Notes on the induced field from an evolving Harris current sheet

Zach Beever

1 Time-evolution of the Harris model

The Harris current sheet model takes the simple form

$$\mathbf{B} = B_0 \tanh\left(\frac{z}{L}\right)\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}}.$$

If L changes with time there will be an induced electric field described by

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Signaling the dependence of L on t via $L(t)$, we find

$$\frac{\partial \mathbf{B}}{\partial t} = B_0 \operatorname{sech}^2\left(\frac{z}{L(t)}\right) \cdot \left(-\frac{z}{L^2(t)}\right) \cdot \frac{\mathrm{d}L(t)}{\mathrm{d}t}\,\hat{\mathbf{i}}.$$

The relevant Maxwell equation then becomes

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -B_0 \operatorname{sech}^2\left(\frac{z}{L(t)}\right) \cdot \frac{z}{L^2(t)} \cdot \frac{\mathrm{d}L(t)}{\mathrm{d}t}.$$

Because the Harris model extends infinitely along the y-axis, $E_z = 0$. If this were not so, we would see infinite growth as we moved in this direction. There is no such restriction for E_y, though, and so[1]

$$E_y(z) = \int_{-\infty}^{z} B_0 \operatorname{sech}^2\left(\frac{z}{L(t)}\right) \cdot \frac{z}{L^2(t)} \cdot \frac{\mathrm{d}L(t)}{\mathrm{d}t}\,\mathrm{d}z.$$

Setting $u = z/L(t)$ and $\mathrm{d}u = \mathrm{d}z/L(t)$, this simplifies to

$$E_y(z) = B_0 \frac{\mathrm{d}L(t)}{\mathrm{d}t} \int_{-\infty}^{z/L(t)} \operatorname{sech}^2(u) \cdot u\,\mathrm{d}u.$$

We can integrate this by parts to get

$$E_y(z) = B_0 \frac{\mathrm{d}L(t)}{\mathrm{d}t}\left(u\tanh(u)\Big|_{-\infty}^{z/L(t)} - \int_{-\infty}^{z/L(t)} \tanh(u)\,\mathrm{d}u\right).$$

This remaining integral can be rewritten as

$$\int_{-\infty}^{z/L(t)} \tanh(u)\,\mathrm{d}u = \int_{-\infty}^{z/L(t)} \frac{\sinh(u)}{\cosh(u)}\,\mathrm{d}u = \int_{-\infty}^{z/L(t)} \mathrm{d}(\ln\cosh(u)) = \ln\cosh u\Big|_{-\infty}^{z/L(t)}.$$

Using the above, $E_y(z)$ becomes

$$E_y(z) = B_0 \frac{\mathrm{d}L(t)}{\mathrm{d}t}\left[\frac{z}{L(t)}\tanh\left(\frac{z}{L(t)}\right) - \ln\cosh\left(\frac{z}{L(t)}\right) + C\right]$$

[1] The limits of integration are chosen so that the entire field contributes to E_y at the given z. As we will see, we find the same result whether we set the lower limit to ∞ or $-\infty$.

where C is given by

$$C = \lim_{u \to -\infty} \ln \cosh u - u \tanh u.$$

Since, by defintion,

$$\cosh u = \frac{e^u + e^{-u}}{2} \qquad \text{and} \qquad \tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}}$$

we can rewrite C as

$$\begin{aligned} C &= \lim_{u \to -\infty} \ln \frac{e^{-u}}{2} + u \frac{e^{-u}}{e^{-u}} \\ &= \lim_{u \to -\infty} -u + u - \ln 2 \\ &= -\ln 2 \end{aligned}$$

The final expression for the induced field is given by

$$\mathbf{E}(z,t) = B_0 \frac{\mathrm{d}L(t)}{\mathrm{d}t} \left[\frac{z}{L(t)} \tanh\left(\frac{z}{L(t)}\right) - \ln\cosh\left(\frac{z}{L(t)}\right) - \ln 2 \right] \hat{\mathbf{j}}$$

One major question remains: the Harris model is formulated as an equilibrium solution to the Vlasov equation. Is a time-dependent model physical? Certainly the above result captures the real y-directed current found in the magnetosphere, so it is at least on the right track.