1 A Brief Review of Linear Algebra

1.1. Let $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and M be a 2-by-2 matrix. Show that the matrix EM is obtained from M by interchanging the first and second rows.

By direct calculation,

$$\begin{aligned} EM &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \\ &= \begin{pmatrix} 0 \cdot M_{11} + 1 \cdot M_{21} & 0 \cdot M_{12} + 1 \cdot M_{22} \\ 1 \cdot M_{11} + 0 \cdot M_{21} & 1 \cdot M_{12} + 0 \cdot M_{22} \end{pmatrix} \\ &= \begin{pmatrix} M_{21} & M_{22} \\ M_{11} & M_{12} \end{pmatrix} \end{aligned}$$

which is simply M with its rows interchanged.

1.2. Let $E = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$ and M be a 2-by-2 matrix. Show that the matrix EM is obtained from M by multiplying the elements in the first row by s_1 and the elements in the second row by s_2.

By direct calculation,

$$\begin{aligned} EM &= \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \\ &= \begin{pmatrix} s_1 \cdot M_{11} + 0 \cdot M_{21} & s_1 \cdot M_{12} + 0 \cdot M_{22} \\ 0 \cdot M_{11} + s_2 \cdot M_{21} & 0 \cdot M_{12} + s_2 \cdot M_{22} \end{pmatrix} \\ &= \begin{pmatrix} s_1 M_{11} & s_1 M_{12} \\ s_2 M_{21} & s_2 M_{22} \end{pmatrix} \end{aligned}$$

which is simply M with its first row multiplied by s_1 and its second row multiplied by s_2.

1.3. Let $E_1 = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ and M by a 2-by-2 matrix. Show that the matrix $E_1 M$ is obtained from M by multiplying the first row by s and adding it to the second row. Similarly for the matrix $E_2 = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$: $E_2 M$ is obtained from M by multiplying the second row by s and adding it to the first row.

By direct calculation,

$$\begin{aligned} E_1 M &= \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot M_{11} + 0 \cdot M_{21} & 1 \cdot M_{12} + 0 \cdot M_{22} \\ s \cdot M_{11} + 1 \cdot M_{21} & s \cdot M_{12} + 1 \cdot M_{22} \end{pmatrix} \\ &= \begin{pmatrix} M_{11} & M_{12} \\ sM_{11} + M_{21} & sM_{12} + M_{22} \end{pmatrix} \end{aligned}$$

which is simply M with its first row multiplied by s and added to its second row. Similarly,

$$\begin{aligned}
E_1 M &= \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \\
&= \begin{pmatrix} 1 \cdot M_{11} + s \cdot M_{21} & 1 \cdot M_{12} + s \cdot M_{22} \\ 0 \cdot M_{11} + 1 \cdot M_{21} & 0 \cdot M_{12} + 1 \cdot M_{22} \end{pmatrix} \\
&= \begin{pmatrix} sM_{21} + M_{11} & sM_{22} + M_{12} \\ M_{21} & M_{22} \end{pmatrix}
\end{aligned}$$

is just M with its second row multiplied by s and added to its first row.

1.4. The three operations effected in exercises 1-3 are known as elementary row operations. The Es defined here are called elementary matrices. How do you effect the corresponding there elementary column operations on an arbitrary 2-by-2 matrix M?

The elementary column operations are the transpose of the elementary row operations. To see this, note that we can enact column operations through

$$(EM^T)^T = (M^T)^T E^T = ME^T$$

In particular, we see that we must act on M from the right with E^T to enact column operations.

1.5. Let $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Show that multiplying a 3-by-3 matrix M from the right with e interchanges two columns in M and that multiplying M from the left with e interchanges two rows in M.

Denote $M = \begin{pmatrix} \vec{M}_1 & \vec{M}_2 & \vec{M}_3 \end{pmatrix}$, where each $\vec{M}_i$ is a column vector. Then

$$Me = \begin{pmatrix} \vec{M}_1 & \vec{0} & \vec{0} \end{pmatrix} + \begin{pmatrix} \vec{0} & \vec{0} & \vec{M}_2 \end{pmatrix} + \begin{pmatrix} \vec{0} & \vec{M}_3 & \vec{0} \end{pmatrix} = \begin{pmatrix} \vec{M}_1 & \vec{M}_3 & \vec{M}_2 \end{pmatrix},$$

i.e. postmultiplying by e exchanges two columns in M.

Now let's change our labeling so that $\vec{M}_i^T$ refers to a row in M. That is

$$M = \begin{pmatrix} \vec{M}_1^T \\ \vec{M}_2^T \\ \vec{M}_3^T \end{pmatrix}.$$

Premultiplying M by e gives

$$eM = \begin{pmatrix} \vec{M}_1^T \\ \vec{0}^T \\ \vec{0}^T \end{pmatrix} + \begin{pmatrix} \vec{0}^T \\ \vec{0}^T \\ \vec{M}_2^T \end{pmatrix} + \begin{pmatrix} \vec{0}^T \\ \vec{M}_3^T \\ \vec{0}^T \end{pmatrix} = \begin{pmatrix} \vec{M}_1^T \\ \vec{M}_3^T \\ \vec{M}_2^T \end{pmatrix},$$

i.e. it exchanges two rows in M.

1.6. Generalize the elementary row and column operations to n-by-n matrices.

To exchange two rows, simply exchange the desired two rows in the identity matrix before right-multiplying M. You can pull off a similar trick with columns: just switch the desired two columns in the identity before left-multiplying by M.

Scaling rows and columns is easily achieved by placing scaling factors on the diagonal of an empty matrix. Multiplying by M on the right scales the nth row by the nth value, while multiplying by M on the left scales the nth column correspondingly.

Now consider a 'near-identity matrix' with some off-diagonal element given by s. If this is in the E_{ij}th spot, left-multiplying by M will add s times the jth column to the ith column. By contrast, right-multiplying by M will add s times the jth row to the ith row.

1.7. Let $c = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and d be a 3-by-3 diagonal matrix. Show that the similarity transformation $c^{-1}dc$ permutes the diagonal elements of d cyclically.

To start,

$$c^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

which shows us that

$$\begin{aligned} c^{-1}dc &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & d_1 \\ d_2 & 0 & 0 \\ 0 & d_3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} d_2 & 0 & 0 \\ 0 & d_3 & 0 \\ 0 & 0 & d_1 \end{pmatrix} \end{aligned}$$

Repeating this yields

$$\begin{aligned} c^{-1}(c^{-1}dc)c &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_2 & 0 & 0 \\ 0 & d_3 & 0 \\ 0 & 0 & d_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & d_2 \\ d_3 & 0 & 0 \\ 0 & d_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} d_3 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_2 \end{pmatrix} \end{aligned}$$

and

$$
\begin{aligned}
c^{-1}(c^{-1}c^{-1}dcc)c &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_3 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & d_3 \\ d_1 & 0 & 0 \\ 0 & d_2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}
\end{aligned}
$$

As this is the matrix we began with, we see that repeatedly applying $c^{-1}dc$ yields a cycle of permutations.

1.8. Show that the determinant of antisymmetric n-by-n matrices vanishes if n is odd.

Since all antisymmetric matrices satisfy $M^T = -M$, we have $\det M^T = -\det M$ for n odd and $\det M^T = \det M$ for n even. This follows from viewing M^T as an operation that multiplies each column by -1, and hence $\det M^T = (-1)^n \det M$.

Using the fact that $\det M = \det M^T$ for all matrices, we see that an antisymmetric odd-dimensional matrix must have $\det M = -\det M$. This is only possible if $\det M$ vanishes.

1.9. Diagonalize the matrix $M = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$. Show that M is not diagonalizable if $a = b$.

The eigenvalues of M are a and b (to see this, consider the invariant nature of eigenvalues under the transpose operation), and so its eigenvectors are

$$
\eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \eta_2 = \begin{pmatrix} 1 \\ b-a \end{pmatrix}
$$

We could go through the process of performing a similarity transformation, but it's more useful to think of diagonalization as a change of basis to these eigenvectors. In that case, our matrix can be expressed as

$$
M_{\text{diag}} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},
$$

where $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to η_1 in our new basis and $x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ corresponds to η_2.

Clearly, when $a = b$, we have $\eta_1 = \eta_2$. This degeneracy makes diagonalization impossible. To see this explicitly, remember that the similarity transformation we need to perform to diagonalize our matrix is $S^{-1}MS$ where

$$
S = \begin{pmatrix} 1 & 1 \\ 0 & b-a \end{pmatrix}.
$$

When $a = b$ this has no inverse.

1.10. Write the polynomial equation satisfied by the eigenvalues of a matrix M that is diagonalizable as $\lambda^n + \sum_{k=1}^n c_k \lambda^{n-k} = 0$. Show that the matrix M satisfies the equation $M^n + \sum_{k=1}^n c_k M^{n-k} = 0$.

> It is easiest to exponentiate M after it has been diagonalized. Since $S^{-1}MS = D$, we have $M = SDS^{-1}$. Exponentiating this gives $M^k = SDS^{-1}S \cdots SDS^{-1} = SD^kS^{-1}$. Inserting this into our matrix's characteristic equation yields
>
> $$SD^nS^{-1} + \sum_{k=1}^{n} c_k SD^{n-k}S^{-1} = S(D^n + \sum_{k=1}^{n} c_k D^{n-k})S^{-1} = S \cdot 0 \cdot S^{-1} = 0,$$
>
> where the middle equality follows from the fact that each entry in D is one of M's eigenvalues, and so D^n satisfies $D^n + \sum_{k=1}^n c_k D^{n-k} = 0$.

1.11. Show by explicit computation that the eigenvalues of the traceless matrix $M = \begin{pmatrix} c & a-b \\ a+b & -c \end{pmatrix}$ with a, b, c real have the form $\lambda = \pm w$, with w either real or imaginary. Show that for $b = 0$, the eigenvalues are real. State the theorem that this result verifies.

> The characteristic equation of M is given by
>
> $$-(c+\lambda)(c-\lambda) - (a+b)(a-b) = \lambda^2 - c^2 + b^2 - a^2 = 0$$
>
> i.e. $\lambda = \pm\sqrt{a^2 + c^2 - b^2}$. This value is real when $b^2 \leq a^2 + b^2$ and imaginary otherwise. When $b = 0$, we have $\lambda = \pm\sqrt{a^2 + c^2}$. This makes sense: eigenvalues give the values by which our matrix stretches the plane. M rotates our standard Cartesian unit vectors while stretching by a in one direction and c in another. Our eigenvalues are simply the total distance stretched. We have just verified the Pythagorean theorem.

1.12. Let the matrix in exercise 11 be complex. Find its eigenvalues. When do they become real?

> Our characteristic equation does not change, and so $\lambda = \pm\sqrt{a^2 + c^2 - b^2}$. From this, it is clear that we obtain real eigenvalues when $a^2 - b^2 + c^2$ lands on the positive half of the real line.

1.13. There were of course quite a few nineteenth-century theorems about matrices. Given A and B an m-by-n matrix and an n-by-m matrix, respectively, Sylvester's theorem states that $\det(I_m + AB) = \det(I_n + BA)$. Prove this for A and B square and invertible.

> A particularly nice way to prove this involves finding two matrices M and N such that $\det MN = \det(I + AB)$ and $\det NM = \det(I + BA)$. Then the proof follows from the fact that $\det MN = \det M \det N = \det N \det M = \det NM$.
>
> Consider the matrices
>
> $$M = \begin{pmatrix} 1 & -A \\ B & 1 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix},$$
>
> where each 1 denotes a fitting identity submatrix. We have $\det MN = \det(I + AB)$ and $\det NM = \det(I + BA)$, and so $\det(I_m + AB) = \det(I_n + BA)$. As indicated in the punchline of the last sentence, this proof works for all matrices that are compatible, not just those that are square and invertible.

1.14. Show that $(M^{-1})^T = (M^T)^{-1}$ and $(M^{-1})^* = (M^*)^{-1}$. Thus, $(M^{-1})^\dagger = (M^\dagger)^{-1}$.

Starting with an identity, we see

$$\begin{aligned} MM^{-1} &= I \\ (MM^{-1})^T &= I \\ (M^{-1})^T M^T &= I \\ (M^{-1})^T &= (M^T)^{-1} \end{aligned}$$

Similarly,

$$\begin{aligned} MM^{-1} &= I \\ M^*(M^{-1})^* &= I \\ (M^{-1})^* &= (M^*)^{-1} \end{aligned}$$

As $M^\dagger = M^{*T}$, the above results show $(M^{-1})^\dagger = (M^\dagger)^{-1}$.

1.15. The n-by-n matrix M defined by $M_{ij} = x_j^{i-1}$ plays an important role in random matrix theory, for example. Show that $\det M$, known as the Vandermonde determinant, is equal to (up to an overall sign) $\prod_{i<j}(x_i - x_j)$.

Consider the Vandermonde matrix

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$

Let us diagonalize this matrix by use of elementary column operations. Subtracting the first column from all others, we see $\det M = \det A$, where A is the $(n-1)$-by-$(n-1)$ matrix given by

$$A = \begin{pmatrix} x_2 - x_1 & \cdots & x_n - x_1 \\ \vdots & \ddots & \vdots \\ x_2^{n-1} - x_1^{n-1} & \cdots & x_n^{n-1} - x_1^{n-1} \end{pmatrix}.$$

From this, it is clear that

$$A_{ij} = (x_{j+1}^i - x_1^i) = (x_{j+1} - x_1) \sum_{k=0}^{i-1} x_{j+1}^k x_1^{i-1-k}.$$

Dividing each column by its associated $(x_{j+1} - x_1)$, we find $\det M = \det A = \prod_{j=2}^n (x_j - x_1) \det B$, where

$$B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \sum_{k=0}^{1} x_2^k x_1^{1-k} & \sum_{k=0}^{1} x_3^k x_1^{1-k} & \cdots & \sum_{k=0}^{1} x_n^k x_1^{1-k} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=0}^{n-2} x_2^k x_1^{n-2-k} & \sum_{k=0}^{n-2} x_3^k x_1^{n-2-k} & \cdots & \sum_{k=0}^{n-2} x_n^k x_1^{n-2-k} \end{pmatrix}$$

Defining $a_{ij} = \sum_{k=0}^{i-1} x_{j+1}^k x_1^{i-1-k}$, we see $a_{ij} = x_{j+1}^{i-1} + x_1 \sum_{k=0}^{i-2} x_{j+1}^k x_1^{i-2-k} = x_{j+1}^{i-1} + x_1 a_{i-1,j}$. Thus we can rewrite our new matrix as

$$B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ x_2 + x_1 a_{11} & x_3 + x_1 a_{12} & \cdots & x_n + x_1 a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-2} + x_1 a_{n-2,1} & x_3^{n-2} + x_1 a_{n-2,2} & \cdots & x_n^{n-2} + x_1 a_{n-2,n-1} \end{pmatrix}.$$

Noting that $B_{ij} = a_{ij}$, we can start at the bottom of this matrix and subtract x_1 times the $(i-1)$th row from the ith row to determine B'

$$B' = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \end{pmatrix},$$

which is a $(n-1) \times (n-1)$ Vandermonde matrix with $\det B' = \det B$. We can continue in this fashion until we reach a 1×1 matrix, at which point we see the determinant of A is given by

$$\det A = \prod_{i<j} (x_i - x_j).$$

1.16. Let the Cartesian coordinates of the three vertices of a triangle be given by (x_i, y_i) for $i = 1, 2, 3$. Show that the area of a triangle is given by

$$\text{Area of triangle} = \frac{1}{2} \det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}$$

Interestingly, this expression is the beginning of the notion of projective geometry and plays a crucial role in recent development in theoretical physics (see, for example, Henriette Elvang and You-tin Huang, Scattering Amplitudes in Gauge Theory and Gravity, p. 203).

Consider the bounding box of an arbitrary triangle with one vertex being shared between the two objects. Label the coordinates of this vertex (x_1, y_1). Take the other two vertices to be (x_2, y_2) and (x_3, y_3). Obviously, these lie on the edges of the bounding box.

We will arbitrarily take the second point to be further from the first in the x-coordinate, with the third point being further from the first in the y-coordinate. The argument can be repeated for different variations.

The total area of our triangle is given by the area of its bounding box minus the area of its surrounding triangles. In our case, these quantities are given by

$$
\begin{aligned}
A &= (x_2 - x_1)(y_3 - y_1) = x_2y_3 - x_2y_1 - x_1y_3 + x_1y_1 \\
B &= \frac{1}{2}(x_3 - x_1)(y_3 - y_1) = \frac{1}{2}(x_3y_3 - x_3y_1 - x_1y_3 + x_1y_1) \\
C &= \frac{1}{2}(x_2 - x_3)(y_3 - y_2) = \frac{1}{2}(x_2y_3 - x_2y_2 - x_3y_3 + x_3y_2) \\
D &= \frac{1}{2}(x_2 - x_1)(y_2 - y_1) = \frac{1}{2}(x_2y_2 - x_2y_1 - x_1y_2 + x_1y_1)
\end{aligned}
$$

and so our triangle's area is

$$
A - B - C - D = \frac{1}{2}(x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)) = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}.
$$

2 Symmetry and Groups

2.1. The center of a group G (denoted by Z) is defined to be the set of elements $\{z_1, z_2, \cdots\}$ that commute with all elements of G, that is $z_i g = g z_i$ for all g. Show that Z is an abelian subgroup of G.

We must show that Z contains the identity, is closed under multiplication, and contains the inverses of all its elements.

Because the left inverse and right inverse are the same, we have $Ig = gI$, and so, by the definition of Z, $I \in Z$.

Assume that a product of two elements in Z produces an element not in Z. That is,

$$z_i z_j = g_l \notin Z.$$

This would imply $g_l g_k \neq g_k g_l$, or $z_i z_j g_k \neq g_k z_i z_j$. But, by definition of the elements of Z, $z_i z_j g_k = z_i g_k z_j = g_k z_i z_j$, which contradicts the previous result. We see Z is closed.

Now, assume there exists an out-of-set inverse z_i^{-1} of an element z_i. Then $z_i^{-1} g_j \neq g_j z_i^{-1}$. Right multiplying by z_i gives us

$$z_i^{-1} g_j z_i \neq g_j z_i^{-1} z_i = g_j I = g_j,$$

or $z_i^{-1} g_j z_i \neq g_j$. But, by virtue of the nature of Z,

$$z_i^{-1} g_j z_i = z_i^{-1} z_i g_j = I g_j = g_j,$$

and so our above result becomes $g_j \neq g_j$. By contradiction, Z must be abelian subgroup of G.

2.2. Let $f(g)$ be a function of the elements in a finite group G, and consider the sum $\sum_{g \in G} f(g)$. Prove the identity $\sum_{g \in G} f(g) = \sum_{g \in G} f(gg') = \sum_{g \in G} f(g'g)$ for g' an arbitrary element of G. We will need this identity again and again in chapters II.1 and II.2.

This identity holds if both gg' and $g'g$ cycle through all elements of G as g goes through all available values. Imagine creating a multiplication table for our group. Cycling through g, we see the products gg' and $g'g$ move one-by-one through a row or column. As each row and column contains every element of G, the products gg' and $g'g$ touch upon every element in the group. That

$$\sum_{g \in G} f(g) = \sum_{g \in G} f(gg') = \sum_{g \in G} f(g'g)$$

follows.

2.3. Show that $Z_2 \otimes Z_4 \neq Z_8$.

From the discussion preceding this problem set, we know that the possibility of $Z_2 \otimes Z_4$ being isomorphic to Z_8 would require 2 and 4 to be coprime (which they clearly aren't). Geometrically, we can think of Z_8 of lying on a circle, while $Z_2 \otimes Z_4$ lies on a torus. These shapes are not homeomorphic, and so it is not a surprise that our groups aren't isomorphic.

To give an explicit example of their difference, consider the elements that square to the identity. In Z_8, we have $(-1)^2 = 1$ and $(1)^2 = 1$, whereas $Z_2 \otimes Z_4$ gives us

$$(1,1)^2 = (1,-1)^2 = (-1,1)^2 = (-1,-1)^2.$$

2.4. Find all groups of order 6.

By Lagrange's theorem, all subgroups of a group of order 6 must be Z_2, Z_3, or Z_6. Clearly, if we take Z_6 to be a subgroup, then that subgroup *is* the group. What if we take Z_2 to be a subgroup? Forming the direct product of this with Z_3 is the only possible construction that gives us 6 elements: $Z_3 \otimes Z_3$ is of order 9 and $Z_2 \otimes Z_2 \otimes Z_2$ is of order 8.

To restate our findings more concisely: there are two groups of order 6, Z_6 and $Z_2 \otimes Z_3$.

3 Finite Groups

3.1. Show that for 2-cycles $(1a)(1b)(1a) = (ab)$.

We can show this by brute force by seeing where $(1a)(1b)(1a)$ takes a and b. Using arrows to represent the alterations made by successive cycles (where we multiply right to left), we see

$$a \to 1 \to b \to b \quad \text{and} \quad b \to b \to 1 \to a.$$

That is, $(1a)(1b)(1a)$ exchanges a and b, and so is equivalent to (ab).

3.2. Show that A_n for $n \geq 3$ is generated by 3-cycles, that is, any element can be written as the product of 3-cycles.

We know that all permutations may be broken into a product of 2-cycles. In the case of A_n, these products always consist of an even number of such cycles (being as they form the group of even permutations).

If there are two 2-cycles that share both numbers, i.e. there exists a term like $(ab)(ba)$, they can be removed from the product (being as they form the identity permutation). For 2-cycles sharing one element, such as $(ab)(cb)$, we can simply combine these into (abc). Lastly, 2-cycles sharing no elements may be rewritten by use of the identity,

$$(ab)(cd) = (ab)(bc)(cb)(cd) = (abc)(cdb).$$

All pairs in our 2-cycle representation of A_n have thus been converted into products of 3-cycles. A_1 is the identity and A_2 is not a group, being as there are no even permutations that make only one exchange by definition. Thus our above result holds only for $n \geq 3$.

3.3. Show that S_n is isomorphic to a subgroup of A_{n+2}. Write down explicitly how S_3 is a subgroup of A_5.

In order to show that S_n is isomorphic to a subgroup of A_{n+2}, we must produce a bijective homomorphism from one to the other.

It is clear that all even permutations within S_n can be mapped to themselves, as A_{n+2} contains these elements. For odd permutations, consider the mapping $\sigma \to \phi(\sigma) = \sigma(n+1, n+2)$. When applied to an even permutation τ, we define $\phi(\tau) = \tau$.

To check if this is a homomorphism, look at $\phi(\sigma\tau)$, where $\sigma, \tau \in S_n$. If σ and τ are both even, their product is even and we have $\phi(\sigma\tau) = \sigma\tau = \phi(\sigma)\phi(\tau)$. If they are both odd, their product is also even. In that case, we have

$$\phi(\sigma\tau) = \sigma\tau = \sigma(n+1, n+2)\tau(n+1, n+2) = \phi(\sigma)\phi(\tau).$$

In the case where one permutation is odd and the other even, their product is also odd. Their mapping becomes

$$\phi(\sigma\tau) = \sigma\tau(n+1, n+2) = \phi(\sigma)\phi(\tau).$$

Being as a homomorphism preserves the group structure, our image (contained within A_{n+2}) is a group.

Using the above, we see S_3 can be mapped to

$$\begin{aligned} I &\to I \\ (12) &\to (12)(45) \\ (13) &\to (13)(45) \\ (23) &\to (23)(45) \\ (123) &\to (123) \\ (132) &\to (132) \end{aligned}$$

where we have denoted the identity permutation by I.

3.4. List the partitions of 5. (We will need this later.)

The seven partitions of 5 are given by

$$\begin{gathered}1+1+1+1+1\\1+1+1+2\\1+2+2\\1+1+3\\2+3\\1+4\\5\end{gathered}$$

3.5. Count the number of elements with a given cycle structure.

We'll use a specific cycle structure to keep track of everything. The methods used are easily generalized to any cycle structure.

Consider the structure given on page 59,

$$(xxxxx)(xxxxx)(xxxx)(xx)(xx)(xx)(x)(x)(x)(x),$$

or $n_5 = 2$, $n_4 = 1$, $n_3 = 0$, $n_2 = 3$, and $n_1 = 4$ (with $n = 24$). How many cycles can represented using a similar structure?

There are $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20$ ways to populate the first cycle, but this is over counting by a factor of 5 (being as cycles such as (12345) and (23451) are really the same). The possibilities for the second cycle are found similarly: there are $(19 \cdot 18 \cdot 17 \cdot 16 \cdot 15)/5$ ways to fill it.

If we continued on and began counting the possible $(xxxx)$ cycles, we'd be missing another source of error: the first two cycles can be exchanged without altering our permutation. Indeed, if we continue without changing anything we'll be over counting each group of j elements by a factor of n_j! Fixing this, we see there are

$$\frac{24 \cdot 23 \cdots 15}{5^2 \cdot 2!}$$

different ways of choosing the first two cycles. We can continue this to find the expression for the total number of elements with a given cycle structure is

$$\frac{n!}{\prod_j j^{n_j} \cdot n_j!}.$$

3.6. List the possible cycle structures in S_5 and count the number of elements with each structure.

Our answer to question 4 comes in handy here, as the partitions of n are related to the cycle structures of S_n. There are seven possible cycle structures,

$$\begin{gathered}(x)(x)(x)(x)(x)\\(x)(x)(x)(xx)\\(x)(xx)(xx)\\(x)(x)(xxx)\\(xx)(xxx)\\(x)(xxxx)\\(xxxxx)\end{gathered}$$

Using the above formula, we see that these have, respectively, 1, 60, 15, 40, 20, 30, and 24 associated permutations.

3.7. Show that $\mathcal{Q}$ forms a group.

Clearly, by Hamilton's multiplication rules our set is closed. It also has the identity, 1. Each element's inverse is given by

$$\begin{aligned}1^{-1} &= 1\\-1^{-1} &= -1\\i^{-1} &= -i\\-i^{-1} &= i\\j^{-1} &= -j\\-j^{-1} &= j\\k^{-1} &= -k\\-k^{-1} &= k\end{aligned}$$

Being as $\mathcal{Q}$ obeys these three properties, it forms a group.

3.8. Show that A_4 is not simple.

To show that A_4 is not simple, we need to find an invariant (or normal) subgroup. The subgroup $Z_2 \otimes Z_2$ is given as an example of one in the text, so we need only verify this. Explicitly, this subgroup takes the form

$$\{I, (12)(34), (13)(24), (14)(23)\} = Z_2 \otimes Z_2 \subset A_4,$$

i.e. it is the identity paired with all combinations of disjoint 2-cycles. The remaining elements in A_4 take the (disjoint) form of individual 3-cycles. Obviously, $g^{-1}Ig = I$ for all $g \in A_4$, so let's concentrate on the nontrivial elements.

Label $Z_2 \otimes Z_2$'s 2-cycles by σ_i and consider a 3-cycle in A_4 (denoted by τ). We must show that

$$\tau^{-1}\sigma_i\sigma_j\tau \subset Z_2 \otimes Z_2.$$

Inserting the identity between σ's leaves us with $\tau^{-1}\sigma_i\sigma_j\tau = \tau^{-1}\sigma_i\tau\tau^{-1}\sigma_j\tau$. Consider just $\tau^{-1}\sigma_i\tau$. If $\sigma_i : j \to k$, then

$$\tau^{-1}\sigma_i\tau : \tau^{-1}(j) \to \tau^{-1}(k).$$

This is because

$$\tau^{-1}\sigma_i\tau(\tau^{-1}(j)) = \tau^{-1}\sigma_i(j) = \tau^{-1}(k).$$

Because τ is injective, τ^{-1} defines a unique mapping. The result is that $\tau^{-1}\sigma_i\tau$ is disjoint from $\tau^{-1}\sigma_j\tau$ when $i \neq j$. But the product of two disjoint 2-cycles is a defining feature of $Z_2 \otimes Z_2$, and so $Z_2 \otimes Z_2$ is normal. This, of course, implies (by definition) that A_4 is not simple.

3.9. Show that A_4 is an invariant subgroup (in fact, maximal) of S_4.

The same argument can be made as above for general cycle structures. In particular, we can think of transformations like $g^{-1}hg$ for $g \in S_4$ and $h \in A_4$ as changes of basis (or relabeling procedures). That is, g renames element i to j, which is acted upon by h, which is then taken back to its original set. The elements of A_4 remain even permutations no matter what is fed to them.

3.10. Show that the kernel of a homomorphic map of a group G into itself is an invariant subgroup of G.

There are two parts to this. Given a set $\{g \in G | \phi(g) = e\}$, we must first show that the elements g form a group. Secondly, we must show that the given group is normal.

The set given contains the identity. Consider $\phi(e) = h$ for some $h \in G$. Then

$$e = \phi(e)\phi(e)^{-1} = \phi(e)\phi(e^{-1}) = \phi(e)\phi(e) = \phi(e \cdot e) = \phi(e) = h,$$

where $\phi(g)^{-1} = \phi(g^{-1})$ can be seen from the fact that $\phi(gg^{-1}) = \phi(g)\phi(g^{-1}) = e$. Furthermore, it is closed under multiplication: given g, h in our subset,

$$\phi(gh) = \phi(g)\phi(h) = e \cdot e = e.$$

Taken together, we see our set forms a group, and so is a subgroup of G. Now consider performing a similarity transformation on it by an element $h \in G$ that's *not* in our subgroup. Is this still in our subgroup? We have

$$\phi(h^{-1}gh) = \phi(h^{-1})\phi(g)\phi(h) = \phi(h)^{-1} \cdot e \cdot \phi(h) = \phi(h)^{-1}\phi(h) = e.$$

3.11. Calculate the derived subgroup of the dihedral group.

As detailed in the text,

$$D_n = \{I, R, R^2, \cdots, R^{n-1}, r, Rr, R^2r, \cdots, R^{n-1}r\}.$$

The derived subgroup of this is given by all elements of the form $\langle a, b\rangle = a^{-1}b^{-1}ab$ where $a, b \in D_n$, as well as products of these elements.

When a, b are pure rotations, $\langle a, b\rangle = I$. This is because

$$\langle R^i, R^j\rangle = R^{n-i}R^{n-j}R^iR^j = R^{2n-i-j+i+j} = R^{2n} = I.$$

When a is a rotation and b is a rotation and reflection, we have

$$\langle R^i, R^jr\rangle = R^{n-i}rR^{n-j}R^iR^jr = R^{n-i}rR^ir = R^{2(n-i)},$$

using the fact that $rRr = R^{-1}$. Swapping the order of these, we see

$$\langle R^jr, R^i\rangle = rR^{n-j}R^{n-i}R^jrR^i = rR^{-i}rR^i = R^{2i}.$$

Finally,

$$\langle R^ir, R^jr\rangle = rR^{n-i}rR^{n-j}R^irR^jr = R^iR^{n-j}R^iR^{n-j} = R^{2(n-j+i)},$$

so all elements of the form $\langle a, b\rangle$ are rotations. Clearly, products of these elements are also rotations. So our derived subgroup is given by

$$D = \{I, R, R^2, \cdots, R^{n-1}\}.$$

3.12. Given two group elements f and g, show that, while in general $fg \neq gf$, fg is equivalent to gf (That is, they are in the same equivalence class).

We have defined our equivalence classes as objects that can be related by similarity transformations, i.e. $g' \sim g$ if $g' = h^{-1}gh$ for some $h \in G$. In the case of fg, we see

$$gf \sim g^{-1}gfg = fg.$$

3.13. Prove that groups of even order contain at least one element (which is not the identity) that squares to the identity.

3.14. Using Cayley's theorem, map V to a subgroup of S_4. List the permutation corresponding to each element of V. Do the same for Z_4.

3.15. Map a finite group G with n elements into S_n a la Cayley. The map selects n permutations, known as "regular permutations," with various special properties, out of the $n!$ possible permutations of n objects.

(a) Show that no regular permutations besides the identity leaves an object untouched.

(b) Show that each of the regular permutations takes object 1 (say) to a different object.

(c) Show that when a regular permutation is resolved into cycles, the cycles all have the same length. Verify that these properties hold for what you got in exercise 14.

3.16. In a Coxeter group, show that if $n_{ij} = 2$, then a_i and a_j commute.

3.17. Show that for an invariant subgroup H, the left coset gH is equal to the right coset Hg.

A normal subgroup H obeys $g^{-1}Hg = H$ for all $g \in G$ (where $H \subset G$). Multiplying this condition by g from the left shows

$$Hg = gH.$$

3.18. In general, a group H can be embedded as a subgroup into a larger group G in more than one way. For example, A_4 can be naturally embedded into S_6 by following the route $A_4 \subset S_4 \subset S_5 \subset S_6$. Find another way of embedding A_4 into S_6. Hint: Think geometry!

3.19. Show that the derived subgroup of S_n is A_n. (In the text, with the remark about even permutations we merely showed that it is a subgroup of S_n.)

3.20. A set of real-valued functions f_i of a real variable x can also define a group if we define multiplication as follows: given f_i and f_j, the product $f_i \cdot f_j$ is defined as the function $f_i(f_j(x))$. Show that the functions $I(x) = x$ and $A(x) = (1-x)^{-1}$ generate a three-element group. Furthermore, including the function $C(x) = x^{-1}$ generates a six-element group.

With multiplication defined this way, it is clear that $I(x)$ is the identity, as

$$I(f(x)) = f(x) \quad \text{and} \quad f(I(x)) = f(x)$$

for an arbitrary $f(x)$. So, given $I(x)$ and $A(x)$, only $A(x)$ acts meaningfully change a group element. Composing $A(x)$ with itself gives

$$A(A(x)) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{1-x-1} = \frac{x-1}{x}.$$

Denoting this by $B(x)$ and composing it with $A(x)$ gives

$$B(A(x)) = \frac{\frac{1}{1-x}-1}{\frac{1}{1-x}} = \frac{1-(1-x)}{1} = x = I(x)$$

while composing $B(x)$ with itself gives

$$B(B(x)) = \frac{\frac{x-1}{x}-1}{\frac{x-1}{x}} = \frac{x-1-x}{x-1} = \frac{1}{1-x} = A(x).$$

Finally, composing $A(x)$ with $B(x)$ results in

$$A(B(x)) = \frac{1}{1-\frac{x-1}{x}} = \frac{x}{x-(x-1)} = x = I(x)$$

The complete multiplication table is

	$I(x)$	$A(x)$	$B(x)$
$I(x)$	$I(x)$	$A(x)$	$B(x)$
$A(x)$	$A(x)$	$B(x)$	$I(x)$
$B(x)$	$B(x)$	$I(x)$	$A(x)$

and it is consistent with the properties of a 3 element group.

Let us investigate the effect of the inclusion of $C(x) = x^{-1}$ in the original set of functions. Clearly, $C(x)$ is its own inverse. Other possible compositions are

$$A(C(x)) = \frac{1}{1-\frac{1}{x}} = \frac{x}{x-1} = \frac{1}{B(x)} \equiv D(x)$$

$$B(C(x)) = \frac{\frac{1}{x}-1}{\frac{1}{x}} = 1-x = \frac{1}{A(x)} \equiv E(x)$$

$$C(A(x)) = \frac{1}{A(x)} = E(x)$$

$$C(B(x)) = \frac{1}{B(x)} = D(x)$$

All that is left is to check that the inclusion of the newly defined functions $D(x)$ and $E(x)$ leave the set closed under composition. With $D(x)$ on the left, we have

$$
\begin{aligned}
D(A(x)) &= \frac{\frac{1}{1-x}}{\frac{1}{1-x}-1} = \frac{1}{1-(1-x)} = \frac{1}{x} = C(x) \\
D(B(x)) &= \frac{\frac{x-1}{x}}{\frac{x-1}{x}-1} = \frac{x-1}{x-1-x} = 1-x = E(x) \\
D(C(x)) &= \frac{\frac{1}{x}}{\frac{1}{x}-1} = \frac{1}{1-x} = A(x) \\
D(D(x)) &= \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1} = \frac{x}{x-(x-1)} = x = I(x) \\
D(E(x)) &= \frac{1-x}{1-x-1} = \frac{x-1}{x} = B(x)
\end{aligned}
$$

While a similar analysis of $E(x)$ reveals

$$
\begin{aligned}
E(A(x)) &= 1 - \frac{1}{1-x} = \frac{x}{x-1} = D(x) \\
E(B(x)) &= 1 - \frac{x-1}{x} = \frac{1}{x} = C(x) \\
E(C(x)) &= 1 - \frac{1}{x} = \frac{x-1}{x} = B(x) \\
E(D(x)) &= 1 - \frac{x}{x-1} = \frac{1}{1-x} = A(x) \\
E(E(x)) &= 1-(1-x) = x = I(x)
\end{aligned}
$$

With $D(x)$ and $E(x)$ as the inner function, we find

$$
\begin{aligned}
A(D(x)) &= \frac{1}{1-\frac{x}{x-1}} = \frac{x-1}{x-1-x} = 1-x = E(x) \\
A(E(x)) &= \frac{1}{1-(1-x)} = \frac{1}{x} = C(x) \\
B(D(x)) &= \frac{\frac{x}{x-1}-1}{\frac{x}{x-1}} = \frac{x-(x-1)}{x} = \frac{1}{x} = C(x) \\
B(E(x)) &= \frac{(1-x)-1}{1-x} = \frac{x}{x-1} = D(x) \\
C(D(x)) &= \frac{1}{\frac{x}{x-1}} = B(x) \\
C(E(x)) &= \frac{1}{1-x} = A(x)
\end{aligned}
$$

The complete multiplication table of this new group is

	$I(x)$	$A(x)$	$B(x)$	$C(x)$	$D(x)$	$E(x)$
$I(x)$	$I(x)$	$A(x)$	$B(x)$	$C(x)$	$D(x)$	$E(x)$
$A(x)$	$A(x)$	$B(x)$	$I(x)$	$D(x)$	$E(x)$	$C(x)$
$B(x)$	$B(x)$	$I(x)$	$A(x)$	$E(x)$	$C(x)$	$D(x)$
$C(x)$	$C(x)$	$E(x)$	$D(x)$	$I(x)$	$B(x)$	$A(x)$
$D(x)$	$D(x)$	$C(x)$	$E(x)$	$A(x)$	$I(x)$	$B(x)$
$E(x)$	$E(x)$	$D(x)$	$C(x)$	$B(x)$	$A(x)$	$I(x)$

4 Rotations and the Notion of Lie Algebra

4.1. Suppose you are given two vectors $\overrightarrow{p}$ and $\overrightarrow{q}$ in ordinary 3-dimensional space. Consider this array of three numbers:

$$\begin{pmatrix} p^2q^3 \\ p^3q^1 \\ p^1q^2 \end{pmatrix}$$

Prove that it is not a vector, even though it looks like a vector. (Check how it transforms under rotation!) In contrast,

$$\begin{pmatrix} p^2q^3 - p^3q^2 \\ p^3q^1 - p^1q^3 \\ p^1q^2 - p^2q^1 \end{pmatrix}$$

does transform like a vector. It is in fact the vector cross product $\overrightarrow{p} \otimes \overrightarrow{q}$.

To prove that this is not a vector, we merely need to find one rotation for which it does not transform like a vector. We will take the test rotation to be about the x-axis,

$$R_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix}.$$

Under this transformation, the vectors $\overrightarrow{p}$ and $\overrightarrow{q}$ transform as

$$\overrightarrow{p} = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix} \to \begin{pmatrix} p^1 \\ p^2\cos\theta_x + p^3\sin\theta_x \\ -p^2\sin\theta_x + p^3\cos\theta_x \end{pmatrix}$$
$$\overrightarrow{q} = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix} \to \begin{pmatrix} q^1 \\ q^2\cos\theta_x + q^3\sin\theta_x \\ -q^2\sin\theta_x + q^3\cos\theta_x \end{pmatrix}$$

and so the given array of numbers transform as

$$\begin{aligned} \begin{pmatrix} p^2q^3 \\ p^3q^1 \\ p^1q^2 \end{pmatrix} &\to \begin{pmatrix} (p^2\cos\theta_x + p^3\sin\theta_x)(-q^2\sin\theta_x + q^3\cos\theta_x) \\ (-p^2\sin\theta_x + p^3\cos\theta_x)q^1 \\ p^1(q^2\cos\theta_x + q^3\sin\theta_x) \end{pmatrix} \\ &= \begin{pmatrix} p^2q^3\cos^2\theta_x + (p^3q^3 - p^2q^2)\cos\theta_x\sin\theta_x - p^3q^2\sin^2\theta_x \\ -p^2q^1\sin\theta_x + p^3q^1\cos\theta_x \\ p^1q^2\cos\theta_x + p^1q^3\sin\theta_x \end{pmatrix} \end{aligned}$$

This is clearly not the transformation law characterizing vectors, and so the given array is not a vector.

By contrast, if we compute the transformation of a similar array,

$$\begin{pmatrix} p^3q^2 \\ p^1q^3 \\ p^2q^1 \end{pmatrix} \to \begin{pmatrix} (-p^2\sin\theta_x + p^3\cos\theta_x)(q^2\cos\theta_x + q^3\sin\theta_x) \\ p^1(-q^2\sin\theta_x + q^3\cos\theta_x) \\ (p^2\cos\theta_x + p^3\sin\theta_x)q^1 \end{pmatrix}$$
$$= \begin{pmatrix} p^3q^2\cos^2\theta_x + (p^3q^3 - p^2q^2)\cos\theta_x\sin\theta_x - p^2q^3\sin^2\theta_x \\ -p^1q^2\sin\theta_x + p^1q^3\cos\theta_x \\ p^2q^1\cos\theta_x + p^3q^1\sin\theta_x \end{pmatrix}$$

we can subtract this from the first array to find that their combination does transform as a vector,

$$\begin{pmatrix} p^2q^3 - p^3q^2 \\ p^3q^1 - p^1q^3 \\ p^1q^2 - p^2q^1 \end{pmatrix} \to \begin{pmatrix} p^2q^3 - p^3q^2 \\ (p^3q^1 - p^1q^3)\cos\theta_x + (p^3q^1 - p^1q^3)\sin\theta_x \\ -(p^1q^2 - p^2q^1)\sin\theta_x + (p^1q^2 - p^2q^1)\cos\theta_x \end{pmatrix}$$

4.2. Verify that $R \simeq I + A$, with A given by $A = \theta_x\mathcal{J}_x + \theta_y\mathcal{J}_y + \theta_z\mathcal{J}_z$, satisfies the condition $\det R = 1$.

We assume that A is infinitesimal, as this is not true otherwise. Let us make this explicit by redefining $R = I + \epsilon A$ and denote the eigenvalues of A by λ_i. Then the eigenvalues of R are given by $1 + \epsilon\lambda_i$ and the determinant is

$$\begin{aligned} \det(R) = \det(I + A) &= (1+\epsilon\lambda_1)(1+\epsilon\lambda_2)(1+\epsilon\lambda_3) \\ &= 1 + \epsilon(\lambda_1 + \lambda_2 + \lambda_3) + \mathcal{O}(\epsilon^2) \\ &= 1 + \epsilon\mathrm{Tr}(A) + \mathcal{O}(\epsilon^2) \end{aligned}$$

Since A is traceless (being the sum of the traceless generating matrices), this becomes

$$\det(R) = 1.$$

4.3. Using (14), show that a rotation around the x-axis through angle θ_x is given by

$$R_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix}$$

Write down $R_y(\theta_y)$. Show explicitly that $R_x(\theta_x)R_y(\theta_y) \neq R_y(\theta_y)R_x(\theta_x)$.

To go from the Lie algebra of $SO(3)$ to the Lie group element $R_x(\theta_x)$, group theory tells us to exponentiate the associated element of the former by θ_x. Before doing so, it will be useful to calculate

$$\mathcal{J}_x^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \equiv -I_x.$$

Note that $\mathcal{J}_x I_x = I_x \mathcal{J}_x = \mathcal{J}_x$. Exponentiating J_x through an angle θ_x yields

$$\begin{aligned}
R_x(\theta_x) &= \exp(\theta_x \mathcal{J}_x) \\
&= 1 + \theta_x \mathcal{J}_x + \frac{1}{2!}(\theta_x \mathcal{J}_x)^2 + \frac{1}{3!}(\theta_x \mathcal{J}_x)^3 + \frac{1}{4!}(\theta_x \mathcal{J}_x)^4 + \frac{1}{5!}(\theta_x \mathcal{J}_x)^5 + \cdots \\
&= 1 + \theta_x \mathcal{J}_x - \frac{1}{2!}\theta_x^2 I_x - \frac{1}{3!}\theta_x^3 \mathcal{J}_x + \frac{1}{4!}\theta_x^4 I_x + \frac{1}{5!}\theta_x^5 \mathcal{J}_x + \cdots \\
&= (I - I_x) + \left(1 - \frac{1}{2!}\theta_x^2 + \frac{1}{4!}\theta_x^4 - \cdots\right) I_x + \left(\theta_x - \frac{1}{3!}\theta_x^3 + \frac{1}{5!}\theta_x^5 - \cdots\right)\mathcal{J}_x \\
&= (I - I_x) + \cos\theta_x I_x + \sin\theta_x \mathcal{J}_x
\end{aligned}$$

or, in component form,

$$R_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix}$$

Following the same procedure for the y-axis gives

$$R_y(\theta_y) = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix}$$

To check their commutation, we compute

$$\begin{aligned}
R_x(\theta_x)R_y(\theta_y) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ \sin\theta_x \sin\theta_y & \cos\theta_x & \sin\theta_x \cos\theta_y \\ \cos\theta_x \sin\theta_y & -\sin\theta_x & \cos\theta_x \cos\theta_y \end{pmatrix} \\
R_y(\theta_y)R_x(\theta_x) &= \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta_y & \sin\theta_y \sin\theta_x & -\sin\theta_y \cos\theta_x \\ 0 & \cos\theta_x & \sin\theta_x \\ \sin\theta_y & -\cos\theta_y \sin\theta_x & \cos\theta_y \cos\theta_x \end{pmatrix}
\end{aligned}$$

Clearly, $R_x(\theta_x)R_y(\theta_y) \neq R_y(\theta_y)R_x(\theta_x)$.

4.4. Use the hermiticity of J to show that c_{ijk} in (18) are real numbers.

Since each J_i is hermitian, we have

$$\begin{aligned}
[J_i, J_j]^\dagger &= (J_i J_j - J_j J_i)^\dagger \\
&= J_j^\dagger J_i^\dagger - J_i^\dagger J_j^\dagger \\
&= J_j J_i - J_i J_j \\
&= [J_j, J_i] \\
&= -[J_i, J_j] \\
&= -i c_{ijk} J_k
\end{aligned}$$

Simultaneously, (18) implies

$$\begin{aligned}
[J_i, J_j]^\dagger &= (i c_{ijk} J_k)^\dagger \\
&= -i c_{ijk}^\dagger J_k^\dagger \\
&= -i c_{ijk}^\dagger J_k
\end{aligned}$$

Subtracting one from the other gives

$$0 = -i(c_{ijk} - c_{ijk}^\dagger) J_k.$$

Since J_k is not zero, this implies

$$c_{ijk} = c_{ijk}^\dagger,$$

i.e. $c_{ijk} \in \mathbb{R}$.

4.5. Calculate $[J_{(mn)}, J_{(pq)}]$ by brute force using (24).

4.6. Of the six 4-by-4 matrices J_{12}, J_{23}, J_{31}, J_{14}, J_{24}, J_{34} that generate $SO(4)$, what is the maximum number that can be simultaneously diagonalized?

From eq. 25, we see that

$$[J_{(mn)}, J_{(pq)}] = 0$$

if and only if mn has no indices in common with pq. So we could choose to simultaneously diagonalize either $J_{(12)}$ and $J_{(34)}$, $J_{(13)}$ and $J_{(24)}$, or $J_{(14)}$ and $J_{(23)}$. After diagonalizing such a set we can go no further, as all other matrices in this representation of $SO(4)$ do not commute with one of the diagonalized matrices. So the maximum number of simultaneously diagonalizable matrices is 2.

4.7. Verify (31).

We first compute $e^{i\phi J_3}$ and $e^{-i\phi J_3}$, noting that

$$J_3^2 = J_{12}^2 = (-i)^2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv I_{12}$$

and so

$$
\begin{aligned}
e^{i\phi J_3} &= I + i\phi J_3 + \frac{1}{2!}(i\phi J_3)^2 + \frac{1}{3!}(i\phi J_3)^3 + \frac{1}{4!}(i\phi J_3)^4 + \frac{1}{5!}(i\phi J_3)^5 + \cdots \\
&= I + i\phi J_3 - \frac{1}{2!}\phi^2 I_{12} - \frac{1}{3!}i\phi^3 J_3 + \frac{1}{4!}\phi^4 I_{12} + \frac{1}{5!}i\phi^5 J_3 - \cdots \\
&= (I - I_{12}) + \Big(1 - \frac{1}{2!}\phi^2 + \frac{1}{4!}\phi^4 - \cdots\Big) I_{12} + i\Big(\phi - \frac{1}{3!}\phi^3 + \frac{1}{5!}\phi^5 - \cdots\Big) J_3 \\
&= (I - I_{12}) + \cos\phi I_{12} + i\sin\phi J_3
\end{aligned}
$$

We can find $e^{-i\phi J_3}$ by exchanging $\phi \to -\phi$,

$$
e^{-i\phi J_3} = (I - I_{12}) + \cos\phi I_{12} - i\sin\phi J_3.
$$

In component form, these become

$$
e^{i\phi J_3} = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad e^{-i\phi J_3} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Altogether, this gives

$$
\begin{aligned}
e^{-i\phi J_3} K_1 e^{i\phi J_3} &= \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\cos\phi & i\sin\phi & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & -i\cos\phi \\ 0 & 0 & 0 & -i\sin\phi \\ 0 & 0 & 0 & 0 \\ i\cos\phi & i\sin\phi & 0 & 0 \end{pmatrix} \\
&= -i \begin{pmatrix} 0 & 0 & 0 & \cos\phi \\ 0 & 0 & 0 & \sin\phi \\ 0 & 0 & 0 & 0 \\ -\cos\phi & -\sin\phi & 0 & 0 \end{pmatrix} \\
&= \cos\phi \times -i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \sin\phi \times -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\
&= \cos\phi K_1 + \sin\phi K_2
\end{aligned}
$$

5 Representation Theory

5.1. Show that the identity is in a class by itself.

We know that two elements g and g' are in the same class when

$$g' = f^{-1}gf.$$

In the case of $g = e$, any arbitrary f will result in $g' = e$, as

$$g' = f^{-1}ef = f^{-1}f = e,$$

i.e. the identity element is the only member of its equivalence class.

5.2. Show that in an abelian group, every element is in a class by itself.

In an abelian group, the condition for two elements to belong to the same equivalence class becomes

$$g' = f^{-1}gf = f^{-1}fg = eg = g,$$

i.e. each element belongs to a unique equivalence class.

5.3. These days, it is easy to generate finite groups at will. Start with a list consisting of a few invertible d-by-d matrices and their inverses. Generate a new list by adding to the old list all possible pairwise products of these matrices. Repeat. Stop when no new matrices appear. Write such a program. (In fact, a student did write such a program for me once.) The problem is of course that you can't predict when the process will (or if it will ever) end. But if it does end, you've got yourself a finite group together with a d-dimensional representation.

5.4. In chapter I.2, we worked out the equivalence classes of S_4. Calculate the characters of the 4-dimensional representation of S_4 as a function of its classes.

Given that character is a function of class, we need only find a representation of one element from each class. By inspection, we can write

$$I \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(12)(34) \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$(123) \to \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(132) \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where we immediately see

$$\begin{aligned} \chi^{(4)}(I) &= 4 \\ \chi^{(4)}\big((12)(34)\big) &= 0 \\ \chi^{(4)}\big((123)\big) = \chi^{(4)}\big((132)\big) &= 1 \end{aligned}$$

6 Schur's Lemma and the Great Orthogonality Theorem

6.1. Show that in the 3-dimensional vector space, the three vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}1\\ \omega\\ \omega^*\end{pmatrix}$, $\begin{pmatrix}1\\ \omega^*\\ \omega\end{pmatrix}$ (where $\omega = e^{j2\pi/3}$) are orthogonal to one another. Furthermore, a vector $\begin{pmatrix}u\\v\\w\end{pmatrix}$ orthogonal to all three must vanish. Prove that in d-dimensional complex vector space there can be at most d mutually orthogonal vectors.

6.2. Determine the multiplication table of the class algebra for $D_5 = C_{5v}$.

6.3. Show that $f(c, d, I) = \delta_{\bar{c}d}/n_c$.

6.4. Show that $f(c, d, e) = f(\bar{c}, \bar{d}, \bar{e})$.

6.5. Prove (30).

6.6. Use Schur's lemma to prove the almost self-evident fact that all irreducible representations of an abelian group are 1-dimensional.