Notes on quantizing the massive scalar field

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1 From coordinates to fields

Consider the familiar simple harmonic oscillator. It has the Lagrangian

$$\mathcal{L}(x, \dot{x}) = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

From mechanics, we know the equations of motion for this system are revealed by taking the functional derivative of the action $S = \int \mathcal{L}(x, \dot{x})\,\mathrm{d}t$ and setting it equal to 0 (i.e. finding an extremal trajectory of the action). Doing so reveals

$$\begin{aligned}
\frac{\delta S[x(t)]}{\delta x(t)} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\int_a^b \frac{1}{2}m(\dot{x}+\varepsilon\dot{\eta})^2 - \frac{1}{2}k(x+\varepsilon\eta)^2\,\mathrm{d}t - \int_a^b \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\,\mathrm{d}t\Big)\\
&= \lim_{\varepsilon\to 0}\Big(\int_a^b m\dot{x}\dot{\eta} - kx\eta\,\mathrm{d}t + \mathcal{O}(\varepsilon)\Big)\\
&= -\int_a^b (m\ddot{x} + kx)\eta\,\mathrm{d}t\\
&= 0
\end{aligned}$$

which implies $m\ddot{x} = -kx$, as we expect.[1] More generally we have

$$\begin{aligned}
\frac{\delta S[x(t)]}{\delta x(t)} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\int_a^b \mathcal{L}(x+\varepsilon\eta, \dot{x}+\varepsilon\dot{\eta})\,\mathrm{d}t - \int_a^b \mathcal{L}(x, \dot{x})\,\mathrm{d}t\Big)\\
&= \lim_{\varepsilon\to 0}\Big(\int_a^b \frac{\partial\mathcal{L}}{\partial x}\eta + \frac{\partial\mathcal{L}}{\partial\dot{x}}\dot{\eta}\,\mathrm{d}t + \mathcal{O}(\varepsilon)\Big)\\
&= \int_a^b \Big(\frac{\partial\mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}}\Big)\eta\,\mathrm{d}t
\end{aligned}$$

which shows that valid trajectories must satisfy

$$\frac{\partial\mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}} = 0.$$

This can immediately be generalized to a system with multiple coordinates,

$$\frac{\partial\mathcal{L}}{\partial q^a} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}^a} = 0.$$

What is the equivalent principle for a covariant field $\phi(x)$? We know that a relativistic system must treat both temporal and spatial derivatives similarly, so the Lagrangian for such a field will have dependencies on ϕ and $\partial_\mu\phi$. By similar considerations, the action must be promoted to an integral over spacetime. Finding the extrema of this new action yields

[1] In the above series of steps, $\eta(t)$ is an arbitrary function that vanishes at the endpoints of the integral. It perturbs the action from the true path $x(t)$.

$$\begin{aligned}
\frac{\delta S[\phi(x)]}{\delta \phi(x)} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\int_\Gamma \mathcal{L}(\phi+\varepsilon\varphi, \partial_\mu\phi+\varepsilon\partial_\mu\varphi)\,\mathrm{d}^4x - \int_\Gamma \mathcal{L}(\phi,\partial_\mu\phi)\,\mathrm{d}^4x\Big) \\
&= \lim_{\varepsilon\to 0}\Big(\int_\Gamma \frac{\partial\mathcal{L}}{\partial\phi}\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\varphi\,\mathrm{d}^4x + \mathcal{O}(\varepsilon)\Big) \\
&= \int_\Gamma \Big(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big)\varphi\,\mathrm{d}^4x \\
&= 0
\end{aligned}$$

or[2]

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0.$$

This is the covariant form of the Euler-Lagrange equation. We have found it by assuming the action principle holds for fields.

2 The massive scalar field

Consider the relativistic classical field described by[3]

$$\begin{aligned}
\mathcal{L}(\phi,\partial_\mu\phi) &= \frac{1}{2}\partial_\mu\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \\
&= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)\cdot(\nabla\phi) - \frac{1}{2}m^2\phi^2
\end{aligned}$$

where, by analogy to the Lagrangian of the simple harmonic oscillator, we identify $\frac{1}{2}\dot{\phi}^2$ with the kinetic energy of the field and $\frac{1}{2}(\nabla\phi)\cdot(\nabla\phi) + \frac{1}{2}m^2\phi^2$ with its potential energy. Recognizing that the Hamiltonian should be described by the sum of both quantities, we define the momentum conjugate to ϕ to be[4]

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}.$$

With this definition, we have

$$\mathcal{H}(\phi,\pi) = \pi\dot{\phi} - \mathcal{L}(\phi,\partial_\mu\phi),$$

which is a nice generalization of the Hamiltonian to fields.

3 From classical to quantum

We would like to quantize this field. For single particle quantum mechanics, we know that the quantization process involves promoting observable variables to operators. In particular, we know that a variable q^a and its conjugate momentum p_a have the commutation relation

$$[q^a, p_b] = i\delta_{ab}.$$

Inspired by this, we do the same for fields. Continuing with our simple relativistic field expression, we try to find an expression for ϕ. Following the Euler-Lagrange equation, we arrive at

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = 0$$

where we have shown the dependence of ϕ on x^μ. It is easiest to work in energy-momentum space, so we carry out a Fourier transform a la

[2]As an aside, we can easily deal with vector and tensor fields by tacking an index onto ϕ.

[3]In many ways this is the first classical field one would suggest: the first term is the simplest way to construct a Lorentz scalar out of $\partial_\mu\phi$, and the second represents a potential dependent on the absolute value of the field. The factor of m^2 is there to make the units work out (where $\hbar = c = 1$). It is conveniently labeled to suggest mass, as it turns out the quanta of this field have a mass m.

[4]From this definition, we can see that an 'upstairs' index in ϕ gets converted to a 'downstairs' index in π. The converse holds true, too.