1 Atoms in Motion

1.1. If heat is merely molecular motion, what is the difference between a hot, stationary baseball and a cool, rapidly moving one?

> The hot, stationary baseball's atoms are vibrating rapidly; the ball is stationary because the motion of its atoms has no preferred direction. By contrast, the cool baseball's atoms *do* have a preferred direction. It is cool because its atoms are jiggling less vigorously than those of the hot baseball.

1.2. If the atoms of all objects are perpetually in motion, how can there be any permanent objects, such as fossil imprints?

> In the end, there *are* no permanent objects, only very long-lasting ones. The ones that have lasted longer have done so by virtue of the forces between the atoms being strong enough to overcome their vibrations.

1.3. Explain qualitatively why and how friction in a moving machine produces heat. Explain also, if you can, why heat cannot produce useful motion by the reverse process.

> When a machine moves, the parts in contact with one another rub against each other by exchanging atoms and knocking others loose, releasing energy that is imparted to nearby atoms in the form of vibrations.
>
> Heat cannot produce useful motion because such atomic motion is random: it has no preferred direction, and so cannot produce useful work.

1.4. Chemists have found that the molecules of rubber consist of long criss-crossed chains of atoms. Explain why a rubber band becomes warm when it is stretched.

> As a rubber band is stretched, these chains pass by each other. The forces present in each push against the others, causing atomic motion, i.e. heat.

1.5. What should happen to a rubber band which is supporting a given weight, if it is heated? (To find out, try it.)

> When a rubber band is heated, the additional motion within each chain will create more knots and twists, shrinking the band.

1.6. Can you explain why there are no crystals that have the shape of a regular pentagon?

> A regular pentagon cannot be used to tile the plane, and therefore a layer of crystal cannot be made from atoms arranged in such a way.

1.7. You are given a large number of steel balls of equal diameter d and a container of known volume V. Every dimension of the container is much greater than the diameter of a ball. What is the greatest number of balls, N, that can be placed in the container?

This is a sphere packing problem, which can become quite complicated the deeper you want to go. As a first approximation, we may treat each ball as a box, giving

$$N = \frac{V}{d^3}.$$

We can do better than this. Consider packing the balls into a lattice and zooming in on a small question of the structure, centered at one ball. There are eight other balls touching this one: we'll take the studied volume to be the cube containing the corners of these balls. That is, our volume contains eight eighth-spheres (one at each corner) touching the central ball. The diagonal of this cube is clearly $2d$, giving a side length of $l = \frac{2}{\sqrt{3}}d$ and a volume of $V_B = \frac{8}{3\sqrt{3}}d^3$. The volume occupied by our steel balls is equivalent to two of them, or $V_S = 2 \times \frac{\pi}{6}d^3$. Taking the ratio of these gives us the percentage of volume available to be occupied by our steel balls,

$$\frac{V_S}{V_B} = \frac{\pi}{3}d^3 \cdot \frac{3\sqrt{3}}{8d^3} = \frac{\pi\sqrt{3}}{8}.$$

The number of balls we can fit in our space with this packing method is given by

$$N = V\frac{\pi\sqrt{3}}{8} \cdot \frac{6}{\pi d^3} = \frac{3\sqrt{3}}{4}\frac{V}{d^3}.$$

But we can do even better! Instead of the previous imagined volume, consider a cube with, once again, its corners filled with eighth-spheres. But now, instead of a ball at the center, there are half-spheres emanating from the middle of each face. The diagonal of each face of this cube is clearly $2d$, giving a side length of $l = \sqrt{2}d$ and a volume of $V_B = 2\sqrt{2}d^3$. The volume occupied by our steel balls is equivalent to four of them (one from the eight eighths, and three from the six halves), or $V_S = \frac{2}{3}\pi d^3$. The ratio of these is

$$\frac{V_S}{V_B} = \frac{2}{3}\pi d^3 \cdot \frac{1}{2\sqrt{2}d^3} = \frac{\pi}{3\sqrt{2}}.$$

This, the optimal result (as proven by Gauss), gives us

$$N = V\frac{\pi}{3\sqrt{2}} \cdot \frac{6}{\pi d^3} = \sqrt{2}\frac{V}{d^3}.$$

1.8. How should the pressure P of a gas vary with n, the number of atoms per unit volume, and $\langle v \rangle$, the average speed of an atom? (Should P be proportional to n and/or $\langle v \rangle$, or should it vary more, or less, rapidly than linearly?

If we double the number of atoms per unit volume, twice as many atoms will be colliding with each side of our container, suggesting $P \propto n$. If we increase the average speed of our atoms, not only will they be colliding against each wall with greater force (their change in momentum will be larger, and $F = \frac{dp}{dt}$), but collisions will also happen more frequently (as each atom takes less time to traverse its bounding volume). Therefore, we expect $P \propto \langle v \rangle^2$. This makes sense from an energy standpoint, too: we expect P to be proportional to the energy of the gas, which is proportional to $\langle v \rangle^2$.

1.9. Ordinary air has a density of about $\rho_G = 0.001\,\mathrm{g\,cm^{-3}}$, while liquid air has a density of about $\rho_L = 1.0\,\mathrm{g\,cm^{-3}}$.

(a) Estimate the number of air molecules per cm^3 in ordinary air, n_G, and in liquid air, n_L.

(b) Estimate the mass m of an air molecule.

(c) Estimate the average distance l an air molecule should travel between collisions at normal temperature and pressure (NTP, 20° C at 1 atm). This distance is called the *mean free path*.

(d) Estimate at what pressure P, in normal atmospheres, a vacuum system should be operated in order that the mean free path be about one meter.

(a) The atmosphere is approximately 80% N_2 and 20% O_2, with the former having an atomic mass of $\approx 2 \times 14$ u and the latter having an atomic mass of $\approx 2 \times 16$ u. Noting that Avogadro's number is $N_A = 6 \times 10^{23}\,\mathrm{mol^{-1}}$, and defining $\alpha_N = 1/28\,\mathrm{mol\,g^{-1}}$ and $\alpha_O = 1/32\,\mathrm{mol\,g^{-1}}$, we can perform dimensional analysis to find

$$n_G = N_A(0.8\alpha_N + 0.2\alpha_O)\rho_G = 2 \times 10^{19}\,\mathrm{cm^{-3}}$$
$$n_L = N_A(0.8\alpha_N + 0.2\alpha_O)\rho_L = 2 \times 10^{22}\,\mathrm{cm^{-3}}$$

(b) We can find m by dividing the density by the number of air molecules in a given volume

$$m = \frac{\rho_G}{n_G} = 5 \times 10^{-23}\,\mathrm{g}.$$

(c) To estimate the mean free path, imagine the trajectory of a single molecule. If we idealize this molecule as a sphere, its motion sweeps out a cylinder of volume $\pi r^2 l$, where r is the molecule radius and l is the length of the cylinder. A 'free' path corresponds to a volume containing exactly one such molecule, or

$$n_G \pi r^2 l = 1.$$

(d) Since each air molecule consists of two atoms, we'll guess a radius of 2 Å, or 2×10^{-8} cm. This gives us an estimated mean free path of

$$l = \frac{1}{n_G \pi r^2} = 4 \times 10^{-5}\,\mathrm{cm}.$$

Now, we estimated earlier that $P \propto n_G T$ (as $T \propto \langle v \rangle^2$). We can explicitly include the constant of proportionality B and solve for it using our given values of 1 atm and 20 °C,

$$B = \frac{P}{n_G T} = 2.5 \times 10^{-21}\,\mathrm{atm/cm^3/^\circ C}.$$

With this, we can then express n_G in terms of P and solve for it in the mean free path equation. At $l = 100$ cm, we find

$$P = \frac{BT}{l \pi r^2} = 4 \times 10^{-7}\,\mathrm{atm}.$$

1.10. The intensity of a collimated, parallel beam of potassium atoms is reduced 3.0% by a layer of argon gas 1.0 mm thick at a pressure of 6.0×10^{-4} mmHg. Calculate the effective target area A per argon atom.

> If the argon reduces the potassium beam's intensity by 3%, we may guess that, if the thickness of the gas layer were larger, more reduction would occur. Total attenuation would correspond to the thickness of the gas layer equaling the mean free path of the argon, as this is when, on average, atoms passing through the gas would collide with something every time. (Here, we are relying on the fact that both types of atoms have a similar size and that the potassium beam is coherent enough to avoid self-collisions.) Put another way, if l is the mean free path, we have $0.03l = 0.1\,\text{cm}$, or $l = 3\,\text{cm}$.
>
> With this information, we can rearrange (1.12) by identifying $A = \pi r^2$ and swapping it with P, resulting in
>
> $$A = \frac{BT}{lP} = 2 \times 10^{-14}\,\text{cm}^2.$$
>
> where we have made the conversion from mmHg to atm implicit.

1.11. X-ray diffraction studies show that NaCl crystals have a cubic lattice, with a spacing of 2.820 Å between nearest neighbors. Look up the density and molecular weight of NaCl and calculate Avogadro's number N_A. (This is one of the most precise experimental methods for determining N_A.)

> The density of NaCl is $\rho = 2.16\,\text{g}\,\text{cm}^{-3}$ and its molecular weight is $m = 58.44\,\text{g}\,\text{mol}^{-1}$. If we were to cube the length given to us, we would have a volume that, when combined with these two quantities, would allow us to compute Avogadro's number.
>
> But that isn't quite right. The spacing given is between nearest *atoms*, not molecules, and so our cube only contains half as many molecules as we initially expected it to have. That is
>
> $$\frac{1}{2}\frac{m}{\rho}\frac{1}{l^3} = 6.03 \times 10^{23}\,\text{mol}^{-1}.$$

1.12. Boltwood and Rutherford found that radium in equilibrium with its disintegration products produced 13.6×10^{10} helium atoms per second per gram of radium. They also measured that the disintegration of 192 mg of radium produced $0.0824\,\text{mm}^3$ of helium per day at standard temperature and pressure (STP, 0 °C at 1 atm). Use these data to calculate:

(a) The number of helium atoms N_H per cm^3 of gas at STP.

(b) Avogadro's number N_A.

> (a) Jumping right to it, we estimate
>
> $$N_H = 13.6 \times 10^{10}\,\text{g}^{-1}\,\text{s}^{-1} \times \frac{192\,\text{mg}}{0.0824\,\text{mm}^3/\text{day}} = 2.7 \times 10^{19}\,\text{cm}^{-3}.$$
>
> (b) Looking up the atomic weight of helium, as well as its density at STP, gives $m = 4\,\text{g}\,\text{mol}^{-1}$ and $\rho = 1.786 \times 10^{-4}\,\text{g}\,\text{cm}^{-3}$. Combining these with N_H gives
>
> $$N_A = N_H\frac{m}{\rho} = 6.04 \times 10^{23}\,\text{mol}^{-1}.$$

1.13. Rayleigh found that 0.81 mg of olive oil on a water surface produced a mono-molecular layer 84 cm in diameter. What value of Avogadro's number N_A results, assuming the approximate composition $H(CH_2)_{18}COOH$ in a linear chain, with density $0.8\,\mathrm{g\,cm^{-3}}$?

1.14. About 1860, Maxwell showed that the viscosity of a gas is given by

$$\eta = \frac{1}{3}\rho v l.$$

where ρ is the density, v is the mean molecular speed, and l, the mean free path. The latter quantity he had earlier shown to be $l = 1/(\sqrt{2}\pi N_g \sigma^2)$, where σ is the diameter of the molecule. Loschmidt (1865) used the measured value of η, ρ (gas), and ρ (solid) together with Joule's calculated v to determine N_g, the number of molecules per $\mathrm{cm^3}$ in a gas at STP. He assumed the molecules to be hard spheres, tightly packed in a solid. Given $\eta = 2.0 \times 10^{-4}\,\mathrm{g\,cm^{-1}\,s^{-1}}$ for air at STP, ρ (liquid) $\approx 1.0\,\mathrm{g\,cm^{-3}}$, ρ (gas) $\approx 1.0 \times 10^{-3}\,\mathrm{g\,cm^{-3}}$ and $v \approx 500\,\mathrm{m\,s^{-1}}$, calculate N_g.

Making sure to use ρ (gas), we can substitute in the expression for the mean free path into η and solve for N_g. Estimating the diameter of a molecule to be $2 \times 2\,\text{Å}$, we find

$$N_g = \frac{1}{3}\frac{\rho v}{\sqrt{2}\pi\eta\sigma^2} = 1.2 \times 10^{-19}.$$

1.15. A glass full of water is left standing on an average outdoor windowsill in California.

(a) How much time T do you think it would take to evaporate completely?

(b) How many molecules J per $\mathrm{cm^2}$ and per s would be leaving the water glass at this rate?

(c) Briefly discuss the connection, if any, between your answer to part (a) above and the average rainfall over the earth.

1.16. A raindrop of an afternoon thundershower fell upon a Paleozoic mud flat and left an imprint which is later dug up as a fossil by a hot, thirsty geology student. As he drains his canteen, the student idly wonders how many molecules of water, N, of that ancient raindrop he has just drunk. Estimate N using only data which you already know. (Make reasonable assumptions regarding necessary information which you do not know.)

Let's estimate a radius for the raindrop of 0.5 cm, giving it a volume of $V_R = 0.5\,\mathrm{cm^3}$. The density of water is $\rho = 1.0\,\mathrm{g\,cm^{-3}}$, its molar mass is $m = 18\,\mathrm{g\,mol^{-1}}$, and Avogadro's number is $N_A = 6 \times 10^{23}\,\mathrm{mol^{-1}}$, so there are

$$N_M = V_R N_A \frac{\rho}{m} = 2 \times 10^{22}$$

molecules in our prehistoric raindrop. How many raindrops are there on earth? With a radius of $r = 6 \times 10^8$ cm, the earth has a surface area of $A = 4.5 \times 10^{18}\,\mathrm{cm^2}$. The ocean occupies approximately 30% of this area and has a mean depth of $D = 4 \times 10^5$ cm, giving us

$$N_R = \frac{AD}{V_R} = 4 \times 10^{24}$$

raindrops on the earth. Since our original raindrop first hit the ground a long time ago, it's reasonable to assume its molecules have spread out evenly across the globe, giving us approximately

$$\frac{N_M}{N_R} = 5 \times 10^{-3}$$

ancient molecules per raindrop's worth of water. If the canteen the student drank from holds a liter of water, then it can be filled with 2000 raindrops. From this, we estimate the student drank approximately 10 molecules of water that were once in the raindrop he's observing.

2 Conservation of Energy, Statics

2.1. Use the principle of virtual work to establish the formula for an unequal-arm balance, as shown in Fig. 2-1, $W_1 l_1 = W_2 l_2$. (Neglect the weight of the cross-beam.)

By tilting the balance by an angle $\Delta\theta$, we lift W_2 up through the amount $l_2 \sin\Delta\theta$, while lowering W_1 by $l_1 \sin\Delta\theta$. Since the system is in static equilibrium, this should result in not net change in energy. That is,

$$W_2 l_2 \sin\Delta\theta - W_1 l_1 \sin\Delta\theta = 0.$$

Or, dividing through by $\sin\Delta\theta$,

$$W_1 l_1 = W_2 l_2.$$

2.2. Extend the formula obtained in Ex. 2.1 to include a number of weights hung at various distances from the pivot point,

$$\sum_i W_i l_i = 0.$$

(Distances on one side of the fulcrum are considered positive and on the other side, negative.)

If we count lengths to the left side of the fulcrum as negative, then, when moving the balance by an amount $\Delta\theta$, we find

$$\sum_i W_i l_i \sin\Delta\theta = 0.$$

Once again, dividing through by $\sin\Delta\theta$, we see

$$\sum_i W_i l_i = 0.$$

2.3. A body is acted upon by n forces and is in static equilibrium. Use the principle of virtual work to prove that:

(a) If $n = 1$, the magnitude of the force must be zero. (A trivial case.)

(b) If $n = 2$, the forces must be equal in magnitude, opposite in direction, and collinear.

(c) If $n = 3$, the forces must be coplanar and their lines of action must pass through a single point.

(d) For any n, the sum of the products of the magnitude of a force F_i times the cosine of the angle Δ_i between the force and any fixed line, is zero:

$$\sum_{i=1}^{n} F_i \cos\Delta_i = 0..$$

(a) We can employ the principle of virtual work by imagining moving the body by an amount $\mathrm{d}\mathbf{s}$. In the first case, we must have
$$\mathbf{F} \cdot \mathrm{d}\mathbf{s} = 0.$$
which is only possible if $\mathbf{F} = \mathbf{0}$.

(b) In the second, we have
$$\mathbf{F}_1 \cdot \mathrm{d}\mathbf{s} + \mathbf{F}_2 \cdot \mathrm{d}\mathbf{s} = (\mathbf{F}_1 + \mathbf{F}_2) \cdot \mathrm{d}\mathbf{s} = 0.$$
which is only possible if $\mathbf{F}_1 = -\mathbf{F}_2$.

(c) Continuing the pattern, we see
$$(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3) \cdot \mathrm{d}\mathbf{s} = 0.$$
implies the forces are coplanar (any one can be written in terms of the other two) and pass through a single point (as otherwise there would be no cancellation).

(d) In the final case, we find
$$\left(\sum_i \mathbf{F}_i\right) \cdot \mathrm{d}\mathbf{s} = s \sum_i F_i \cos \Delta_i = 0.$$
which, as s is arbitrary, shows that the sum of the products of the magnitude of each force, times the cosine of the angle that force makes with a fixed line, is 0.

2.4. Problems involving static equilibrium in the absence of friction may be reduced, using the *Principle of Virtual Work*, to problems of mere geometry: Where does one point move when another moves a given small distance? In many cases this question is easily answered if the following properties of a triangle are used (referring to Fig. 2-2):

(a) If the sides d_1 and d_2 remain fixed in length, but the angle α changes by a small amount $\Delta\alpha$, the opposite side L changes by an amount
$$\Delta L = \frac{d_1 d_2}{L} \sin \alpha \Delta\alpha.$$

(b) If the three sides a, b, c of a right triangle change in length by small amounts Δa, Δb, and Δc, then
$$a\Delta a + b\Delta b = c\Delta c \qquad \text{(where } c \text{ is the hypotenuse)}.$$

(a) In the first case, we can express the length L in terms of d_1, d_2, and α by
$$L^2 = (d_1 \sin \alpha)^2 + (d_2 - d_1 \cos \alpha)^2.$$
Taking the differential of each side (considering only L and α as mutable), yields
$$\begin{aligned} 2L\Delta L &= 2(d_1 \sin \alpha)(d_1 \cos \alpha)\Delta\alpha + 2(d_2 - d_1 \cos \alpha)(d_1 \sin \alpha)\Delta\alpha && (1)\\ &= 2d_1^2 \sin \alpha \cos \alpha \Delta\alpha + 2d_2 d_1 \sin \alpha \Delta\alpha - 2d_1^2 \sin \alpha \cos \alpha \Delta\alpha && (2)\\ &= 2d_2 d_1 \sin \alpha \Delta\alpha && (3) \end{aligned}$$
Dividing by $2L$ gives the final result,
$$\Delta L = \frac{d_1 d_2}{L} \sin \alpha \Delta\alpha.$$

(b) The second relation can be found by taking the differential of the Pythagorean theorem, $a^2 + b^2 = c^2$,

$$2a\Delta a + 2b\Delta b = 2c\Delta c$$

and dividing by 2,

$$a\Delta a + b\Delta b = c\Delta c.$$

2.5. A uniform plank 1.5 m long and weighing 3.00 kg is pivoted at one end. The plank is held in equilibrium in a horizontal position by a weight and pulley arrangement, as shown in Fig. 2-3. Find the weight W needed to balance the plank. Neglect friction.

We can find the weight by using torque. Setting our axis of rotation at the plank's pivot point (and considering a counterclockwise torque as positive), we find

$$(0.75\,\mathrm{m})(3.00\,\mathrm{kg}) - (1.50\,\mathrm{m})W \sin 45^\circ = 0.$$

Replacing $\sin 45^\circ$ with $1/\sqrt{2}$, we see

$$W = \frac{(0.75\,\mathrm{m})(3.00\,\mathrm{kg})\sqrt{2}}{(1.50\,\mathrm{m})} \tag{4}$$

$$= \frac{3\sqrt{2}}{2}\,\mathrm{kg} \tag{5}$$

2.6. A ball of radius 3.0 cm and weight 1.00 kg rests on a plane tilted at an angle α with the horizontal and also touches a vertical wall, as shown in Fig. 2-4. Both surfaces have negligible friction. Find the force with which the ball presses on the wall F_W and on the plane F_P.

The force that the wall and inclined plane must counteract is gravity, which always points downward. The force exerted by the wall acts parallel to the ground, while the force exerted by the plane is perpendicular to it. So we have

$$\mathbf{F}_W + \mathbf{F}_P + \mathbf{F}_g = 0.$$

Or, split into the x and y directions,

$$F_W - \sin\alpha F_P = 0 \tag{6}$$

$$\cos\alpha F_P + F_g = 0. \tag{7}$$

Solving this yields

$$F_W = F_g \tan\alpha \tag{8}$$

$$F_P = \frac{F_g}{\cos\alpha}. \tag{9}$$

2.7. The jointed parallelogram frame $AA'BB'$ is pivoted (in a vertical plane) on the pivots P and P', as shown in Fig. 2-5. There is negligible friction in the pins at A, A', B, B', P, and P'. The members $AA'CD$ and $B'BGH$ are rigid and identical in size. $AP = A'P' = \frac{1}{2}PB = \frac{1}{2}P'B'$. Because of the counterweight w_c, the frame is in balance without the loads W_1 and W_2. If a 0.50 kg weight W_1 is hung from D, what weight W_2, hung from H, is needed to produce equilibrium?

Since the frame is in static equilibrium before we hang W_1 and W_2, we may neglect the extra information and consider only their effects on each other: if they balance, then the system will balance. If we move W_1 down by an amount Δy, we move weight W_2 up by an amount $2\Delta y$ on account the PB being twice the length of AP. So

$$2W_2\Delta y - W_1\Delta y = 0.$$

Substituting in 0.50 kg for W_1 and solving for W_2 yields

$$W_2 = \frac{W_1}{2} = 0.25\,\text{kg}.$$

2.8. The system shown is in static equilibrium. Use the principle of virtual work to find the weights A and B. Neglect the weight of the strings and the friction in the pulleys.

We must make two independent displacements to generate the two equations necessary to solve for A and B. If we move A to the left by an amount Δx, then the 1 kg weight will move down and B will rise. By how much?

We can use the second relationship we derived in 2.4, $x\Delta x + y\Delta y = l\Delta l$, or, since y is held fixed, $\Delta l = x\Delta x/l$. For the leftmost weight, $l_1 = x/\cos 30°$, while for B we have $l_B = x/\cos 45°$. Using the principle of virtual work, we see

$$\begin{aligned} 0 &= B\Delta l_B - (1\,\text{kg})\Delta l_1 && (10)\\ &= B\frac{x\Delta x}{l_B} - (1\,\text{kg})\frac{x\Delta x}{l_1} && (11)\\ &= B\Delta x\cos 45° - (1\,\text{kg})\Delta x\cos 30° && (12) \end{aligned}$$

Solving for B yields

$$B = (1\,\text{kg})\frac{\cos 30°}{\cos 45°} = \sqrt{\frac{3}{2}}\,\text{kg}.$$

If we now displace A downward by Δy, we move both the 1 kg weight and B upward. Now, our relationships are $\Delta l = y\Delta y/l$, $l_1 = y/\sin 30°$, and $l_B = y/\sin 45°$. Writing out our new equation gives

$$\begin{aligned} 0 &= (1\,\text{kg})\Delta l_1 - A\Delta y + B\Delta l_B && (13)\\ &= (1\,\text{kg})\frac{y\Delta y}{l_1} - A\Delta y + B\frac{y\Delta y}{l_B} && (14)\\ &= (1\,\text{kg})\Delta y\sin 30° - A\Delta y + B\Delta y\sin 45° && (15) \end{aligned}$$

Solving for A and substituting in our found value for B gives

$$A = \frac{1}{2}(1+\sqrt{3})\,\text{kg}.$$

2.9. A weight $W = 50\,\text{lb}$ is suspended from the midpoint of a wire ACB as shown in Fig. 2-7. $AC = CB = 5\,\text{ft}$. $AB = 5\sqrt{2}\,\text{ft}$. Find the tension T_1 and T_2 in the wire.

By symmetry, $T_1 = T_2$. To counteract the force of the weight, we must have

$$T1 \sin\alpha + T_2 \sin\alpha = 2T_1 \sin\alpha = 2T_2 \sin\alpha = W.$$

By simple trigonometry, $\alpha = \arccos \frac{5\sqrt{2}/2}{5} = 45°$, and so

$$T_1 = T_2 = 25\sqrt{2}\,\text{lb}.$$

2.10. The truss shown in Fig. 2-8 is made of light aluminum struts pivoted at each end. At C is a roller which rolls on a smooth plate. When a workman heats up member AB with a welding torch, it is observed to increase in length by an amount x, and the load W is thereby moved vertically an amount y.

(a) Is the motion of W upward or downward?

(b) What is the force F in the member AB (including the sense, i.e., tension or compression)?

(a) By conservation of energy, $Fx + Wy = 0$, since the force exerted by the bar is internal to the system. Thus, W moves downward.

(b) Solving the previous equation for F gives

$$F = -W\frac{y}{x}$$

which is a positive quantity when y is negative.

2.11. What horizontal force F (applied at the axle) is required to push a wheel of weight W and radius R over a block of height h, as shown in Fig. 2-9?

By the principle of virtual work, the distance the wheel moves horizontally times our applied force must equal its change in potential energy, or $F\Delta d = W\Delta h$.

Drawing a right triangle from the axle to the corner of the block, we can identify the sought after displacement as shortening the horizontal side while lengthening the vertical one, or $\Delta d = -\Delta x$ and $\Delta h = \Delta y$. Once again using the relationship we derived between the sides of a right triangle, we can write $x\Delta x + y\Delta y = r\Delta r$. The radius of the wheel cannot change, and so we find $\Delta x/\Delta y = -y/x = -\Delta d/\Delta h$

We can find the at-rest horizontal side length, x, by simple trigonometry

$$R^2 = (R-h)^2 + d^2.$$

When solved, this gives $x = \sqrt{h(2R-h)}$. Meanwhile, the vertical side length of this triangle is simply $R-h$. Substituting these into our expression relating work to the change in potential energy yields

$$\begin{aligned} F &= W\frac{\Delta h}{\Delta d} && (16)\\ &= W\frac{x}{y} && (17)\\ &= W\frac{\sqrt{h(2R-h)}}{R-h} && (18) \end{aligned}$$

2.12. A horizontal turntable of diameter D is mounted on bearings with negligible friction. Two horizontal forces in the plane of the turntable of equal magnitude F, parallel to each other but pointing in opposite directions, act on the rim of the turntable on opposite ends of the diameter, as shown in Fig. 2-10.

(a) What force F_B acts on the bearing?

(b) What is the torque (= moment of this force couple) τ_O about a vertical axis through the center O?

(c) What would be the moment τ_P about a vertical axis through an arbitrary point P in the same plane?

(d) Is the following statement correct or false? Explain. "Any two forces acting on a body can be combined into a single resultant force that would have the same effect." In framing your answer, consider the case where the two forces are opposite in direction but not quite equal in magnitude.

(a) By symmetry, there is no net force F_B on the bearing.

(b) The torque about it, choosing a counterclockwise rotation as positive torque, is

$$F\frac{D}{2} + F\frac{D}{2} = FD.$$

(c) By writing the torque about an arbitrary axis in vector notation, we find

$$\mathbf{F} \times (\mathbf{x} + \frac{\mathbf{D}}{2}) - \mathbf{F} \times (\mathbf{x} + \frac{\mathbf{D}}{2}) = \mathbf{F} \times \mathbf{D} = FD.$$

where there is no $\sin\theta$ in the last term by virtue of θ being 90° with respect to the force and $\mathbf{D}$.

(d) As long as the body is rigid we may treat it as a point particle, in which case the force vectors are acting at a single point and may be combined into a resultant force.

2.13. A flat steel plate floating on mercury is acted upon by three forces at three corners of a square of side 0.100 m, as shown in Fig. 2-11. Find a *single* fourth force $\mathbf{F}$ which will hold the plate in equilibrium. Give the magnitude, direction, and point of application of $\mathbf{F}$ along the line AB.

By setting all forces equal to 0, we obtain two equations

$$50 - 50\sin 45^\circ + y = 0 \tag{19}$$
$$50 - 50\cos 45^\circ + x = 0. \tag{20}$$

Solving this gives components $F_x = F_y = 50(1/\sqrt{2} - 1)$ N (orienting the vector 45° from the horizontal axis in the same manner as the vector at O) and a magnitude of $F = \sqrt{F_x^2 + F_y^2} = 50(\sqrt{2} - 1)$ N. To find the point of application along AB, we write out the torque about the bottom left corner,

$$\left(\frac{50}{\sqrt{2}}\right)\,\mathrm{N} \cdot \sqrt{2(0.1)^2}\,\mathrm{m} + xF_y = 0.$$

Or, solving for x,

$$x = -\frac{5}{F_y} = \frac{5}{50(1 - 1/\sqrt{2})} \approx 0.34\,\mathrm{m}.$$

2.14. In the absence of friction, at what speed v will the weights W_1 and W_2 in Fig. 2-12 be moving when they have traveled a distance D, starting from rest? ($W_1 > W_2$)

By the conservation of energy, we can write

$$\text{P.E.} + \text{K.E.} = -W_1 D \sin\theta + W_2 D \sin\theta + \frac{1}{2}\frac{W_1}{g}v^2 + \frac{1}{2}\frac{W_2}{g}v^2 = 0.$$

Solving for v gives

$$v = \sqrt{2Dg\frac{W_1 - W_2}{W_1 + W_2}\sin\theta}.$$

2.15. In Fig. 2-13, the weights are equal, and there is negligible friction. If the system is released from rest, at what speed v will the weights be moving when they have traveled a distance D?

Once again,

$$\text{P.E.} + \text{K.E.} = WD\sin\theta - WD\sin\phi + \frac{1}{2}\frac{W}{g}v^2 + \frac{1}{2}\frac{W}{g}v^2 = 0.$$

Solving for v gives

$$v = \sqrt{2Dg(\sin\phi - \sin\theta)}.$$

This makes sense. The larger ϕ is, the bigger the component of the gravitational pull along the ramp is.

2.16. A mass M_1 slides on a 45° inclined plane of height H as shown in Fig. 2-14. It is connected by a flexible cord of negligible mass over a small pulley (neglect its mass) to an equal mass M_2 hanging vertically as shown. The length of the cord is such that the masses can be held at rest both at height $H/2$. The dimensions of the masses and the pulley are negligible compared to H. At time $t = 0$ the two masses a released.

(a) For $t > 0$ calculate the vertical acceleration a of M_2.

(b) Which mass will move downward?

(c) At what time t_1 will the mass identified in part b) strike the ground?

(d) If the mass identified in part b) stops when it hits the ground, but the other mass keeps moving, will it strike the pulley?

Mass M_2 will move downward, and so the acceleration of both blocks will be the same. Denoting $M_1 = M_2 = m$ and the tension in the cord T, we can write

$$T - m\frac{g}{\sqrt{2}} = ma \tag{21}$$

$$mg - T = ma \tag{22}$$

where the top equation is for M_1 and the bottom is for M_2. Removing T, we find $a = g(1-1/\sqrt{2})/2$. Integrating this twice gives us the distance both weights have moved, $d = at^2/2$. Solving for t and noting that M_2 hits the ground when $d = H/2$, we find $t = \sqrt{H/a}$.

The kinetic energy M_1 has at the instant M_2 hits the ground will be entirely converted to potential energy when M_1 comes to a stop, i.e. $mv^2/2 = mg\Delta h$.

When M_2 strikes the ground, M_1 is at a height $H/2 + \sin 45°H/2$, or $H(1 + 1/\sqrt{2})/2 \approx 0.85H$. Solving for Δh in the previous equation gives

$$\Delta h = \frac{1}{2}\frac{v^2}{g} = \frac{1}{2}\frac{(at)^2}{g} = \frac{1}{2}\frac{aH}{g} = H\frac{1}{4}(1 - \frac{1}{\sqrt{2}}) \approx 0.07H.$$

Adding these together gives a final height of $\approx 0.92H$, which is less than H, ergo M_1 does not strike the pulley.

2.17. A derrick is made of a uniform boom of length L and weight w, pivoted at its lower end, as shown in Fig. 2-15. It is supported at an angle θ with the vertical by a horizontal cable attached at a point a distance x from the pivot, and a weight W is slung from its upper end. Find the tension T in the horizontal cable.

Using the principle of virtual work, we recognize that $T\Delta x_T + W\Delta y_W + w\Delta y_w = 0$. How can we relate these displacements?

Consider increasing the angle of the derrick by an amount $\Delta\theta$. Using the first relationship we derived in 2.4, we see $\Delta x' = (y'l'/x')\sin\theta\Delta\theta$, where here x', y', and l' refer to the horizontal side, vertical side, and hypotenuse of any right triangle drawn along the derrick and the wall. For the cable, we find

$$\Delta x_T = \frac{x\cos\theta x}{x\sin\theta}\sin\theta\Delta\theta = x\cos\theta\Delta\theta.$$

For w and W, we must use the second relationship we derived in 2.4, $y'\Delta y' + x'\Delta x' = l'\Delta l'$. Here, l is immutable and so $y'/x' = -\Delta x'/\Delta y'$. Using this, we find

$$\Delta x_W = \frac{y_W L}{x_W}\sin\theta\Delta\theta = -\frac{\Delta x_W L}{\Delta y_W}\sin\theta\Delta\theta \tag{23}$$

$$\Delta x_w = \frac{y_w(L/2)}{x_w}\sin\theta\Delta\theta = -\frac{\Delta x_w(L/2)}{\Delta y_w}\sin\theta\Delta\theta \tag{24}$$

Solving for Δy_W and Δy_w gives

$$\Delta y_W = -L\sin\theta\Delta\theta \tag{25}$$

$$\Delta y_w = -\frac{L}{2}\sin\theta\Delta\theta \tag{26}$$

Putting these quantities into our original relationship and solving for T gives

$$T = -\frac{1}{\Delta x_T}(W\Delta y_W + w\Delta y_w) = \frac{L}{x}(W + \frac{W}{2})\tan\theta.$$

This makes physical sense: a smaller angle requires less force, as does a cable attachment point closer to the end of the derrick.

2.18. A uniform ladder 10 ft long with rollers at the top end leans against a smooth vertical wall, as shown in Fig. 2-16. The ladder weighs 30 lb. A weight $W = 60$ lb is hung from a rung 2.5 ft from the top end. Find

(a) the force F_R with which the rollers push on the wall.

(b) the horizontal and vertical forces F_h and F_v with which the ladder pushes on the ground.

(a) We can find F_R by setting the total torque equal to 0, taking the ladder's contact point with the floor as the axis of rotation. We have

$$5wg\sin\arctan\frac{6}{8} + 7.5Wg\sin\arctan\frac{6}{8} + 10F_R\sin\arctan -\frac{8}{6} = 0.$$

Solving this gives $F_R = 441\,\mathrm{N}$. If we wish to express this in pounds, we may divide by $g = 9.8\,\mathrm{m\,s^{-2}}$ to get $F_R = 45\,\mathrm{lb}$.

(b) By drawing a force diagram, it is clear that $\mathbf{F}_h = -\mathbf{F}_R$, so $F_h = F_R$. Meanwhile, F_v must cancel both the hanging weight and the weight of the ladder,

$$F_v - wg - Wg = 0.$$

From this, we see $F_v = 882\,\mathrm{N}$, or $F_v = 90\,\mathrm{lb}$.

2.19. A plank of weight W and length $\sqrt{3}R$ lies in a smooth circular trough of radius R, as shown in Fig. 2-17. At one end of the plank is a weight $W/2$. Calculate the angle θ at which the plank lies when it is in equilibrium.

The plank forms a chord across the circular trough. From this, we know that the segment connecting its center of mass to the center of our trough lies perpendicular to the plank. This allows us to form two right triangles with their hypotenuses connecting the center of our trough to the ends of our plank. This allows us to solve for the perpendicular distance D from our trough's midpoint to our plank's center of mass,

$$D^2 + \left(\frac{\sqrt{3}}{2}R\right)^2 = R^2$$

or $D = R/2$. Using this, we can form two more right triangles to obtain the vertical distance between the horizontal line crossing our trough's center and our plank's center of mass, $H_1 = R\cos\theta/2$. We can form a similar triangle between the center of mass of our plank and the lower end of it, with its hypotenuse given by $\sqrt{3}R/2$. This triangle has a height of $H_a = \sqrt{3}R\sin\theta/2$, giving our plank's lower end point a distance of $H_2 = H_1 + H_a = R(\cos\theta + \sqrt{3}\sin\theta)/2$ from the horizontal line intersecting our trough's center.

Now that we have the distances of these two points, we can use the principle of virtual work to obtain the angle our system rests at. Considering the horizontal line passing through the trough's center as $y = 0$, we have a potential energy of

$$V = -WH_1 - \frac{W}{2}H_2 = -\frac{WR}{4}(3\cos\theta + \sqrt{3}\sin\theta).$$

Under a small change in angle, this varies as

$$\Delta V = -\frac{WR}{4}(-3\sin\theta + \sqrt{3}\cos\theta)\Delta\theta.$$

Setting this to zero gives

$$\tan\theta = \frac{1}{\sqrt{3}}$$

which has the solution $\theta = 30^\circ$.

2.20. A uniform bar of length l and weight W is supported at its ends by two inclined planes as shown in Fig. 2-18. From the principle of virtual work find the angle α at which the bar is in equilibrium. (Neglect friction.)

Consider describing the coordinates of the endpoints of our bar using unit vectors pointed along the planes, $\mathbf{i}'$ along the one oriented at 30° to the horizontal and $\mathbf{j}'$ along the one oriented 120° to the horizontal. By simple trigonometry, the left and right endpoints are given by

$$\mathbf{l} = l\cos\alpha\mathbf{j}' \tag{27}$$
$$\mathbf{r} = l\sin\alpha\mathbf{i}' \tag{28}$$

The location of the rod's center of mass is the midway point of both of these, $(\mathbf{l}+\mathbf{r})/2$. We can express this quantity in terms of our standard coordinates by noting that

$$\mathbf{i}' = \cos 30°\mathbf{i} + \sin 30°\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} \tag{29}$$
$$\mathbf{j}' = -\cos 120°\mathbf{i} + \sin 120°\mathbf{j} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} \tag{30}$$

Substituting these into our expression for the location of the bar's center of mass gives

$$\frac{\mathbf{l}+\mathbf{r}}{2} = l(\frac{\sqrt{3}}{2}\sin\alpha - \frac{1}{2}\cos\alpha)\mathbf{i} + l(\frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha)\mathbf{j}.$$

By the principle of virtual work, our potential energy will be at an extremum when our system is in equilibrium. For the current system, this corresponds to when the height is at a minimum. Taking the derivative of the coefficient of $\mathbf{j}$ in the above expression and setting it equal to 0 gives

$$-\sqrt{3}\sin\alpha + \cos\alpha = 0$$

or $\tan\alpha = 1/\sqrt{3}$. This is the equation we arrived at in our last problem, having an answer of $\alpha = 30°$. This is interesting, as it seems the angle is independent of the length of the rod.

2.21. A small solid sphere of radius 4.5 cm and weight W, is to be suspended by a string from the ends of a smooth hemispherical bowl of radius 49 cm, as sown in Fig. 2-19. It is found that if the string is any shorter than 40 cm, it breaks. Use the principle of virtual work to find the breaking strength F of the string.

2.22. An ornament for a courtyard at a World's Fair is to be made up of four identical, frictionless metal spheres, each weighing $2\sqrt{6}$ ton-wt. The spheres are to be arranged as shown in Fig. 2-20, with three resting on a horizontal surface and touching each other; the fourth is to rest freely on the other three. The bottom three are kept from separating by spot welds at the points of contact with each other. Allowing for a factor of safety of 3, how much tension T must the spot welds withstand?

2.23. A rigid wire frame is formed in a right triangle, and set in a vertical plane as shown in Fig. 2-21. Two beads of masses $m_1 = 100$ g, $m_2 = 300$ g slide without friction on the wires, and are connected by a cord. When the system is in static equilibrium, what is the tension T in the cord, and what angle α does it make with the first wire?

2.24. Find the tension T needed to hold the cart shown in Fig. 2-22 in equilibrium, if there is no friction.

(a) Using the principle of virtual work.

(b) Using force components.

2.25. A bobbin of mass $M = 3$ kg consists of a central cylinder of radius $r = 5$ cm and tow end plates of radius $R = 6$ cm. It is placed on a slotted incline on which it will roll but not slip, and a mass $m = 4.5$ kg is suspended from a cord wound around the bobbin, as shown in Fig. 2-23. It is observed that the system is in static equilibrium. What is the angle of tilt θ of the incline?

2.26. A loop of flexible chain, of total weight W, rests on a smooth right circular cone of base radius r and height h, as shown in Fig. 2-24. The chain rests in a horizontal circle on the cone, whose axis is vertical. Find the tension T in the chain. Neglect friction.

By the principle of virtual work, $T\Delta l + W\Delta h = 0$. Given that the chain rests at a location along the cone with a certain radius r', a small change in length can be related to a change in radius by $\Delta l = 2\pi\Delta r'$. Similarly, by noting that the slope of the cylinder is $-h/r$, a small change in height can be related to a change in radius by $\Delta h = -\Delta r' \cdot h/r$. Putting this together, we see

$$2\pi T\Delta r' = W\Delta r'\frac{h}{r}$$

or $T = (Wh)/(2\pi r)$.

2.27. A cart on an inclined plane is balanced by the weight w as shown in Fig. 2-25. All parts have negligible friction. Find the weight W of the cart.

2.28. A bridge truss is constructed as shown in Fig. 2-26. All joints may be considered frictionless pivots and all members rigid, weightless, and of equal length. Find the reaction forces F_1 and F_2 and the force F_{DF} in the member DF.

2.29. In the truss shown in Fig. 2-27, all diagonal struts are of length 5 units and all horizontal ones are of length 6 units. All joints are freely hinged, and the weight of the truss is negligible.

(a) Which of the members could be replaced with flexible cables, for the load position shown?

(b) Find the forces in struts BD and DE.

2.30. In the system shown in Fig. 2-28, a pendulum bob of weight w is initially held in a vertical position by a thread A. When this thread is burned, releasing the pendulum, it swings to the left and barely reaches the ceiling at its maximum swing. Find the weight W. (Neglect friction, the radius of the pulley, and the finite sizes of the weights.)

Let us measure potential energy from the ceiling. Then, as there is only gravity acting on the system after w is released, we have

$$-3w - h_0W = 0w - (h_0 + 4)W$$

where h_0 denotes the initial distance W is hung from the ceiling, and the value of 4 can be arrived at by examining the figure: when the weight barely reaches the ceiling, the cable of length 3 will occupy most of the ceiling of length 4. The length 5 cable will take up the additional unit of length and have its remaining 4 units draped over the pulley. Kinetic energy does not enter this conservation of energy equation due to the weights having no velocity at the points of consideration.

Returning to the above equation and solving for W yields $W = 3w/4$.

2.31. Two equal masses m are attached to a third mass $2m$ by equal lengths of fine thread and the thread is passed over two small pulleys with negligible friction situated 100 cm apart, as shown in Fig. 2-29. The mass $2m$ is initially held level with the pulleys midway between them, and is then released from rest. When it has descended a distance of 50 cm it strikes a table top. What is it's speed v when it reaches the table top?

We can solve this using conservation of energy, specifically $\Delta T = -\Delta V$. By the time the central mass has descended 50 cm, the two smaller masses have each ascended $(50\sqrt{2} - 50)$ cm. This can be seen by drawing a right triangle between one of the pulleys and the spot on the table where the central mass strikes: because the horizontal and vertical legs of the right triangle are 50 cm, the hypotenuse is $50\sqrt{2}$ cm. The change in height, then, is given by the change in the hypotenuse length. Putting this all together, we see

$$\Delta V = mg(50\sqrt{2} - 50) - 2mg \cdot 50 + mg(50\sqrt{2} - 50) = 100(\sqrt{2} - 2)mg.$$

The kinetic energy is slightly trickier, as the smaller masses do not have the same velocity as the larger mass. Noting that the horizontal component of the previously studied right triangle does not change, we can related the two velocities by taking the time derivative of the Pythagorean theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) = \frac{\mathrm{d}}{\mathrm{d}t}(l^2) \tag{31}$$

$$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 2l\frac{\mathrm{d}l}{\mathrm{d}t} \tag{32}$$

$$y\frac{\mathrm{d}y}{\mathrm{d}t} = l\frac{\mathrm{d}l}{\mathrm{d}t} \tag{33}$$

So, denoting $\mathrm{d}y/\mathrm{d}t$ by v_{2m}, we see that the velocities of our small masses are related to the velocity of the large mass by $v_m = yv_{2m}/l$. At the moment of impact, $y = 50$ cm and $l = 50\sqrt{2}$ cm, and so $v_m = v_{2m}/\sqrt{2}$. Putting this all together gives

$$\Delta T = \frac{1}{2}mv_m^2 + \frac{1}{2}(2m)v_{2m}^2 + \frac{1}{2}mv_m^2 = \frac{3}{2}mv_{2m}^2.$$

Finally, combining our kinetic and potential energies yields

$$\Delta T = \frac{3}{2}mv_{2m}^2 = 100(2 - \sqrt{2})mg = -\Delta V.$$

or

$$v_{2m} = \sqrt{\frac{200}{3}(2 - \sqrt{2})g}.$$

Putting in the explicit value of $g = 980\,\mathrm{cm\,s^{-2}}$ gives $v_{2m} = 196\,\mathrm{cm\,s^{-1}}$.

2.32. A tank of cross-sectional area A contains a liquid having density ρ. The liquid squirts freely from a small hole of area a distance H below the free surface of the liquid, as shown in Fig. 2-30. If the liquid has no internal friction (viscosity), with what speed v does it emerge?

By conservation of energy,

$$\frac{m}{V}gH = \frac{1}{2}\frac{m}{V}v^2$$

where v is the velocity of the water at the hole, $m/V = m/(AH) = \rho$ is the density of the water, and H is the change in height from the top of the liquid to the hole. Solving for v gives $v = \sqrt{2gH}$.

2.33. Smooth, identical logs are piled in a stake truck. The truck is forced off the highway and comes to a rest on an even keel lengthwise but with the bed at an angle θ with the horizontal, as shown in Fig. 2-31. As the truck is unloaded, the removal of the log shown dotted leaves the remaining three in a condition where they are just ready to slide, that is, if θ were any smaller, the logs would fall down. Find θ.

2.34. A spool of weight w and radii r and R is wound with cord, and suspended from a fixed support by two cords wound on the smaller radius; a weight W is then suspended from two cords wound on the larger radius, as shown in Fig. 2-32. W is chosen so that the spool is just balanced. Find W.

2.35. A suspension bridge is to span a deep gorge 54 m wide. The roadway will consist of a steel truss supported by six pairs of vertical cables spaced 9.0 m apart, as shown in Fig. 2-33. Each cable is to carry an equal share of the 4.80×10^4 kg weight. The two pairs of cables nearest the center are to be 2.00 m long. Find the proper lengths of the remaining vertical cables A and B and the maximum tension T_{max} in the two longitudinal cables, if the latter are to be at a 45° angle with the horizontal at their ends.

2.36. The insulating support structure of a Tandem Van de Graaff may be represented, as shown in Fig. 2-34: two blocks of about uniform density, length L, height h and weight W, supported from vertical bulkheads by pivot joints (A and B) and forced apart by a screw jack (F) at the center. Since the material of the blocks cannot support tension, the jack must be adjusted to give zero force on the upper pivot.

(a) What force F is required?

(b) What is the total force (magnitude and direction) $\mathbf{F}_A$ on one of the lower pivots A?

3 Kepler's Laws and Gravitation

3.1. Some properties of the ellipse. The size and shape of an ellipse are determined by specifying the values of any two of the following quantities (as shown in Fig. 3-1):

a : the semi-major axis

b : the semi-minor axis

c : the distance from the center to one focus

e : the eccentricity

r_p : the perihelion (or perigee) distance (the closest distance from a focus to the ellipse)

r_a : the aphelion (or apogee) distance (the farthest distance from a focus to the ellipse)

The relationships of these various quantities are

$$\begin{aligned} a^2 &= b^2 + c^2, \\ e &= \frac{c}{a} \quad \text{(definition of } e\text{)}, \\ r_p &= a - c = a(1-e), \\ r_a &= a + c = a(1+e). \end{aligned}$$

Show that the area of an ellipse is given by $A = \pi ab$.

Imagine a circle in the plane with a radius of a, having an area of πa^2. To transform this into an ellipse, we may apply a linear transformation that stretches an arbitrary axis by $\frac{b}{a}$, thereby transforming our circle into an ellipse with a semi-major axis of b and a semi-minor axis of a. This linear transformation has a determinant of $\frac{b}{a}$, and thus transforms our circle's array by the same amount, giving an area of

$$\pi a^2 \frac{b}{a} = \pi ab.$$

3.2. The distance of the moon from the center of the earth varies from 363 300 km at perigee to 405 500 km at apogee, and its period is 27.322 days. A certain artificial earth satellite is orbiting so that its perigee height from the surface of the Earth is 225 km and its apogee height is 710 km. The mean diameter of the Earth is 12 756 km. What is the sidereal period T of the satellite?

We know that $T \propto a^{3/2}$, where a is the semi-major axis of the ellipse of a body's orbit. We also know that the semi-major axis is given by $\frac{r_p + r_a}{2}$.

Given the apogee, perigee, and period of the moon, we can determine the constant of proportionality relating T to $a^{3/2}$,

$$C = \frac{T}{\left(\frac{r_p+r_a}{2}\right)^{3/2}} = \frac{27.322\,\text{days}}{\left(\frac{363\,300\,\text{km}+405\,500\,\text{km}}{2}\right)^{3/2}} = 1.146 \times 10^{-7}\,\text{days/km}^{3/2}.$$

From this, we can find the period of the satellite via

$$T = C\left(\frac{r_p + r_a}{2}\right)^{3/2} = 1.146 \times 10^{-7}\,\text{days/km}^{3/2}\left(\frac{225\,\text{km} + 6378\,\text{km} + 710\,\text{km} + 6378\,\text{km}}{2}\right)^{3/2},$$

which is approximately 1.56 hours.

3.3. The eccentricity of the earth's orbit is 0.0167. Find the ratio v_{max}/v_{min} of its maximum speed in its orbit to its minimum speed.

The earth's maximum speed is at perihelion, while its slowest occurs at aphelion. Equating the area swept out at each moment (imagining a triangle with a base stretching from the earth to the sun with another side tangential to the orbit), we see

$$\frac{1}{2} r_p v_{max} = \frac{1}{2} r_a v_{min},$$

or $v_{max}/v_{min} = r_a/r_p$. This ratio, in turn, is given by

$$\frac{r_a}{r_p} = \frac{1+e}{1-e} = \frac{1+0.0167}{1-0.0167} = 1.034.$$

3.4. The radii of the earth and the moon are 6378 km and 1738 km, respectively, and their masses are in the ratio 81.3 to 1.000. Calculate the acceleration of gravity $g_☾$ at the surface of the moon. ($g_♁ = 9.81\,\text{m}\,\text{s}^{-2}$.)

The force of gravity is given by

$$\mathbf{F} = m\mathbf{a} = G\frac{Mm}{r^2}\hat{\mathbf{r}},$$

and so the magnitude of the gravitational acceleration at a given point from a body is

$$a = G\frac{M}{r^2}.$$

We can solve for G in terms of the parameters of the Earth, and substitute this into the equation for the gravitational acceleration of the moon,

$$G = \frac{gr_♁^2}{M_♁}, \qquad a_☾ = \frac{gr_♁^2}{M_♁}\frac{\frac{1}{81.3}M_♁}{r_☾^2} = \left(\frac{r_♁}{r_☾}\right)^2 \frac{g}{81.3} = \left(\frac{6378\,\text{km}}{1738\,\text{km}}\right)^2 \cdot \frac{9.8\,\text{m}\,\text{s}^{-2}}{81.3} = 1.62\,\text{m}\,\text{s}^{-2}.$$

3.5. In 1986, Halley's comet is expected to return on its seventh trip around the sun since the days in 1456 when people were so frightened that they offered prayers in the church "to be saved from the Devil, the Turk, and the comet." In its most recent perihelion on April 19, 1910, it was observed to pass near the sun at a distance 0.60 AU.

(a) How far r_a does it go from the sun at the outer extreme of its orbit?

(b) What is the ratio v_{max}/v_{min} of its maximum orbital speed to its minimum speed?

(a) The constant found in Exercise 3.2 is dependent on the mass of the system under consideration. Since this was the Earth-Moon system, it is *not* appropriate to use it in this problem. Instead, we must make an approximation. Given that the mass of the sun dwarfs everything else in the solar system, we suppose that this constant is roughly the same for Halley's comet as it is for the Earth-Sun system. In the Earth-Sun system, we have

$$1\,\text{year} \propto (1\,\text{AU})^{3/2}.$$

This simple relationship sets the constant of proportionality equal to unity. Rearranging the formula derived in Exercise 3.2, we see

$$r_a = 2\left(\frac{T}{C}\right)^{2/3} - r_p = 2\left(\frac{76\,\text{years}}{1\,\text{year}/\text{AU}^{3/2}}\right)^{2/3} - 0.60 = 35.3\,\text{AU}.$$

3.6. A satellite in a circular orbit near the earth's surface has a typical period of about 100 minutes. What should be the radius r of its orbit (in Earth radii, $r_{♁}$) for a period of 24 hours?

We can once again solve for the constant of proportionality in Kepler's third law by noting

$$C = \frac{100\,\text{minutes}}{(1.1\,r_{♁})^{3/2}},$$

where we have chosen the orbital radius of the $T = 100$ minute satellite to be slightly more than the earth's radius. For a period of 24 hours, or 1440 minutes, a satellite must orbit at a radius of

$$r = \left(\frac{T}{C}\right)^{2/3} = 6.5\,r_{♁}.$$

3.7. Consider two earth satellites of equal orbital radius, one of them in a polar orbit, the other in an orbit in the equatorial plane. Which satellite needed the larger booster rocket and why?

The one in the polar orbit needed a larger booster rocket as the energy it received from the earth's rotation had to be fought against.

3.8. A true "Syncom" satellite rotates synchronously with the earth. It always remains in a fixed position with respect to a point P on the earth's surface.

(a) Consider the straight line connecting the center of the earth with the satellite. If P lies on the intersection of this line with the earth's surface, can P have any geographic latitude λ or what restrictions do exist? Explain.

(b) What is the distance r_s from the earth's center to a Syncom satellite of mass m? Express r_s in units of the earth-moon distance $r_{\earth\leftmoon}$

Note: Consider the earth a uniform sphere. You may use $T_{\leftmoon} = 27$ days for the moon's period.

(a) P must lie along the equator, for if it did not our satellite would be orbiting about a point not coinciding with the system's center of mass. This is most clearly seen when P is taken to be one of the poles.

(b) Once again returning to Kepler's third law, we estimate the constant associate with an Earth orbit to be

$$C = \frac{27\,\text{days}}{(r_{\earth\leftmoon})^{3/2}}.$$

Requiring our Syncom satellite have a period of 1 days allows us to estimate an orbital distance of

$$r = \left(\frac{T}{C}\right)^{2/3} = \frac{1}{9}\,r_{\earth\leftmoon}.$$

3.9. (a) Comparing data describing the earth's orbital motion about the sun with data for the moon's orbital motion about the earth, determine the mass of the sun $m_{\odot}$ relative to the mass of the earth $m_{\earth}$.

(b) Io, a moon of Jupiter, has an orbital period of revolution of 1.769 days and an orbital radius of 421 800 km. Determine the mass of Jupiter $m_{\jupiter}$ in terms of the mass of the earth.

(a) We will need improve our methods used in previous problems and find the explicit way in which the mass of a body enters into the constant of proportionality in Kepler's third law. Assuming the orbits of interest are nearly circular, we may use Newton's law of gravitation to write

$$m\omega^2 r^2 = \frac{GMm}{r}.$$

We can cancel the mass of the orbiting body from both sides, exchange ω for $2\pi/T$, and solve for the mass M of the central body to find

$$M = \frac{4\pi^2 r^3}{GT^2}.$$

This allows us to find the requested mass ratio $m_{\odot}/m_{\earth}$,

$$\frac{m_{\odot}}{m_{\earth}} = \frac{(389\,r_{\earth\leftmoon})^3/(365\,\text{days})^2}{(1\,r_{\earth\leftmoon})^3/(27\,\text{days})^2} = 322000\ m_{\earth},$$

where we have expressed the distance from the earth to the sun in terms of earth-lunar distances.

(b) Utilizing the above method, we see

$$\frac{m_{\jupiter}}{m_{\earth}} = \frac{(421\,800\,\text{km})^3/(1.769\,\text{days})^2}{(384\,400\,\text{km})^3/(27\,\text{days})^2} = 308\ m_{\earth}.$$

3.10. Two stars, a and b, move around one another under the influence of their mutual gravitational attraction. If the semi major axis of their relative orbit is observed to be R, measured in astronomical units

(AU) and their period of revolution is T years, find an expression for the sum of the mass, $m_a + m_b$, in terms of the mass $m_\odot$ of the sun.

> In this case, the stars are orbiting their common center of mass. We can use the method of the previous exercise (where the expression for the sun's mass uses the earth's orbit) to quickly find
>
> $$\frac{m_a + m_b}{m_\odot} = \frac{(R\,\mathrm{AU})^3/(T\,\mathrm{years})^2}{(1\,\mathrm{AU})^3/(1\,\mathrm{year})^2} = \frac{R^3}{T^2}\,m_\odot.$$

3.11. If the attractive gravitational force between a very large central sphere M and a satellite m in orbit about it were actually $\mathbf{F} = -GMm\mathbf{R}/R^{(3+a)}$, (where $\mathbf{R}$ is the radial vector between them) how would Kepler's second and third law be modified? (In discussing the third law, assume a circular orbit.)

> Clearly, Kepler's third law would be amended to read
>
> $$T^2 \propto R^{3+a},$$
>
> as can be seen in the explicit form of the third law present in the solution to Exercise 3.9. Working in polar coordinates, we see that the second law implies
>
> $$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\cdot r\cdot r\theta\right) = r\frac{\mathrm{d}r}{\mathrm{d}\theta}\theta + \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}s + \frac{1}{2}rv = 0,$$
>
> where s is the arclength of the orbit and v is the orbiting body's tangential velocity. We know that, given a circular orbit, $r' = 0$ and the radius is explicitly dependent on the period (which is a constant). Thus, areas swept out in equal times are still equivalent.

3.12. In making laboratory measurements of g, how precise does one have to be to detect diurnal variations Δg due to the moon's gravitation? For simplicity, assume that your laboratory is so located that the moon passes through zenith and nadir. Also, neglect earth-tide effects.

> When the moon is 'below' the laboratory, the total acceleration due to the gravitational field will be
>
> $$a_{max} = G\frac{M_{\text{♁}}}{r_{\text{♁}}^2} + G\frac{M_{\text{☾}}}{(r_{\text{♁☾}} + r_{\text{♁}})^2}.$$
>
> When the moon is 'above' the laboratory, the total acceleration will now be
>
> $$a_{min} = G\frac{M_{\text{♁}}}{r_{\text{♁}}^2} - G\frac{M_{\text{☾}}}{(r_{\text{♁☾}} - r_{\text{♁}})^2}.$$
>
> Subtracting these two gives us the difference in the gravitational acceleration, Δg,
>
> $$a_{max} - a_{min} = G\frac{M_{\text{☾}}}{(r_{\text{♁☾}} + r_{\text{♁}})^2} + G\frac{M_{\text{☾}}}{(r_{\text{♁☾}} - r_{\text{♁}})^2} = 6.64\times10^{-5}\,\mathrm{m\,s^{-2}}.$$

3.13. An eclipsing binary star system is one whose orbital plane nearly contains the line of sight, so that one star eclipses the other periodically. The relative orbital velocity of the two components can be measured from the Doppler shift of their spectral lines. If T is the observed period in days, and V is the orbital velocity in $\mathrm{km\,s^{-1}}$, what is the total mass M of the binary system in solar masses?

Note: The mean distance from the earth to the sun is 1.50×10^8 km.

Be equating the orbital velocity to the star's angular velocity, we find

$$V = \omega R = \frac{2\pi}{T} R.$$

Solving for R and cubing this equation allows us to plug in the explicit form of Kepler's third law found in an earlier exercise,

$$\frac{V^3 T^3}{8\pi^3} = G \frac{M_{total} T^2}{4\pi^2}$$

which can be solved for the mass of the system

$$M_{total} = \frac{V^3 T}{2\pi G}.$$

To express this in terms of the mass of the sun, we may simply divide everything by Kepler's third law, solved for mass, in terms of the Earth-Sun orbit,

$$\frac{M_{total}}{M_\odot} = \frac{V^3 T}{2\pi G} \cdot \frac{G T^2_{\text{♁}\odot}}{4\pi^2 R^3_{\text{♁}\odot}} = \frac{V^3 T T^2_{\text{♁}\odot}}{8\pi^3 R^3_{\text{♁}\odot}}.$$

3.14. A comet rounds the sun at a perihelion distance of $r_p = 1.00 \times 10^6$ km. At this point its velocity is $v = 500.0\,\mathrm{km\,s^{-1}}$.

(a) What is the radius of curvature R_c of the orbit at perihelion (in km)?

(b) For an ellipse with semi-major axis a and semi-minor axis b, the radius of curvature at perihelion is $R_c = \frac{b^2}{a}$. If you know R_c and r_p you should be able to write a relation involving a and only these quantities. Do so, and find a.

(c) If you were able to solve for a from the above information, you should be able to calculate the period T_c of the comet. Do so.

3.15. Using the idea that two mutually gravitating bodies each "fall" toward the other, and thus move about some fixed common point (their center of mass), show that their period in an orbit in which they remain a given fixed distance apart depends only upon the sum of their masses $M + m$ and not at all upon the ratio of their masses. This is also true for elliptical orbits. Assuming that the semi-major axes of the ellipses in which the bodies move are R and r, find the period T of their orbit.

3.16. How can one find the mass of the moon?

3.17. The trigonometric parallax of Sirius (i.e., the angle subtended at Sirius by the radius of the Earth's orbit) is 0.378 degrees arc. Using this and the data contained in Fig. 3-2, deduce as best you can the mass M of the Sirius system in terms of that of the sun, and

(a) assuming that the orbital plane is perpendicular to the line of sight, and

(b) allowing for the actual tilt of the orbit.

Is your value in part (b) above an upper or lower limit (or either)?

4 Kinematics

4.1. (a) A body travels in a straight line with a constant acceleration. At $t = 0$, it is located at $x = x_0$ and has a velocity $v_x = v_{x_0}$. Show that its position and velocity at time t are

$$\begin{aligned} x(t) &= x_0 + v_{x0}t + \frac{1}{2}at^2, \\ v_x(t) &= v_{x0} + at. \end{aligned}$$

(b) Eliminate t from the preceding equations, and thus show that, at any time,

$$v_x^2 = v_{x0}^2 + 2a(x - x_0)$$

(a) We can integrate acceleration to find the velocity,

$$v_x(t) = \int a\mathrm{d}t = at + v_{x0},$$

and integrate this again to find the position

$$x(t) = \int at + v_x(0)\mathrm{d}t = \frac{1}{2}at^2 + v_{x0}t + x_0.$$

(b) From the velocity, we see that time can be rewritten as

$$t = \frac{v_x(t) - v_{x0}}{a}.$$

Substituting this into the position yields

$$\begin{aligned} x(t) &= \frac{1}{2}a\left(\frac{v_x(t) - v_{x_0}}{a}\right)^2 + v_{x0}\left(\frac{v_x(t) - v_{x0}}{a}\right) + x_0 \\ &= \frac{1}{2}\frac{v_x(t)^2}{a} - \frac{v_{x0}v_x(t)}{a} + \frac{1}{2}\frac{v_{x0}^2}{a} + \frac{v_{x0}v_x(t)}{a} - \frac{v_{x0}^2}{a} + x_0 \\ &= \frac{1}{2}\frac{v_x(t)^2}{a} - \frac{1}{2}\frac{v_{x0}^2}{a} + x_0. \end{aligned}$$

Suppressing the argument of our functions and solving for v_x yields

$$v_x^2 = v_{x0}^2 + 2a(x - x_0).$$

4.2. Generalize the preceding problem to the case of three dimensional motion with constant acceleration components a_x, a_y, a_z, along the three coordinate axis Show that

(a)

$$\begin{aligned}
x(t) &= x_0 + v_{x0}t + \frac{1}{2}a_x t^2 \\
y(t) &= y_0 + v_{y0}t + \frac{1}{2}a_y t^2 \\
z(t) &= z_0 + v_{z0}t + \frac{1}{2}a_z t^2 \\
v_x(t) &= v_{x0} + a_x t \\
v_y(t) &= v_{y0} + a_y t \\
v_z(t) &= v_{z0} + a_z t
\end{aligned}$$

.

(b)

$$v^2 = v_x^2 + v_y^2 + v_z^2 = v_0^2 + 2a[a_x(x - x_0) + a_y(y - y_0) + a_z(z - z_0)],$$

where

$$v_0^2 = v_{x0}^2 + v_{y0}^2 + v_{z0}^2.$$

(a) As each coordinate axis is orthogonal to the other two, our descriptions of the velocity and position of a particle undergoing uniform acceleration in the x direction hold analogously for the y and z directions.

(b) Because of the above, and the fact that $v^2 = v_x^2 + v_y^2 + v_z^2$, we may simply add together the three versions of the derived formula relating the square of the velocity in one direction with the position, acceleration, and initial velocity in that direction,

$$v_x^2 + v_y^2 + v_z^2 = v_{x0}^2 + v_{y0}^2 + v_{z0}^2 + 2a[a_x(x - x_0) + a_y(y - y_0) + a_z(z - z_0)].$$

4.3. An angle may be measured by the length of arc of a circle that the angle subtends, with the vertex of the angle subtends, with the vertex of the angle at the center of the circle. If s is the arc length and R is the radius of the circle, as shown in Fig. 4-1, then the subtended angle θ, in radians, is

$$\theta = \frac{s}{R}.$$

(a) Show that, if $\theta \ll 1$ radian, $\sin\theta \approx \theta$, and $\cos\theta \approx 1$.

(b) With the above result, and the formulas for the sine and cosine of the sum of two angles, find the derivatives of $\sin x$ and $\cos x$, using the fundamental formula

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}.$$

(a) When θ is small, the arclength spanned can be seen as the side to a right triangle, where both its hypotenuse and adjacent sidelengths are given by R. In this case, we see

$$\cos\theta \approx \frac{R}{R} = 1, \qquad \sin\theta \approx \frac{s}{R} = \theta\frac{R}{R} = \theta.$$

(b) For $\sin x$, we have

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x} &= \lim_{\Delta x\to 0}\frac{\sin x\cos\Delta x+\cos x\sin\Delta x-\sin x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\sin x+\Delta x\cos x-\sin x}{\Delta x}\\ &= \cos x\end{aligned}$$

.

While the derivative of $\cos x$ is

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos x}{\Delta x} &= \lim_{\Delta x\to 0}\frac{\cos x\cos\Delta x-\sin x\sin\Delta x-\cos x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\cos x-\sin x\Delta x-\cos x}{\Delta x}\\ &= -\sin x\end{aligned}$$

.

4.4. An object is moving counterclockwise in a circle of radius R at constant speed V, as shown in Fig. 4-2. The center of the circle is at the origin of rectangular coordinates (x, y), and at time $t = 0$ the particle is at $(R, 0)$. Show that

(a)

$$\begin{aligned}x &= R\cos\omega t,\\ y &= R\sin\omega t,\\ v_x &= -V\sin\omega t,\\ v_y &= V\cos\omega t,\\ a_x &= -\frac{V^2}{R}\cos\omega t,\\ a_y &= -\frac{V^2}{R}\sin\omega t,\\ a &= \frac{V^2}{R},\end{aligned}$$

(b)

$$\begin{aligned}\ddot{x}+\omega^2 x &= 0\\ \ddot{y}+\omega^2 y &= 0.\end{aligned}$$

(a) A point traveling around a circle may be decomposed into x and y coordinates by means of trigonometry: draw a right triangle from the origin of the circle to the point. The hypotenuse of this triangle is of length R, while its vertical side is our point's y coordinates and its horizontal side is our point's x coordinate. These are related to R by

$$\frac{x}{R}=\cos\theta \qquad \text{and} \qquad \frac{y}{R}=\sin\theta,$$

where θ is the angle our particle makes with the x-axis. If this angle is changing at ω radians per second, the coordinates of our point are

$$x = R\cos\omega t \qquad y = R\sin\omega t.$$

Now, recognizing that the arclength is given by $s = r\theta$, we have $V = R\omega$, or $\omega = V/R$. Differentiating our coordinates, then, gives

$$v_x = -V\sin\omega t \qquad v_y = V\cos\omega t$$

Performing another round of differentiation yields

$$a_x = -\frac{V^2}{R}\cos\omega t \qquad a_y = -\frac{V^2}{R}\sin\omega t$$

The total acceleration is given by the hypotenuse of the triangle whose legs are formed by a_x and a_y, and so

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{\frac{V^4}{R^2}\cos^2\omega t + \frac{V^4}{R^2}\sin^2\omega t} = \sqrt{\frac{V^4}{R^2}} = \frac{V^2}{R}$$

(b) For this part, notice that multiplying our x and y coordinates by $\omega^2 = V^2/R^2$ gives

$$\omega^2 x = \frac{V^2}{R}\cos\omega t \qquad \omega^2 y = \frac{V^2}{R}\sin\omega t$$

This is precisely the opposite of our coordinate accelerations, and so we obtain

$$\ddot{x} + \omega^2 x = 0$$
$$\ddot{y} + \omega^2 y = 0$$

4.5. A Skyhook balloon with a scientific payload rises at a rate of 1000 feet per minute. At an altitude of 30000 feet the balloon bursts and the payload free-falls. (Such disasters *do* occur!)

(a) For what length of time t was the payload off the ground?

(b) What was the payload's speed v at impact?

Neglect air-drag.

(a) Clearly, at 1000 feet per minute, the balloon ascended for 30 minutes before bursting at 30000 feet. At that point, its position thereafter is given by

$$x(t) = 30000 + \frac{100}{6}t - 16t^2$$

where t is measured in seconds. We can find the time at which the balloon hits the ground by solving for t. This is a standard quadratic equation with a non-negative root of $t = 43.825$ seconds. Adding this to our rise time gives an air time of 30 minutes and 43.825 seconds.

(b) Differentiating our position coordinate gives

$$v(t) = \frac{100}{6} - 32t.$$

At $t = 43.825$ seconds, this comes out to be -1385 feet per second. The speed at which the balloon hits the ground is the absolute value of this, so 1385 feet/s.

4.6. Consider a train that can accelerate with an acceleration of $20\,\mathrm{cm\,s^{-2}}$ and slow down with a deceleration of $100\,\mathrm{cm\,s^{-2}}$ Find the minimum time t for the train to travel between two stations 2 km apart.

The fastest route between two points will clearly be the one in which the train is always accelerated. We can then setup two conditions—corresponding to the times during which each acceleration is effective—and solve for the total time. The first is that the train must begin and end its journey at rest, i.e.

$$v_{\mathrm{end}} = a_1 t_1 + a_2 t_2 = 0\,\mathrm{cm\,s^{-1}}.$$

The second is that the total distance covered must be 2 km, or, equivalently, 2×10^5 cm. The distance covered by the first part of the trip is given by $\frac{1}{2}a_1 t_1^2$, while the distance covered by the latter half is given by $(a_1 t_1)t_2 + \frac{1}{2}a_2 t_2^2$, and so this condition simplifies to

$$s_{\mathrm{end}} = \frac{1}{2}a_1 t_1^2 + a_1 t_1 t_2 + \frac{1}{2}a_2 t_2^2 = 2 \times 10^5\,\mathrm{cm}$$

Solving for t_1 in the first equation gives the requirement that $t_1 = -\frac{a_2}{a_1}t_2$. Using this to eliminate t_1 from the second equation yields

$$\frac{1}{2}a_1\Big(-\frac{a_2}{a_1}t_2\Big) - a_1\frac{a_2}{a_1}t_2^2 + \frac{1}{2}a_2 t_2^2 = t_2^2\Big(\frac{1}{2}\frac{a_2^2}{a_1} - a_2 + \frac{1}{2}a_2\Big) = 2 \times 10^5\,\mathrm{cm}$$

Isolating t_2 and substituting in $a_1 = 20\,\mathrm{cm\,s^{-1}}$ and $a_2 = -100\,\mathrm{cm\,s^{-1}}$ gives $t_2 \approx 25.82\,\mathrm{s}$. Combining this with our first requirement, tells us that $t_1 = 5t_2 \approx 129.1\,\mathrm{s}$. Addding these together gives a total time of approximately 155 s.

4.7. If you throw a small ball vertically upward in real air with drag, does it take longer to go up or come down?

When throwing the ball up, drag complements the force of gravity in pulling the ball down. Conversely, drag *opposes* the force of gravity when the ball is falling. This opposition guarantees a slower acceleration from rest at the apex of the trajectory, ensuring its fall time is longer than its rise time.

4.8. Consider a point on the surface of the earth at the equator:

(a) What is its speed v relative to the center of the earth?

(b) What is its angular frequency ω?

(c) What is the ratio of its radial acceleration a due to angular motion and its gravitational acceleration g?

(a) The period of Earth's rotation is 24 hours, or 86400 seconds, giving it an angular velocity of $2\pi/T = 7.27 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$. The speed of a point on the equator is given by $v = r\omega$, which, in the case of $r = 6.38 \times 10^6$ m, is approximately $464\,\mathrm{m\,s^{-1}}$.

(b) As stated above, the angular frequency of this point is $7.27 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$.

(c) The radial acceleration of this point is given by $a = v^2/r = 0.0337\,\mathrm{m\,s^{-2}}$. Dividing this by the accepted gravitationl accleration at the earth's surface, $9.8\,\mathrm{m\,s^{-2}}$ gives a value of 0.00344.

4.9. A Corporal rocket fired vertically was observed to have a constant upward acceleration of $2g$ during the burning of the rocket motor, which lasted for 50 seconds. Neglecting air resistance and variation of g with altitude,

(a) Draw a v-t diagram for entire flight of rocket.

(b) Calculate the maximum height attained H_{max}.

(c) Calculate the total elapsed time T from the firing of the rocket to its return to Earth.

(a) The v-t diagram for the flight of the rocket consists of a straight line of slope $2g$ until $t = 50$, at which point a line of slope $-g$ continues onward.

(b) The maximum height attained can be found by writing out the rocket's position function after its upward acceleration has stopped,

$$x(t) = \frac{1}{2}(2g)(50)^2 + (2g)(50)t - \frac{1}{2}gt^2.$$

We can find the maximum of this function by differentiating with respect to time and setting the result to zero, giving $t = 100\,\text{s}$. Plugging this back into our position function gives a maximum height of $73\,500\,\text{m}$, or approximately 46 miles.

(c) We can find the total elapsed time by solving for the positive zero of our above position function. This is given by

$$t = 100 + 50\sqrt{6},$$

or approximately 222.5 seconds. Adding this to our thrust time of 50 seconds gives a total air time of 275.5 seconds.

4.10. In a lecture demonstration a small steel ball bounces on a steel plate. On each bounce the downward speed of the ball arriving at the plate is reduced by a factor e in the rebound, i.e. $v_{\text{upward}} = ev_{\text{downward}}$. If the ball was initially dropped from a height of 50 cm above the plate at time $t = 0$, and if 30 seconds later the silencing of a microphone sound indicated all bouncing had ceased, what was the value of e?

On initially dropping the ball, its velocity starts at 0 and it is acted upon solely by the gravitational force, allowing us to write its position at $h(t) = \frac{1}{2}gt_1^2$ (where we are counting the downward direction as positive). This can be solved for t_1 to find $t_1 = \sqrt{\frac{2h}{g}}$.

Upon rebounding with velocity ev_1, the ball's ascension time can be solved for by finding the point at which its velocity attains a minimum, $ev_1 - gt_2 = 0$, or $t_2 = \frac{ev_1}{g}$. To find v_1, we may invoke conservation of energy on the initial drop: $\frac{1}{2}mv_1^2 = mg\Delta x$. This gives us an impact velocity of $v_1 = \sqrt{2gh}$, which, when substituting in our previous equation, gives us a rise time of the second bound of $t_2 = e\sqrt{\frac{2h}{g}}$. As the trajectory of each bounce is symmetric about its midpoint, t_2 is the ball's subsequent fall time as well.

The total time must add up to 30 s, at which point the ball will have undergone and infinite number of ever smaller rebounds. It is easy to see that each rebound decreases the air time by an additional factor of e, and so we have

$$t_1 + 2t_2 + 2t_3 + \cdots = \sqrt{\frac{2h}{g}} + \sum_{n=1}^{\infty} 2e^n\sqrt{\frac{2h}{g}} = 30\,\text{s}$$

The geometric series in the rightmost term can be easily solved to find

$$\sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2h}{g}}\frac{1}{1-e} = 30$$

Isolating e gives

$$e = 1 - \frac{2\sqrt{\frac{2h}{g}}}{30 - \sqrt{\frac{2h}{g}}}$$

Plugging in $g = 9.8\,\mathrm{m\,s^{-2}}$ and $h = 0.5\,\mathrm{m}$ gives $e \approx 0.978$.

4.11. A projectile is fired over level terrain at an initial speed v_0, at an angle θ with the horizontal. (Neglect air resistance.)

(a) Find the maximum height attained $H_{\max}$ and the range R.

(b) At what angle should the above projectile be fired in order to attain the maximum range?

(a) In the vertical direction, the projectile's initial velocity is $v_o \sin\theta$ and it experiences a constant deceleration due to gravity. Its position and velocity functions are given by

$$y(t) = v_0 \sin\theta t - \frac{1}{2}gt^2 \qquad v_y(t) = v_0 \sin\theta - gt.$$

In the horizontal direction, the projectile experiences no force, starting at a velocity of $v_o \cos\theta$. Its relevant functions are

$$x(t) = v_0 \cos\theta t \qquad v_x(t) = v_0 \cos\theta.$$

The point at which our projectile attains maximum height is the point at which its vertical velocity is zero. This occurs at a time $t = \frac{v_0 \sin\theta}{g}$. Plugging this into $y(t)$ gives us our maximum height,

$$H_{\max}(\theta) = \frac{1}{2}\frac{v_0^2 \sin^2\theta}{g}.$$

The range of our projectile can be determined from $x(t)$ by finding the time at which the projectile hits the ground. By symmetry, this occurs at $t = 2\frac{v_0 \sin\theta}{g}$, giving it a range of

$$R(\theta) = \frac{v_0^2 \sin 2\theta}{g}.$$

(b) $R(\theta)$ clearly attains a maximum when the argument of the sin function is $\pi/2$, so $\theta = \pi/4$.

4.12. A champion archer hits a bullseye in a target mounted on a wall a distance L away and situated at a height h above his bow. Deduce the relation between the speed V at which the arrow left his bow, the arrow's initial angle θ with the horizontal, the height, and the distance to the target, whose solution the archer evidently knew. *Note*: The archer did not neglect air resistance, but you may have to.

The arrow's horizontal distance will be given by its horizontal velocity times time, or $L = V\cos\theta t$, giving a total flight time of $t = \frac{L}{V}\sec\theta$.

Meanwhile, the arrow's vertical distance will be given by $h = V \sin\theta t - \frac{1}{2}gt^2$. Putting in the value of t we solved for above gives the relation

$$L \tan\theta - \frac{1}{2} g \frac{L^2}{V^2} \sec^2\theta = h$$

4.13. A boy throws a ball upward at an angle of 70° with the horizontal, and it passes neatly through an open window, 32 feet above his shoulder, moving horizontally.

(a) What speed v did the ball have as it left his hand?

(b) What was the radius of curvature of its path, R, as it passed over the windowsill?

Can you find the radius of curvature of its path at any given time?

(a) We may use our results from Exercise 4.11 to solve for v when the angle is 70° and the maximum height is 32 feet. This gives a speed of

$$v = \frac{\sqrt{2gh}}{\sin\theta} = 48.3\,\text{ft/s},$$

where we have used the value $g = 32.17\,\text{ft/s}^2$.

(b) The radius of curvature of the ball at a point is the radius of a circle (on the inside of the ball's trajectory) whose tangent and curvature exactly match the ball's trajectory. Put another way, it is the radius of a circle that—if we were to look at the ball for only a split second—we would imagine the ball moving along if it were undergoing circular motion. That is, the radius of curvature is such that it obeys, at a given point

$$a_N = \frac{v_T^2}{R},$$

where a_N is the acceleration normal to the trajectory and v_T is the tangential velocity of the ball along its path. At the ball's apex, this is quite easy, as $a_N = g$ and $v_T = v_x$, so

$$R = \frac{v_x^2}{g} = \frac{(48.3\,\text{ft/s}\cos 70^\circ)^2}{32.17\,\text{ft/s}^2} \approx 8.5\,\text{ft}.$$

4.14. A small pebble is lodged in the tread of a tire of radius R. If this tire is rolling at speed V without slipping on a horizontal road, and the pebble touches the road at time $t = 0$, where coordinates x (horizontal) and y (vertical) are zero, find equations for the x and y components of

(a) the position of the pebble,

(b) its velocity $\mathbf{v}$,

(c) and its acceleration, $\mathbf{a}$

as functions of the time.

(a) Consider a point going around a circle of radius R starting at $(1, 0)$. This can be described by the familiar pair of equations

$$x(t) = R\cos\theta \qquad y(t) = R\sin\theta.$$

To change this to a clockwise rotation, we may simply replace each argument by its negative. We must then rotate the coordinates by $\pi/2$, which can be achieved by subtracting this amount from each argument. Together, these two operations send $\cos(-\theta-\pi/2)$ to $-\sin\theta$ and $\sin(-\theta-\pi/2)$ to $-\cos\theta$.

We must now offset this system by an amount R in the y-direction, and note that it moves with a constant velocity V in the x-direction (adding a term Vt to $x(t)$). Finally, using $\theta = \omega t$, we find the position functions of

$$x(t) = -R\sin\omega t + vt \qquad y(t) = R(1-\cos\omega t)$$

This is the equation of a cycloid.

(b) The velocity of the pebble is given by the time derivative of its position,

$$v_x(t) = -V\cos\omega t + V \qquad v_y(t) = V\sin\omega t$$

where we have used the fact that $wR = V$ in the above.

(c) The acceleration of the pebble is given by the time derivative of its velocity,

$$a_x(t) = \frac{V^2}{R}\sin\omega t \qquad a_y(t) = \frac{V^2}{R}\cos\omega t.$$

4.15. The driver of a car is following a truck when he suddenly notices that a stone is caught between two of the rear tires of the truck. Being a safe driver (and a physicist too), he immediately increases his distance to the truck to 22.5 meter, so as not to be hit by the stone in case it comes loose. At what speed v was the truck traveling? (Assume the stone does not bounce after hitting the ground.)

The key is to realize that the car and truck are traveling at the same speed, and so only the stone's rotational velocity enters into the situation. If the rock is launched at the point guaranteed to maximize its range, it will travel—by Exercise 4.11—a distance $R = v^2/g$, where v is the velocity of the truck. Putting in the driver's change of distance and solving for v gives $v = 14.85\,\mathrm{m\,s^{-1}}$.

4.16. A circus performer was devising a new act. He wanted to combine the Human Cannon Ball with a trapeze stunt. He had a cannon out of which he came with a muzzle velocity V. He wanted to get high enough so that he could grab the trapeze ($r = 2\,\mathrm{m}$) and then continue on up to the platform located at $h = 20\,\mathrm{m}$ above the floor, as shown in Fig. 4-3. (The trapeze should not go slack, i.e., his vertical velocity must be zero at both r and h).

(a) At what angle θ must the cannon be set?

(b) How far down the tent from the platform, x, should he put the cannon?

(c) What value of V must he choose?

(a) At the point at which the performer grab's the trapeze, he should have enough horizontal velocity to carry him up the remaining $2\,\mathrm{m}$, i.e.

$$\frac{1}{2}mv_x^2 = 2mg,$$

or $v_x = V\cos\theta = 2\sqrt{g}$. To ensure he has no vertical velocity at this point, 18 m must be the maximum height he achieves from launching out of the cannon. Using the results of Exercise 4.11, this fixes the relationship

$$18 = \frac{1}{2}\frac{V^2\sin^2\theta}{g}.$$

Substituting $2\sqrt{g}/\cos\theta$ for V in the second equation allows us to solve for θ, giving a value of $\theta = \arctan(3)$.

(b) The performer's maximum height from the cannon launch occurs at half the range (if he were not to be stopped in the horizontal direction), so

$$x = \frac{V^2\sin 2\theta}{2g} = \frac{V^2\sin\theta\cos\theta}{g} = \frac{4g}{\cos^2\theta}\frac{\sin\theta\cos\theta}{g} = 4\tan\theta = 12$$

where we have substitued in $2\sqrt{g}/\cos\theta$ for V and used $\tan\arctan 3 = 3$.

(c) The velocity of the cannon is given by $V = 2\sqrt{g}/\cos\theta = 19.8\,\mathrm{m\,s^{-1}}$.

4.17. A mortar emplacement is set 27 000 ft horizontally from the edge of a cliff that drops 350 ft down from the level of the mortar, as shown in Fig. 4-4 It is desired to shell objects concealed on the ground behind the cliff. What is the smallest horizontal distance d from the cliff face that shells can reach if fired at a muzzle speed of 1000 ft/s?

The range of our mortar *on level ground* is given by

$$R = 27000 = \frac{v^2\sin 2\theta}{g} = \frac{10^6\sin 2\theta}{g}.$$

Solving for the angle of launch gives us $\theta \approx 30.15°$, but this is *not* the optimal angle to achieve a minimum shelling distance. The range as a function of θ is symmetric about 45°—and a larger angle will give us a steeper drop after the cliff's edge. Therefore, $\theta \approx 59.85°$, which can be used to find the horizontal position as a function of time, $x(t) \approx 502.26t$. The smallest distance the mortar can shell from the cliff face is given by the this equation when t represents the change in time from the projectile's passing the cliff edge to hitting the ground. That is, the sought after t satisifes

$$-v\sin\theta t - \frac{1}{2}gt^2 = -350,$$

where the first term is negative because, at the point at which the projectile passes the cliff's edge, its vertical velocity is the negative of its starting vertical velocity. Solving for t gives approximately 0.4 seconds, yielding a minimum horizontal shelling distance of 201 feet.

4.18. A Caltech freshman, inexperienced with suburban traffic officers, has just received a ticket for speeding. Thereafter, when he comes upon one of the "Speedometer Test" sections on a level stretch of highway, he decides to check his speedometer reading. As he passes the "0" start of the marked section, he presses on his accelerator and for the entire period of the test he holds his car at constant acceleration He notices that he passes the 010 mile post 16 s after starting the test, and 8.0 s later he passes the 0.20 mile post.

(a) What speed v should his speedometer have read at the 0.20 mile post?

(b) What was his acceleration a?

4.19. On the long horizontal test track at Edwards AFB, both rocket and jet motors can be tested. On a certain day, a rocket motor, started from rest, accelerated constantly until its fuel was exhausted, after which it ran at constant speed. It was observed that this exhaustion of rocket fuel took place as the rocket passed the midpoint of the measured test distance Then a jet motor was started from rest down the track, with a constant acceleration for the entire distance It was observed that both rocket and jet motors covered the test distance in exactly the same time. What was the ratio of the acceleration a_J of the jet motor to that of the rocket motor, a_R?

5 Newton's Laws

5.1. Two blocks of mass $m_1 = 1\,\mathrm{kg}$, $m_2 = 2\,\mathrm{kg}$ on a horizontal surface, connected by a string, are being pulled by another string which is attached to a mass $m_3 = 2\,\mathrm{kg}$ hanging ove ra pulley, as shown in Fig. 5-1. Neglect friction and the masses of the pulley and the strings.

(a) Sketch free-body diagrams for all masses, showing the forces acting.

(b) Find the acceleration a of the masses.

(c) Find the tensions T_1 and T_2 in the strings.

(a) The hanging mass has two forces acting upon it: gravity in the downward direction and T_2 in the upward one. The mass m_2 has four forces: gravity acting downward, the normal force acting upward, T_2 acting to the left, and T_1 acting to the right. Finally, mass m_3 has three forces: gravity acting downward, the normal force acting upward, and T_1 acting to the left.

(b) We can find the acceleration by noting that it is the same throughout the system. From m_3, we see $m_3 a = m_3 g - T_2$, from m_2, we see $m_2 a = T_2 - T_1$, and from m_1 we see $m_1 a = T_1$. Putting everything in terms of the masses of our system gives

$$m_3 a = m_3 g - (m_2 a + T_1) = m_3 g - m_2 a - m_1 a.$$

Solving for a yields

$$a = g\frac{m_3}{m_1 + m_2 + m_3} = \frac{2}{5}g.$$

(c) Substituting in our above answer for a gives us

$$\begin{aligned} T_1 &= \frac{m_1 m_3}{m_1 + m_2 + m_3} = \frac{2}{5}\,\mathrm{N} \\ T_2 &= \frac{m_2 m_3 + m_1 m_3}{m_1 + m_2 + m_3} = \frac{6}{5}\,\mathrm{N} \end{aligned}$$

5.2. A mass m (kg) hangs on a cord suspended from an elevator which is descending with an acceleration $0.1g$. What is the tension T in the cord in newtons?

We are given the acceleration, and we know that

$$ma = mg - T.$$

Solving for T yields $T = \frac{9}{10}mg = 8.82m\,\mathrm{N}$.

5.3. Two objects of mass $m = 1\,\mathrm{kg}$ each, connected by a taut string of length $L = 2\,\mathrm{m}$, move in a circular orbit with constant speed $V = 5\,\mathrm{m\,s^{-1}}$ about their common center C in a zero-g environment, as shwon in Fig. 5-2. What is the tension T in the string in newtons?

The acceleration felt by the masses is in the inward radial direction and it is given by

$$a = \frac{v^2}{r}.$$

The only force that can provide such a centripetal acceleration is the tension in the string, and so we must have

$$m\frac{v^2}{r} = T,$$

or $T = 25\,\mathrm{N}$.

5.4. Referring to Fig. 5-3: What horizontal force F must be constantly applied to M so that M_1 and M_2 do not move relative to M? Neglect friction.

When the force is applied, it will accelerate the system as whole, and so

$$F = (M + M_1 + M_2)a.$$

Meanwhile, both M_1 and M_2 can undergo acceleration in addition to that from the force. Specifically,

$$\begin{aligned} M_1(a + a_{M_2}) &= T \\ M_2 a_{M_2} &= M_2 g - T \\ M_2 a &= F_N. \end{aligned}$$

The condition that our subsystem does not move relative to M is a_{M_2} (the additional acceleration) is zero, or $M_2 g = T$. Putting this into the first of the above three equations, substituting in a, and solving for F yields

$$F = g\frac{M_2}{M_1}(M + M_1 + M_2).$$

5.5. Reffering to Fig. 5-4: What horizontal force F must be constantly applied to $M = 21\,\mathrm{kg}$ so that $m_1 = 5\,\mathrm{kg}$ does not move relative to $m_2 = 4\,\mathrm{kg}$? Neglect friction.

In order for m_1 and m_2 to not move relative to each other, the system as a whole must undergo uniform acceleration. From this fact, we can write

$$\begin{aligned} F &= (M + m_1 + m_2)a \\ T &= m_1 a \\ T_x &= m_2 a \\ T_y - m_2 g &= 0. \end{aligned}$$

We may use the last two equations to eliminate T from the second one, finding

$$\sqrt{m_2^2 a^2 + m_2^2 g^2} = m_1 a,$$

or

$$a = \frac{m_2 g}{\sqrt{m_1^2 - m_2^2}}.$$

Feeding this into our first equations gives us a force of

$$F = (M + m_1 + m_2)\frac{m_2 g}{\sqrt{m_1^2 - m_2^2}} = 392\,\text{N}.$$

5.6. In the system shown in Fig. 5-5, M_1 slides without friction on the inclined plane. $\theta = 30^\circ$, $M_1 = 400\,\text{g}$, $M_2 = 200\,\text{g}$. Find the acceleration $\mathbf{a}$ of M_2 and the tension T in the cords.

Drawing a force diagram for the entire system, we find that

$$\begin{aligned} M_1 a_1 &= T_2 - M_1 g \sin\theta \\ T_1 &= 2T_2 \\ M_2 a_2 &= M_2 g - T_2 \end{aligned}$$

where positive a_1 and a_2 corresponds to the situation where the hanging mass is falling. Here, T_1 corresponds to the rightmost cord and T_2 to the leftmost.

These relations leave us with three unknowns, T_1, a_1, and a_2. To eliminate one more, observe that when M_2 falls a distance Δx, the pulley attached to M_1 rises by the same amount: this shortens the cord wrapped around the pulley, causing the mass M_1 to rise by an additional Δx. That is,

$$2\Delta x_{M_2} = \Delta x_{M_1}.$$

Differentiating both sides twice gives us the same relationship between their respective accelerations, or $2a_{M_2} = a_{M_1}$.

Solving for a_2 yields

$$a_2 = \frac{1}{1 + 4\frac{M_1}{M_2}}\Big(g - 2\frac{M_1}{M_2} g \sin\theta\Big),$$

which comes out to be $-g/9$, or an acceleration of $g/9$ upward. Substituting this value into the equation relating M_2's acceleration to its force yields a tension of

$$T_2 = M_2(g - a_2) = \frac{10}{9} M_2 g = 2.18\,\text{N}.$$

5.7. A simple crane is made of two parts, "A" with mass M_A, length D, height H, and distance $D/2$ between wheels of radius r; and part "B," a uniform rod or boom of length L and mass M_B. The crane is shown assembled in Fig. 5-6, with the pivot point P at midpoint of top of A. The center of gravity of A is midway between the wheels.

(a) With the rod or boom B set at angle θ with teh horizontal, what is the maximum mass $M_{\max}$ that the crane can lift without tipping over?

(b) If there is a mass $M' = (4/5)M_{\max}$ at the end of the rope, what is the minimum time t necessary to raise this load M' a distance $(L\sin\theta)$ from the ground? (The angle θ remains fixed, and the mass of the rope may be neglected.)

(a) The crane will tip over if its center of mass is in front of its wheels. That is,

$$\frac{M_A \cdot \frac{D}{2} + M_B \cdot (\frac{D}{2} + \frac{L}{2}\cos\theta) + M' \cdot (\frac{D}{2} + L\cos\theta)}{M_A + M_B + M'} \leqslant \frac{3}{4}D.$$

Solving for M' and taking the edge case where the center of mass is exactly on the front wheel gives a maximum mass of

$$M_{\text{max}} = \frac{(M_A + M_B)D - 2M_B L\cos\theta}{4L\cos\theta - D}.$$

(b) The maximum force our crane can exert when lifting an object must be equal to the force generated by the maximum weight (due to action and reaction),

$$T_{\text{max}} = M_{\text{max}}g.$$

The largest acceleration M' can undergo, then, is

$$M'a = \frac{4}{5}M_{\text{max}}a = T_{\text{max}} - \frac{4}{5}M_{\text{max}}g = \frac{1}{5}M_{\text{max}}g,$$

or $a = g/4$. Assuming M' starts from rest, the total time it takes to raise M' to $L\sin\theta$ is given by solving $\Delta x = \frac{1}{2}at^2$ for t and substituting in our value for a, i.e.

$$t = \sqrt{\frac{2L\sin\theta}{a}} = \sqrt{\frac{8L\sin\theta}{g}}.$$

5.8. An early arrangement for measuring the acceleration of gravity, called Atwood's Machine, is shown in Fig. 5-7. The pulley P and cord C have negligible mass and friction. The system is balanced with equal masses M on each side as shown (solid line), and then a small rider m is added to one side. The combined masses accelerate through a certain distance h, the rider is caught on a ring and the two equal masses then move on with constant speed, v. Find the value of g that corresponds to the measured valeus of m, M, h, and v.

We can find the acceleration of the system by drawing force diagrams for each weight, coming up with the relations

$$Ma = T - Mg$$
$$(M + m)a = (M + m)g - T.$$

Solving for g gives us

$$g = \frac{m + 2M}{m}a.$$

Now, as the system starts from rest, the final velocity can be related to the acceleration by $v = a\Delta t$, and the total distance (up to striking the ring) can be related to the acceleration by $\frac{1}{2}a\Delta t^2 = h$. Removing Δt from the second equations reveals

$$a = \frac{v^2}{2h},$$

and so

$$g = \frac{m + 2M}{2mh}v^2$$

5.9. An elevator of mass M_2 has hanging from its ceiling a mass M_1, as shown in Fig. 5-8. The elevator is being accelerated upward by a constant force F. (F is greater than $(M_1 + M_2)/g$.) The mass M_1 is initially a distance s above the elevator floor.

(a) Find the acceleration a_0 of the elevator.

(b) What is the tension T in the string connecting the mass M_1 to the elevator?

(c) If the string suddenly breaks, what is the acceleration a of the elevator immediately after, and wha tis the acceleration a' of mass M_1?

(d) How much time t does it take for M_1 to hit the bottom of the elevator?

(a) The elevator total acceleration can be found by solving

$$(M_1 + M_2)a_0 = F - (M_1 + M_2)g,$$

giving a value of

$$a_0 = \frac{F - (M_1 + M_2)g}{M_1 + M_2)}.$$

(b) Because M_1 is undergoing the same acceleration as the elevator, we have

$$M_1 a_0 = T - M_1 g,$$

or

$$T = \frac{M_1}{M_1 + M_2} F.$$

(c) If the string breaks, the elevator will continue upward with an acceleration of

$$a = \frac{F}{M_2} - g,$$

while M_1 will only feel the force of gravity, giving an acceleration of

$$a' = g.$$

(d) The distance the elevator will have moved upward in a time t is given by

$$\frac{1}{2}\left(\frac{F}{M_2} - g\right)t^2 = \Delta s_2,$$

while the distance the hanging mass will have moved (relative to the elevator) is given by

$$\frac{1}{2}gt^2 = \Delta s_1.$$

The two will meet when $\Delta s_1 + \Delta s_2 = s$, so

$$\begin{aligned} s &= \Delta s_1 + \Delta s_2 \\ &= \frac{1}{2}gt^2 + \frac{1}{2}\left(\frac{F}{M_2} - g\right)t^2 \\ &= \frac{1}{2}\frac{F}{M_2}t^2. \end{aligned}$$

Solving for t gives a value of

$$t = \sqrt{\frac{2M_2 s}{F}}.$$

5.10. Given the system shown in FIg. 5-9, consider all surfaces frictionless. If $m = 150\,\mathrm{g}$ is released when it is $d = 1\,\mathrm{m}$ above the base of $M = 1650\,\mathrm{g}$, how long after release, Δt, will m strike the base of M?

5.11. None of the identical gondolas on teh Martian canal Rimini is quite able to support the load of both Paolo and Francesca, two affectionate marsupials who refuse to go in separate boats. The enterprising gondolier, Giuseppe, collects their fare by rigging them up from the mast as shown in Fig. 5-10, using the massless ropes and massless, frictionless pulleys characteristic of Martian construction. Giuseppe ferries them across before they hit either the mast or the deck. Assuming Paulo's mass is 90 kg and Francesca's is 60 kg, how much load W does Giuseppe save?

Hint: Remember that the tension in a massless cord htat passes over a massless, frictionless pulley is the same on both sides of the pulley.

5.12. A painter working from a "bosun's" chair is hung down the side of a tall building, as shown in Fig. 5-11. Wishing to move in a hurry, the 180 lb painter pulls down on the fall rope so hard that he presses against the chair with a force of only 100 lb. The chair itself weighs 30.0 lb.

(a) What is the acceleration **a** of the painter and the chair?

(b) What is the total force F supported by the pulley?

5.13. A space traveler about to leave for the moon has a spring balance and a 1.0 kg mass A, which when hung on teh balance on the Earth gives the reading of 9.8 N. Arriving at the moon at a place where the acceleration of gravity is not known exactly but has a value of about one sixth the acceleration of gravity at the Earth's surface, he picks up a stone B which gives a reading of 9.8 N when weighed on teh spring balance. He then hangs A and B ove ra pulley as shown in Fig. 5-12 and observes that B falls with an acceleration of $1.2\,\mathrm{m\,s^{-2}}$. What is the mass m_B of stone B?

5.14. A mass usspended from a spring hangs motionless, and is then given an upward blow such that it moves initially at unit speed. If the mass and spring constant are such that the equation of motion is $\ddot{x} = -x$, find the maximum height $x_{\max}$ attained by numerical integration of the equation of motion.

5.15. A particle of mass m moves along a straight line. Its motion is resisted by a force proportional to its velocity, $F = -kv$. It starts with speed $v = v_0$ at $x = 0$ and $t = 0$.

(a) Find x as a function of t by numerical integration.

(b) Find the time $t_{1/2}$ required to lose half its speed, and the maximum distance $x_{\max}$ attained.

Notes:

(a) Adjust he scales of x and t so that the equation of motion has simple numerical coefficients.

(b) Invent a scheme to attain good accuracy with a relatively coarse interval for Δt.

(c) Use dimensional analysis to deduce how $t_{1/2}$ and $x_{\max}$ shoudl depend upon v_0, k, and m, and solve for the actual motion only for a single convenient value of v_0, say $v_0 = 1.00$ (in the modified x and t units).

5.16. A certain charged particle moves in an electric and magnetic field according to the equations,

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = -2v_y,$$
$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = 1 + 2v_x.$$

At $t = 0$ the particle starts at $x = 0$, $y = 0$ with velocity $v_x = 1.00$, $v_y = 0$. Determine th enature of the motion by numerical integration.

5.17. A shell is fired with a muzzle velocity $v = 1000\,\mathrm{ft/s}$ at an angle of 45° with teh horizontal. Its motion is resisted by a force proporitonal to the cube of its velocity ($F = -kv^3$). The coefficient k is such that the resisting force is equal to twie the weight of the shell when $v = 1000\,\mathrm{ft/s}$. Find the approximate maximum height attained $h_{\max}$, and the horizontal range R by numerical integration, and compare these with the values expected in the absence of resistance.

6 Conservation of Momentum

6.1. When two bodies move along a line, there is a special system of coordinates in which the momentum of one body is equal and opposite to that of the other. That is, the total momentum of the two bodies is zero. This frame of reference is called the center-of-mass system (abbreviated CM). If the bodies have masses m_1 and m_2 and are moving at speeds $v_!$ and v_2, show that the CM system is moving at speed

$$v_{\mathrm{CM}} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}.$$

Given this definition, we may find the speed of such a coordinate system by requiring

$$m_1(v_1 - v_{\mathrm{CM}}) + m_2(v_2 - v_{\mathrm{CM}}) = 0.$$

Solving for v_{CM} gives us the speed

$$v_{\mathrm{CM}} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}.$$

6.2. Generalize Ex. 6.1 for any number of masses moving along a line, i.e., show that the speed of the coordinate system, in which the total momentum is zero, is given by

$$v_{\mathrm{CM}} = \frac{\sum m_i v_i}{\sum m_i}.$$

Our requirement now becomes

$$\sum m_i(v_i - v_{\mathrm{CM}}) = 0.$$

This sum can be split into two and v_{CM} can be brought outside (being constant), to find

$$\sum m_i v_i - v_{\mathrm{CM}} \sum m_i = 0.$$

Solving for v_{CM} gives us the speed

$$v_{\mathrm{CM}} = \frac{\sum m_i v_i}{\sum m_i}.$$

6.3. If T is the total kinetic energy of the two masses in Ex. 6.1, and T_{CM} is their total kinetic energy in the CM system, show that

$$T = T_{\mathrm{CM}} + \left(\frac{m_1 + m_2}{2}\right) v_{\mathrm{CM}}^2.$$

If v_1' and v_2' are the velocities of the two masses in the center of mass coordinate system, then

$$T_{\mathrm{CM}} = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2.$$

Their velocities in the 'stationary' coordinate system are simply given by $v_1 = v_1' + v_{\mathrm{CM}}$ and $v_2 = v_2' + v_{\mathrm{CM}}$, giving a Their velocities in the 'stationary' coordinate system are simply given by $v_1 = v_1' + v_{\text{CM}}$ and $v_2 = v_2' + v_{\text{CM}}$, giving a total kinetic energy of

$$\begin{aligned}
T &= \frac{1}{2}m_1(v_1' + v_{\text{CM}})^2 + \frac{1}{2}m_2(v_2' + v_{\text{CM}})^2 \\
&= \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2 + m_1 v_1' v_{\text{CM}} + m_2 v_2' v_{\text{CM}} + \frac{1}{2}m_1 v_{\text{CM}}^2 + \frac{1}{2}m_2 v_{\text{CM}}^2 \\
&= T_{\text{CM}} + v_{\text{CM}}(m_1 v_1' + m_2 v_2') + \left(\frac{m_1 + m_2}{2}\right)v_{\text{CM}}^2 \\
&= T_{\text{CM}} + \left(\frac{m_1 + m_2}{2}\right)v_{\text{CM}}^2,
\end{aligned}$$

where the vanishing of the second term in the second to last line occurs by virtue of $m_1 v_1' = -m_2 v_2'$ (we are in the center of mass coordinates).

6.4. Generalize the result of Ex. 6.3 to any number of masses. Show that

$$T = T_{\text{CM}} + \frac{\sum m_i}{2} v_{\text{CM}}^2.$$

Using similar notation to above, the kinetic energy in the center of mass system is given by

$$T_{\text{CM}} = \sum \frac{1}{2} m_i v_i'^2$$

while the kinetic energy of the system in the 'stationary' frame is given by

$$\begin{aligned}
T &= \sum \frac{1}{2} m_i (v_i' + v_{\text{CM}})^2 \\
&= \sum \frac{1}{2} m_i v_i'^2 + \sum m_i v_i' v_{\text{CM}} + \sum \frac{1}{2} m_i v_{\text{CM}}^2 \\
&= T_{\text{CM}} + v_{\text{CM}} \sum m_i v_i + \frac{\sum m_i}{2} v_{\text{CM}}^2 \\
&= T_{\text{CM}} + \frac{\sum m_i}{2} v_{\text{CM}}^2.
\end{aligned}$$

6.5. Two gliders with masses m_1 and m_2 are free to move on a horizontal air track. m_2 is stationary and m_1 collides with it perfectly elastically. They rebound with equal and opposite velocities. What is the ratio m_2/m_1 of their masses?

Because this collision is perfectly elastic, both momentum *and energy* are conserved. That is,

$$\begin{aligned}
m_1 v_{1_i} + m_2 v_{2_i} &= m_1 v_{1_f} + m_2 v_{2_f}, \\
\frac{1}{2} m_1 v_{1_i}^2 + \frac{1}{2} m_2 v_{2_i}^2 &= \frac{1}{2} m_1 v_{1_f}^2 + \frac{1}{2} m_2 v_{2_f}^2.
\end{aligned}$$

We are given that $v_{2_i} = 0$ and $v_{2_f} = -v_{1_f}$, and so the above requirements simplify to

$$\begin{aligned}
m_1 v_{1_i} &= (m_1 - m_2) v_{1_f}, \\
\frac{1}{2} m_1 v_{1_i}^2 &= \frac{1}{2}(m_1 + m_2) v_{1_f}^2.
\end{aligned}$$

Removing the factor of $\frac{1}{2}$ from the second relation, squaring the first, and dividing them both gives

$$m_1 = \frac{(m_1 - m_2)^2}{m_1 + m_2} = \frac{m_1^2 - 2m_1 m_2 + m_2^2}{m_1 + m_2}.$$

By multiplying both sides by $m_1 + m_2$, canceling like terms, and dividing by m_2, we get

$$\frac{m_2}{m_1} = 3.$$

6.6. A neutron having a kinetic energy E collides head-on with a stationary nucleus of C^{12} and rebounds perfectly elastically in the direction from which it came. What is its final kinetic energy E'?

This is most easily handled if we derive the relationship between our particles' initial and final velocities in a perfectly elastic collision. We have the two requirements

$$\begin{aligned} m_1 v_{1_i} + m_2 v_{2_i} &= m_1 v_{1_f} + m_2 v_{2_f} \\ \frac{1}{2} m_1 v_{1_i}^2 + \frac{1}{2} m_2 v_{2_i}^2 &= \frac{1}{2} m_1 v_{1_f}^2 + \frac{1}{2} m_2 v_{2_f}^2. \end{aligned}$$

Removing the common factor of 1/2 from the second line and collecting velocities by particle, we find

$$\begin{aligned} m_1 (v_{1_i} - v_{1_f}) &= m_2 (v_{2_f} - v_{2_i}) \\ m_1 (v_{1_i}^2 - v_{1_f}^2) &= m_2 (v_{2_f}^2 - v_{2_i}^2). \end{aligned}$$

Dividing the second equation by the first gives a particularly simple result,

$$v_{1_i} + v_{1_f} = v_{2_i} + v_{2_f},$$

or $v_{1_i} - v_{2_i} = -v_{1_f} + v_{2_f}$. This, combined with the equation for conservation of momentum, allows us to solve for the final velocities in terms of the initial ones by inverting the system

$$\begin{bmatrix} 1 & -1 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} v_{1_f} \\ v_{2_f} \end{bmatrix} = \begin{bmatrix} -v_{1_i} + v_{2_i} \\ m_1 v_{1_i} + m_2 v_{2_i} \end{bmatrix}.$$

This yields

$$\begin{aligned} v_{1_f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1_i} + \frac{2m_2}{m_2 + m_1} v_{2_i} \\ v_{2_f} &= \frac{2m_1}{m_1 + m_2} v_{1_i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2_i}. \end{aligned}$$

For the given problem, we know that $v_{c_i} = 0\,\mathrm{m\,s^{-1}}$, and so

$$v_{n_f} = \frac{m_n - m_c}{m_n + m_c} v_{n_i}.$$

If we square both sides and multiply them by $m_n/2$, we find

$$E' = \left(\frac{m_n - m_c}{m_n + m_c}\right)^2 E$$

. Working in units of amu, where $m_n = 1\,\mathrm{amu}$ and $m_c = 12\,\mathrm{amu}$ gives

$$E' = 0.716E.$$

6.7. A projectile of mass $m = 10\,\mathrm{kg}$ is shot vertically upward from the earth with an initial velocity $v_p = 500\,\mathrm{m\,s^{-1}}$.

(a) Calculate the recoil velocity of the earth v_E.

(b) Calculate the ratio of the kinetic energy of the earth T_E to that of the projectile T_p at the moment of their separation.

(c) Sketch qualitatively the velocity and kinetic energy of the projectile and of the earth versus time.

Neglect air resistance and the orbital motion of the earth.

(a) By conservation of momentum

$$m_p v_p + m_E v_E = 0,$$

or

$$v_E = -\frac{m_p}{m_E} v_p.$$

Using $m_E = 5.98 \times 10^{24}\,\mathrm{kg}$ gives a rebound velocity of

$$v_E = -8.36 \times 10^{-22}\,\mathrm{m\,s^{-1}}.$$

(b) The ratio of T_E to T_p is

$$\frac{T_E}{T_p} = \frac{\frac{p_E^2}{2m_E}}{\frac{p_p^2}{2m_p}} = \frac{m_p}{m_e} = 1.67 \times 10^{-24},$$

where the factors of momentum have gone away by virtue of $p_E = -p_p$, and so $p_E^2 = p_p^2$.

(c) Both bodies undergo projectile motion, though the earth moves *much* less than the projectile.

6.8. A particle of mass $m = 1.0\,\mathrm{kg}$, traveling at a speed $V = 10\,\mathrm{m\,s^{-1}}$, strikes a particle at rest of mass $M = 4.0\,\mathrm{kg}$ and rebounds in the direction from which it came, with a speed V_F. If an amount of heat $h = 20\,\mathrm{J}$ is produced in the collision, what is V_F? (Define all introduced quantities and state clearly from what physical laws your initial equations are derived.)

By conservation of momentum,

$$mV = mV_F + Mu.$$

By conservation of energy,

$$\frac{1}{2}mV^2 = \frac{1}{2}mV_F^2 + \frac{1}{2}Mu^2 + 20,$$

where the last term is the energy converted into heat. We know all variables except V_F and u, so let us eliminate u from the second equation (removing the factor of 1/2 as well),

$$\begin{aligned} mV^2 &= mV_F^2 + M\left(\frac{mV - mV_F}{M}\right)^2 + 40 \\ &= mV_F^2 + \frac{m^2}{M}V^2 - 2\frac{m^2}{M}VV_F + \frac{m^2}{M}V_F^2 + 40. \end{aligned}$$

This can be arranged as a quadratic equation in V_F like so

$$\left(m + \frac{m^2}{M}\right)V_F^2 + \left(-2\frac{m^2}{M}V\right)V_F + \left(\frac{m^2}{M}V^2 + 40 - mV^2\right) = 0,$$

which has the solutions

$$V_F = \frac{2\frac{m^2}{M}V \pm \sqrt{4\frac{m^4}{M^2}V^2 - 4\left(m + \frac{m^2}{M}\right)\left(\frac{m^2}{M}V^2 + 40 - mV^2\right)}}{2\left(m + \frac{m^2}{M}\right)}.$$

Substituting in all the necessary values and recognizing that V_F should have the opposite sign as V gives $V_F = -3.66\,\mathrm{m\,s^{-1}}$.

6.9. A machine gun mounted on the north end of a 10 000 kg, 5 m long platform, free to move on a horizontal air-bearing, fires bullets into a thick target mounted on the south end of the platform. The gun fires 10 bullets of mass 100 g each every second at a muzzle velocity of $500\,\mathrm{m\,s^{-1}}$. Does the platform move? If so, in what direction and at what speed v?

Every time the gun fires a bullet, the platform begins moving in the opposite direction by conservation of momentum. When the bullet hits the target, this momentum is reabsorbed and the platform stops—but it has *still moved a distance*. If we can find the distance it moves over one second we can find its velocity.

The momentum of each bullet is

$$p_b = 0.1\,\mathrm{kg} \times 500\,\mathrm{m\,s^{-1}} = 50\,\mathrm{kg\,m\,s^{-1}}..$$

This is equal in magnitude to the platform's momentum immediately after the bullet is fired, and so the platforms velocity is given by

$$v_p = \frac{p_b}{10\,000\,\mathrm{kg}} = 5 \times 10^{-3}\,\mathrm{m\,s^{-1}}.$$

The time between the firing of a bullet and its impact is its distance divided by its velocity, or $1 \times 10^{-2}\,\mathrm{s}$. So, for a given bullet, the platform moves a total of $5 \times 10^{-5}\,\mathrm{m}$. This occurs 10 times per second for a final velocity of $5 \times 10^{-4}\,\mathrm{m\,s^{-1}}$ to the north.

6.10. A mass m_1 connected by a cable over a pulley to a container of water, which initially has a mass $m_2(t = 0) = m_0$, as shown in Fig. 6-1. The system is then released and $m2$ (with help of an internal pump) ejects water in the downward direction at a constant rate $dm/dt = r_0$ with a velocity v_0 relative to the container. Find the acceleration $\mathbf{a}$ of m_1, as a function of time. Neglect the masses of cable and pulley.

The dynamics of the first mass are governed by

$$m_1 a = m_1 g - T,$$

where a positive acceleration is counted as downward. The second mass, in contrast, is varying in time as $m_2(t) = m_0 - r_0 t$. The water ejected out the bottom of the container exerts an upward force equal to its change in momentum over time, which is

$$\frac{dp_2}{dt} = \frac{dm_2}{dt}v_0 - m_2\frac{dv_0}{dt} = r_0 v_0$$

and so we have, for the second mass,

$$(m_0 - r_0 t)a = T + r_0 v_0 - (m_0 - r_0 t)g.$$

Removing T from the first equation and solving for a gives an acceleration of

$$a = \frac{(m_1 - m_0)g + r_0(gt + v_0)}{m_1 + m_0 - r_0 t}.$$

This, of course, is valid only for the time during which the container ejects water, or $0 \leqslant t < \frac{m_0}{r}$.

6.11. A toboggan slides down an essentially frictionless, snow covered slope, scooping up snow along the path. if the slope is 30° and the toboggan picks up 0.50 kg of snow per meter of travel, calculate its acceleration a at an instant when its speed is $4.0\,\mathrm{m\,s^{-1}}$ and its mass (including content) is 9.0 kg.

There is only one force acting on our toboggan: gravity. Its magnitude in the downward direction of the slope is $mg\sin\theta$, where θ is the slope's angle. This must be equal to the toboggan's rate of change of momentum, or

$$\frac{dp}{dt} = \frac{dm}{dt}v + ma = mg\sin\theta.$$

Solving for a gives us a changing acceleration of

$$a = g\sin\theta - \frac{dm}{dt}\frac{v}{m}.$$

This makes sense: if our toboggan is gaining mass, it should accelerate more slowly down the hill. To work with the information we have (which is the change of mass per *distance*, not time) we can use $dm/dt = dm/dx \cdot dx/dt$, or

$$a = g\sin\theta - \frac{dm}{dx} fracv^2 m.$$

Putting in the values given in the problem statement gives us an acceleration of $4.01\,\mathrm{m\,s^{-2}}$ at the instant the sled's speed is $4.0\,\mathrm{m\,s^{-1}}$.

6.12. The end of a chain, of mass per unit length μ, at rest on a table top at $t = 0$, is lifted vertically at a constant speed v, as shown in Fig. 6-2. Evaluate the upward lifting force F as a function of time.

We can imagine our chain as a vertical rod that can 'grow' upon lifting it. (Really, this is just the part of the chain that has left the table.) This 'growth' rate is given by

$$\frac{dm}{dt} = \mu v.$$

There are only two forces acting on the chain: that which is lifting it, and gravity. The gravitational force is due to the part of the chain in the air, or $F_g = \mu v t g$. Newton's second law then says

$$\frac{dp}{dt} = \frac{dm}{dt}v + m\frac{dv}{dt} = F - \mu v t g.$$

Recognizing that the velocity is constant allows us to solve for the upward force,

$$F = \mu v(v + gt).$$

6.13. The speed of a rifle bullet may be measured by means of a ballistic pendulum: The bullet, of known mass m and unknown speed V, embeds itself in a stationary wooden block of mass M, suspended by a pendulum of length L, as shown in Fig. 6-3. This sets the block to swinging. The amplitude x of swing may be measured and, using conservation of energy, the velocity of the block immediately after impact may be found. Derive an expression for the speed of the bullet in terms of m, M, L, and x.

By conservation of momentum, the block and bullet obey

$$mV = (M+m)v,$$

where v is the speed of the pendulum (with the embedded bullet) after the impact. By conservation of energy, the pendulum's kinetic energy at this point must equal its potential energy at the height of its swing. We can find the height by focusing on the angle the pendulum makes with the vertical, having the relation

$$\cos\theta = \frac{L-h}{L},$$

or $h = L(1-\cos\theta)$. This comes from noticing the right triangle hidden in the accompanying figure, with a hypotenuse of L, a long side of $L-h$, and a short side of x. To put this angle in terms of x, we will invoke the small angle approximation of $\cos\theta$, so $\cos\theta \approx 1-\frac{\theta^2}{2}$. Then, using $L\theta \approx x$, we can simplify this to $\cos\theta \approx 1-\frac{x^2}{2L^2}$. Using all this, and relating the kinetic and potential energy, gives

$$\frac{1}{2}\frac{m^2}{M+m}V^2 = (M+m)gL\frac{x^2}{2L^2}.$$

We may solve this for V to find

$$V = \Big(\frac{M+m}{m}\Big)x\sqrt{\frac{g}{L}}.$$

6.14. Two gliders A and A' are connected rigidly together and have a combined mass M and are separated by a distance $2L$. Another glider B of mass m, length L, is constrained to move between A and A', as shown in Fig. 6-4. All gliders move on a very long linear air track without friction. All collisions between (A, A') and B are perfectly elastic. Originally the whole system is at rest and glider B is in contact with glider A. A cap between A and B then is exploded, giving a total kinetic energy T to the system.

(a) Show the *qualitative* features of B's motion, i.e., position x on the track, velocity v with respect to the track, by sketching x and v as functions of time. Use the *same* time scale for both sketches.

(b) Calculate the period τ_0 in terms of T, L, m, and M.

Hint: The relative velocity of B with respect to (A, A') is

$$\mathbf{v}_{\text{rel}} = \mathbf{v}_B - \mathbf{v}_{(A,A')}.$$

6.15. Two equally massive gliders, moving on a level air track at equal and opposite, velocities, $\mathbf{v}$ and $-\mathbf{v}$, collide almost elastically, and rebound with slightly smaller speeds. They lose a fraction $f \ll 1$ of their kinetic energy in the collision. If these same gliders collide with one of them initially at rest, with what speed will the second glider move after the collision? (This small residual speed Δv may easily be measured in terms of the final speed v of the originally stationary glider, and thus the elasticity of the spring bumpers may be determined.) *Note*: If $x \ll 1$, $\sqrt{1+x} \approx 1-\frac{x}{2}$.

6.16. A rocket of initial mass $m = M_0$ ejects its burnt fuel at a constant rate $dm/dt = -r_0$ and at a velocity V_0 (relative to the rocket).

(a) Calculate the initial acceleration a of the rocket (neglect gravity).

(b) If $V_0 = 2.0\,\mathrm{km\,s^{-1}}$, at what rate r_0 must fuel be ejected to develop $10\times10^5\,\mathrm{kg-wt}$ of thrust?

(c) Write a differential equation which connects the speed v of the rocket with its residual mass $m = M$, and solve the equation, if you can.

(a) Since we are neglecting gravity, there are no external forces acting on the rocket, and

$$\frac{dp}{dt} = \frac{dm}{dt}v + m\frac{dv}{dt} = 0,$$

or $a = -\frac{dm}{dt}\frac{v}{m}$. Putting in the initial values for our rocket, we find

$$a = r_0 \frac{V_0}{M_0}.$$

(b) Multiplying the previously derived equation by M_0 and dividing the left side by 9.8 (to convert from kg-wt to N), we find

$$r_0 = \frac{10 \times 10^5\,\mathrm{kg-wt}}{2000\,\mathrm{m\,s^{-1}} 9.8\,\mathrm{kg-wt/N}} = 490\,\mathrm{kg\,s^{-1}}.$$

(c) Looking at the initial equation from part (a), we have

$$\frac{dv}{dt} = -\frac{V_0}{m}\frac{dm}{dt}.$$

Integrating both sides with respect to t gives

$$v = -V_0 \int_{M_0}^{M} \frac{dm}{m} = V_0 \ln \frac{M_0}{M}.$$

6.17. An earth satellite of mass 10 kg and average cross-sectional area $0.50\,\mathrm{m}^2$ is moving in a circular orbit at 200 km altitude where the molecular mean free paths are many meters and the air density is about $1.6 \times 10^{-10}\,\mathrm{kg\,m^{-3}}$. Under the crude assumption that the molecular impacts with the satellite are effectively inelastic (but that the molecules do not literally stick tot he satellite but drop away from it at low relative velocity),

(a) Calculate the retarding force F_R that the satellite would experience due to air friction.

(b) How should such a frictional force vary with the satellite's velocity v? Would the satellite's speed decrease as a result of the net force on it? (Check the speed of a circular satellite orbit vs. height.)

7 Vectors

Generalize Exs. 6.1 through 6.4 to three dimensional motion using vector notation.

7.1. If two bodies have masses m_1 and m_2 and are moving at velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, show that the CM system is moving at velocity

$$\mathbf{v}_{\mathrm{cm}} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}.$$

If we transform to the center of mass system, we have, by definition

$$m_1(\mathbf{v}_1 - \mathbf{v}_{\mathrm{cm}}) + m_2(\mathbf{v}_2 - \mathbf{v}_{\mathrm{cm}}) = 0.$$

Rearranging this gives

$$\mathbf{v}_{\rm cm} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}.$$

7.2. Show that for N bodies of masses m_i and velocities $\mathbf{v}_i$ the velocity of the coordinate system, in which the total momentum is zero, is given by

$$\mathbf{v}_{\rm cm} = \frac{\sum_{i=1}^N m_i\mathbf{v}_i}{\sum_{i=1}^N m_i}.$$

For the same reasons given in the previous answer, we have

$$\sum_{i=1}^N m_i(\mathbf{v}_i - \mathbf{v}_{\rm cm}) = 0,$$

which can be rearranged to get

$$\mathbf{v}_{\rm cm} = \frac{\sum_{i=1}^N m_i\mathbf{v}_i}{\sum_{i=1}^N m_i}.$$

7.3. If T is the total kinetic energy of the two masses in Ex. 7.1 and $T_{\rm CM}$ their total kinetic energy in the CM system, show that

$$T = T_{\rm CM} + \left(\frac{m_1 + m_2}{2}\right)\|\mathbf{v}_{\rm cm}\|^2.$$

Following the argument used in the answer to Ex. 6.3, we have

$$T_{\rm CM} = \frac{1}{2}m_1\mathbf{v}_1' \cdot \mathbf{v}_1' + \frac{1}{2}m_2\mathbf{v}_2' \cdot \mathbf{v}_2'.$$

We can write the expression for the energy of the original system using $\mathbf{v}_1' = \mathbf{v}_1 - \mathbf{v}_{\rm CM}$ and $\mathbf{v}_2' = \mathbf{v}_2 - \mathbf{v}_{\rm CM}$,

$$\begin{aligned}
T &= \frac{1}{2}m_1(\mathbf{v}_1' + \mathbf{v}_{\rm CM}) \cdot (\mathbf{v}_1' + \mathbf{v}_{\rm CM}) + \frac{1}{2}m_2(\mathbf{v}_2' + \mathbf{v}_{\rm CM}) \cdot (\mathbf{v}_2' + \mathbf{v}_{\rm CM}) \\
&= \frac{1}{2}m_1\mathbf{v}_1' \cdot \mathbf{v}_1' + m_1\mathbf{v}_1' \cdot \mathbf{v}_{\rm CM} + \frac{1}{2}m_1\mathbf{v}_{\rm CM} \cdot \mathbf{v}_{\rm CM} + \frac{1}{2}m_2\mathbf{v}_2' \cdot \mathbf{v}_2' + m_2\mathbf{v}_2' \cdot \mathbf{v}_{\rm CM} + \frac{1}{2}m_2\mathbf{v}_{\rm CM} \cdot \mathbf{v}_{\rm CM} \\
&= T_{\rm CM} + (m_1\mathbf{v}_1' + m_2\mathbf{v}_2') \cdot \mathbf{v}_{\rm CM} + \left(\frac{m_1 + m_2}{2}\right)\mathbf{v}_{\rm CM} \cdot \mathbf{v}_{\rm CM} \\
&= T_{\rm CM} + \left(\frac{m_1 + m_2}{2}\right)\|\mathbf{v}_{\rm CM}\|^2
\end{aligned}$$

where the middle term in the second to last equation disappears because the momentum of the center of mass system is zero.

7.4. Generalize the result of Ex. 7.3 to N masses. Show that

$$T = T_{\rm CM} + \frac{\sum_{i=1}^N m_i}{2}\|\mathbf{v}_{\rm cm}\|^2.$$

In this case,

$$T_{\mathrm{CM}} = \sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{v}'_i \cdot \mathbf{v}'_i,$$

and so we have

$$\begin{aligned}
T &= \sum_{i=1}^{N} \frac{1}{2} m_i (\mathbf{v}'_i + \mathbf{v}_{\mathrm{CM}}) \cdot (\mathbf{v}'_i + \mathbf{v}_{\mathrm{CM}}) \\
&= \sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{v}'_i \cdot \mathbf{v}'_i + \sum_{i=1}^{N} m_i \mathbf{v}'_i \cdot \mathbf{v}_{\mathrm{CM}} + \sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{v}_{\mathrm{CM}} \cdot \mathbf{v}_{\mathrm{CM}} \\
&= T_{\mathrm{CM}} + \mathbf{v}_{CM} \cdot \sum_{i=1}^{N} m_i \mathbf{v}'_i + \frac{\sum_{i=1}^{N} m_i}{2} \|\mathbf{v}_{\mathrm{CM}}\|^2 \\
&= T_{\mathrm{CM}} + \frac{\sum_{i=1}^{N} m_i}{2} \|\mathbf{v}_{\mathrm{CM}}\|^2,
\end{aligned}$$

where the middle term in the second to last equation disappears for the same reason as it did in the previous exercise.

7.5. A particle is initially at a point $\mathbf{r}_0$, and is moving under gravity with an initial velocity $\mathbf{v}_0$. Find the subsequent motion $\mathbf{r}(t)$.

Assuming the gravitational field remains roughly constant over the particle's trajectory, we may denote its acceleration by $\mathbf{a}$ and integrate twice to find

$$\begin{aligned}
\mathbf{r}(t) &= \int \int \mathbf{a} \, \mathrm{d}t \mathrm{d}t \\
&= \int \mathbf{a}t + \mathbf{v}_0 \, \mathrm{d}t \\
&= \frac{1}{2} \mathbf{a}t^2 + \mathbf{v}_0 t + \mathbf{r}_0.
\end{aligned}$$

7.6. You are given three vectors,

$$\begin{aligned}
\mathbf{a} &= 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \\
\mathbf{b} &= 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \\
\mathbf{c} &= \mathbf{i} + 3\mathbf{j}
\end{aligned}$$

Find

(a) $\mathbf{a} + \mathbf{b}$

(b) $\mathbf{a} - \mathbf{b}$

(c) $\mathbf{a}_x$

(d) $\mathbf{a} \cdot \mathbf{i}$

(e) $\mathbf{a} \cdot \mathbf{b}$

(f) $(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

(a) $(3+2)\mathbf{i} + (2-1)\mathbf{j} + (-1+1)\mathbf{k} = 5\mathbf{i} - \mathbf{j}$

(b) $(3-2)\mathbf{i} + (2--1)\mathbf{j} + (-1-1)\mathbf{k} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

(c) 3

(d) $3 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 = 3$

(e) $3 \cdot 2 + 2 \cdot -1 + -1 \cdot 1 = 3$

(f) $(3 \cdot 1 + 2 \cdot 3 + -1 \cdot 0)(2\mathbf{i} - \mathbf{j} + \mathbf{k}) - (3)(\mathbf{i} + 3\mathbf{j}) = (19-3)\mathbf{i} + (-9+9)\mathbf{j} + (9+0)\mathbf{k} = 16\mathbf{i} + 9\mathbf{k}$

7.7. A particle of mass 1 kg is moving in such a way that its position is described by the vector

$$\mathbf{r}(t) = t\mathbf{i} + (t + t^2/2)\mathbf{j} - (4/\pi^2)\sin(\pi t/2)\mathbf{k}.$$

(a) Find the position, velocity $\mathbf{v}(t)$, acceleration $\mathbf{a}(t)$, and kinetic energy $T(t)$ of the particle at $t = 0$ and $t = 1$ second.

(b) Find the force $\mathbf{F}(t)$ that will produce this motion.

(c) Find the radius of curvature $R(t)$ of the particle's path at $t = 1$ second.

(a) The position vector is already given. Its derivative is the sought velocity vector,

$$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \mathbf{i} + (1+t)\mathbf{j} - (2/\pi)\cos(\pi t/2)\mathbf{k}.$$

The derivative of this is the acceleration,

$$\mathbf{a}(t) = \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = \mathbf{j} + \sin(\pi t/2)\mathbf{k}.$$

The particle's kinetic energy at time t is given by $m\|v(t)\|^2/2$, or

$$T(t) = \frac{1}{2}\Big(1 + (1+t)^2 + (2/\pi)^2\cos^2(\pi t/2)\Big) = 1 + t + t^2/2 + (2/\pi^2)\cos(\pi t/2).$$

At 0 and 1 seconds, these values are

$$\mathbf{r}(0) = \mathbf{0}\,\mathrm{m} \qquad \mathbf{v}(0) = \mathbf{i} + \mathbf{j} - (2/\pi)\mathbf{k}\,\mathrm{m\,s^{-1}} \qquad \mathbf{a}(0) = \mathbf{j}\,\mathrm{m\,s^{-2}} \qquad T(0) = 1 + (2/\pi^2)\,\mathrm{J}$$

and

$$\mathbf{r}(1) = \mathbf{i} + (3/2)\mathbf{j} - (4/\pi^2)\mathbf{k}\,\mathrm{m} \qquad \mathbf{v}(1) = \mathbf{i} + 2\mathbf{j}\,\mathrm{m\,s^{-1}} \qquad \mathbf{a}(1) = \mathbf{j} + \mathbf{k}\,\mathrm{m\,s^{-2}} \qquad T(1) = 5/2\,\text{joule}$$

(b) The force capable of producing such a motion is simply the acceleration vector times the mass of the particle, or

$$\mathbf{F}(t) = m\mathbf{a}(t) = \mathbf{j} + \sin(\pi t/2)\mathbf{k}.$$

(c) We can find the radius of curvature by solving $\|\mathbf{a}_\perp(t)\| = \|\mathbf{v}(t)\|^2/R$ for R, where $\mathbf{a}_\perp(t)$ is the acceleration perpendicular to the velocity vector. This can be found by subtracting off the part of the acceleration vector tangential to the velocity vector,

$$\mathbf{a}_\perp(t) = \mathbf{a}(t) - \frac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{\|\mathbf{v}(t)\|^2}\mathbf{v}(t).$$

At 1 second, this becomes

$$\mathbf{a}_\perp(1) = -(2/5)\mathbf{i} + (1/5)\mathbf{j} + \mathbf{k}\,\mathrm{m\,s^{-2}},$$

while $\|\mathbf{v}(1)\|^2 = 5\,\mathrm{m\,s^{-1}}$. The magnitude of $\mathbf{a}_\perp(1)$ is $sqrt(6/5)$, and so the radius of curvature at 1 second is

$$R(1) = 5/\sqrt{6/5} \approx 4.56\,\mathrm{m}.$$

7.8. A pilot flying at an air speed of 100 knots wishes to travel due north. He knows, from talking to the airport meteorologist, that there is a 25 knot wind from west to east at his flight altitude.

(a) In what direction should he head his plane?

(b) What will be the duration T of his flight, if his destination is 100 land miles away? (Neglect the time for landing and take-off, and note that 1 knot = 1.15 miles per hour.)

(a) The desired velocity vector, the planes velocity vector, and the wind's velocity vector will combine to make a right triangle with a hypotenuse of 100 knots and a short side of 25 knots. From north, then, the pilot will have to face

$$\arcsin\frac{25}{100} \approx 14.48^\circ$$

towards the east

(b) The pilot's northward velocity is given by $v_N = \sqrt{100^2 - 25^2} \approx 96.82$ knots, or $111.35\,\mathrm{mi/h}$. So the duration of the flight will be

$$T = \frac{d}{v_N} = 53.9\,\mathrm{min}.$$

7.9. A cyclist rides at $10\,\mathrm{mi/h}$ due north and the wind, which is blowing at $6\,\mathrm{mi/h}$ from a point between N and E, appears to the cyclist to come from a point 15°E of N.

(a) Find the true direction of the wind.

(b) Find the direction in which the wind will appear to meet the cyclist on his return if he rides at the same speed.

(a) We can find the true direction of the wind by observing that the apparent velocity $\mathbf{v}_A$ is related to the true velocity $\mathbf{v}_T$ and the cyclist's velocity $\mathbf{v}_C$ by $\mathbf{v}_T - \mathbf{v}_C = \mathbf{v}_A$. In Cartesian coordinates, the three vectors are given by

$$\begin{aligned}\mathbf{v}_C &= 10\mathbf{j}\\ \mathbf{v}_A &= -A\sin(15^\circ)\mathbf{i} - A\cos(15^\circ)\mathbf{j}\\ \mathbf{v}_T &= -6\sin(\theta)\mathbf{i} - 6\cos(\theta)\mathbf{j}\end{aligned}$$

If we equate the $\|\mathbf{v}_T\|^2 = 36$ to the squared magnitude of $\mathbf{v}_A + \mathbf{v}_C$, we find $A \approx 15.072\,\mathrm{mi/h}$, and so the angle the true velocity vector makes with respect to north is

$$\angle\mathbf{v}_T = \arctan\left(\frac{-15.072\sin(15^\circ)}{10 - 15.072\cos(15^\circ)}\right) = 40.5^\circ.$$

(b) Using the relationship above, we find

$$\mathbf{v}_A = \mathbf{v}_T - \mathbf{v}_C = (-6\sin(40.5^\circ))\mathbf{i} + (10 - 6\cos(40.5^\circ))\mathbf{j}$$

which makes an angle of 35.6°E of S.

7.10. A man standing on the bank of a river 1.0 mi wide wishes to get to a point directly opposite him on the other bank. He can do this in two ways:

(a) head somewhat upstream, so that his resultant motion is straight across,

(b) head toward the opposite bank and then walk up along the bank from the point downstream to which the current has carried him.

If he can swim 2.5 mi/h and walk 4.0 mi/h, and if the current is 2.0 mi/h which is the faster way to cross, and by how much?

In the first case, the velocity vectors for the stream and his independent motion are given by

$$\begin{aligned} \mathbf{v}_S &= -2\hat{\mathbf{i}} \\ \mathbf{v}_M &= 2.5\cos\theta\hat{\mathbf{i}} + 2.5\sin\theta\hat{\mathbf{j}}, \end{aligned}$$

where θ is such that the $\hat{\mathbf{i}}$ component of the vector sum of these velocities is zero, i.e. $\theta = \cos^{-1}(4/5)$. In this case, the man's overall velocity is given by

$$\mathbf{v}_T = 2.5\sin\cos^{-1}\frac{4}{5}\hat{\mathbf{j}} = 1.5\hat{\mathbf{j}}.$$

The total time taken to cross the river is then $t = \|\mathbf{d}\|/\|\mathbf{v}_T\|$, or 40 minutes.

In the second case, the velocity vectors for the stream and the man are

$$\begin{aligned} \mathbf{v}_S &= -2\hat{\mathbf{i}} \\ \mathbf{v}_M &= 2.5\hat{\mathbf{j}}. \end{aligned}$$

The man will cross the stream once he has gone a distance of one mile in the $\hat{\mathbf{j}}$ direction, which occurs at the 24 minute mark. At this point, he will have moved a distance -0.8 miles downstream. At a speed of 4 mi/h, he will cover this distance in 12 minutes to arrive at his destination, giving a total travel time of 36 minutes. Hence, method two is faster by approximately 4 minutes.

7.11. A motorboat that runs at a constant speed V relative to the water is operated in a straight river channel where the water is flowing smoothly with a constant speed R. The boat is first sent on a round trip from its anchor point to a point a distance d directly upstream. It is then sent on a round trip from its anchor point to a point a distance d away directly across the stream. For simplicity assume that the boat runs the entire distance in each case at full speed and that no time is lost in reversing course at the end of the outward lap. If t_V is the time the boat took to make the round trip in line with the stream flow, t_A the time the boat took to make the round trip across the stream, and t_L the time the boat would take to go a distance $2d$ on a lake,

(a) What is the ratio t_V/t_A?

(b) What is the ratio t_A/t_L?

(a) The first case is particularly simple, as the magnitude of the boat's total velocity moving upstream is $V - R$, while the magnitude of its velocity downstream is $V + R$. This gives a total time of

$$t_V = \frac{d}{V - R} + \frac{d}{V + R} = \frac{2d}{V^2 - R^2}V.$$

For the second case, if the boat is to move in a straight line, then the boat's must be at an angle to the channel's edge such that $V \cos\theta = R$, which, after geometric considerations, gives a cross-stream velocity of

$$V \sin\theta = V \sin\cos^{-1}\frac{R}{V} = \sqrt{V^2 - R^2}.$$

By symmetry, this velocity is the same for both legs of the trip. The total time taken is

$$t_A = \frac{2d}{\sqrt{V^2 - R^2}},$$

and thus the desired ratio is

$$\frac{t_V}{t_A} = \frac{V}{\sqrt{V^2 - R^2]}}.$$

This makes sense, as if R were to be 0, the two times would be equal.

(b) The time taken to go a distance of $2d$ on a lake is simply

$$t_L = \frac{2d}{V}$$

and from this we see that $t_A/t_L = t_V/t_A$.

7.12. Use vectors to find the great circle distance D between two points on the earth (radius $= r_{♁}$), whose latitudes and longitudes are (λ_1, ϕ_1) and (λ_2, ϕ_2).

Note: Use a system of rectangular coordinates with the origin at the center of the earth, one axis along the earth's axis, another pointed toward $\lambda = 0$, $\phi = 0$, and the third axis pointed toward $\lambda = 0$, $\phi = 90°$ W. Let longitudes vary from 0° westward to 360°.

7.13. What is the magnitude and direction of the acceleration $\mathbf{a}$ of the moon at

(a) New moon?

(b) Quarter moon?

(c) Full moon?

Note:

$$R_{♁\odot} = 1.50 \times 10^8\,\mathrm{km}$$
$$R_{♁☾} = 3.85 \times 10^5\,\mathrm{km}$$
$$M_\odot = 3.33 \times 10^5\,\mathrm{M}_{♁}$$

7.14. Two identical 45° wedges M_1 and M_2, with smooth faces and $M_1 = M_2 = 8\,\mathrm{kg}$, are used to move a smooth-faced mass $M = 384\,\mathrm{kg}$, as shown in Fig. 7-1. Both wedges rest upon a smooth horizontal plane; one wedge is butted against a vertical wall, and to the other wedge a force $F = 592\,\mathrm{kg-wt}$ is applied horizontally.

(a) What is the magnitude and direction of the acceleration $\mathbf{a}_1$ of the movable wedge M_1?

(b) What is the magnitude and direction of the acceleration $\mathbf{a}$ of the larger wedge M?

(c) What force F_2 does the stationary wedge M_2 exert on the heavy mass M?

Neglect friction.

7.15. A mass m is suspended from a frictionless pivot at the end of a string of arbitrary length, and is set to whirling in a horizontal circular path whose plane is a distance H below the pivot point, as shown in Fig. 7-2. Find the period of revolution T of the mass in its orbit.

There are only two forces acting on the mass, friction and tension. If the mass is to move in a plane, the vertical components of both forces must cancel, and so

$$A \sin\theta = mg,$$

where A is the tension and θ is the angle made between the string and the horizontal. The requirement that the mass move in a circular manner introduces the additional restriction of

$$A \cos\theta = \frac{mv^2}{R}.$$

We may divide the first equation by the second to find

$$\tan\theta = \frac{gR}{v^2},$$

and, using $\tan\theta = H/R$, we obtain

$$\frac{H}{R} = \frac{gR}{v^2}.$$

The period of the mass's revolution is given by the total distance it traverses in one revolution divided by its velocity, or $T = 2\pi R/v$. Substituting this into the above equation and simplifying gives

$$T = 2\pi\sqrt{\frac{H}{g}};$$

the greater the distance between the ceiling and the mass, the longer its period.

7.16. Two small, sticky, putty balls a and b, each of mass 1 gram, travel under the influence of gravity with acceleration $-9.8\hat{\mathbf{k}}\,\mathrm{m\,s^{-2}}$. Given the initial condition at $t = 0$,

$$\begin{aligned}
\mathbf{r}_a(0) &= 7\hat{\mathbf{i}} + 4.9\hat{\mathbf{k}},\\
\mathbf{v}_a(0) &= 7\hat{\mathbf{i}} + 3\hat{\mathbf{j}},\\
\mathbf{r}_b(0) &= 49\hat{\mathbf{i}} + 4.9\hat{\mathbf{k}},\\
\mathbf{v}_b(0) &= -7\hat{\mathbf{i}} + 3\hat{\mathbf{j}},
\end{aligned}$$

find $\mathbf{r}_a(t)$ and $\mathbf{r}_b(t)$ for all times $t > 0$.

By simple integration, one can immediately find

$$\begin{aligned}
\mathbf{r}_a(t) &= (7 + 7t)\hat{\mathbf{i}} + (3t)\hat{\mathbf{j}} + (4.9 - 4.9t^2)\hat{\mathbf{k}}\\
\mathbf{r}_b(t) &= (49 - 7t)\hat{\mathbf{i}} + (3t)\hat{\mathbf{j}} + (4.9 - 4.9t^2)\hat{\mathbf{k}}
\end{aligned}$$

which, from inspection, breaks down at $t = 3$ seconds, as this is when both position vectors are equal: the two balls will stick together and continue on as one. To find the resulting position, we may use conservation of momentum at the instant of impact to obtain

$$m\mathbf{v}_a(3) + m\mathbf{v}_b(3) = 2m\mathbf{v_c}(3),$$

or

$$\mathbf{v}_c(3) = 3\hat{\mathbf{j}} - 29.4\hat{\mathbf{k}}.$$

Then, for $t > 3$, we have

$$\begin{aligned}\mathbf{r}_c(t) &= \mathbf{r}_a(3) + \mathbf{v}_c(3)(t-3) + \frac{1}{2}\mathbf{a}(3)(t-3)^2 \\ &= (28)\hat{\mathbf{i}} + (9 + 3(t-3))\hat{\mathbf{j}} + (-39.2 - 29.4(t-3) - 4.9(t-3)^2)\hat{\mathbf{k}} \\ &= (28)\hat{\mathbf{i}} + (3t)\hat{\mathbf{j}} + (4.9 - 4.9t^2)\hat{\mathbf{k}}.\end{aligned}$$

7.17. You are on a ship traveling steadily east at 15 knots. A ship o a steady course whose speed is known to be 26 knots is observed 6.0 miles due south of you; it is later observed to pass behind you, its distance of closest approach being 3.0 miles.

(a) What was the course of the other ship?

(b) What was the time T between its position south of you and its position of closest approach?

8 Non-Relativistic Two-Body Collisions in Three Dimensions

8.1. Analogous to the above discussion, derive the results for a three-dimensional non-relativisti collision $(m_1 + m_2 = m_3 + m_4)$ for the case $m_1, m_2 \neq m_3, m_4$, e.g., show that in the collision of two bodies with initial momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, the final momenta are given byalign

$$\begin{aligned}\mathbf{p}_3 &= \mathbf{P}_3 + m_3\mathbf{v}_{CM} \\ \mathbf{p}_4 &= \mathbf{P}_4 + m_4\mathbf{v}_{CM},\end{aligned}$$

where $\mathbf{p}_i = m_i\mathbf{v}_i$ is the momentum of mass m_i in laboratory system, and $\mathbf{P}_i = \mathbf{p}_i - m_i\mathbf{v}_{CM}$ is the momentum of mass m_i in the CM system, and

$$\begin{aligned}|\mathbf{P}_1| &= |\mathbf{P_2}| = \sqrt{2m_rT_{CM}}, \\ |\mathbf{P}_3| &= |\mathbf{P_4}| = \sqrt{2m_r'T_{CM}'}.\end{aligned}$$

From simple geometry (see Figure 8-2), we know that $\mathbf{u}_i = \mathbf{v}_i - \mathbf{v}_{CM}$, or $\mathbf{v}_i = \mathbf{u}_i + \mathbf{v}_{CM}$. Since $\mathbf{p}_i = m_i\mathbf{v}_i$, we find

$$\begin{aligned}\mathbf{p}_3 &= \mathbf{P}_3 + m_3\mathbf{v}_{CM}, \\ \mathbf{p}_4 &= \mathbf{P}_4 + m_4\mathbf{v}_{CM},\end{aligned}$$

where $\mathbf{P}_i = m_i\mathbf{u}_i$. To find the magnitude of $\mathbf{P}_i$, we may follow the steps of the preceeding discussion to find

$$T_{CM} = \frac{|\mathbf{P}_{1,2}|^2}{2m_r} \qquad T_{CM}' = \frac{|\mathbf{P}_{3,4}|^2}{2m_r'}$$

which may be easily inverted to obtain

$$|\mathbf{P}_1| = |\mathbf{P_2}| = \sqrt{2m_r T_{CM}},$$
$$|\mathbf{P}_3| = |\mathbf{P_4}| = \sqrt{2m_r' T_{CM}'}.$$

8.2. A moving particle collides perfectly elastically with an equally massive particle initially at rest. Show that the two particles move at right angles to one another after the collision.

When all masses are the same, conservation of momentum reduces to

$$\mathbf{v}_1 = \mathbf{v}_2 + \mathbf{v}_3.$$

Since energy is proportional to the square of the speed, we are encouraged to take the squared magntiude of this expression,

$$|\mathbf{v}_1|^2 = |\mathbf{v}_2|^2 + 2\mathbf{v}_2 \cdot \mathbf{v}_3 + |\mathbf{v}_3|^2.$$

This is an elastic collision, and so

$$\frac{1}{2}m|\mathbf{v}_1|^2 = \frac{1}{2}m|\mathbf{v}_2|^2 + \frac{1}{2}m|\mathbf{v_3}|^2,$$

or $|\mathbf{v}_1|^2 = |\mathbf{v}_2|^2 + |\mathbf{v}_3|^2$. Taken together with the second equation, this implies $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$, which shows that the particles depart at a right angle to each other.

8.3. A moving particle of mass M collides perfectly elastically with a stationary particle of mass $m < M$. Find the maximum possible angle $\theta_{\max}$ through which the incident particle can be deflected.

To the larger particle, assign incoming and outgoing velocities of $\mathbf{v}_0$ and $\mathbf{v}_1$, respectively. The smaller particle is stationary prior to the collision and has a velocity $\mathbf{u}$ afterwards.

By conservation of momentum and energy,

$$M\mathbf{v}_0 = M\mathbf{v}_1 + m\mathbf{u}$$
$$\frac{1}{2}M|\mathbf{v}_0|^2 = \frac{1}{2}M|\mathbf{v}_1|^2 + \frac{1}{2}m|\mathbf{u}|^2.$$

The angle at which the incident particle is deflected is given by the angle between $\mathbf{v}_0$ and $\mathbf{v}_1$, so we will eliminate $\mathbf{u}$ from the second equation. This results in

$$\begin{aligned} M|\mathbf{v}_0|^2 &= M|\mathbf{v}_1|^2 + m\frac{M^2}{m^2}(\mathbf{v}_0 - \mathbf{v}_1)\cdot(\mathbf{v}_0 - \mathbf{v}_1) \\ &= M|\mathbf{v}_1|^2 + \frac{M^2}{m}|\mathbf{v}_0|^2 - 2\frac{M^2}{m}\mathbf{v}_0 \cdot \mathbf{v}_1 + \frac{M^2}{m}|\mathbf{v}_1|^2 \\ &= M|\mathbf{v}_1|^2 + \frac{M^2}{m}|\mathbf{v}_0|^2 - 2\frac{M^2}{m}|\mathbf{v}_0||\mathbf{v}_1|\cos\theta + \frac{M^2}{m}|\mathbf{v}_1|^2, \end{aligned}$$

where θ is the angle of deflection. Isolating the cosine term yields

$$\cos\theta = \frac{1}{2}\Big(\frac{M+m}{M}\Big)\frac{|\mathbf{v}_1|}{|\mathbf{v}_0|} + \frac{1}{2}\Big(\frac{M-m}{M}\Big)\frac{|\mathbf{v}_0|}{|\mathbf{v}_1|}.$$

Defining the ratio $r = |\mathbf{v}_1|/|\mathbf{v}_0|$, our angle is given by

$$\theta = \cos^{-1}\Big(\frac{1}{2}\Big[\frac{M+m}{M}\Big]r + \frac{1}{2}\Big[\frac{M-m}{M}\Big]\frac{1}{r}\Big).$$

We may find the extremal point of this function by setting the derivative with respect to r equal to 0, which yields

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = -\frac{1}{\sqrt{1-\cos^2\theta}}\Big(\frac{1}{2}\Big[\frac{M+m}{M}\Big] - \frac{1}{2}\Big[\frac{M-m}{M}\Big]\frac{1}{r^2}\Big) = 0.$$

This is only zero when the term in parentheses vanishes, and so we must have

$$\frac{M+m}{M} = \frac{M-m}{M}\frac{1}{r^2},$$

or

$$r = \sqrt{\frac{M-m}{M+m}}.$$

Substituting this back into the expression for our angle simplifies the expression to

$$\theta = \cos^{-1}\Big(\frac{\sqrt{M^2-m^2}}{M}\Big),$$

which, by simple geometric considerations, is equivalent to

$$\theta = \sin^{-1}\Big(\frac{m}{M}\Big).$$

8.4. A particle of mass m_1 and velocity $\mathbf{v}_1$ collides perfectly elastically with another particle of mass $m_2 = 3m_1$ which is at rest ($\mathbf{v}_2 = 0$). After the collision, m_2 moves at angle $\theta_2 = 45^\circ$ with respect to the original direction of m_1, as shown in Fig. 8-5. Find θ_1, the final angle of motion of m_1, and v_1', v_2', the final velocities.

Conservation of energy and momentum give us a total of 3 equations (split component-wise along the parallel and perpendicular direction to the incident particle's trajectory),

$$\frac{1}{2}m_1|\mathbf{v}_1|^2 = \frac{1}{2}m_1|\mathbf{v}_1'|^2 + \frac{1}{2}m_2|\mathbf{v}_2'|^2 = \frac{1}{2}m_1|\mathbf{v}_1'|^2 + \frac{3}{2}m_1|\mathbf{v}_2'|^2,$$
$$m_1|\mathbf{v}_1| = m_1|\mathbf{v}_1'|\cos\theta + m_2|\mathbf{v}_2'|\cos(-45^\circ) = m_1|\mathbf{v}_1'|\cos\theta + \frac{3}{\sqrt{2}}m_1|\mathbf{v}_2'|,$$
$$0 = m_1|\mathbf{v}_1'|\sin\theta + m_2|\mathbf{v}_2'|\sin(-45^\circ) = m_1|\mathbf{v}_1'|\sin\theta - \frac{3}{\sqrt{2}}m_1|\mathbf{v}_2'|.$$

Using the last equation, we may solve for $|\mathbf{v}_2'|$ in terms of $|mathbf{v}_1'$ and θ, obtaining,

$$|\mathbf{v}_2'| = |\mathbf{v}_1'|\frac{\sqrt{2}}{3}\sin\theta,$$

which, when substituted into the second equation (and cancelling like terms), yields

$$|\mathbf{v}_1| = |\mathbf{v}_1'|(\cos\theta + \sin\theta).$$

This and the previous relationship can then be placed into the first equation to find

$$|\mathbf{v}_1'|^2(\cos\theta+\sin\theta)^2 = |\mathbf{v}_1'|^2(1+\frac{2}{3}\sin^2\theta),$$

which we may use to solve for θ as follows:

$$\begin{aligned}
\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta &= 1 + \frac{2}{3}\sin^2\theta \\
1 + 2\cos\theta\sin\theta &= 1 + \frac{2}{3}\sin^2\theta \\
2\cos\theta\sin\theta &= \frac{2}{3}\sin^2\theta \\
\cos\theta &= \frac{1}{3}\sin\theta \\
3 &= \tan\theta,
\end{aligned}$$

or $\theta = \arctan 3$. With this we can solve for $|\mathbf{v}_1'|$,

$$|\mathbf{v}_1'| = \frac{|\mathbf{v}_1|}{\cos\arctan 3 + \sin\arctan 3} = \frac{\sqrt{10}}{4}|\mathbf{v}_1|,$$

and $|\mathbf{v}_2'|$,

$$|\mathbf{v}_2'| = |\mathbf{v}_1'|\frac{\sqrt{10}}{4}\frac{\sqrt{2}}{3}\sin\arctan 3 = \frac{\sqrt{2}}{4}|\mathbf{v}_1|$$

8.5. Two particles of equal mass m are shot at on eanother from perpendicular directions with equal speeds. After they collide, it is found that one particle was deflected 60° from its initial direction, towards the initial direction of the other particle, as shwon in Fig. 8-6. Determine the angle α by which the second particle gets deflected towards the initial direction of the first if the collision is elastic.

By conservation of energy and momentum, we have

$$\begin{aligned}
m\mathbf{v}_1 + m\mathbf{v}_2 &= m\mathbf{u}_1 + m\mathbf{u}_2, \\
\frac{1}{2}m|\mathbf{v}_1|^2 + \frac{1}{2}m|\mathbf{v}_2|^2 &= \frac{1}{2}m|\mathbf{u}_1|^2 + \frac{1}{2}m|\mathbf{u}_2|^2.
\end{aligned}$$

Removing common terms (such as the mass) and recognizing that $|\mathbf{v}_1| = |\mathbf{v}_2|$, these simplify to

$$\begin{aligned}
\mathbf{v}_1 + \mathbf{v}_2 &= \mathbf{u}_1 + \mathbf{u}_2, \\
2|\mathbf{v}_1|^2 &= |\mathbf{u}_1|^2 + |\mathbf{u}_2|^2.
\end{aligned}$$

Dotting the first equation with itself, we find the relation

$$|\mathbf{v}_1|^2 + 2\mathbf{v}_1\cdot\mathbf{v}_2 + |\mathbf{v}_2|^2 = |\mathbf{u}_1|^2 + 2\mathbf{u}_1\cdot\mathbf{u}_2 + |\mathbf{u}_2|^2,$$

which, noting that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal, collapses to

$$2|\mathbf{v}_1|^2 = |\mathbf{u}_1|^2 + 2\mathbf{u}_1\cdot\mathbf{u}_2 + |\mathbf{u}_2|^2.$$

In order for this to be true and for conservation of energy to hold, $\mathbf{u}_1\cdot\mathbf{u}_2$ must be zero, or the final velocities are at a right angle to one another. In the case, the angle $\alpha = 120^\circ$.

8.6. Two particles of equal mass are travelling on courses at right angles to each other with speeds of $v_1 = 8\,\mathrm{m\,s^{-1}}$ and $v_2 = 6\,\mathrm{m\,s^{-1}}$, respectively. They collide elastically. After the collision, m_1 is observed to be traveling in a path that makes an angle $\theta = \arctan(1/2)$ with respect to the direction of its path before the collision, as shown in Fig. 8-7.

(a) What is the vector velocity $\mathbf{v}_{CM}$ of the center of mass? Give Cartesian components.

(b) What are the magnitudes u_1, u_2 of the final velocities in the CM system?

(c) What is the final velocity $\mathbf{v}'_1$ of particle 1 in the lab system?

(a) In cartesian coordinates, $\mathbf{v}_1 = 8\hat{\mathbf{i}}$ while $\mathbf{v}_2 = -6\hat{\mathbf{j}}$. The center of mass velocity is simply

$$\mathbf{v}_{CM} = \frac{8m\hat{\mathbf{i}} - 6m\hat{\mathbf{j}}}{m+m} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}.$$

(b) In an elastic collision, $|\mathbf{u}_1| = |\mathbf{u}_3|$ and $|\mathbf{u}_2| = |\mathbf{u}_4|$. Since the initial velocities in the CM system are given by

$$\begin{aligned}\mathbf{u}_1 &= 8\hat{\mathbf{i}} - (4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}) = 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}},\\ \mathbf{u}_2 &= -6\hat{\mathbf{j}} - (4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}) = -4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}.\end{aligned}$$

Both of these have a magnitude of $5\,\mathrm{m\,s^{-1}}$, and thus

$$u_1 = |\mathbf{u}_3| = u_2 = |\mathbf{u}_4| = 5\,\mathrm{m\,s^{-1}}.$$

(c) We know that $\mathbf{v}'_1 = \mathbf{u}_3 + \mathbf{v}_{CM}$. We also know the angle $\mathbf{v}'_1$ makes with the coordinate axes, suggesting we dot this equation, in turn, with both unit vectors. Doing so reveals

$$\begin{aligned}\mathbf{v}'_1 \cdot \hat{\mathbf{i}} = |\mathbf{v}'_1| \cos\arctan\frac{1}{2} = \frac{2}{\sqrt{5}}|\mathbf{v}'_1| &= \mathbf{u}_3 \cdot \hat{\mathbf{i}} + \mathbf{v}_{CM} \cdot \hat{\mathbf{i}}\\ &= |\mathbf{u}'_1|\cos\theta + 4\\ &= 5\cos\theta + 4\\ \mathbf{v}'_1 \cdot \hat{\mathbf{j}} = |\mathbf{v}'_1|\cos(-(\frac{\pi}{2} - \arctan\frac{1}{2})) = |\mathbf{v}'_1|\sin\arctan\frac{1}{2} = \frac{1}{\sqrt{5}}|\mathbf{v}'_1| &= \mathbf{u}'_1 \cdot \hat{\mathbf{j}} + \mathbf{v}_{CM} \cdot \hat{\mathbf{j}}\\ &= |\mathbf{u}'_1|\cos(-(\frac{\pi}{2} - \theta)) - 3\\ &= 5\sin\theta - 3,\end{aligned}$$

where θ is the angle $\mathbf{u}'_1$ makes with the x-axis. Dividing the second equation by the first gives

$$\frac{1}{2} = \frac{5\sin\theta - 3}{5\cos\theta + 4},$$

which may be rearranged to obtain

$$\cos\theta = 2(\sin\theta - 1).$$

This equation is satisfied when $\theta = \pi/2$, in which case we have $\mathbf{u}'_1 = 5\hat{\mathbf{j}}$. Adding this to the center of mass velocity yields

$$\mathbf{v}'_1 = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}}.$$

8.7. A proton moving along the x-axis with a speed of $v_0 = 1.00 \times 10^7\,\mathrm{m\,s^{-1}}$ collides elastically with a stationary proton. After the collision, one proton moves in the xy-plane at an angle of 30° with the x-axis. Find the velocities $\mathbf{v}'_1$ and $\mathbf{v}'_2$ (speed and direction!) of both protons after the collision.

We know, from problem 8.2, that a collision between two particles of equal mass—with one at rest—results in ending velocities orthogonal to each other. Therefore, if $\angle \mathbf{v}_1' = 30°$, we must have $\angle \mathbf{v}_2' = -60°$ (the other possible choice would not conserve momentum). Then, from conservation of momentum, we know

$$
\begin{aligned}
v_1 &= \frac{\sqrt{3}}{2} v_1' + \frac{1}{2} v_2' \\
0 &= \frac{1}{2} v_1' - \frac{\sqrt{3}}{2} v_2' \\
v_1^2 &= v_1'^2 + v_2'^2
\end{aligned}
$$

where the first two equations are the component representations of $\mathbf{v}_1 = \mathbf{v}_1' + \mathbf{v}_2'$. From the above, we immediately see that $v_1' = \sqrt{3} v_2'$, and thus $v_1 = 2v_2'$. So $v_1' = \frac{\sqrt{3}}{2} v_1$, making an angle of 30° with the x-axis. Meanwhile, $v_2' = \frac{1}{2} v_1$, making an angle of $-60°$ with the x-axis.

8.8. A proton moving along the x-axis with a speed of $v_0 = 1.00 \times 10^7\,\mathrm{m\,s^{-1}}$ collides elastically with a stationary beryllum (Be) nucleus. After the collision the Be nucleus is observed to move in the xy-plane at an angle 30° with the x-axis. Find:

(a) The speed v_2 of the Be nucleus in the lab system,

(b) The final velocity $\mathbf{v}_1'$ of the proton in the lab system,

(c) The final velocity $\mathbf{u}_1'$ of the proton in the CM system.

Note: Assume the relative masses of the Be nucleus and proton to be 9 : 1.

8.9. A circular air puck of mass 100 g and radius 2.00 cm is initially moving at a speed of $150\,\mathrm{cm\,s^{-1}}$ on a horizontal table, when it collides elastically with a stationary air puck of mass 200 g and radius 3.00 cm. At the instant of collision, the line joining the centers of the two pucks makes an angle of 60° with the original line of motion of the 100 g puck. If there is no friction, either with the table or between the pucks, find the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of each puck after the collision.

8.10. An object of mass m_1, moving with a linear speed v in a laboratory system, collides with an object of mass m_2 which is at rest in the laboratory. After the collision it is observed tha ta fraction $|\Delta T/T|_{CM} = 1 - \alpha^2$ of the kinetic energy in the CM system was lost in the collision. What was the fraction $|\Delta T/T|_{\mathrm{lab}}$ of energy lost in the *laboratory* system?

8.11. (a) A particle of mass m collides perfectly elastically with a stationary particle of mass $M > m$. THe incident particle is deflected through a 90° angle. At what angle θ with the roiginal direction of m does the more massive particle recoil?

(b) If in the collision a fraction $(1-\alpha^2)$ of the CM energy is lost, what is the recoil angle of the originally stationary particle?

8.12. A proton with kinetic energy 1 MeV collie selastically with a stationary nucleus and is deflected through 90°. If the proton's energy is now 0.80 MeV, what was the mass M of the target nucleus in units of the proton mass m_P?

8.13. A puck of mass 1 kg moving at a speed of $v_1 = 6\,\mathrm{m\,s^{-1}}$ due N collides with a stationary puck of mass 2 kg. After the collision the 1 kg puck is moving at 45° NE of its original direction at a speed of $v_1' = 2\sqrt{2}\,\mathrm{m\,s^{-1}}$.

(a) What is the velocity $\mathbf{v}_2'$ of the 2 kg puck after impact?

(b) What fraction α of the kinetic energy was lost in the CM system?

(c) Through what angle θ was the 1 kg deflected in the CM system?

8.14. A "particle" of mass $m_1 = 2\,\mathrm{kg}$, which is moving with a velocity $\mathbf{v}_1 = (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})\,\mathrm{m\,s^{-1}}$ collides inelastically wiht a second particle of mass $m_2 = 3\,\mathrm{kg}$, moving with a velocity $\mathbf{v}_2 = (-2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}})\,\mathrm{m\,s^{-1}}$.

(a) Find the velocity $\mathbf{v}$ of the composite particle.

(b) Find the total kinetic energy T_{CM} of the above particles in the CM system, *before* impact.