1 Lagrangians

1.1. Use Fermat's principle of least time to derive Snell's law.

Consider the path of a photon as it changes medium. It is emitted at (x_1, y_1) and arrives at (x_2, y_2) by way of point (x_0, y_0) at the material boundary. The total distance covered is

$$\Delta d = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2} + \sqrt{(x_2 - x_0)^2 + (y_2 - y_0)^2}.$$

We can find the time by dividing each term by the velocity of light in the respective material, $v_i = c/n_i$,

$$\Delta t = \frac{n_1}{c}\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2} + \frac{n_2}{c}\sqrt{(x_2 - x_0)^2 + (y_2 - y_0)^2}.$$

Let us orient our coordinate system so that the boundary between the two media is parallel to the y-axis and centered at (x_0, y_0), i.e. identifying the single degree of freedom as y_0. Fermat's principle suggests we should minimize the above expression, and so, upon taking its derivative with respect to y_0 and setting it equal to 0, we find

$$\frac{\mathrm{d}\Delta t}{\mathrm{d}y_0} = \frac{n_1}{c}\frac{y_0 - y_1}{\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}} - \frac{n_2}{c}\frac{y_2 - y_0}{\sqrt{(x_2 - x_0)^2 + (y_2 - y_0)^2}} = 0.$$

or, equivalently,

$$n_1\frac{y_0 - y_1}{\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}} = n_2\frac{y_2 - y_0}{\sqrt{(x_2 - x_0)^2 + (y_2 - y_0)^2}}.$$

If we denote each angle between the normal of the boundary and its closest light path endpoint (i.e. (x_1, y_1) or (x_2, y_2)) by θ_i, this becomes

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

which is precisely Snell's law.

1.2. Consider the functionals

$$H[f] = \int G(x, y)f(y)\mathrm{d}y,$$
$$I[f] = \int_{-1}^{1} f(x)\mathrm{d}x,$$
$$J[f] = \int \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}y,$$

of the function f. Find the functional derivatives

$$\frac{\delta H[f]}{\delta f(z)}, \frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)}, \frac{\delta J[f]}{\delta f(x)}.$$

A direct application of the (physicist's) definition of the functional derivative yields

$$
\begin{aligned}
\frac{\delta H[f]}{\delta f(z)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\int G(x,y)\big(f(y)+\epsilon\delta(z-y)\big)\mathrm{d}y - \int G(x,y)f(y)\mathrm{d}y\Big),\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int G(x,y)\epsilon\delta(z-y)\mathrm{d}y,\\
&= \int G(x,y)\delta(y-z)\mathrm{d}y,\\
&= G(x,z).
\end{aligned}
$$

The second case is slightly more complicated. The first functional derivative gives

$$
\begin{aligned}
\frac{\delta I[f^3]}{\delta f(x_1)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\int_{-1}^{1}\big(f(x)+\epsilon\delta(x_1-x)\big)^3\mathrm{d}x - \int_{-1}^{1} f^3(x)\mathrm{d}x\Big),\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\int_{-1}^{1} f^3(x)+3f^2(x)\epsilon\delta(x_1-x)+\mathcal{O}(\epsilon^2)\mathrm{d}x - \int_{-1}^{1} f^3(x)\mathrm{d}x\Big),\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{-1}^{1} 3f^2(x)\epsilon\delta(x_1-x)\mathrm{d}x,\\
&= \int_{-1}^{1} 3f^2(x)\delta(x_1-x)\mathrm{d}x,\\
&= 3f^2(x_1) \quad \text{for } -1\le x_1\le 1.
\end{aligned}
$$

The second application of the functional derivative gives,

$$
\begin{aligned}
\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(3\big(f(x_1)+\epsilon\delta(x_0-x_1)\big)^2 - 3f^2(x_1)\Big),\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(3f^2(x_1)+6f(x_1)\epsilon\delta(x_0-x_1)+\mathcal{O}(\epsilon^2)-3f^2(x_1)\Big),\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(6f(x_1)\epsilon\delta(x_0-x_1)\big),\\
&= 6f(x_1)\delta(x_0-x_1).
\end{aligned}
$$

For the third case, we have

$$
\begin{aligned}
\frac{\delta J[f]}{\delta f(x)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\int\Big(\frac{\partial f}{\partial y}+\epsilon\frac{\partial\delta(x-y)}{\partial y}\Big)^2\mathrm{d}y - \int\Big(\frac{\partial f}{\partial y}\Big)^2\mathrm{d}y\Big),\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\int\Big(\frac{\partial f}{\partial y}\Big)^2+2\frac{\partial f}{\partial y}\epsilon\frac{\partial\delta(x-y)}{\partial y}+\mathcal{O}(\epsilon^2)\mathrm{d}y - \int\Big(\frac{\partial f}{\partial y}\Big)^2\mathrm{d}y\Big),\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int 2\frac{\partial f}{\partial y}\epsilon\frac{\partial\delta(x-y)}{\partial y}\mathrm{d}y,\\
&= -\int 2\frac{\partial^2 f}{\partial y^2}\delta(x-y)\mathrm{d}y,\\
&= -2\frac{\partial^2 f}{\partial x^2},
\end{aligned}
$$

where we have assumed the boundary terms vanish upon integration by parts.

1.3. Consider the functional $G[f] = \int g(y, f)\mathrm{d}y$. Show that

$$\frac{\delta G[f]}{\delta f(x)} = \frac{\partial g(x,f)}{\partial f}.$$

Now consider the functional $H[f] = \int g(y, f, f')\mathrm{d}y$ and show that

$$\frac{\delta H[f]}{\delta f(x)} = \frac{\partial g}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial g}{\partial f'},$$

where $f' = \partial f/\partial y$. For the functional $J[f] = \int g(y, f, f', f'')\mathrm{d}y$ show that

$$\frac{\delta J[f]}{\delta f(x)} = \frac{\partial g}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial g}{\partial f'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial g}{\partial f''},$$

where $f'' = \partial^2 f/\partial y^2$.

Functionally differentiating G gives

$$\begin{aligned}
\frac{\delta G[f]}{\delta f(x)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\int g\big(y, f(y) + \epsilon\delta(x-y)\big)\mathrm{d}y - \int g(y,f)\mathrm{d}y\Big),\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\int g(y,f) + \frac{\partial g(y,f)}{\partial f}\epsilon\delta(x-y) + \mathcal{O}(\epsilon^2)\mathrm{d}y - \int g(y,f)\mathrm{d}y\Big),\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \frac{\partial g(y,f)}{\partial f}\epsilon\delta(x-y)\mathrm{d}y,\\
&= \int \frac{\partial g(y,f)}{\partial f}\delta(x-y)\mathrm{d}y,\\
&= \frac{\partial g(x,f)}{\partial f},
\end{aligned}$$

where we have expanded g in a Taylor series about f in the second step.

For H, the integrand's Taylor expansion takes the form

$$g(y, f(y) + \epsilon\delta(x-y), f'(y) + \epsilon\delta'(x-y)) = g(y,f) + \frac{\partial g}{\partial f}\epsilon\delta(x-y) + \frac{\partial g}{\partial f'}\epsilon\delta'(x-y) + \mathcal{O}(\epsilon^2).$$

Using this in the functional derivative of H gives

$$\begin{aligned}
\frac{\delta H[f]}{\delta f(x)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \frac{\partial g}{\partial f}\epsilon\delta(x-y) + \frac{\partial g}{\partial f'}\epsilon\delta'(x-y)\mathrm{d}y,\\
&= \int\Big(\frac{\partial g}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial g}{\partial f'}\Big)\delta(x-y)\mathrm{d}y,\\
&= \frac{\partial g}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial g}{\partial f'}.
\end{aligned}$$

To compute the functional derivative of J, we simply need to recognize that the Taylor expansion of g now has the additional term

$$\epsilon\frac{\partial g}{\partial f''}\epsilon\delta''(x-y).$$

If we integrate this by parts twice, we see it adds the term

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial g}{\partial f''},$$

to the final expression. That is, our functional derivative becomes

$$\frac{\delta J[f]}{\delta f(x)} = \frac{\partial g}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial g}{\partial f'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial g}{\partial f''}.$$

1.4. Show that

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \delta(x-y),$$

and

$$\frac{\delta\dot{\phi}(t)}{\delta\phi(t_0)} = \frac{\mathrm{d}}{\mathrm{d}t}\delta(t-t_0).$$

If we write

$$\phi(x) = \int \phi(z)\delta(x-z)\mathrm{d}z,$$

we can take the functional derivative of this to find

$$\begin{aligned}
\frac{\partial\phi(x)}{\partial\phi(y)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\int \big(\phi(z)+\epsilon\delta(y-z)\big)\delta(x-z)\mathrm{d}z - \int \phi(z)\delta(x-z)\mathrm{d}z\Big),\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \epsilon\delta(y-z)\delta(x-z)\mathrm{d}z,\\
&= \int \delta(y-z)\delta(x-z)\mathrm{d}z,\\
&= \delta(x-y).
\end{aligned}$$

Following a similar strategy, we write

$$\dot{\phi}(t) = \int \frac{\mathrm{d}\phi(t')}{\mathrm{d}t'}\delta(t-t')\mathrm{d}t',$$

in which case we have

$$\begin{aligned}
\frac{\delta\dot{\phi}(t)}{\delta\phi(t_0)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\int \frac{\mathrm{d}}{\mathrm{d}t'}\big(\phi(t')+\epsilon\delta(t_0-t')\big)\delta(t-t')\mathrm{d}t' - \int \frac{\mathrm{d}\phi(t')}{\mathrm{d}t'}\delta(t-t')\mathrm{d}t'\Big),\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \frac{\mathrm{d}}{\mathrm{d}t'}\epsilon\delta(t_0-t')\delta(t-t')\mathrm{d}t',\\
&= \int \frac{\mathrm{d}}{\mathrm{d}t'}\delta(t_0-t')\delta(t-t')\mathrm{d}t',\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\delta(t-t_0).
\end{aligned}$$

1.5. For a three-dimensional elastic medium, the potential energy is

$$V = \frac{\mathcal{T}}{2}\int \mathrm{d}^3x(\boldsymbol{\nabla}\psi)^2,$$

and the kinetic energy is

$$T = \frac{\rho}{2}\int \mathrm{d}^3x\Big(\frac{\partial\psi}{\partial t}\Big)^2.$$

Use these results, and the functional derivative approach, to show that ψ obeys the wave equation

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}.$$

The Lagrangian of the system is given by

$$L = T - V = \int \frac{\rho}{2}\Big(\frac{\partial \psi}{\partial t}\Big)^2 - \frac{\mathcal{T}}{2}(\boldsymbol{\nabla}\psi)^2 \mathrm{d}^3x.$$

We know that this must be stationary, and so we functionally differentiate and set the result equal to zero:

$$\begin{aligned}
\frac{\delta L[\psi]}{\delta \psi} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \rho\Big(\frac{\partial \psi}{\partial t}\Big)\Big(\epsilon\frac{\partial \delta(\mathbf{x}-\mathbf{x}')}{\partial t}\Big) - \mathcal{T}(\boldsymbol{\nabla}\psi)\big(\epsilon\boldsymbol{\nabla}\delta(\mathbf{x}-\mathbf{x}')\big) + \mathcal{O}(\epsilon^2)\mathrm{d}^3x', \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \Big(\mathcal{T}\nabla^2\psi - \rho\frac{\partial^2\psi}{\partial t^2}\Big)\epsilon\delta(\mathbf{x}-\mathbf{x}')\mathrm{d}^3x', \\
&= \int \Big(\mathcal{T}\nabla^2\psi - \rho\frac{\partial^2\psi}{\partial t^2}\Big)\delta(\mathbf{x}-\mathbf{x}')\mathrm{d}^3x', \\
&= \mathcal{T}\nabla^2\psi - \rho\frac{\partial^2\psi}{\partial t^2} = 0.
\end{aligned}$$

Keeping mind that $\mathcal{T}/\rho = v^2$, this can be written as

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}.$$

1.6. Show that if $Z_0[J]$ is given by

$$Z_0[J] = \exp\Big(-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y J(x)\Delta(x-y)J(y)\Big),$$

where $\Delta(x) = \Delta(-x)$ then

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = -\Big[\int \mathrm{d}^4y\Delta(z_1-y)J(y)\Big]Z_0[J].$$

A straightforward application of the functional derivative gives

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[\exp\Big(-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\big(J(x)+\epsilon\delta(z_1-x)\big)\Delta(x-y)\big(J(y)+\epsilon\delta(z_1-y)\big)\Big) - Z_0[J]\Big].$$

The exponent of the first term in brackets can be expanded to yield

$$-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\Big(J(x)\Delta(x-y)J(y) + \epsilon\delta(z_1-x)\Delta(x-y)J(y) + J(x)\Delta(x-y)\epsilon\delta(z_1-y) + \mathcal{O}(\epsilon^2)\Big),$$

which can used to rewrite the first term as a product of three exponentials,

$$Z_0[J]\exp\Big(-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\epsilon\delta(z_1-x)\Delta(x-y)J(y)\Big)\exp\Big(-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4yJ(x)\Delta(x-y)\epsilon\delta(z_1-y)\Big).$$

These can be expanded in a Taylor series in ϵ, giving

$$Z_0[J]\Big(1 - \frac{\epsilon}{2}\int \mathrm{d}^4x\mathrm{d}^4y\delta(z_1-x)\Delta(x-y)J(y) - \frac{\epsilon}{2}\int \mathrm{d}^4x\mathrm{d}^4yJ(x)\Delta(x-y)\delta(z_1-y) + \mathcal{O}(\epsilon^2)\Big).$$

Substituting this into our expression for the functional derivative gives

$$\begin{aligned}
\frac{\delta Z_0[J]}{\delta J(z_1)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon} Z_0[J]\Big(-\frac{\epsilon}{2}\int \mathrm{d}^4x\mathrm{d}^4y\delta(z_1-x)\Delta(x-y)J(y) - \frac{\epsilon}{2}\int \mathrm{d}^4x\mathrm{d}^4yJ(x)\Delta(x-y)\delta(z_1-y)\Big),\\
&= Z_0[J]\Big(-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\delta(z_1-x)\Delta(x-y)J(y) - \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4yJ(x)\Delta(x-y)\delta(z_1-y)\Big),\\
&= Z_0[J]\Big(-\frac{1}{2}\int \mathrm{d}^4y\Delta(z_1-y)J(y) - \frac{1}{2}\int \mathrm{d}^4xJ(x)\Delta(x-z_1)\Big),\\
&= Z_0[J]\Big(-\frac{1}{2}\int \mathrm{d}^4y\Delta(z_1-y)J(y) - \frac{1}{2}\int \mathrm{d}^4y\Delta(z_1-y)J(y)\Big),\\
&= -\Big[\int \mathrm{d}^4y\Delta(z_1-y)J(y)\Big]Z_0[J].
\end{aligned}$$

2 Simple Harmonic Oscillators

2.1. For the one-dimensional harmonic oscillator, show that with creation and annihilation operators defined as in eqns 2.9 and 2.10, $[\hat{a},\hat{a}]=0$, $[\hat{a}^\dagger,\hat{a}^\dagger]=0$, $[\hat{a},\hat{a}^\dagger]=1$, and $\hat{H}=\hbar\omega(\hat{a}^\dagger\hat{a}+\frac{1}{2})$.

Both $[\hat{a},\hat{a}]=0$ and $[\hat{a}^\dagger,\hat{a}^\dagger]=0$ follow from the fact that any operator commutes with itself. For $[\hat{a},\hat{a}^\dagger]$, we have

$$\begin{aligned}
[\hat{a},\hat{a}^\dagger] &= \frac{m\omega}{2\hbar}\Big[\hat{x}+\frac{i}{m\omega}\hat{p},\hat{x}-\frac{i}{m\omega}\hat{p}\Big]\\
&= \frac{m\omega}{2\hbar}\Big(-\frac{2i}{m\omega}\Big)[\hat{x},\hat{p}]\\
&= \frac{m\omega}{2\hbar}(\frac{2\hbar}{m\omega})\\
&= 1
\end{aligned}$$

To answer the last part of this question, we can solve for $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$, then substitute these into our Hamiltonian. We have

$$\hat{x}=\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger) \qquad \hat{p}=-i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}-\hat{a}^\dagger)$$

so our Hamiltonian becomes

$$\begin{aligned}
\hat{H} &= \frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2\\
&= -\Big(\frac{\hbar\omega}{4}\Big)(\hat{a}-\hat{a}^\dagger)^2+\Big(\frac{\hbar\omega}{4}\Big)(\hat{a}+\hat{a}^\dagger)^2\\
&= \Big(\frac{\hbar\omega}{2}\Big)(\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a})\\
&= \hbar\omega\Big(\hat{a}^\dagger\hat{a}+\frac{1}{2}\Big)
\end{aligned}$$

In the last step we made use of $[\hat{a},\hat{a}^\dagger]=1$, and so $\hat{a}\hat{a}^\dagger=1+\hat{a}^\dagger\hat{a}$.

2.2. For the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\hat{x}^4,$$

where λ is small, show by writing the Hamiltonian in terms of creation and annihilation operators and using perturbation theory, that the energy eigenvalues of all the levels are given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega + \frac{3\lambda}{4}\left(\frac{\hbar}{m\omega}\right)^2(2n^2 + 2n + 1).$$

Let's consider our base Hamiltonian to be the one coinciding with our simple harmonic oscillator,

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega(\hat{n} + \frac{1}{2}).$$

Our perturbation, then, is $\hat{H}_1 = \hat{x}^4 = (\frac{\hbar}{2m\omega})^2(\hat{a} + \hat{a}^\dagger)^4$. We can express our total Hamiltonian as $\hat{H} = \hat{H}_0 + \lambda\hat{H}_1$.

If λ is sufficiently small, our perturbed Hamiltonian's energy eigenstate should be expressible as a Taylor series in λ centered around our initial one. This includes both the eigenvalue *and* the eigenvector, as, being Hermitian, a different eigenvalue of our Hamiltonian necessitates a different eigenvector. Putting this all together gives

$$(\hat{H}_0 + \lambda\hat{H}_1)(|n^0\rangle + \lambda|n^1\rangle + \cdots) = (E_0 + \lambda E_1 + \cdots)(|n^0\rangle + \lambda|n^1\rangle + \cdots).$$

Multiplying out and collecting factors of λ yields

$$\hat{H}_0|n^0\rangle + \lambda(\hat{H}_0|n^1\rangle + \hat{H}_1|n^0\rangle) + \mathcal{O}(\lambda^2) = E_0|n^0\rangle + \lambda(E_0|n^1\rangle + E_1|n^0\rangle) + \mathcal{O}(\lambda^2).$$

If $\lambda = 0$, we are left with our unperturbed Hamiltonian, and so $E_0 = \hbar\omega(n + \frac{1}{2})$. To find E_1, we can focus on the first order part of our equation, as the coefficients of each power of λ on either side must be equal. That is,

$$\hat{H}_0|n^1\rangle + \hat{H}_1|n^0\rangle = E_0|n^1\rangle + E_1|n^0\rangle.$$

Now, we may assume that our initial eigenvector is normalized, $\langle n^1|n^1\rangle = 1$. Imposing a normalization constraint on our first order approximation gives

$$(\langle n^0| + \lambda\langle n^1|)(|n^0\rangle + \lambda|n^1\rangle) = 1 + \lambda(\langle n^0|n^1\rangle + \langle n^1|n^0\rangle) + \mathcal{O}(\lambda^2) = 1.$$

In other words, $\langle n^0|n^1\rangle = -\langle n^1|n^0\rangle = 0$. So we may operate on the left by $\langle n^0|$ to isolate our desired term.

$$\langle n^0|\hat{H}_1|n^0\rangle = E_1.$$

It's easiest to carry out this computation by expanding the right-most term of $\hat{H}_1$ as $(\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger)^2$. Let's compute $(\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger)|n^0\rangle$.

$$(\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger)|n^0\rangle = (\sqrt{n(n-1)}|n^0 - 2\rangle + (2n+1)|n^0\rangle + \sqrt{(n+1)(n+2)}|n^0 + 2\rangle$$

We may operate on this again with $(\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger)$, but first notice that any term of the form $|n^0 + k\rangle$ for nonzero $k \in \mathbb{Z}$ will be annihilated upon operating on the left with $\langle n^0|$, as all eigenvectors of the Hamiltonian are orthogonal. So the only contributing factors will be the first term, raised by $\hat{a}^\dagger\hat{a}^\dagger$, the second term, operated on by $\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}$, and the third term, lowered by $\hat{a}\hat{a}$. Carrying this all out yields

$$(n(n-1) + (2n+1)^2 + (n+1)(n+2))|n^0\rangle,$$

which, after operating by $\langle n^0|$ on the left and simplifying, gives $3(2n^2+2n+1)$. So

$$\langle n^0|\hat{H}_1|n^0\rangle = \frac{3}{4}\Big(\frac{\hbar}{m\omega}\Big)^2(2n^2+2n+1) = E_1.$$

Putting this all together, we see that our Hamiltonian has eigenvalues of

$$E_n = \Big(n+\frac{1}{2}\Big)\hbar\omega + \frac{3\lambda}{4}\Big(\frac{\hbar}{m\omega}\Big)^2(2n^2+2n+1).$$

2.3. Use eqns 2.46 and 2.62 to show that

$$\hat{x}_j = \frac{1}{\sqrt{N}}\Big(\frac{\hbar}{m}\Big)^{1/2}\sum_k \frac{1}{(2\omega_k)^{1/2}}[\hat{a}_k e^{ikja} + \hat{a}_k^\dagger e^{-ikja}].$$

Substituting 2.62 into 2.46 gives us

$$\hat{x}_j = \frac{1}{\sqrt{N}}\sum_k \sqrt{\frac{\hbar}{2m\omega_k}}(\hat{a}_k + \hat{a}_{-k}^\dagger)e^{ikja}.$$

By taking $\frac{\hbar}{m}$ outside the summation, distributing e^{ikja}, and reindexing the second sum, we arrive at

$$\hat{x}_j = \frac{1}{\sqrt{N}}\Big(\frac{\hbar}{m}\Big)^{\frac{1}{2}}\sum_k \frac{1}{(2\omega_k)^{\frac{1}{2}}}[\hat{a}_k e^{ikja} + \hat{a}_k^\dagger e^{-ijka}].$$

2.4. Using $\hat{a}|0\rangle = 0$ and eqns 2.9 and 2.10 together with $\langle x|\hat{p}|\psi\rangle = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\langle x|\psi\rangle$, show that

$$0 = \Big(x + \frac{\hbar}{m\omega}\frac{\mathrm{d}}{\mathrm{d}x}\Big)\langle x|0\rangle,$$

and hence

$$\langle x|0\rangle = \Big(\frac{m\omega}{\pi\hbar}\Big)^{1/4} e^{-m\omega x^2/2\hbar}.$$

As $\hat{a}$ annihilates the lowest eigenstate, we have

$$\hat{a}|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\Big(\hat{x} + \frac{i}{m\omega}\hat{p}\Big)|0\rangle = 0,$$

and so

$$\Big(\hat{x} + \frac{i}{m\omega}\hat{p}\Big)|0\rangle = 0.$$

Operating on the left by $\langle x|$ gives

$$\begin{aligned}\langle x|\Big(\hat{x}+\frac{i}{m\omega}\hat{p}\Big)|0\rangle &= x\langle x|0\rangle + \frac{i}{m\omega}\langle x|\hat{p}|0\rangle \\ &= x\langle x|0\rangle + \frac{i}{m\omega}\cdot(-i\hbar)\frac{\mathrm{d}}{\mathrm{d}x}\langle x|0\rangle \\ &= \Big(x+\frac{\hbar}{m\omega}\frac{\mathrm{d}}{\mathrm{d}x}\Big)\langle x|0\rangle \\ &= 0\end{aligned}$$

This is a linear, separable differential equation. Let's rewrite it in an easier-to-tackle form.

$$\frac{1}{\langle x|0\rangle}\frac{\mathrm{d}\langle x|0\rangle}{\mathrm{d}x} = -x\frac{m\omega}{\hbar}.$$

Integrating both sides with respect to x gives us

$$\begin{aligned}\int \frac{1}{\langle x|0\rangle}\frac{\mathrm{d}\langle x|0\rangle}{\mathrm{d}x}\mathrm{d}x &= \int -x\frac{m\omega}{\hbar}\mathrm{d}x \\ \ln|\langle x|0\rangle| &= -x^2\frac{m\omega}{2\hbar}+C \\ \langle x|0\rangle &= Ce^{-m\omega x^2/2\hbar}\end{aligned}$$

We can find C by normalizing our state,

$$\int_{-\infty}^{\infty} C^2 e^{-m\omega x^2/\hbar}\mathrm{d}x = 1.$$

This is a Gaussian integral. It has the solution

$$\int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar}\mathrm{dx} = \Big(\frac{\pi\hbar}{m\omega}\Big)^{\frac{1}{2}}.$$

Substituting this into our normalization constraint yields

$$C = \Big(\frac{m\omega}{\pi\hbar}\Big)^{\frac{1}{4}}.$$

This gives us a final solution of

$$\langle x|0\rangle = \Big(\frac{m\omega}{\pi\hbar}\Big)^{\frac{1}{4}} e^{-m\omega x^2/2\hbar}.$$