1 Introduction

1.1. Suppose that f is a C^2 function and x^* is a point of its domain at which we have $\nabla f(x^*) \cdot d \geq 0$ and $d^T \nabla^2 f(x^*) d > 0$ for every nonzero feasible direction d. Is x^* necessarily a local minimum of f? Prove or give a counterexample.

Recall that a local minimum x^* of f is defined as a point in the domain D where there exists an $\varepsilon > 0$ such that for all $x \in D$ satisfying $|x - x^*| < \varepsilon$ we have

$$f(x^*) \leq f(x).$$

Let us take the function given in the problem statement and expand it in a Taylor series about x^*,

$$f(x) = f(x^* + \alpha d) = f(x^*) + \alpha \nabla f(x^*) \cdot d + \frac{1}{2}\alpha^2 d^T \nabla^2 f(x^*) d + o(\alpha^2).$$

Here we have chosen $|d| = 1$, and so, with $x = x^* + \alpha d$, we have

$$|x - x^*| = |x^* + \alpha d - x^*| = |\alpha d| = \alpha.$$

Choose an α such that $|o(\alpha^2)| \leq \frac{1}{2}\alpha^2 d^T \nabla^2 f(x^*) d$. With this choice, $f(x)$ has the minimum value

$$f(x) \geq f(x^*) + \left(\frac{1}{2}\alpha^2 d^T \nabla^2 f(x^*) d - |o(\alpha)^2|\right)$$

which satisfies $f(x^*) < f(x)$ by virtue of the positive nature of the parenthesized term. And so, with $\varepsilon < \alpha$, x^* is necssarily a local minimum of f.