^{i\pi/3} & 0 & 0 \\ 0 & e^{i\pi/3} & 0 \\ 0 & 0 & e^{i\pi/3} \end{pmatrix}, \qquad \begin{pmatrix} e^{i2\pi/3} & 0 & 0 \\ 0 & e^{i2\pi/3} & 0 \\ 0 & 0 & e^{i2\pi/3} \end{pmatrix}$$

or I, $e^{i\pi/3}I$, and $e^{i2\pi/3}I$. It is important to note that this works because the matrices in this representation of SU(3) are of dimension 3×3. If we tried to embed $\mathbb{Z}_4$ in the same way the resulting matrices would not have unit determinant.

A normal subgroup $N \subset G$ is defined by

$$gng^{-1} \in N, \quad \forall g \in G \,\text{and}\, \forall n \in N.$$

By the rules of matrix multiplication, we have

$$gIg^{-1} = gg^{-1} = I \in N$$
$$g(e^{i\pi/3}I)g^{-1} = e^{i\pi/3}gg^{-1} = e^{i\pi/3} \in N$$
$$g(e^{i2\pi/3}I)g^{-1} = e^{i2\pi/3}gg^{-1} = e^{i2\pi/3} \in N$$

Not only is $\mathbb{Z}_3$ a normal subgroup—each element of $\mathbb{Z}_3$ gets mapped to itself when a similarity transformation is applied to it.

1.4. Check that the charge values, indicated by the superfixes in the first table, come out right.

In terms of the unit charge e we have $q_u = \frac{2}{3}e$, $q_d = -\frac{1}{3}e$, and $q_s = -\frac{1}{3}e$.

1.5. Explain this more cmopletely, using the 2-spinor index description for the quark spins,a s described in §22.8, and using a new 3-dimensional ‘SU(3) index’ which takes 3 values u, d, s.

1.6. See if you can explain all this in some appropriate detail. Care is needed for the treatment of the 2-spinor spin indices, if you wish to use them. An antisymmetry in a pair of them allows that pair to be removed (as when representing a spin 0 state in terms of a pair of spin $\frac{1}{2}$ particles, as in §23.4). Yet there is a (hidden) symmetry also, because there are only two independent spin states for each quark.